Non–dangerous risky investments for insurance companies

Manfred Schäl

Inst. Angew. Math., Univ. Bonn, D–53115 Bonn, Wegelerstr. 6, (e–mail: schael@uni–bonn.de)

Key words. ruin probability, investment, financial market, stochastic dynamic programming

The problem of controlling ruin probabilities is studied in a **Cramér–Lundberg model** where the claim process is described by a compound Poisson process with *claim size* Y_n at claim time T'_n. The number N'_t of claims in $(0,t]$ is a Poisson process with *claim intensity* λ. There is a *premium (income) rate* c which is fixed.

The insurance company can invest the capital (surplus/risk reserve) in a financial market where 2 assets can be traded. One of them is called the bond and is described by the interest rate which here is assumed w.l.o.g. to be zero.

The other asset is called stock. It is described by a 1–dimensional *price process* $\{S_n, n \geq 1\}$ where S_n is the price of one share of the stock at time T''_n. The price process will be driven by a compound Poisson process which can be defined by the sequence $\{T''_n, n\geq 1\}$ of market jump times and the sequence of returns $\{R_n, n\geq 1\}$, where $1 + R_n > 0$ and

(1) $S_n = S_{n-1} \cdot (1 + R_n)$.

We write N''_t for the number of market jumps in $(0,t]$ where $\{N''_t\}$ is a Poisson process with *market intensity* ν. In general, ν will be much larger than λ.

Thus the price process is driven by a Lévy process as in the Black–Scholes model. However, a compound Poisson process is chosen in place of a Wiener process. Moreover, only a moment condition is assumed for the distribution of R_n. Thus, the model for the financial market is quite general and flexible. The main advantage of the Black–Scholes model is the completeness of the financial market. But this property is not needed in the present control problem.

We define the Poisson process $\{N_t\}$ by **superposition**:

(2) $N_t := N'_t + N''_t$ is a Poisson process with parameter $\lambda + \nu$ and jump times T_n, $n \geq 1$.

We write $K_n = 1$ if the jump at T_n is caused by the financial market and $K_n = 0$ if the jump is caused by a claim. Then we make the following assumption:

Model Assumption:

All random variables $Z_n := T_n - T_{n-1}$, Y_n, R_n, K_n, $n\geq 1$, *are independent. The* (Z_n) *are iid and have an exponential distribution with parameter* $\lambda+\nu$; *the* (Y_n) *are iid and positive; the* (R_n) *are iid as well as the* (K_n). *We assume*

$$P[R_n < 0] > 0,\ E[R_n] > 0 \ \textit{and}\ E[R_n^2] < \infty,$$

$$(3)\qquad P[K_n = 1] = \frac{\nu}{\lambda+\nu} = 1 - P[K_n = 0],$$

$$q := \frac{\lambda}{c} \cdot E[Y_n] < 1 \ \textit{for the classical ruin probability } q \textit{ with start in } 0.$$

In the **joint model of insurance and finance**, $\{T_n, n \geq 0\}$ are the decision times and the real–valued discrete–time process $\{X_n, n \geq 0\}$ describes the *risk process* (surplus process) immediately after time T_n. A dynamic portfolio specifies a portfolio $\theta_n \in \mathbb{R}$ at any time T_n. There θ_n represents the amount of capital invested in the stock. We have the following law of motion:

$$
\begin{aligned}
(4) \qquad & X_{n+1} = X_n + c \cdot Z_{n+1} + \theta_n \cdot R_{n+1} && \text{for } K_{n+1} = 1,\ X_n \geq 0. \\
& X_{n+1} = X_n + c \cdot Z_{n+1} - Y_{n+1} && \text{for } K_{n+1} = 0,\ X_n \geq 0.
\end{aligned}
$$

For technical reasons we set $X_{n+1} = -\infty$ for $X_n < 0$. Once in state $x \in (-\infty,0)$, the system moves to the absorbing state $-\infty$ in the next step.

We write $\Theta(x)$ for the set of all portfolios θ *admissible* at x which is assumed to be

$$\Theta(x) = [0,x]\ ,\ x \geq 0.$$

We define $\Theta(x) := \{0\}$ for $x < 0$. For $\theta \in \Theta(x)$, $x - \theta$ represents the amount of the capital which is invested in the bond, i.e., which is not invested in the stock. In this model we do not allow for negative amounts θ_n, thus excluding short selling of the stock.

