Markets with convex transaction costs

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The Market Model

- The market consists of d assets traded at $t = 0, \ldots, T$.
- Filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$.
- The price of a portfolio is a non-linear function of the amount due to transaction costs, other illiquidity effects... $\rightarrow$ Modeling portfolio processes becomes an issue.
- Kabanov (1999): Portfolios are vectors in $\mathbb{R}^d$, expressing the number of physical units of assets (or values of assets in terms of some numéraire).
- The set of all portfolios that can be transformed to a vector in $\mathbb{R}^d_+$ is a random subset of $\mathbb{R}^d$: solvency region.
  The form of the solvency region is determined by the current price and transaction costs.

The Market Model

- A market model is a sequence $(C_t)_{t=0}^T$ of $\mathcal{F}_t$-measurable set-valued mappings $\Omega \rightrightarrows \mathbb{R}^d$ such that each $C_t(\omega)$ is a closed subset of $\mathbb{R}^d$ with $\mathbb{R}^d_- \subseteq C_t(\omega)$.
- For each t and ω $C_t(\omega)$ denotes the set of all portfolios that are freely available in the market.
- A market model is called convex, if each $C_t(\omega)$ is convex.
- A convex market model is called conical, if each $C_t(\omega)$ is a cone.

Example 1: Frictionless market

If $(S_t)_{t=0}^T$ is an adapted price process with values in $\mathbb{R}^d_+$, then

$$C_t(\omega) = \{x \in \mathbb{R}^d \,|\, S_t(\omega) \cdot x \leq 0\}, \quad t = 0, \ldots, T$$

defines a conical market model.

Example 2: Proportional transaction costs

• [Kabanov (1999)]: If $(S_t)_{t=0}^T$ is an adapted price process and $(\Lambda_t)_{t=0}^T$ an adapted matrix of transaction costs coefficients, the solvency regions are defined as

$$\hat{K}_t := \{x \in \mathbb{R}^d \,|\, \exists a \in \mathbb{R}_+^{d\times d} :\; x^i S_t^i + \sum_{j=1}^d (a^{ji} - (1+\lambda_t^{ij})a^{ij}) \geq 0,\; 1 \leq i \leq d\}.$$

• One can also define solvency regions directly in terms of bid-ask matrices $(\Pi_t)_{t=0}^T$ as in [Schachermayer (2004)]:

$$\hat{K}_t = \{x \in \mathbb{R}^d \,|\, \exists a \in \mathbb{R}_+^{d\times d} :\; x^i + \textstyle\sum_{j=1}^d (a^{ji} - \pi_t^{ij} a^{ij}) \geq 0,\; 1 \leq i \leq d\}.$$

• For each ω and t the set $\hat{K}_t(\omega)$ is a polyhedral cone and

$$C_t(\omega) := -\hat{K}_t(\omega), \quad t = 0, \ldots, T$$

defines a conical market model.

Example 3: Convex price processes

[Astic and Touzi (2007)], [Pennanen (2006)]

- A convex price process is a sequence $(S_t)_{t=0}^T$ of $\mathbb{R} \cup \{+\infty\}$-valued functions on $\mathbb{R}^d \times \Omega$ such that for each t the function S_t is $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable and for each ω the function $S_t(\cdot, \omega)$ is lower semicontinuous, convex and vanishes at 0.

