Backward SDEs with constrained jumps and Quasi-Variational Inequalities: applications to impulse controls in finance

Huyên PHAM

PMA Université Paris 7, and Institut Universitaire de France

Joint work with : I. Kharroubi (PMA, CREST), J. Ma and J. Zhang (USC)

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Introduction

Consider the parabolic **Quasi-Variational Inequality** (QVI) :

$$\min\left[-\partial_t v - \mathcal{L}v - f \;,\; v - \mathcal{H}v\right] = 0 \;, \quad v(T,.) \;=\; g, \qquad (1)$$

where $\mathcal{L}$ is the **second order local operator**

$$\mathcal{L}v(t,x) = b(x).D_x v(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma'(x)D_x^2 v(t,x))$$

and $\mathcal{H}$ is the nonlocal operator

$$\mathcal{H}v(t,x) = \sup_{e\in E} \mathcal{H}^e v(t,x)$$

with

$$\mathcal{H}^e v(t,x) = v(t, x+\gamma(x,e)) + c(x,e).$$

Introduction (II)

The QVI (1) is the **dynamic programming equation of the impulse control problem** (see Bensoussan-Lions 82 or Øksendal-Sulem 06) :

$$v(t,x) = \sup_{\alpha} \mathbf{E}\Big[g(X_T^\alpha) + \int_t^T f(X_s^\alpha)ds + \sum_{t<\tau_i\leq s} c(X_{\tau_i^-}^\alpha, \xi_i)\Big]$$

with

- controls : $\alpha = (\tau_i, \xi_i)_i$ where
  - $(\tau_i)_i$ **time decisions** : nondecreasing sequence of stopping times
  - $(\xi_i)_i$ **action decisions** : sequence of r.v. s.t. $\xi_i \in \mathcal{F}_{\tau_i}$ valued in E,
- controlled process X^α defined by

$$X_s^\alpha = x + \int_t^s b(X_u^\alpha)du + \int_t^s \sigma(X_u^\alpha)dW_u + \sum_{t<\tau_i\leq s} \gamma(X_{\tau_i^-}^\alpha, \xi_i)$$

Introduction (III)

Various applications of impulse controls :

• Financial modelling with discrete transaction dates, due e.g. to fixed transaction costs or liquidity constraints

• Optimal multiple stopping : swing options

• Firm's investment and real options : management of power plants, valuation of gas storage, ...

⋮

• More generally to models with control policies that do not accumulate in time.

→ Many papers !

Introduction (IV)

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  - The obstacle term contains the solution itself
  - It is nonlocal

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▶ Classical approach : Decouple the QVI (1) by defining by iteration the sequence of functions $(v_n)_n$:

$$\min\left[-\partial_t v_{n+1} - \mathcal{L} v_{n+1} - f \ , \ v_{n+1} - \mathcal{H} v_n\right] = 0 \ , \ v_{n+1}(T,.) = g \qquad (2)$$

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→ associated to a sequence of optimal stopping time problems (reflected BSDEs)

→ Furthermore, to compute v_{n+1}, we need to know v_n on the whole domain → heavy computations : **numerically challenging !**

Introduction (V)

- **Our basic motivation** :

▶ Find a probabilistic representation of QVI using BSDE, i.e. nonlinear Feynman-Kac formula

▶ We hope to use such a representation for deriving a direct numerical procedure for QVI

Idea of the approach

• Instead of viewing the obstacle term as a reflection of v onto $\mathcal{H}v$ (or v_{n+1} onto $\mathcal{H}v_n$)

▶ consider it as a constraint on the jumps of $v(t, X_t)$ for some suitable forward jump process X :

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▶ consider it as a constraint on the jumps of $v(t, X_t)$ for some suitable forward jump process X :

• Let us introduce the uncontrolled jump diffusion X :

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \int_E \gamma(X_{t^-}, e)\mu(dt, de), \tag{3}$$

where μ is a Poisson random measure whose intensity λ is finite and supports the whole space E.