A number $x < 0$ represents a state of ruin. A *stationary* (*investment*) *plan* is a measurable function φ such that $\varphi(x) \subset \Theta(x)$ for all x. Then $\varphi(X_n)$ specifies the portfolio $\theta_n \in \Theta(X_n)$ for the period $(T_n,T_{n+1}]$. We will sometimes write for the state (risk) process

$$(5) \qquad X_n = X_n^{x,\varphi}\ .$$

Our performance criterion is the *ruin probability*:

$$(6) \qquad \psi(x,\varphi) := P[X_n^{x,\varphi} < 0 \ \text{ for some } n]\ .$$

At first view, the ruin probability is not a classical performance criterion for control problems. However, one can write the ruin probability as some total cost in an embedded discrete–stage model where one has to pay one unit of cost when entering the ruin state. After this simple observation, results from discrete–time dynamic programming for minimizing costs apply.

Lundberg inequalities will be derived for the controlled model which extend the classical inequalities for the uncontrolled model.

The present paper is related to Gaier, Grandits & Schachermayer (2003) where a continuous–time control model is studied for a Black–Scholes market by different methods.

If $\psi(x)$ is the classical ruin probability for an initial reserve x, then $\psi(x) = \psi(x,\varphi_0)$ where $\varphi_0(x) \equiv 0$. If $r_o > 0$ is the classical Lundberg exponent then a classical theorem says

$$0 < C_o \cdot e^{-r_o \cdot x} \leq \psi(x) = \psi(x,\varphi_0) \leq e^{-r_o \cdot x} \quad \text{for some constant } C_o > 0.$$

Now let $\tilde{\varphi}$ be a stationary investment plan such that the decision maker invests a constant fraction γ of capital at any (decision) time, i.e. $\tilde{\varphi}(x) = \gamma \cdot x$, then it was shown by Paulsen & Gjessing (1997) and Frovola, Kabanov & Pergamenshchikov (2002) that the asymptotic behaviour of the ruin probability is completely different under the investment plan $\tilde{\varphi}$. In fact, in the latter case the ruin probability has a polynomial decay (as function of the initial reserve) even if the financial market is described by a stock price process with high returns. Therefore, this plan is called **dangerous** in Frovola, Kabanov & Pergamenshchikov (2002).

However, if $\hat{\varphi}$ is an investment plan such that a constant amount A is invested in the stock independently of the current risk reserve, i.e. $\hat{\varphi}(x) \equiv A$, then one can find some $\hat{A}$ and some exponent $\hat{r} > r_o$ such that

(7) $\psi(x,\hat{\varphi}) \le e^{-\hat{r}\cdot x}$

(8) $\psi(x,\varphi) \ge \hat{C}\cdot e^{-\hat{r}\cdot x}$ for some $0 < \hat{C} \le 1$ and for every investment plan φ

(see Gaier, Grandits & Schachermayer 2003, Schäl 2005). A plan with (7) for some $\hat{r} > r_o$ will be call *profitable*. The plan $\hat{\varphi}$ may however be not admissible when the insurance company is poor, since $\hat{\varphi}(x) = A \notin \Theta(x)$ for $x < A$. In the present paper we study stationary investment plans φ^* such that

(9) $\varphi^*(x) = \min(x, A)$ for some $A > 0$.

Such a plan is **admissible**.

THEOREM *There exist a plan* φ^* *of the form* (9) *and some* $r^* > r_o$ *such that*

(9) $\psi(x,\varphi^*) \le e^{-r^*\cdot x}$.

Then φ^* *is admissible and profitable.*

Of course in view of (8), we have $r^* \le \hat{r}$ in (10).

THEOREM *One even can choose* $r^* = \hat{r}$ *if the market intensity* ν *is high.*

If ν is high which is a natural condition, then the model is close to a continuous–time model.