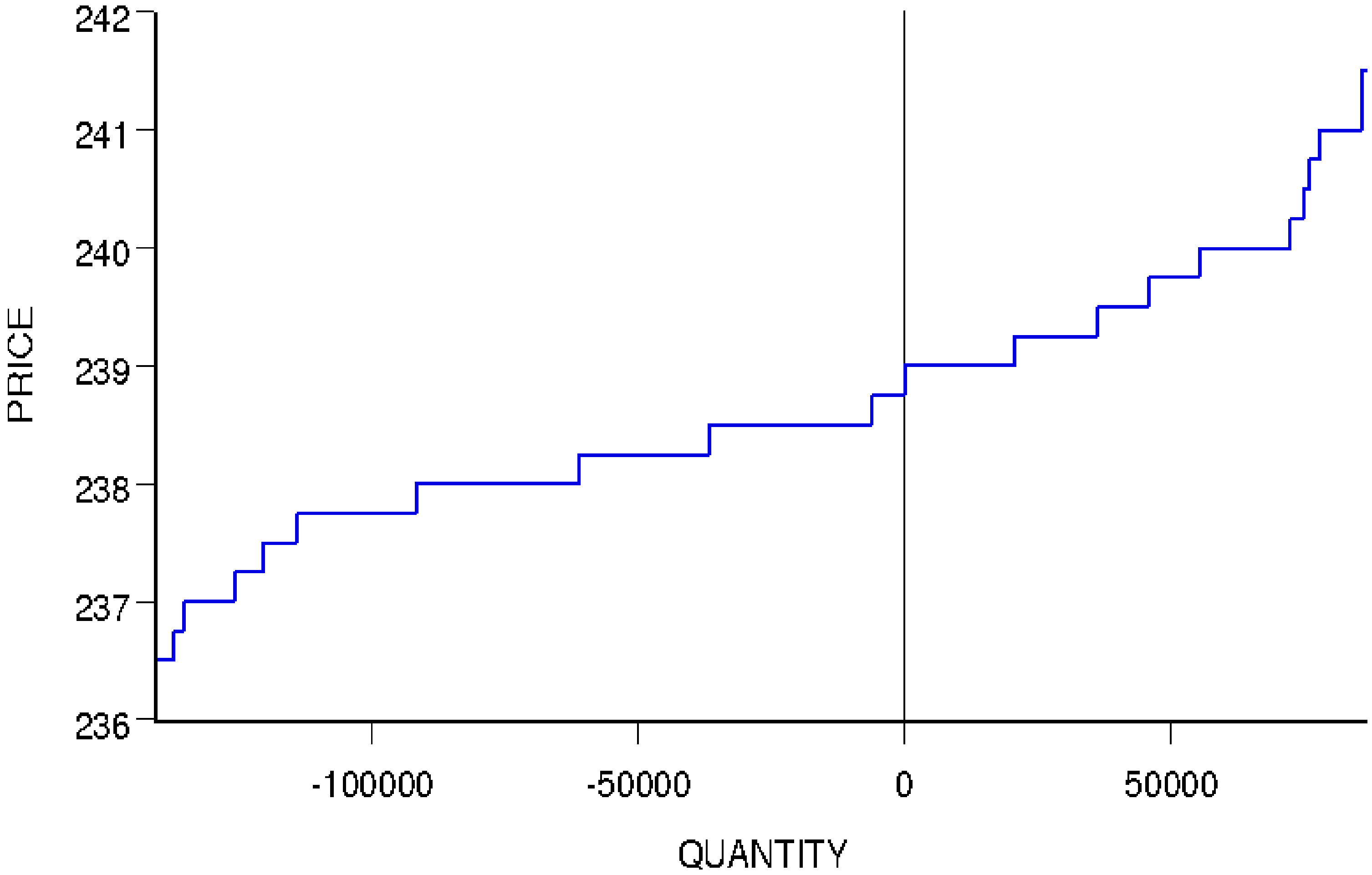
- $S_t(x, \omega)$ denotes the total price of buying a portfolio x at time t and scenario ω.

- If $(S_t)_{t=0}^T$ is a convex price process, then

$$C_t(\omega) = \{x \in \mathbb{R}^d \,|\, S_t(x, \omega) \leq 0\}, \quad t = 0, \dots, T$$

  defines a convex market model.

Example 3: Convex price processes



Example 3: Convex price processes

- [Çetin and Rogers (2007)]: A market with one riskfree and one risky asset. The convex price process is given by

  $$S_t((y,x),\omega) = y + s_t(\omega)\varphi(x)$$

  for a strictly positive adapted price process of a risky asset $(s_t)_{t=0}^T$ and a strictly convex and increasing function $\varphi : \mathbb{R} \to (-\infty, \infty]$. (Example: $\varphi(x) = \frac{e^{\alpha x}-1}{\alpha}$.)

- [Çetin, Jarrow and Protter (2004)]: A supply curve $s_t(x,\omega)$ gives a price per unit of x units of a risky asset. Then the total price is given by

  $$S_t((y,x),\omega) = y + s_t(x,\omega)x.$$

  No assumptions about convexity, smoothness required.

