Backward SDEs with constrained jumps and Quasi-Variational Inequalities: applications to impulse controls in finance

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Consider the parabolic **Quasi-Variational Inequality** (QVI):

\[
\begin{align*}
\min \{ -\partial_t v - \mathcal{L}v - f, \ v - \mathcal{H}v \} &= 0, \\
v(T, \cdot) &= g,
\end{align*}
\]

(1)

where \( \mathcal{L} \) is the **second order local operator**

\[
\mathcal{L}v(t, x) = b(x).D_x v(t, x) + \frac{1}{2} \text{tr}(\sigma \sigma'(x) D_x^2 v(t, x))
\]

and \( \mathcal{H} \) is the **nonlocal operator**

\[
\mathcal{H} v(t, x) = \sup_{e \in \mathcal{E}} \mathcal{H}^e v(t, x)
\]

with

\[
\mathcal{H}^e v(t, x) = v(t, x + \gamma(x, e)) + c(x, e).
\]
Introduction (II)

The QVI (1) is the **dynamic programming equation of the impulse control problem** (see Bensoussan-Lions 82 or Øksendal-Sulem 06):

\[ v(t, x) = \sup_{\alpha} \mathbb{E} \left[ g(X_T^\alpha) + \int_t^T f(X_s^\alpha)ds + \sum_{t < \tau_i \leq s} c(X_{\tau_i}^\alpha, \xi_i) \right] \]

with

- **controls** : \( \alpha = (\tau_i, \xi_i) \)
  - \( (\tau_i) \): **time decisions** : nondecreasing sequence of stopping times
  - \( (\xi_i) \): **action decisions** : sequence of r.v. s.t. \( \xi_i \in F_{\tau_i} \) valued in \( E \)
- **controlled process** \( X^\alpha \) defined by

\[ X_s^\alpha = x + \int_t^s b(X_u^\alpha)du + \int_t^s \sigma(X_u^\alpha)dW_u + \sum_{t < \tau_i \leq s} \gamma(X_{\tau_i}^\alpha, \xi_i) \]
Various applications of impulse controls:

- Financial modelling with discrete transaction dates, due e.g. to fixed transaction costs or liquidity constraints
- Optimal multiple stopping: swing options
- Firm’s investment and real options: management of power plants, valuation of gas storage, ...
- More generally to models with control policies that do not accumulate in time.

→ Many papers!
Main theoretical and numerical difficulty in the QVI (1):
- The obstacle term contains the solution itself
- It is nonlocal
Introduction (IV)

- **Main theoretical and numerical difficulty** in the QVI (1):
  - The obstacle term contains the solution itself
  - It is nonlocal

  ➤ Classical approach: **Decouple** the QVI (1) by defining by iteration the sequence of functions \((v_n)_n:\)

\[
\min \left[ -\partial_t v_n + \mathcal{L} v_n + f, \ v_n +\mathcal{H} v_n \right] = 0, \quad v_n(T, .) = g \quad (2)
\]

→ associated to a sequence of optimal stopping time problems (reflected BSDEs)
• **Main theoretical and numerical difficulty** in the QVI (1):
  - The obstacle term contains the solution itself
  - It is nonlocal

▶ Classical approach: **Decouple** the QVI (1) by defining by iteration the sequence of functions \( (v_n)_n \):

\[
\min \left[ -\partial_t v_{n+1} - \mathcal{L} v_{n+1} - f, \ v_{n+1} - \mathcal{H} v_n \right] = 0, \ v_{n+1}(T, \cdot) = g \tag{2}
\]

→ associated to a sequence of optimal stopping time problems (reflected BSDEs)

→ Furthermore, to compute \( v_{n+1} \), we need to know \( v_n \) on the whole domain → heavy computations: **numerically challenging**!
• **Our basic motivation:**

  - Find a probabilistic representation of QVI using BSDE, i.e. nonlinear Feynman-Kac formula

  - We hope to use such a representation for deriving a direct numerical procedure for QVI
Idea of the approach

- Instead of viewing the obstacle term as a reflection of \( v \) onto \( \mathcal{H}v \) (or \( v_{n+1} \) onto \( \mathcal{H}v_n \))