Idea of the approach (II)

Take some smooth function $v(t,x)$ and define :

$$Y_t := v(t, X_t), \quad Z_t := \sigma(X_{t^-})' D_x v(t, X_{t^-}),$$

$$\begin{aligned} U_t(e) &:= v(t, X_{t^-} + \gamma(X_{t^-}, e)) - v(t, X_{t^-}) + c(X_{t^-}, e) \\ &= (\mathcal{H}^e v - v)(t, X_{t^-}) \end{aligned}$$

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- Apply Itô's formula :

$$\begin{aligned} Y_t = Y_T + \int_t^T f(X_s) ds + K_T - K_t - \int_t^T Z_s . dW_s \\ + \int_t^T \int_E [U_s(e) - c(X_{s^-}, e)] \mu(ds, de), \end{aligned}$$

where

$$K_t \;:=\; \int_0^t (-\partial_t v - \mathcal{L} v - f)(s, X_s) ds$$

Idea of the approach (III)

- Now, suppose that $\min[-\partial_t v - \mathcal{L}v - f, v - \mathcal{H}v] \geq 0$, and $v(T,.) = g$:

▶ Then (Y, Z, U, K) satisfies

$$Y_t = g(X_T) + \int_t^T f(X_s)ds + K_T - K_t - \int_t^T Z_s.dW_s + \int_t^T \int_E [U_s(e) - c(X_{s^-}, e)]\mu(ds, de), \quad (4)$$

K is a nondecreasing process, and U satisfies the nonpositivity constraint :

$$-U_t(e) \geq 0, \quad 0 \leq t \leq T, \; e \in E. \quad (5)$$

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▶ View (4)-(5) as a Backward Stochastic Equation (BSE) with jump constraints

▶ We expect to retrieve the solution to the QVI (1) by solving the minimal solution to this constrained BSE.

Remark : Another look at this BSE.
The nonnegativity jump-constraint : $-U_t(e) \geq 0$ can be removed by defining another nondecreasing process :

$$\bar{K}_t := K_t - \int_0^t \int_E U_s(e)\mu(ds, de),$$

so that the BSE for Y becomes (for simplicity, take $c = 0$) :

$$Y_t + \int_t^T Z_s.dW_s = g(X_T) + \int_t^T f(X_s)ds + \bar{K}_T - \bar{K}_t$$

→ The minimal solution to this BSE corresponds to the **superreplication problem** of the payoff $g(X_T) + \int_t^T f(X_s)ds$ by means of W in a jump-diffusion model. (Bouchard 06).

▶ Here, we shall keep explicitly the jump-constraint → more general jump-constraint on U

▶ Moreover, by considering general dependence on f, c, we introduce a class of BSDE with constrained jumps.

Outline

1. Backward SDEs with constrained jumps
   - Formulation of the problem
   - Existence and approximation via penalization
2. Connection with quasi-variational inequalities
3. Numerical issues
   - Probabilistic method based on BSDE representation of QVI
4. Conclusion

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Definition

Minimal Solution : find a solution $(Y, Z, U, K) \in \mathcal{S}^2 \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{A}^2$ to

$$Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s.dW_s - \int_t^T \int_E (U_s(e) - c(X_{s^-}, Y_{s^-}, Z_s, e))\mu(ds, de) \quad (6)$$

with

$$h(U_t(e), e) \geq 0, \quad d\mathbf{P} \otimes dt \otimes \lambda(de) \ \ a.e. \quad (7)$$

such that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})$ to (6)-(7) :

$$Y_t \leq \tilde{Y}_t, \quad 0 \leq t \leq T, \ a.s.$$

BSDEs with constrained jumps

Related literature

• Constraints on Y $\longrightarrow$ reflected BSDE : El Karoui et al (97), Hamadène et al, etc ...