Example 4: Convex transaction costs

- Replace a bid-ask matrix $(\Pi_t)_{t=0}^T$ by a matrix of convex price processes $(S_t^{ij})_{t=0}^T$ $(1 \le i, j \le d)$ on $\mathbb{R}_+$.
- $S^{ij}(x, \omega)$ denotes the number of units of asset i for which one can buy x units of asset j. In a market with proportional transaction costs we have $S^{ij}(x, \omega) = \begin{cases} \pi^{ij}(\omega)x & \text{if } x \ge 0, \\ \frac{1}{\pi^{ji}(\omega)}x & \text{if } x \le 0 \end{cases}$.
- If (S_t^{ij}) $(1 \le i, j \le d)$ are sequences of convex price processes on $\mathbb{R}_+$, then

$$C_t(\omega) = \{x \in \mathbb{R}^d \,|\, \exists a \in \mathbb{R}_+^{d \times d} : \; x^i \le \sum_{j=1}^d (a^{ji} - S_t^{ij}(a^{ij}, \omega)), \, 1 \le i \le d\}$$

defines a convex market model.

Notation

- $\mathcal{A}$ denotes the set of all adapted $\mathbb{R}^d$-valued processes.
- A process $x \in \mathcal{A}$ is a self-financing portfolio processes if

$$x_t - x_{t-1} \in C_t \quad P\text{-a.s. for all} \quad t = 0, \dots, T$$

  We always define $x_{-1} := 0$.

- The set of all final values of self-financial portfolio processes (or, equivalently, of all claims that can be replicated at no cost) is denoted by $A_T(C)$

Motivation: Hedging

- We want to give a dual characterization of the set of all initial endowments that allow an investor to hedge a given claim $\rightarrow$ "Hedging theorem".
- A key to the hedging theorem is no-arbitrage condition and FTAP: In a "classical" frictionless model it
  - provides existence of "pricing" martingales (martingale measures)
  - provides closedness of the set $A_T(C)$ of all claims that can be replicated at no cost.

Motivation: Hedging

- In a market with proportional transaction costs several natural generalizations of the notions of arbitrage and martingale measures are possible [Kabanov and Stricker (2001)], [Schachermayer (2004)], [Grigoriev (2005)], [Rásonyi (2008)]...
- In a market with convex structure martingale measures are not sufficient for the dual characterization [Föllmer and Kramkov (1997)]...
- We are interested in a no-arbitrage notion that implies closedness of the set $A_T(C)$.

No-arbitrage notions for conical models

[Kabanov and Stricker (2001)], [Kabanov, Rásonyi and Stricker (2001), (2003)], [Schachermayer (2004)]

- A market model C has the no arbitrage property if

  $$A_T(C) \cap L^0(\mathbb{R}^d_+) = \{0\},$$

  where $A_T(C) = \{x_T \,|\, x \text{ is self-financing}\}$.

- A market model $\tilde{C}$ dominates a conical market model C if

  $$C_t \subseteq \tilde{C}_t \quad \text{and} \quad C_t \setminus C_t^0 \subset \operatorname{ri} \tilde{C}_t \quad \text{for all} \quad t = 0, \ldots, T,$$

  where $C_t^0 = C_t \cap -C_t$.

- A conical market model C has the robust no-arbitrage property if C is dominated by another conical model $\tilde{C}$ which has the no-arbitrage property.

No-arbitrage notions for convex models

- Given a convex market model C, we define a conical market model C^∞ by

$$C_t^\infty(\omega) = \{x \in \mathbb{R}^d \,|\, C_t(\omega) + \alpha x \subset C_t(\omega)\ \forall \alpha > 0\}, \quad t = 0, \dots, T.$$

- $C_t^\infty(\omega)$ is the recession cone of $C_t(\omega)$:

$$C_t^\infty(\omega) = \bigcap_{\alpha>0} \alpha C_t(\omega)$$

  If C is conical then $C^\infty = C$.

- The set $C_t^\infty(\omega)$ describes the behavior of $C_t(\omega)$ infinitely far from the origin.

- We say that a convex market model C has the robust no scalable arbitrage property if the model C^∞ has the robust no-arbitrage property.

No-arbitrage notions for convex models

- Given a convex market model C, one can also consider the conical market model C' given by

$$C'_t(\omega) := \operatorname{cl} \bigcup_{\alpha>0} \alpha C_t(\omega), \quad t = 0, \dots, T.$$

- $C_t(\omega)$ is the tangent cone of $C_t(\omega)$. If C is conical then $C' = C$.
- The set $C'_t(\omega)$ describes the behavior of $C_t(\omega)$ close to the origin.
- We say that a convex market model C has the robust no marginal arbitrage property if the model C' has the robust no-arbitrage property.