- consider it as a constraint on the jumps of \( v(t, X_t) \) for some suitable forward jump process \( X \):

\[
\begin{align*}
\text{Let us introduce the uncontrolled jump diffusion } X : \\
dX_t &= b(X_t) \, dt + \sigma(X_t) \, dW_t + \int \gamma(X_t - e, \mu(\,dt, de), (3)) \\
\text{where } \mu \text{ is a Poisson random measure whose intensity } \lambda \text{ is finite and supports the whole space } E.
\end{align*}
\]
Idea of the approach

• Instead of viewing the obstacle term as a reflection of $v$ onto $\mathcal{H}v$ (or $v_{n+1}$ onto $\mathcal{H}v_n$)

▶ consider it as a constraint on the jumps of $v(t, X_t)$ for some suitable forward jump process $X$ :

• Let us introduce the uncontrolled jump diffusion $X$ :

$$
\begin{align*}
    dX_t &= b(X_t)dt + \sigma(X_t)dW_t + \int_E \gamma(X_t^-, e)\mu(dt, de),
\end{align*}
$$

(3)

where $\mu$ is a Poisson random measure whose intensity $\lambda$ is finite and supports the whole space $E$. 

Idea of the approach (II)

Take some smooth function $v(t, x)$ and define:

$$Y_t := v(t, X_t), \quad Z_t := \sigma(X_t^-)'D_xv(t, X_t^-),$$

$$U_t(e) := v(t, X_t^- + \gamma(X_t^-, e)) - v(t, X_t^-) + c(X_t^-, e)$$

$$= (\mathcal{H}^e v - v)(t, X_t^-)$$
Idea of the approach (II)

Take some smooth function $v(t, x)$ and define:

$$Y_t := v(t, X_t), \quad Z_t := \sigma(X_{t-})' D_x v(t, X_{t-}),$$

$$U_t(e) := v(t, X_{t-} + \gamma(X_{t-}, e)) - v(t, X_{t-}) + c(X_{t-}, e)$$

$$= (\mathcal{H}^e v - v)(t, X_{t-})$$

▶ Apply Itô’s formula:

$$Y_t = Y_T + \int_t^T f(X_s) ds + K_T - K_t - \int_t^T Z_s dW_s$$

$$+ \int_t^T \int_E [U_s(e) - c(X_{s-}, e)] \mu(ds, de),$$

where

$$K_t := \int_0^t (-\partial_t v - \mathcal{L}v - f)(s, X_s) ds$$
Idea of the approach (III)

Now, suppose that $\min[-\partial_t v - \mathcal{L}v - f, v - \mathcal{H}v] \geq 0$, and $v(T, .) = g$:

Then $(Y, Z, U, K)$ satisfies

$$Y_t = g(X_T) + \int_t^T f(X_s)ds + K_T - K_t - \int_t^T Z_s.dW_s$$

$$+ \int_t^T \int_E [U_s(e) - c(X_s^-, e)] \mu(ds, de),$$

(4)

$K$ is a nondecreasing process, and $U$ satisfies the nonpositivity constraint:

$$-U_t(e) \geq 0, \quad 0 \leq t \leq T, \quad e \in E.$$  

(5)
Idea of the approach (III)

- Now, suppose that \( \min[-\partial_t v - \mathcal{L}v - f, v - \mathcal{H}v] \geq 0 \), and \( v(T, .) = g \):
  - Then \((Y, Z, U, K)\) satisfies
    \[
    Y_t = g(X_T) + \int_t^T f(X_s)ds + K_T - K_t - \int_t^T Z_s dW_s \\
    + \int_t^T \int_E [U_s(e) - c(X_{s-}, e)] \mu(ds, de),
    \]
    \(K\) is a nondecreasing process, and \(U\) satisfies the nonpositivity constraint:
    \[-U_t(e) \geq 0, \quad 0 \leq t \leq T, \quad e \in E.\]

- View (4)-(5) as a Backward Stochastic Equation (BSE) with jump constraints
- We expect to retrieve the solution to the QVI (1) by solving the minimal solution to this constrained BSE.
Remark: Another look at this BSE.