• Constraints on Z : Cvitanic et al (98), Hu and Buckdahn (98), Peng (99), Peng and Xu (07)

Assumptions on coefficients

- **Forward SDE** : b and σ Lipschitz continuous, γ bounded and Lipschitz continuous w.r.t. x uniformly in e :

$$|\gamma(x,e) - \gamma(x',e)| \leq k|x - x'| \quad \forall e \in E$$

- **Backward SDE** : f g and c have linear growth, f and g Lipschitz continuous, c Lipschitz continuous w.r.t. y and z uniformly in x and e

$$|c(x,y,z,e) - c(x,y',z',e)| \leq k_c(|y - y'| + |z - z'|)$$

- **Constraint** : h Lipschitz continuous w.r.t. u uniformly in e :

$$|h(u,e) - h(u',e)| \leq k_h|u - u'|$$

and

$$u \mapsto h(u,e) \text{ nonincreasing.} \quad (e.g. \quad h(u,e) = -u)$$

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Penalized BSDEs

Consider for each n the BSDE with jumps :

$$Y_t^n = g(X_T) + \int_t^T f(X_s, Y_s^n, Z_s^n)ds + K_T^n - K_t^n - \int_t^T Z_s^n.dW_s - \int_t^T \int_E [U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)]\mu(ds, de) \quad (8)$$

with a penalization term

$$K_t^n = n \int_0^t \int_E h^-(U_s^n(e), e)\lambda(de)ds$$

where $h^- = \max(-h, 0)$.

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- \int_t^T \int_E [U_s^n(e) - c(X_{s^-}, Y_{s^-}^n, Z_s^n, e)]\mu(ds, de) \qquad (8)
\end{aligned}
$$

with a penalization term

$$
K_t^n = n \int_0^t \int_E h^-(U_s^n(e), e)\lambda(de)ds
$$

where $h^- = \max(-h, 0)$.

$\rightarrow$ For each n, existence and uniqueness of (Y^n, Z^n, U^n) solution to (8) from Tang and Li (94), and Barles et al. (97).

Convergence of the penalized BSDEs

- Convergence of (Y^n) : usually by comparison results
- Convergence of (Z^n, U^n, K^n) : more difficult !

$\longrightarrow$ Moreover, in general, we need some strong convergence to pass to the limit in the nonlinear terms $f(x, y, z)$, $c(x, y, z)$ and $h(u, e)$.

- Uniform boundedness
- Weak convergence method (Peng)

Comparison results

Lemma

The sequence $(Y^n)_n$ *is nondecreasing, i.e.* $\forall n \in \mathbf{N}$, $Y_t^n \leq Y_t^{n+1}$, $0 \leq t \leq T$, *a.s.*

Proof. Based on comparison theorem for BSDEs with jumps in Royer (04). We used the nonincreasing property of h.

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Lemma

For any quadruple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2$ *satisfying (6)-(7), and for all* $n \in \mathbf{N}$, *we have*

$$Y_t^n \leq \tilde{Y}_t, \quad 0 \leq t \leq T, \ a.s.$$

Proof. Suitable change of probability measures.

Uniform boundedness of the penalized BSDEs

Asumption (H1)

There exists a triple $(\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in \mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\mu) \times \mathbf{A}^2$ *satisfying (6)-(7)*

Lemma

Under **(H1)**, *there exists some constant* C *such that*

$$\|Y^n\|_{\mathcal{S}^2} + \|Z^n\|_{\mathbf{H}^2} + \|U^n\|_{\mathbf{L}^2(\tilde{\mu})} + \|K^n\|_{\mathcal{S}^2} \leq C \qquad (9)$$

for all $n \in \mathbf{N}$.

Proof. Classical arguments based on elementary inequality $2ab \leq \frac{a^2}{\eta} + \eta b^2$, Gronwall's lemma and Burkholder-Davis-Gundy's inequality + comparison result of previous lemma.