'PARADOXON' Assume (i) $Y_n \sim E_2$, i.e. the claims have an Erlang distribution;
(ii) the price process is a martingale, i.e. $E[R_n] = 0$; (iii) ν is large.
Then there exist some stationary plan φ such that $\varphi(x) = x$ for small values x and

$$\psi(x,\varphi) \ge \psi(x,\varphi_0) = \psi(x) \text{ and } \psi(x,\varphi) > \psi(x,\varphi_0) = \psi(x) \text{ for small x.}$$

Hence, if the system is close to ruin, it may be good to invest all the capital in a martingale.

The paper builds on methods from discrete–time control / stochastic dynamic programming / Markov decision processes.

References

Frovola A G, Kabanov Yu M, Pergamenshchikov S M (2002). Finance and Stochastics 6: 227–235

Gaier J, Grandits P, Schachermeyer W (2003). Annals of Applied Probability 13: 1054–1076

Paulsen J, Gjessing H K (1997). Advance in Applied Probability 29: 965–985

Schäl M (2004). Scand. Actuarial J. 3: 189–210

Schäl M (2005). Math.Meth.Oper.Res. 62: 141–158

Schmidli H (2002). Annals of Applied Probability 12: 890–907

Not dangerous risky investments in the insurance business

Manfred Schäl
Inst. Angew. Math.
Univ. Bonn

0

Control of ruin probabilities
by discrete-time investments

Manfred Schäl

Inst. Angew. Math.
Univ. Bonn

papers in continuous time:

[Browne, S. (1995)]

Gaier, J., Grandits, P., Schachermayer, W. (2003)

Hipp, C., Plum, M. (2000)

Hipp, C., Schmidli, H. (2003)

Schmidli, H. (2001)

Paulsen & Gjessing (1997/98)

Frovola, Kabanov & Pergamenshchikov (2002)

Fernández et al. (2008)

General framework for financ. market

$S_t = S_0 \, e^{L_t}$ stock price

$\{L_t\}$ Lévy process

Ex 1 **Black-Scholes**

$$L_t = \sigma \cdot W_t + (\mu - \tfrac{1}{2}\sigma^2) \cdot t$$

$$\Longrightarrow S_t^{BS} = S_0 \cdot \mathcal{E}(\sigma W_t + \mu t)$$

Ex 2 **compound Poisson**

$$L_t = \sum_{n=1}^{N_t} B_n$$

$\{N_t\}$ Poisson with rate ν

$$\Rightarrow S_t^{\nu} = S_0 \cdot \prod_{n=1}^{N_t} (1 + R_n)$$

$$= S_0 \, \mathcal{E}\left(\sum_{n=1}^{N_t} R_n\right)$$

weak convergence

$$\{S_t^{\nu}\} \longrightarrow \{S_t^{BS}\} \quad (\nu \to \infty)$$

iff $\left\{\sum_{n=1}^{N_t} B_n^{\nu}\right\} \to \left\{\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2) \cdot t\right\}$

"$\Longleftrightarrow$" $\left\{\sum_{n=1}^{N_t} R_n^{\nu}\right\} \longrightarrow \left\{\sigma W_t + \mu t\right\}$

Necessary conditions for convergence

$$\sum_{n=1}^{N_t} B_n^\nu = M_t^B + A_t^B$$

$$\sum_{n=1}^{N_t} R_n^\nu = M_t^R + A_t^R$$

semi-martingale decomp.

$$A_t^B = \nu \cdot E[B^\nu] \cdot t \xrightarrow[\nu \to \infty]{} (\mu - \tfrac{1}{2}\sigma^2) \cdot t$$

$$\langle M^B \rangle_t = \nu \cdot E[(B^\nu)^2] \cdot t \xrightarrow[\nu \to \infty]{} \sigma^2 \cdot t$$

$$A_t^R = \nu \cdot E[R^\nu] \cdot t \longrightarrow \mu \cdot t$$

$$\langle M^R \rangle_t = \nu \cdot E[(R^\nu)^2] \cdot t \longrightarrow \sigma^2 \cdot t$$

Example Dengler (1996) Cornell Diss.