The nonnegativity jump constraint: \(-U_t(e) \geq 0\) can be removed by defining another nondecreasing process:

\[ \bar{K}_t := K_t - \int_0^t \int_E U_s(e) \mu(ds, de), \]

so that the BSE for \(Y\) becomes (for simplicity, take \(c = 0\)):

\[ Y_t + \int_t^T Z_s.dW_s = g(X_T) + \int_t^T f(X_s)ds + \bar{K}_T - \bar{K}_t \]

\[ \rightarrow \] The minimal solution to this BSE corresponds to the superreplication problem of the payoff \(g(X_T) + \int_t^T f(X_s)ds\) by means of \(W\) in a jump-diffusion model. (Bouchard 06).

\[ \rightarrow \] Here, we shall keep explicitly the jump-constraint \(\rightarrow\) more general jump-constraint on \(U\)

\[ \rightarrow \] Moreover, by considering general dependence on \(f, c\), we introduce a class of BSDE with constrained jumps.
Outline

1. Backward SDEs with constrained jumps
   - Formulation of the problem
   - Existence and approximation via penalization

2. Connection with quasi-variational inequalities

3. Numerical issues
   - Probabilistic method based on BSDE representation of QVI

4. Conclusion
1. **Backward SDEs with constrained jumps**
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Definition

Minimal Solution: find a solution

\((Y, Z, U, K) \in S^2 \times L^2(W) \times L^2(\tilde{\mu}) \times A^2\) to

\[
Y_t = g(X_T) + \int_t^T f(X_s, Y_s, Z_s) \, ds + K_T - K_t - \int_t^T Z_s \, dW_s
\]

\[
- \int_t^T \int_E (U_s(e) - c(X_{s-}, Y_{s-}, Z_s, e)) \mu(ds, de)
\]

(6)

with

\[
h(U_t(e), e) \geq 0, \quad dP \otimes dt \otimes \lambda(de) \text{ a.e.}
\]

(7)

such that for any other solution \((\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K})\) to (6)-(7):

\[
Y_t \leq \tilde{Y}_t, \quad 0 \leq t \leq T, \text{ a.s.}
\]
**Related literature**

- Constraints on $Y \rightarrow$ reflected BSDE: El Karoui et al (97), Hamadène et al, etc ...

- Constraints on $Z$: Cvitanic et al (98), Hu and Buckdahn (98), Peng (99), Peng and Xu (07)
Formulation of the problem
Existence via penalization

Assumptions on coefficients

- **Forward SDE**: $b$ and $\sigma$ Lipschitz continuous, $\gamma$ bounded and Lipschitz continuous w.r.t. $x$ uniformly in $e$:

  $$|\gamma(x, e) - \gamma(x', e)| \leq k|x - x'| \quad \forall e \in E$$

- **Backward SDE**: $f$, $g$, and $c$ have linear growth, $f$ and $g$ Lipschitz continuous, $c$ Lipschitz continuous w.r.t. $y$ and $z$ uniformly in $x$ and $e$

  $$|c(x, y, z, e) - c(x, y', z', e)| \leq k_c(|y - y'| + |z - z'|)$$

- **Constraint**: $h$ Lipschitz continuous w.r.t. $u$ uniformly in $e$:

  $$|h(u, e) - h(u', e)| \leq k_h|u - u'|$$

and

$$u \mapsto h(u, e) \text{ nonincreasing. (e.g. } h(u, e) = -u)$$
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3. **Numerical issues**
   - Probabilistic method based on BSDE representation of QVI

4. Conclusion
Consider for each $n$ the BSDE with jumps:

$$Y_t^n = g(X_T) + \int_t^T f(X_s, Y_s^n, Z_s^n) \, ds + K_T^n - K_t^n - \int_t^T Z_s^n \, dW_s$$

$$- \int_t^T \int_E [U_s^n(e) - c(X_{s-}, Y_{s-}^n, Z_s^n, e)] \mu(ds, de)$$

(8)

with a penalization term

$$K_t^n = n \int_0^t \int_E h^-(U_s^n(e), e) \lambda(de) \, ds$$

where $h^- = \max(-h, 0)$. 