Convergence of the penalized solutions

Theorem

Under **(H1)**, there exists a unique minimal solution

$$(Y, Z, U, K) \in \mathcal{S}^2 \times \mathbf{L}^2(\mathbf{W}) \times \mathbf{L}^2(\tilde{\mu}) \times \mathbf{A}^2$$

with K predictable, to (6)-(7). Y is the increasing limit of (Y^n) and also in $\mathbf{L}^2_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, K is the weak limit of (K^n) in $\mathbf{L}^2_{\mathbb{F}}(\mathbf{0}, \mathbf{T})$, and for any $p \in [1, 2)$,

$$\|Z^n - Z\|_{\mathbf{L}^\mathbf{p}(\mathbf{W})} + \|U^n - U\|_{\mathbf{L}^\mathbf{p}(\tilde{\mu})} \longrightarrow 0,$$

as n goes to infinity.

Proof. Use the weak compactness of (Z^n), (U^n) and $(f(X^n, Y^n, Z^n))$ and (K^n) to get limits Z, U, ϕ and K. Then control jumps of the predictable process K via a random partition of the interval (0,T) and obtain a convergence in measure.

Nonmarkovian case

Remark

Existence and uniqueness results for the minimal solution hold true in a nonmarkovian framework :

$$\begin{aligned} \mathbb{F} &= \text{ filtration generated by } W \text{ and } \mu \\ g(X_T) &= \zeta \\ f(x,y,z) &= f(\omega,y,z) \\ c(x,y,z) &= c(\omega,y,z) \end{aligned}$$

Related semilinear QVIs and viscosity property

- Markov property of X $\longrightarrow$ $Y_t = v(t, X_t)$ for some deterministic function v
- Consider the semilinear QVI :

$$\min\Big[-\partial_t w - \mathcal{L}w - f(., w, \sigma' D_x w), \inf_{e \in E} h(\mathcal{H}^e w - w, e)\Big] = 0 \qquad (10)$$

where $\mathcal{L}$ is the second order local operator as before, and $\mathcal{H}^e$, $e \in E$, are the nonlocal operators

$$\mathcal{H}^e w(t,x) = w(t, x + \gamma(x,e)) + c(x, w(t,x), \sigma'(x) D_x w(t,x), e).$$

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where $\mathcal{L}$ is the second order local operator as before, and $\mathcal{H}^e$, $e \in E$, are the nonlocal operators

$$\mathcal{H}^e w(t,x) = w(t, x + \gamma(x,e)) + c(x, w(t,x), \sigma'(x) D_x w(t,x), e).$$

Asumption (H2)

The function v has linear growth : $\sup_{[0,T]\times\mathbb{R}^d} \frac{v(t,x)}{1+|x|} < \infty$.

Proposition

Under **(H2)**, the function v is a viscosity solution to (10).

Elements of proof : arguments by penalized IPDE

• Markov property of X $\longrightarrow$ $Y_t^n = v_n(t, X_t)$ for some deterministic function v_n.

• From Barles et al (98), we know that v_n is a viscosity solution to the Integral PDE :

$$
\begin{aligned}
& -\partial_t w - \mathcal{L}w - f(., w, \sigma' D_x w) \\
-n \int_E & h^- \left(\mathcal{H}^e w(t,x) - w(t,x), e\right) \lambda(de) = 0
\end{aligned}
\tag{11}
$$

• We then pass to the limit by adapting stability arguments for viscosity solutions.

Terminal condition for v

- Need a terminal condition to complete the PDE characterization of the function v.

- Condition $v(T,.) = g$ is irrelevant : discontinuity in T^- due to constraints

Terminal condition for v

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• Condition $v(T,.) = g$ is irrelevant : discontinuity in T^- due to constraints

▶ Face-lifting terminal data :

$$\min\Big[v(T^-,.) - g, \inf_{e\in E} h(\mathcal{H}^e v(T^-,.) - v(T^-,.), e)\Big] = 0 \qquad (12)$$

Proposition

Under **(H2)**, the function v is a viscosity solution to (12)

Comparison results for semilinear QVIs

Asumption (H3)

There exists a nonnegative function $\Lambda \in \mathcal{C}^2(\mathbb{R}^d)$ *satisfying*

(i) $b \cdot D\Lambda + \frac{1}{2} tr(\sigma\sigma' D^2\Lambda) + f(., \Lambda, \sigma' D\Lambda) \leq \rho\Lambda$ *for some* $\rho > 0$

(ii) $\inf_{e \in E} h(\Lambda(x + \gamma(x,e) + c(x, \Lambda(x), \sigma(x)' D\Lambda(x) - \Lambda(x), e) \geq q(x)$ *for all* $x \in \mathbb{R}^d$ *for some continuous function* $q > 0$ *on* $\mathbb{R}^d$.