$\left.\begin{array}{l}\text{Prigent}\\ \text{Renault}\\ \text{Scaillet}\end{array}\right\}$ (2004) J. Empirical Fin.

$$P(B_n^\nu = b^\nu) = p^\nu \quad , \quad P(B_n^\nu = -b^\nu) = 1 - p^\nu$$

$$\nu \cdot b^2 := \sigma^2$$

$$(2p-1)\,\nu \cdot b := \mu - \tfrac{1}{2}\sigma^2$$

also sufficient

Laplace transform

$$\ell(s) := E[e^{-sR}]$$

$$\nu \cdot [1 - \ell(s)]$$

$$= s \cdot \underbrace{\nu \cdot E[R]}_{\downarrow \; \mu} - \frac{1}{2} s^2 \cdot \underbrace{\nu \cdot E[R^2]}_{\downarrow \; \sigma^2} + o(1)$$

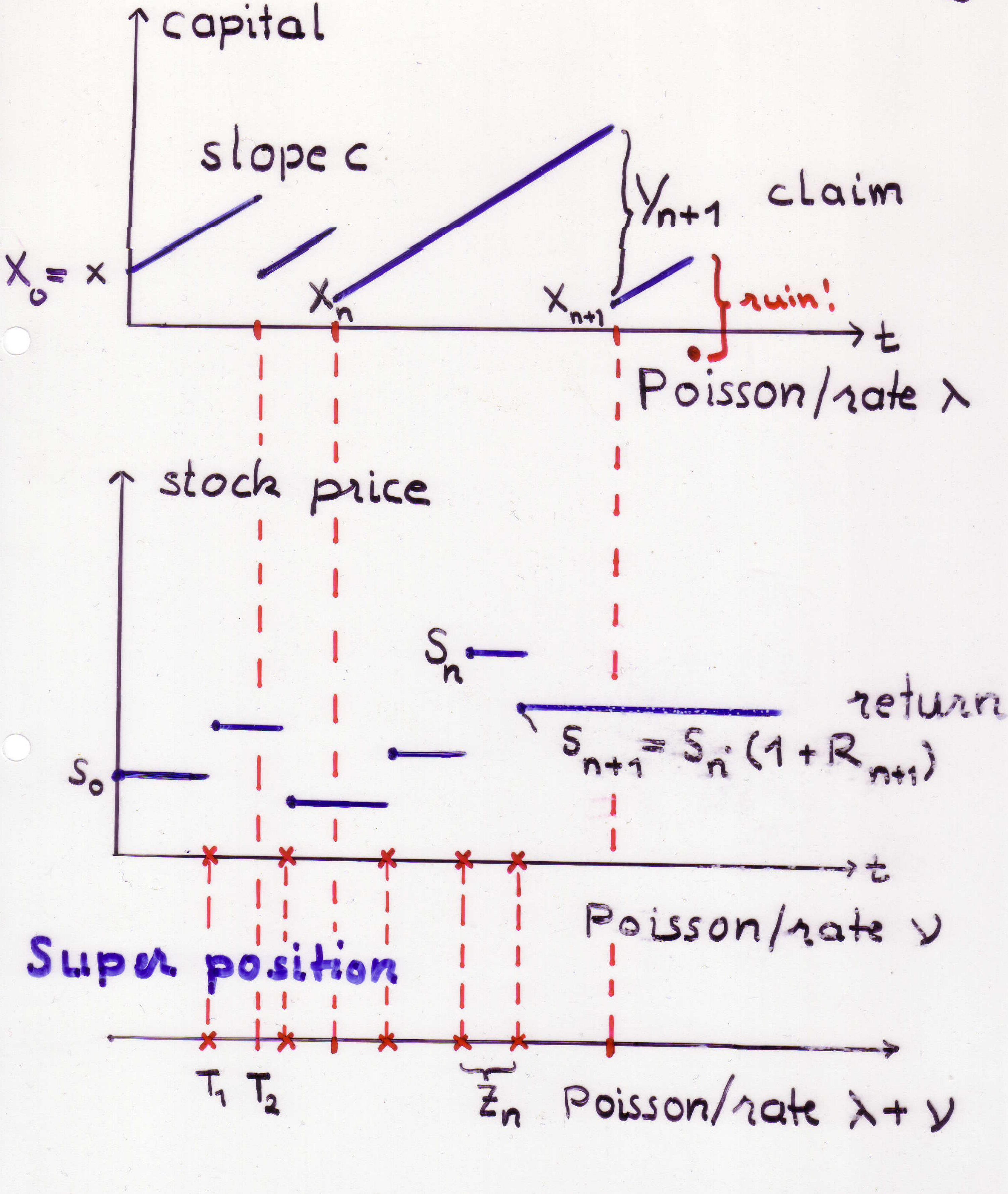
$$\xrightarrow[\nu \to \infty]{} s \cdot \mu - \frac{1}{2} s^2 \sigma^2$$