with a penalization term
Penalized BSDEs

Consider for each $n$ the BSDE with jumps:

$$
Y_t^n = g(X_T) + \int_t^T f(X_s, Y_s^n, Z_s^n) ds + K_T^n - K_t^n - \int_t^T Z_s^n dW_s - \int_t^T \int_E [U_s^n(e) - c(X_s^-, Y_s^n, Z_s^n, e)] \mu(ds, de)
$$

(8)

with a penalization term

$$
K_t^n = n \int_0^t \int_E h^-(U_s^n(e), e) \lambda(de) ds
$$

where $h^- = \max(-h, 0)$.

→ For each $n$, existence and uniqueness of $(Y^n, Z^n, U^n)$ solution to (8) from Tang and Li (94), and Barles et al. (97).
Convergence of the penalized BSDEs

- Convergence of \((Y^n)\): usually by comparison results
- Convergence of \((Z^n, U^n, K^n)\): more difficult!
  → Moreover, in general, we need some strong convergence to pass to the limit in the nonlinear terms \(f(x, y, z)\), \(c(x, y, z)\) and \(h(u, e)\).
  - Uniform boundedness
  - Weak convergence method (Peng)
Comparison results

**Lemma**

The sequence \((Y^n)\) is nondecreasing, i.e. \(\forall n \in \mathbb{N}, Y^n_t \leq Y^{n+1}_t, 0 \leq t \leq T, a.s.\)

**Proof.** Based on comparison theorem for BSDEs with jumps in Royer (04). We used the nonincreasing property of \(h\).
Comparison results

**Lemma**

The sequence \((Y^n)_n\) is nondecreasing, i.e. \(\forall n \in \mathbb{N}, Y^n_t \leq Y^{n+1}_t, 0 \leq t \leq T, a.s.\)

**Proof.** Based on comparison theorem for BSDEs with jumps in Royer (04). We used the nonincreasing property of \(h\).

**Lemma**

For any quadruple \((\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in S^2 \times L^2(W) \times L^2(\tilde{\mu}) \times A^2\) satisfying (6)-(7), and for all \(n \in \mathbb{N}\), we have

\[ Y^n_t \leq \tilde{Y}_t, \ 0 \leq t \leq T, a.s. \]

**Proof.** Suitable change of probability measures.
A assumption (H1)

There exists a triple \((\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{K}) \in S^2 \times L^2(W) \times L^2(\mu) \times A^2\) satisfying (6)-(7)

Lemma

Under (H1), there exists some constant \(C\) such that

\[
\|Y^n\|_{S^2} + \|Z^n\|_{H^2} + \|U^n\|_{L^2(\bar{\mu})} + \|K^n\|_{S^2} \leq C
\]

for all \(n \in \mathbb{N}\).

Proof. Classical arguments based on elementary inequality \(2ab \leq \frac{a^2}{\eta} + \eta b^2\), Gronwall’s lemma and Burkholder-Davis-Gundy’s inequality + comparison result of previous lemma.
Convergence of the penalized solutions

Theorem

Under (H1), there exists a unique minimal solution

\[(Y, Z, U, K) \in S^2 \times L^2(W) \times L^2(\tilde{\mu}) \times A^2\]

with \(K\) predictable, to (6)-(7). \(Y\) is the increasing limit of \((Y^n)\) and also in \(L^2_F(0, T)\), \(K\) is the weak limit of \((K^n)\) in \(L^2_F(0, T)\), and for any \(p \in [1, 2)\),

\[\|Z^n - Z\|_{L^p(W)} + \|U^n - U\|_{L^p(\tilde{\mu})} \to 0,\]

as \(n\) goes to infinity.

