Optimal dividend and reinvestment policies when payments are subject to both fixed and proportional costs

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Talk based on the paper
Optimal dividend payments and reinvestments of diffusion processes with both fixed and proportional costs
1. Introduction to the problem
2. The solution
3. A financial example
The model

Income process without payments

\[ dX_t = \mu(X_t)dt + \sigma(X_t)dW_t. \]

Standing assumptions:

A1. \(|\mu(y)| + |\sigma(y)| \leq K(1 + y)\) for all \(y \geq 0\) and some \(K > 0\).

A2. \(\mu\) and \(\sigma\) are continuously differentiable and the derivatives \(\mu'\) and \(\sigma'\) are Lipschitz continuous for all \(y \geq 0\).

A3. \(\sigma^2(y) > 0\) for all \(y \geq 0\).

A4. \(\mu'(y) \leq r\) for all \(y \geq 0\). Here \(r\) is a discount factor.

Let

\[ L g(y) = \frac{1}{2} \sigma^2(y)g''(y) + \mu(y)g'(y) - rg(y). \]
Comments on Assumption A4

A4: $\mu'(y) \leq r$ for all $y \geq 0$. Here $r$ is a discount factor. Consider the special case

$$dX_t = (\mu_0 + \mu_1 X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$ 

Here $\mu'(x) = \mu_1$ and furthermore

$$E^x[e^{-rt}X_t] = \left(x + \frac{\mu_0}{\mu_1}\right)e^{(\mu_1-r)t} - \frac{\mu_0}{\mu_1}e^{-rt}.$$ 

If $\mu_1 \leq r$ this stabilizes, but if $\mu_1 > r$ it grows to infinity and therefore it is clearly better to wait. The right quantities to compare are therefore $\mu'(x)$ and $r$, one representing the geometric growth rate and the other the geometric discounting rate. The condition $\mu'(x) \leq r$ just says that in no state should growth rate exceed discounting rate.
The problem

Total dividends paid up to time $t$ is $D_t$. When reserves hit zero reinvestments are made, total reinvestments up to time $t$ is $C_t$. Both $C$ and $D$ are nondecreasing and RCLL. Associated costs are

$$d\bar{C}_t = c_0 1_{\{\Delta C_t > 0\}} + c_1 dC_t, \quad 0 \leq c_1 \leq 1,$$

$$d\bar{D}_t = d_0 1_{\{\Delta D_t > 0\}} + d_1 dD_t,$$

where $c_0$, $c_1$, $d_0$ and $d_1$ all are nonnegative constants. Therefore

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t + (1 - c_1)dC_t - (1 + d_1)dD_t$$

$$- c_0 1_{\{\Delta C_t > 0\}} - d_0 1_{\{\Delta D_t > 0\}},$$

with $Y_{0-} = y$. 

Introduction to the problem

The solution

A financial example
For given \((C, D)\) let

\[ V_{C,D}(y) = \limsup_{n \to \infty} E^y \left[ \int_{0-}^{\nu_n} e^{-rt} dA_t \right], \]

where \(A = D - C\) and \(\nu_n = \inf\{t : C_t \lor D_t > n\}\).

We want to find

\[ V^*(y) = \sup_{(C,D)} V_{C,D}(y). \]

and also, if it exists, the optimal policy \((C^*, D^*)\).
Literature

Same model as here, but without fixed costs.
With fixed costs, but only linear Brownian motion.
Avram, Palmowski and Pistorius (2007).
Spectrally negative Lévy process, but no fixed costs.
Discrete time

Papers with absorption at zero

Paulsen (2007).
Same model and expenses as in this paper

Linear Brownian motion.
Papers written for combinations of dividend payments, investment policies and reinsurance policy, but restricted to Brownian motion are

General considerations

Why is it possible to give a complete solution for such a general model?
Consider again the equation $Lg(y) = 0$. Four (or five) basic solutions
General considerations
Consider the variational problem for unknown $V, y^*, \gamma^* \in (0, y^*)$ and $\delta^* \in (0, y^*)$. 
The variational problem

\[ LV(y) = 0, \quad 0 < y < y^*, \]

\[ V(y^*) = V(0) + \frac{y^* + c_0}{1 - c_1}, \]

\[ V'(y^*) = \frac{1}{1 - c_1}, \]

\[ V(y^*) = V(y^* - \delta^*) + \frac{\delta^* - d_0}{1 + d_1}, \]

\[ V'(y^* - \delta^*) = \frac{1}{1 + d_1}, \]

\[ V'(y^*) = \frac{1}{1 + d_1}, \]

\[ V(y) = V(y^*) + \frac{y - y^*}{1 + d_1}, \quad y > y^*. \]
The variational problem

a) If this has a solution this solution is unique and

\[ V(y) = V^*(y), \quad y \geq 0. \]