(iii) $\Lambda \geq g$ *on* $\mathbf{R}^d$

(iv) $\lim_{|x| \to \infty} \frac{\Lambda(x)}{1+|x|} = +\infty$

Assumption **(H3)** essentially ensures the existence of strict supersolution which allows to control the nonlocal term in QVI (10)-(12) via some convex small perturbation. $\Rightarrow$ require some convexity conditions to deal with the dependence of f and c on y, z.

Comparison results for semilinear QVIs (II)

Asumption (H4)

(i) *The function* $f(x,.,.)$ *is convex in* $(y,z) \in \mathbf{R} \times \mathbf{R}^d$ $\forall x \in \mathbf{R}^d$.

(ii) *The function* $h(.,e)$ *is concave in* $u \in \mathbf{R}$ $\forall e \in E$.

(iii) *The function* $c(x,.,.,e)$ *is convex in* $(y,z) \in \mathbf{R} \times \mathbf{R}^d$, $\forall (x,e) \in \mathbf{R}^d \times E$.

(iv) *The function* $c(x,.,z,e)$ *is decreasing in* $y \in \mathbf{R}$, $\forall (x,z,e) \in \mathbf{R}^d \times \mathbf{R}^d \times E$.

Proposition

Assume that **(H3)** and **(H4)** hold. Let U (resp. V) be LSC (resp. USC) viscosity supersolution (resp. subsolution) of (10)-(12) satisfying the linear growth condition

$$\sup_{[0,T]\times\mathbb{R}^d} \frac{|U(t,x)| + |V(t,x)|}{1+|x|} < \infty$$

Then, $U \geq V$ on $[0,T] \times \mathbf{R}^d$.

PDE characterization of the function v

Theorem

Under **(H2)**, **(H3)** and **(H4)**, the function v is the unique viscosity solution to (10)-(12) satisfying the linear growth condition.

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|v(t,x)|}{1+|x|} < \infty.$$

Moreover v is continuous on $[0,T)\times\mathbb{R}^d$.

$\longrightarrow$ Probabilistic representation of semilinear QVIs, and in particular of impulse control problems by means of BSDEs with constrained jumps.

Outline

Approximation by the penalized BSDE

• We set $V_t^n(e) = U_t^n(e) - c(X_t, Y_{t-}^n, Z_s^n, e)$, and we rewrite the penalized BSDE for (Y^n, Z^n, V^n) as :

$$Y_t^n = g(X_T) + \int_t^T \int_E f_n(X_s, Y_s^n, Z_s^n, V_s^n(e), e)\lambda(de)ds - \int_t^T Z_s^n dW_s - \int_t^T \int_E V_s^n(e)\tilde{\mu}(de, ds)$$

where $\tilde{\mu}(dt, de) = \mu(dt, de) - \lambda(de)dt$, and

$$f_n(x, y, z, v, e) := \frac{1}{\lambda(E)} f(x, y, z) - v + nh^-(v + c(x, y, z, e), e).$$

• We assume for simplicity that the state space of jump size E is finite : $E = \{1, \ldots, m\}$ (otherwise discretize E).