Proof. Use the weak compactness of \((Z^n), (U^n)\) and \((f(X^n, Y^n, Z^n))\) and \((K^n)\) to get limits \(Z, U, \phi\) and \(K\). Then control jumps of the predictable process \(K\) via a random partition of the interval \((0, T)\) and obtain a convergence in measure.
Nonmarkovian case

Remark
Existence and uniqueness results for the minimal solution hold true in a nonmarkovian framework:

\[ \mathbb{F} = \text{filtration generated by } W \text{ and } \mu \]

\[ g(X_T) = \zeta \]

\[ f(x, y, z) = f(\omega, y, z) \]

\[ c(x, y, z) = c(\omega, y, z) \]
Related semilinear QVIs and viscosity property

- Markov property of $X \rightarrow Y_t = \nu(t, X_t)$ for some deterministic function $\nu$
- Consider the semilinear QVI:

$$\min \left[ -\partial_t w - \mathcal{L}w - f(. , w, \sigma'D_xw), \inf_{e \in E} h(\mathcal{H}^e w - w, e) \right] = 0 \quad (10)$$

where $\mathcal{L}$ is the second order local operator as before, and $\mathcal{H}^e$, $e \in E$, are the nonlocal operators

$$\mathcal{H}^e w(t, x) = w(t, x + \gamma(x, e)) + c(x, w(t, x), \sigma'(x)D_xw(t, x), e).$$
Related semilinear QVIs and viscosity property

- Markov property of $X \rightarrow Y_t = \nu(t, X_t)$ for some deterministic function $\nu$.
- Consider the semilinear QVI:

$$\min \left[ -\partial_t w - \mathcal{L} w - f(., w, \sigma' D_x w), \inf_{e \in E} h(\mathcal{H}^e w - w, e) \right] = 0 \quad (10)$$

where $\mathcal{L}$ is the second order local operator as before, and $\mathcal{H}^e, e \in E$, are the nonlocal operators

$$\mathcal{H}^e w(t, x) = w(t, x + \gamma(x, e)) + c(x, w(t, x), \sigma'(x)D_x w(t, x), e).$$

**Asumption (H2)**

The function $\nu$ has linear growth: $\sup_{[0,T] \times \mathbb{R}^d} \frac{\nu(t, x)}{1 + |x|} < \infty$.

**Proposition**

Under (H2), the function $\nu$ is a viscosity solution to (10).
• Markov property of $X \to Y^n_t = v_n(t, X_t)$ for some deterministic function $v_n$.

• From Barles et al (98), we know that $v_n$ is a viscosity solution to the Integral PDE:

$$
-\partial_t w - \mathcal{L}w - f(., w, \sigma'D_x w) - n \int_E h^{-}(\mathcal{H}^{-} w(t, x) - w(t, x), e) \lambda(de) = 0
$$

(11)

• We then pass to the limit by adapting stability arguments for viscosity solutions.
Terminal condition for $v$

- Need a **terminal condition** to complete the PDE characterization of the function $v$.
- Condition $v(T,.) = g$ is irrelevant: discontinuity in $T^-$ due to constraints.
Terminal condition for $v$

- Need a terminal condition to complete the PDE characterization of the function $v$.
- Condition $v(T,\cdot) = g$ is irrelevant: discontinuity in $T^-$ due to constraints

- Face-lifting terminal data:

$$\min \left[ v(T^-,\cdot) - g, \inf_{e \in E} h(e) v(T^-,\cdot) - v(T^-,\cdot), e \right] = 0 \quad (12)$$

Proposition

Under (H2), the function $v$ is a viscosity solution to (12)
Asumption (H3)

There exists a nonnegative function \( \Lambda \in C^2(\mathbb{R}^d) \) satisfying

(i) \( b \cdot D\Lambda + \frac{1}{2} \text{tr}(\sigma \sigma' D^2 \Lambda) + f(., \Lambda, \sigma' D\Lambda) \leq \rho \Lambda \) for some \( \rho > 0 \)

(ii) \( \inf_{e \in E} h(\Lambda(x + \gamma(x, e) + c(x, \Lambda(x), \sigma(x)' D\Lambda(x) - \Lambda(x), e)) \geq q(x) \)

for all \( x \in \mathbb{R}^d \) for some continuous function \( q > 0 \) on \( \mathbb{R}^d \).