$$s^* := \arg\max_s \nu \cdot [1 - \ell(s)] \longrightarrow \frac{\mu}{\sigma^2} \geq 0$$

$$\Rightarrow \Delta = \sup_{s \geq 0} \nu \cdot [1 - \ell(s)] \longrightarrow \frac{1}{2} \left(\frac{\mu}{\sigma}\right)^2$$

$\frac{\mu}{\sigma}$ market price of risk

Cramér-Lundberg



$$P[K_n = 1] = \frac{\nu}{\lambda+\nu} = 1 - P[K_n = 0]$$

X_n capital at T_n

θ_n amount of capital invested

$$X_{n+1} = X_n + c \cdot Z_{n+1} + \theta_n \cdot R_{n+1} \cdot K_{n+1} - Y_{n+1} \cdot (1 - K_{n+1})$$

if $X_n \geq 0$

admissible actions θ for x

$$\Theta(x) = \{ 0 \leq \theta \leq \cancel{\alpha} \cdot x \}$$

plan / control

$$\theta_n = \varphi_n(X_n) \in \Theta(X_n)$$

$$\pi = (\varphi_0, \varphi_1, \varphi_2, \dots)$$

stationary

$$\pi = (\varphi, \varphi, \dots) =: \varphi^\infty$$

typical: $\varphi_0 \equiv 0$

$\tilde{\varphi}(x) = \gamma \cdot x$ dangerous!

$\hat{\varphi}(x) \equiv A$ not admissible!

$\varphi(x) = \min(x, A)$

ruin probability

$$\psi(\pi, x) = P[X_n^{x,\pi} < 0 \text{ for some } n]$$

Assumpt. $\lambda \cdot E[Y] < c$

classical

$$0 < C_0 \cdot e^{-r_0 \cdot x} \leq \psi(0^\infty, x) \leq e^{-r_0 \cdot x}$$

for some $C_0 > 0$

$0 < r_0$ Lundberg exponent

Paulsen & Gjessing (1997/98)

Frovola, Kabanov & Pergamenshchikov (2002)

$\tilde{\varphi}(x) = \gamma \cdot x \quad \Rightarrow$

$$\psi(\tilde{\varphi}^\infty, x) \geq \tilde{C} \cdot x^{-s}$$ dangerous!

for some $\tilde{C} > 0$, $s > 0$

Gaier, Grandits, Schachermayer (2003)

Hipp, Schmidli (2003) Sch. (2005)

$\hat{\varphi}(x) \equiv A \Rightarrow$

$\exists A > 0$, $\hat{C} > 0$, $\hat{r} > r_0$ such that

$$\psi(\hat{\varphi}^\infty, x) \leq e^{-\hat{r} \cdot x}$$

$$\psi(\pi, x) \geq \hat{C} \cdot e^{-\hat{r} \cdot x} \quad \forall \pi$$

Here: $\varphi(x) = \min(x, A) \Rightarrow \exists A > 0$, $r > r_0$ s.t.

$\psi(\varphi^\infty, x) \leq e^{-r \cdot x}$ (where $r = \hat{r}$?)