The optimal policy is to pay \( \delta^* \) in dividends whenever \( Y_{t-} = y^* \) and to reinvest \( \gamma^* \) whenever \( Y_{t-} = 0 \).

b) If this has no solution there is no optimal policy, but

\[ V^*(y) = \lim_{\bar{y} \to \infty} V_{\bar{y}, \gamma(\bar{y}), \delta(\bar{y})}(y) \]

and this limit exists and is finite for every \( y \geq 0 \).
Proposition 1

\textbf{a)} Assume there is no optimal solution. Then there exists a solution \( g_2 \) of \( Lg = 0 \) so that

\[
\lim_{y \to \infty} g_2(y) = \lim_{y \to \infty} g'_2(y) = 0.
\]

Furthermore, for any other independent solution \( g_1 \),

\[
\lim_{y \to \infty} g'_1(y) = \lim_{y \to \infty} \frac{g_1(y)}{y} = \bar{g}_1
\]

for some positive and finite \( \bar{g}_1 \).
b) Assume that there are two solutions $g_1$ and $g_2$ of $Lg = 0$ so that

$$\lim_{y \to \infty} g_1'(y) = \bar{g}_1,$$
$$\lim_{y \to \infty} g_2(y) = 0,$$

where $\bar{g}_1$ is finite and nonzero. Assume in addition that

$$\lim_{y \to \infty} \left( \frac{g_1(y)}{\bar{g}_1} - y \right) > \frac{\mu(0)}{r} - d_0.$$

Then there is no optimal solution.
c) Assume there is a solution \( g \) of \( Lg = 0 \) so that
\[
\lim_{y \to \infty} \frac{g(y)}{y} = \infty
\]
or equivalently
\[
\lim_{y \to \infty} g'(y) = \infty.
\]
Then there is an optimal solution.
Let the income process without dividends follow

\[ dX_t = \mu dt + \sigma dW_t, \]

It is easy to verify that \( Lg(y) = 0 \) has the independent solutions

\[ g_i(y) = e^{\theta_i y}, \quad i = 1,2, \]

where

\[ \theta_1 = \frac{1}{\sigma^2} \left( \sqrt{\mu^2 + 2r\sigma^2} - \mu \right) \]

\[ \theta_2 = -\frac{1}{\sigma^2} \left( \sqrt{\mu^2 + 2r\sigma^2} + \mu \right). \]

Clearly \( \theta_1 > 0 \), hence an optimal solution exists by Proposition 1.c. This is the main result of Harrison & al. (1983).
Lemma
Assume A2 and A3. Let $f_i(y), i = 1,2$ solve
\[
\frac{1}{2} \sigma^2(y) f_i''(y) + \mu_i(y) f_i'(y) - rf_i(y) = 0, \quad y \geq 0,
\]
where $\mu_1(y) > \mu_2(y)$ for all $y \geq 0$ and
\[
f_i(0) = f_0 \quad \text{and} \quad f_i'(0) = f_1 \geq 0, \quad i = 1,2.
\]
Then $f_1'(y) < f_2'(y)$ for all $y > 0$, which in turn implies that $f_1(y) < f_2(y)$ for all $y > 0$. 
A useful comparison result

Proposition 2
Assume there is no optimal policy, and let $V$ be the value function. Consider the equation (in $\bar{\gamma}$).

$$V'(\bar{\gamma}) = \frac{1}{1 - c_1}, \quad (1)$$

$$V(\bar{\gamma}) = V(0) + \frac{\bar{\gamma} + c_0}{1 - c_1}. \quad (2)$$

Furthermore, with $g_1$ and $g_2$ as in Proposition 1, write

$$V(y) = a_1 g_1(y) + a_2 g_2(y).$$

a) We have

$$\lim_{y \to \infty} V'(y) = \frac{1}{1 + d_1}.$$
b) If $c_1 + d_1 > 0$ then (1) has a unique solution. Furthermore

$$a_1 = \frac{1}{1 + d_1 \bar{g}_1},$$

$$a_2 = \frac{1}{1 - c_1 \frac{g'_2(\bar{\gamma})}{g_2(\bar{\gamma})}} - \frac{1}{1 + d_1 \bar{g}_1} \frac{g'_1(\bar{\gamma})}{g'_2(\bar{\gamma})}.$$