(iii) \( \Lambda \geq g \) on \( \mathbb{R}^d \)

(iv) \( \lim_{|x| \to \infty} \frac{\Lambda(x)}{1 + |x|} = +\infty \)

Assumption (H3) essentially ensures the existence of strict supersolution which allows to control the nonlocal term in QVI (10)-(12) via some convex small perturbation. \( \Rightarrow \) require some convexity conditions to deal with the dependence of \( f \) and \( c \) on \( y, z \).
Comparison results for semilinear QVIs (II)

**Assumption (H4)**

(i) The function $f(x,.,.)$ is convex in $(y, z) \in \mathbb{R} \times \mathbb{R}^d \ \forall x \in \mathbb{R}^d$.

(ii) The function $h(.,e)$ is concave in $u \in \mathbb{R} \ \forall e \in E$.

(iii) The function $c(x,.,.,e)$ is convex in $(y, z) \in \mathbb{R} \times \mathbb{R}^d, \ \forall (x, e) \in \mathbb{R}^d \times E$.

(iv) The function $c(x,.,z,e)$ is decreasing in $y \in \mathbb{R}, \ \forall (x, z, e) \in \mathbb{R}^d \times \mathbb{R}^d \times E$.

**Proposition**

Assume that (H3) and (H4) hold. Let $U$ (resp. $V$) be LSC (resp. USC) viscosity supersolution (resp. subsolution) of (10)-(12) satisfying the linear growth condition

$$\sup_{[0, T] \times \mathbb{R}^d} \frac{|U(t, x)| + |V(t, x)|}{1 + |x|} < \infty$$

Then, $U \geq V$ on $[0, T] \times \mathbb{R}^d$. 
PDE characterization of the function $v$

Theorem

Under (H2), (H3) and (H4), the function $v$ is the unique viscosity solution to (10)-(12) satisfying the linear growth condition.

$$\sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \frac{|v(t,x)|}{1 + |x|} < \infty.$$  

Moreover $v$ is continuous on $[0, T) \times \mathbb{R}^d$.

$\rightarrow$ Probabilistic representation of semilinear QVIs, and in particular of impulse control problems by means of BSDEs with constrained jumps.
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Approximation by the penalized BSDE

- We set $V^n_t(e) = U^n_t(e) - c(X_t, Y^n_t, Z^n_t, e)$, and we rewrite the penalized BSDE for $(Y^n, Z^n, V^n)$ as:

$$Y^n_t = g(X_T) + \int_t^T \int_E f_n(X_s, Y^n_s, Z^n_s, V^n_s(e), e) \lambda(de)ds$$

$$- \int_t^T Z^n_s dW_s - \int_t^T \int_E V^n_s(e) \tilde{\mu}(de, ds)$$

where $\tilde{\mu}(dt, de) = \mu(dt, de) - \lambda(de)dt$, and

$$f_n(x, y, z, v, e) := \frac{1}{\lambda(E)} f(x, y, z) - v + nh^- (v + c(x, y, z, e), e).$$

- We assume for simplicity that the state space of jump size $E$ is finite: $E = \{1, \ldots, m\}$ (otherwise discretize $E$).
Time discretization of the penalized BSDE

- **Time grid** $\pi = (t_i)$ on $[0, T]$: $t_i = i \Delta t$, $i = 0, \ldots, N$, $\Delta t = T/N$

- **Forward Euler scheme** $X^\pi$ for $X$

  $X_{t_0}^\pi = x$

  $X_{t_{i+1}}^\pi := X_{t_i}^\pi + b(X_{t_i}^\pi) \Delta t + \sigma(X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i}) + \sum_{e=1}^{m} \gamma(X_{t_i}^\pi, e) \mu((t_i, t_{i+1}] \times \{e\})$. 

\[ X_{t_{i+1}}^\pi := X_{t_i}^\pi + b(X_{t_i}^\pi) \Delta t + \sigma(X_{t_i}^\pi)(W_{t_{i+1}} - W_{t_i}) + \sum_{e=1}^{m} \gamma(X_{t_i}^\pi, e) \mu((t_i, t_{i+1}] \times \{e\}). \]
Time discretization of the penalized BSDE