Markov Decision Process for minimizing (total) costs

one-step cost $g(x) = 1_{(-\infty,0)}(x)$

$X_{n+1} = -\infty$ (absorbing) if $X_n < 0$

one-step operator T

for $v : \mathbb{R} \to [0,\infty)$

$$T_\varphi v(x) := E[v(X_{n+1}) \mid X_n = x, \theta_n = \varphi(X_n)], \ x \geq 0$$

$\Rightarrow$ for $(\tilde\varphi, \varphi^\infty) := (\tilde\varphi, \varphi, \varphi, \dots)$

$$\Psi(\tilde\varphi, \varphi^\infty) = T_{\tilde\varphi} \Psi(\varphi^\infty)$$

$$T^* v := \inf_\varphi T_\varphi v$$

Then $T_\varphi \Psi(\varphi^\infty) = \Psi(\varphi^\infty)$

THEOREM $(v \geq g)$

a) $T_{\tilde\varphi} v \leq v \Rightarrow \Psi(\tilde\varphi^\infty) \leq T_{\tilde\varphi} v \leq v$

b) $T^* v \geq v \Rightarrow \Psi(\pi) \geq v \quad \forall \pi$ provided

(*) $E[v(X_n^{x,\pi}) - g(X_n^{x,\pi})] \to 0 \quad \forall x \ \forall \pi$

LEMMA

a) $P[0 \leq X_n^{x,\pi} \leq M \text{ i.o.}] = 0 \quad \forall M > 0, \forall x, \forall \pi$

b) (*) holds if $v(\infty) = 0$, $v(x) = 1 \ \forall x < 0$

Set $v(x) = \min(e^{-rx}, 1)$

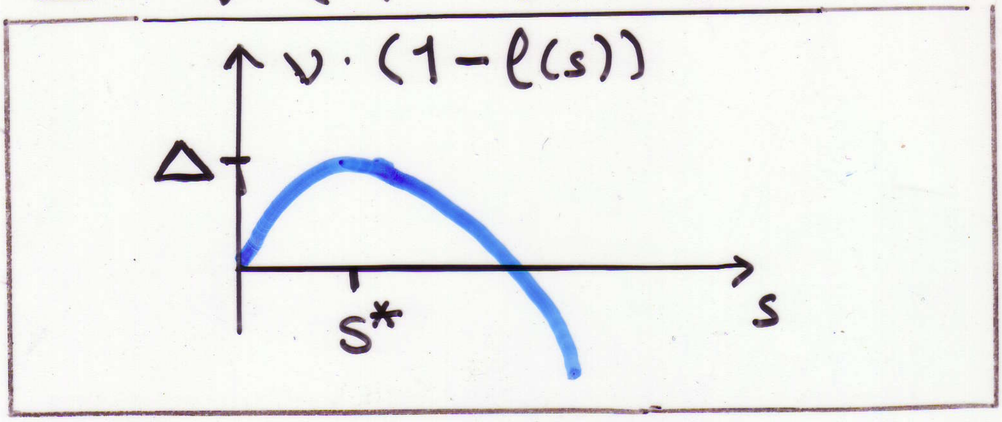
$T_\varphi v \leq v \;\Rightarrow\; \psi(\varphi^\infty) \leq v$

usual ass. on $m(s) := E[e^{s \cdot Y}]$
(small claims case)

Ass. $1 + R > 0$, $P[R < 0] > 0$, $E[R] > 0$

$\ell(s) := E[e^{-sR}]$

$\Delta := \sup_{s \geq 0} \nu \cdot (1 - \ell(s))$
$=: \nu \cdot (1 - \ell(s^*))$



LEMMA $s^* > 0$, $\Delta > 0$

Let $\hat{r}$ **be positive solution to**

(*) $\lambda \cdot (m(r) - 1) = \Delta + c \cdot r$

(**classical:** $\nu = 0 \Rightarrow \Delta = 0 \Rightarrow \hat{r} = r_0$)

LEMMA $\hat{r}$ exists, $\hat{r} > r_0$

DEF. $\hat{A} := s^* / \hat{r}$, $\varphi(x) = \min(x, \hat{A})$

THEOREM

(a) $\exists\ r \in (r_0, \hat{r}]$ such that

$$\psi(\varphi^\infty, x) \leq e^{-rx}, \quad x \geq 0$$

(b) $\psi(\varphi^\infty, x) \leq e^{-\hat{r}(x-\hat{A})} = e^{s^*} \cdot e^{-\hat{r}x}, \quad x \geq \hat{A}$