Main result

Theorem 1 *If the convex market model C has the robust no scalable arbitrage property then the set $A_T(C)$ of all claims that can be replicated with zero initial investment is closed in probability.*

Applications: Hedging

- A contingent claim processes with physical delivery $c = (c_t)_{t=0}^T \in \mathcal{A}$ is a security that gives its owner a random portfolio c_t possibly at each time $t = 0, \dots, T$.

- The set of all claim processes that can be replicated with zero initial investment is

$$A(C) = \{c \in \mathcal{A} \,|\, \exists x \in \mathcal{A} : \; x_t - x_{t-1} + c_t \in C_t, \; t = 0, \dots, T, \; x_T = 0\}.$$

- We call a process $p \in \mathcal{A}$ a super-hedging premium process for a claim process c if $c - p \in A(C)$.

- If $c = (0, \dots, 0, c_T)$ and $p = (p_0, 0, \dots, 0)$, then $c - p \in A(C)$ iff there exists a self-financing portfolio process such that

$$c_T \leq p_0 + x_T.$$

Applications: Hedging

Theorem 2 *[Hedging Theorem] Assume that a market model C is convex and that it has the robust no scalable arbitrage property. Let $c, p \in \mathcal{A}$ be such that $c - p \in L^1(P)$. Then the following are equivalent:*

(i) p *is a super-hedging premium process for* c.

(ii)
$$E\left[\sum_{t=0}^{T}(c_t - p_t) \cdot z_t\right] \leq E\left[\sum_{t=0}^{T}\sigma_{C_t}(z_t)\right]$$

for every $\mathbb{R}^d_+$-valued bounded martingale $(z_t)_{t=0}^T$.

Here $\sigma_{C_t(\omega)}$ denotes the support function of $C_t(\omega)$:

$$\sigma_{C_t(\omega)}(z) := \sup_{x \in C_t(\omega)} x \cdot z, \quad z \in \mathbb{R}^d.$$

Applications: Hedging

- If C is conical, we have

$$\sigma_{C_t(\omega)}(y) = \begin{cases} 0 & \text{if } y \in C_t^*(\omega), \\ +\infty & \text{otherwise.} \end{cases}$$

- An adapted $\mathbb{R}^d \setminus \{0\}$-valued process $z = (z_t)_{t=0}^T$ is called a consistent price system for a conical model C, if z is a martingale such that $z_t \in C_t^*$ almost surely for all t.

- $z = (z_t)_{t=0}^T$ is called a strictly consistent price system for a conical model C if z is a martingale with strictly positive components and such that $z_t \in \operatorname{ri} C_t^*$ almost surely for all t.

[Kabanov, Rásonyi and Stricker (2001), (2003)], [Schachermayer (2004)]

Applications: Hedging

Corollary 3 *Assume that C is a conical market model and that it has the robust no arbitrage property. Assume further that $\mathcal{F}_0$ is trivial and let $c_T \in L^1(P)$ and $p_0 \in \mathbb{R}$. Then the following are equivalent.*

(i) $p = (p_0, 0, \dots, 0)$ *is a super-hedging premium for* $c = (0, \dots, 0, c_T)$.

(ii)
$$E\left[c_T \cdot z_T\right] \leq p_0 \cdot z_0$$
for every bounded consistent price system $(z_t)_{t=0}^T$.

(iii)
$$E\left[c_T \cdot z_T\right] \leq p_0 \cdot z_0$$
for every bounded strictly consistent price system $(z_t)_{t=0}^T$.

[Kabanov, Rásonyi and Stricker (2003)], [Schachermayer (2004)]

Applications: FTAP

Theorem 4 *[FTAP]*

- *A convex market model C has the robust no scalable arbitrage property if and only if there exists a strictly positive martingale z such that $z_t \in \operatorname{ri} \operatorname{dom} \sigma_{C_t}$ for all t. (Equivalently: there exists a strictly consistent price system z for C^∞).*

- *A convex market model C has the robust no marginal arbitrage property if and only if there exists a strictly positive martingale z such that $z_t \in (\operatorname{dom} \sigma_{C_t})'$ for all t. (Equivalently: there exists a strictly consistent price system z for C').*

Similar results in [Kabanov, Rásonyi and Stricker (2003)] and [Schachermayer (2004)] for polyhedral conical models and in [Rásonyi (2007)] and [Rokhlin (2007)] for more general conical models.

Thank you

for your attention!