- Time grid $\pi = (t_i)$ on $[0, T]: t_i = i\Delta t, i = 0, \ldots, N, \Delta t = T/N$

- Forward Euler scheme $X^{\pi}$ for $X$

  \[
  X^{\pi}_{t_0} = x
  \]

  \[
  X^{\pi}_{t_{i+1}} := X^{\pi}_{t_i} + b(X^{\pi}_{t_i})\Delta t + \sigma(X^{\pi}_{t_i})(W^{t_{i+1}} - W^{t_i}) + \sum_{e=1}^{m} \gamma(X^{\pi}_{t_i}, e)\mu((t_i, t_{i+1}] \times \{e\}).
  \]

- Backward Euler scheme $(Y^{n, \pi}, Z^{n, \pi}, V^{n, \pi})$ for $(Y^n, Z^n, V^n)$

  \[
  Y^{n, \pi}_{t_N} = g(X^{\pi}_{t_N})
  \]

  \[
  Y^{n, \pi}_{t_i} = Y^{n, \pi}_{t_{i+1}} + \Delta t \sum_{e=1}^{m} \lambda(e)f_n(X^{\pi}_{t_i}, Y^{n, \pi}_{t_i}, Z^{n, \pi}_{t_i}, V^{n, \pi}_{t_i}, e)
  \]

  \[
  - Z^{n, \pi}_{t_i} \cdot (W^{t_{i+1}} - W^{t_i}) - \sum_{e=1}^{m} V^{n, \pi}_{t_i}(e)\tilde{\mu}((t_i, t_{i+1}] \times \{e\})
  \]
Time discretization of the penalized BSDE (II)

- **Backward Euler scheme** \((Y^n, \pi, Z^n, \pi, V^n, \pi)\) for \((Y^n, Z^n, V^n)\)

\[
Y^n_{t_i} = Y^n_{t_{i+1}} + \Delta t \sum_{e=1}^{m} \lambda(e)f_n(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}, V^n_{t_i}, e)
- Z^n_{t_i} \cdot [W_{t_{i+1}} - W_{t_i}] - \sum_{e=1}^{m} V^n_{t_i} \cdot (e) \tilde{\mu}((t_i, t_{i+1}] \times \{e\})
\]

(13)

- By taking conditional expectation in (13):

\[
Y^n_{t_i} = \mathbb{E}[Y^n_{t_{i+1}} | \mathcal{F}_{t_i}] + \Delta t \sum_{e=1}^{m} \lambda(e)f_n(X^n_{t_i}, Y^n_{t_i}, Z^n_{t_i}, V^n_{t_i}, e)
\]

- By multiplying by \(W_{t_{i+1}} - W_{t_i}\) and taking expectation:

\[
Z^n_{t_i} = \frac{1}{\Delta t} \mathbb{E}[Y^n_{t_{i+1}}(W_{t_{i+1}} - W_{t_i}) | \mathcal{F}_{t_i}]
\]

- By multiplying by \(\tilde{\mu}((t_i, t_{i+1}] \times \{e\})\) and taking expectation:

\[
V^n_{t_i} (e) = \frac{1}{\lambda(e)\Delta t} \mathbb{E}[Y^n_{t_{i+1}} \tilde{\mu}((t_i, t_{i+1}] \times \{e\}) | \mathcal{F}_{t_i}]
\]
Simulation of the penalized BSDE

- For fixed penalization coefficient $n$, the rate of convergence of the time discretization procedure was analyzed in Bouchard and Elie (06).

  ▶ Here, we need to refine the estimation for $n$ large

- Computation of the conditional expectations: Longstaff-Schwarz algorithm, Monte-Carlo method, quantization method, random walk method ...
Conclusion

- New insight into impulse control problems, and more generally into semilinear QVIs by means of BSDEs with constrained jumps
  - This provides direct (without iteration) probabilistic numerical procedure

- Current investigation and further questions
  - Analysis of the convergence of these approximation schemes
  - Numerical implementation.