Time discretization of the penalized BSDE

- Time grid $\pi = (t_i)$ on $[0, T]$: $t_i = i\Delta t$, $i = 0, \ldots, N$, $\Delta t = T/N$
- Forward Euler scheme X^π for X

$$X^\pi_{t_0} = x$$

$$X^\pi_{t_{i+1}} := X^\pi_{t_i} + b(X^\pi_{t_i})\Delta t + \sigma(X^\pi_{t_i})\big(W_{t_{i+1}} - W_{t_i}\big) + \sum_{e=1}^{m} \gamma(X^\pi_{t_i}, e)\mu((t_i, t_{i+1}] \times \{e\}).$$

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- Backward Euler scheme $(Y^{n,\pi}, Z^{n,\pi}, V^{n,\pi})$ for (Y^n, Z^n, V^n)

$$Y^{n,\pi}_{t_N} = g(X^\pi_{t_N})$$

$$Y^{n,\pi}_{t_i} = Y^{n,\pi}_{t_{i+1}} + \Delta t \sum_{e=1}^{m} \lambda(e) f_n(X^\pi_{t_i}, Y^{n,\pi}_{t_i}, Z^{n,\pi}_{t_i}, V^{n,\pi}_{t_i}, e)$$

$$- Z^{n,\pi}_{t_i}.\big(W_{t_{i+1}} - W_{t_i}\big) - \sum_{e=1}^{m} V^{n,\pi}_{t_i}(e)\tilde{\mu}((t_i, t_{i+1}] \times \{e\})$$

Time discretization of the penalized BSDE (II)

- **Backward Euler scheme $(Y^{n,\pi}, Z^{n,\pi}, V^{n,\pi})$ for (Y^n, Z^n, V^n)**

$$
\begin{aligned}
Y_{t_i}^{n,\pi} &= Y_{t_{i+1}}^{n,\pi} + \Delta t \sum_{e=1}^{m} \lambda(e) f_n(X_{t_i}^{\pi}, Y_{t_i}^{n,\pi}, Z_{t_i}^{n,\pi}, V_{t_i}^{n,\pi}, e) \\
&\quad - Z_{t_i}^{n,\pi}.[W_{t_{i+1}} - W_{t_i}] - \sum_{e=1}^{m} V_{t_i}^{n,\pi}(e)\tilde{\mu}((t_i, t_{i+1}] \times \{e\})
\end{aligned} \tag{13}
$$

- By taking conditional expectation in (13) :

$$
Y_{t_i}^{n,\pi} = \mathbf{E}\Big[Y_{t_{i+1}}^{n,\pi}\Big|\mathcal{F}_{t_i}\Big] + \Delta t \sum_{e=1}^{m} \lambda(e) f_n(X_{t_i}^{\pi}, Y_{t_i}^{n,\pi}, Z_{t_i}^{n,\pi}, V_{t_i}^{n,\pi}, e)
$$

- By multiplying by $W_{t_{i+1}} - W_{t_i}$ and taking expectation :

$$
Z_{t_i}^{n,\pi} = \frac{1}{\Delta t}\mathbf{E}\Big[Y_{t_{i+1}}^{n,\pi}(W_{t_{i+1}} - W_{t_i})\Big|\mathcal{F}_{t_i}\Big]
$$

- By multiplying by $\tilde{\mu}((t_i, t_{i+1}] \times \{e\})$ and taking expectation :

$$
V_{t_i}^{n,\pi}(e) = \frac{1}{\lambda(e)\Delta t}\mathbf{E}\Big[Y_{t_{i+1}}^{n,\pi}\tilde{\mu}((t_i, t_{i+1}] \times \{e\})\Big|\mathcal{F}_{t_i}\Big], \quad e = 1, \ldots, m.
$$

Simulation of the penalized BSDE

• For fixed penalization coefficient n, the rate of convergence of the time discretization procedure was analyzed in Bouchard and Elie (06).

▶ Here, we need to refine the estimation for n large

• Computation of the conditional expectations : Longstaff-Schwarz algorithm, Monte-Carlo method, quantization method, random walk method ...

Conclusion

- New insight into impulse control problems, and more generally into semilinear QVIs by means of BSDEs with constrained jumps
  - This provides direct (without iteration) probabilistic numerical procedure
- Current investigation and further questions
  - Analysis of the convergence of these approximation schemes
  - Numerical implementation.