Here $\bar{g}_1 = \lim_{y \to \infty} g'_1(y)$ and $\bar{\gamma}$ is the solution of

$$c_0 = \frac{1 - c_1}{1 + d_1 \bar{g}_1} (g_1(y) - g_1(0))$$

$$+ \left( \frac{1}{g'_2(y)} - \frac{1 - c_1}{1 + d_1 \bar{g}_1} \frac{g'_1(y)}{g'_2(y)} \right) (g_2(y) - g_2(0)) - y.$$
c) If \( c_1 = d_1 = 0 \) there are two possibilities.

(i) The equation (1) has a unique solution and then \( a_1, a_2 \) and \( \bar{\gamma} \) are as in part b above.

(ii) The equation (1) has no solution, but

\[
a_1 = \frac{1}{\bar{g}_1},
\]

\[
a_2 = \lim_{y \to \infty} \left( \frac{g_1(y)}{g_1} - y \right) - \frac{g_1'(0)}{g_1} - c_0 \frac{g_2(0)}{g_2'(0)}.
\]
A financial example

Income process without dividends assumed to be a linear Brownian motion with drift $\mu$ and diffusion $\sigma$, but money can be invested in risk free assets with return $r$. Investment costs are incurred with rate $\alpha(Y_t)$ so that total investment costs have intensity $\alpha(Y_t)Y_t$. Assume that this consists of a fixed part $\alpha_0$ and a part that is proportional with the amount invested $\alpha_1$, i.e.

$$\alpha(y)y = \alpha_0 + \alpha_1 y.$$ 

This gives

$$dX_t = (\mu_0 + (r - \alpha_1)X_t)dt + \sigma dW_t,$$

where $\mu_0 = \mu - \alpha_0$. Assume that $\mu_0 > 0$ and $0 \leq \alpha_1 < r$. When $\alpha_0 = 0$ and $\alpha_1 = r$, this is Brownian motion.

The generator is

$$Lg(y) = \frac{1}{2}\sigma^2 g''(y) + (\mu_0 + (r - \alpha_1)y)g'(y) - rg(y) = 0.$$
A financial example

Assume first that $\alpha_1 = 0$. Two solutions are

$$
g_1(y) = ry + \mu_0,$$

$$
g_2(y) = e^{-k(y)}U(1, \frac{1}{2}, k(y)),
$$

where

$$
k(y) = \frac{r}{\sigma^2} \left( y + \frac{\mu_0}{r} \right)^2,
$$

$$
U(a, b, x) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} e^{-xt} t^{a-1} (1 + t)^{b-a-1} dt, \quad a > 0.
$$

In this case there is no optimal solution, but if $c_1 = d_1 = 0$,

$$
V^*(y) = y + \frac{\mu_0}{r} - \frac{c_0}{U(1, \frac{1}{2}, k(0))} e^{-(k(y)-k(0))} U(1, \frac{1}{2}, k(y)).
$$

The first two terms are the value if there were no costs when reaching zero, i.e. when $c_0 = 0$. 


When $\alpha_1 > 0$, we have the solutions

$$g_1(y) = e^{-k(y)}F(1, \frac{1}{2}, k(y)),\quad g_2(y) = e^{-k(y)}U(1, \frac{1}{2}, k(y)).$$

Also

$$e^{-k(y)}F(a, b, k(y)) \sim \left(y + \frac{\mu_0}{r - \alpha_1}\right)^{-\frac{r}{r-\alpha_1}},$$

hence there is always a solution.
In all tables fixed values are $\sigma^2 = \mu_0 = 1$, $c_0 = d_0 = 0.1$, $c_1 = d_1 = 0.05$, $r = 0.1$ and $\alpha = 0.02$.
Solutions were obtained by using Runge-Kutta for $g_1(0) = 0$, $g'_1(0) = 1$ and $g_2(0) = 1$, $g'_2(0) = 0$, together with the MATLAB function fsolve.
## A financial example

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