(c) $\psi(\varphi^\infty, x) \leq e^{-\hat{r}x}$ if

(**) $$\lambda \cdot \left(1 + \frac{\lambda}{\lambda+\nu} c \cdot \hat{r}\right) e^{s^*} \leq \Delta + \lambda + c \cdot \hat{r}$$

Convergence to Black-Scholes

Then $(*) =: (*)^\nu \longrightarrow$

$(*)^\infty$ $$\lambda \cdot (m(r) - 1) = \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 + c \cdot r$$

is same equation for $\hat{r}$ as for Black-Scholes market

(cp. Gaier / Grandits / Schachermayer)

Moreover $(**) =: (**)^\nu \longrightarrow$

$(**)^\infty$ $$\lambda \cdot \left[e^{\mu/\sigma^2} - 1\right] \leq \hat{r} c + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2$$

$\hat{r} = \hat{r}(\nu) \leq r_\infty$

where $m(t) < \infty \quad \forall\, t < r_\infty$

$m(t) \to \infty \quad$ for $t \to r_\infty$

(usual assumpt. with $0 < r_\infty \leq \infty$)

Here: $r_\infty < \infty$!

Example $Y \sim \Gamma_{\beta\delta} \Rightarrow r_\infty = \beta$

LOWER BOUND $\quad v(x) := C \cdot e^{-\hat{r}x}$

$T^* v \geq v \Rightarrow \psi(\pi) \geq v \quad \forall \pi$

$$\frac{1}{C} := \sup_{y \geq 0} E[\, e^{\hat{r}(Y-y)} \mid Y > y \,] \geq 1$$

THEOREM

$$\psi(\pi, x) \geq C \cdot e^{-\hat{r} \cdot x} \qquad \forall \pi$$

a) $T_{\tilde{\varphi}} v \leq v \Rightarrow \psi(\tilde{\varphi}^\infty) \leq T_{\tilde{\varphi}} v \leq v$

b) $T^* v \geq v \Rightarrow \psi(\pi) \geq v \quad \forall \pi$

APPLICATION: $v = \psi(\varphi^\infty)$

$\Rightarrow \quad T^* v \leq T_\varphi v = v$

Here: $\varphi = \varphi_0 \equiv 0$ ($\rightarrow$ v classical ruin prob.)

$\Rightarrow$

(a) $\Longleftrightarrow$ Howard improvement

(b) $\Longleftrightarrow$ verification theorem

i.e. $T^* v = v \Rightarrow \varphi_0^\infty$ is optimal

DEF. $\varphi^{*\infty}$ optimal Howard improvement of φ_0^∞

$:\Longleftrightarrow \quad T_{\varphi^*} v = T^* v \leq v$

EXAMPLES $\boxed{q := \lambda \cdot E[Y]/c < 1}$

I) $Y \sim E_1$

$$\psi(\varphi_0^\infty, x) = q \cdot e^{-r_0 \cdot x}$$

II) $Y \sim E_2$

$$\psi(\varphi_0^\infty, x) = (q+\Gamma) e^{-r_0 \cdot x} - \Gamma e^{-\gamma \cdot x}$$

$0 < r_0 < \gamma$ [virtual] Lundberg coeff.

$0 < \Gamma$

I) $Y \sim E_1$, $\tilde{\varphi}(x) \le x$

$E[R] \le 0 \Rightarrow \varphi_0^{\infty}$ is optimal

$E[R] > 0 \Rightarrow \exists\, \tilde{\varphi}^{\infty}$ Howard improvement

where [redacted] $\tilde{\varphi}(x) > 0 \quad \forall x > 0$

$\varphi^{*\infty}$ optimal Howard improvement

$\Leftrightarrow \varphi^{*}(x) = \min(x, A^{*})$

$(\exists A^{*} > 0) \quad \hat{A} \le A^{*}$

II) $Y \sim E_2$,

$E[R] = 0$

$\nu \ge \frac{1}{\lambda}[(c-\lambda)^2 - 2\lambda^2] \quad (E[Y] = 2)$

$\Rightarrow \exists$ optimal Howard improvement $\varphi^{*\infty}$

where $\varphi^{*}(x) = x \quad \forall x \le x_0$

$(\exists x_0 > 0)$