# Optimal dividend and reinvestment policies when payments are subject to both fixed and proportional costs

Jostein Paulsen

UNIVERSITETET I BERGEN
*Matematisk institutt*

October 15, 2008

Talk based on the paper
Optimal dividend payments and reinvestments of diffusion processes with both fixed and proportional costs
*SIAM Journal of Control and Optimization*, 2008, Vol. 47, No 5, 2201-2226.

1. Introduction to the problem
2. The solution
3. A financial example

# The model

Income process without payments

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

Standing assumptions:

**A1.** $|\mu(y)| + |\sigma(y)| \leq K(1 + y)$ for all $y \geq 0$ and some $K > 0$.

**A2.** $\mu$ and $\sigma$ are continuously differentiable and the derivatives $\mu'$ and $\sigma'$ are Lipschitz continuous for all $y \geq 0$.

**A3.** $\sigma^2(y) > 0$ for all $y \geq 0$.

**A4.** $\mu'(y) \leq r$ for all $y \geq 0$. Here $r$ is a discount factor.

Let

$$Lg(y) = \frac{1}{2}\sigma^2(y)g''(y) + \mu(y)g'(y) - rg(y).$$

# Comments on Assumption A4

A4: $\mu'(y) \leq r$ for all $y \geq 0$. Here $r$ is a discount factor. Consider the special case

$$dX_t = (\mu_0 + \mu_1 X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x.$$

Here $\mu'(x) = \mu_1$ and furthermore

$$E^x[e^{-rt}X_t] = \left(x + \frac{\mu_0}{\mu_1}\right)e^{(\mu_1 - r)t} - \frac{\mu_0}{\mu_1}e^{-rt}.$$

If $\mu_1 \leq r$ this stabilizes, but if $\mu_1 > r$ it grows to infinity and therefore it is clearly better to wait. The right quantities to compare are therefore $\mu'(x)$ and $r$, one representing the geometric growth rate and the other the geometric discounting rate. The condition $\mu'(x) \leq r$ just says that in no state should growth rate exceed discounting rate.

# The problem

Total dividends paid up to time $t$ is $D_t$. When reserves hit zero reinvestments are made, total reinvestments up to time $t$ is $C_t$. Both $C$ and $D$ are nondecreasing and RCLL. Associated costs are

$$\begin{aligned} d\bar{C}_t &= c_0 1_{\{\Delta C_t > 0\}} + c_1 dC_t, \quad 0 \le c_1 \le 1, \\ d\bar{D}_t &= d_0 1_{\{\Delta D_t > 0\}} + d_1 dD_t, \end{aligned}$$

where $c_0$, $c_1$, $d_0$ and $d_1$ all are nonnegative constants. Therefore

$$\begin{aligned} dY_t = {} & \mu(Y_t)dt + \sigma(Y_t)dW_t + (1 - c_1)dC_t - (1 + d_1)dD_t \\ & -c_0 1_{\{\Delta C_t > 0\}} - d_0 1_{\{\Delta D_t > 0\}}, \end{aligned}$$

with $Y_{0-} = y$.

# The problem

For given $(C, D)$ let

$$V_{C,D}(y) = \limsup_{n\to\infty} E^y \left[ \int_{0-}^{\nu_n -} e^{-rt} dA_t \right],$$

where $A = D - C$ and $\nu_n = \inf\{t : C_t \vee D_t > n\}$.
We want to find

$$V^*(y) = \sup_{(C,D)} V_{C,D}(y).$$

and also, if it exists, the optimal policy $(C^*, D^*)$.

# Literature

**Shreve, Lehoczky and Gaver (1984).**
Same model as here, but without fixed costs.
**Richard (1977), Constantinides and Richard (1978), Harrison, Sellke and Taylor (1983).**
With fixed costs, but only linear Brownian motion.
**Avram, Palmowski and Pistorius (2007).**
Spectrally negative Lévy process, but no fixed costs.
**Porteus (1977).**
Discrete time
*Papers with absorbtion at zero*
**Paulsen (2007).**
Same model and expenses as in this paper
**Jeanblanc-Picqué and Shiryaev (1995)**.
Linear Brownian motion.

# Literature

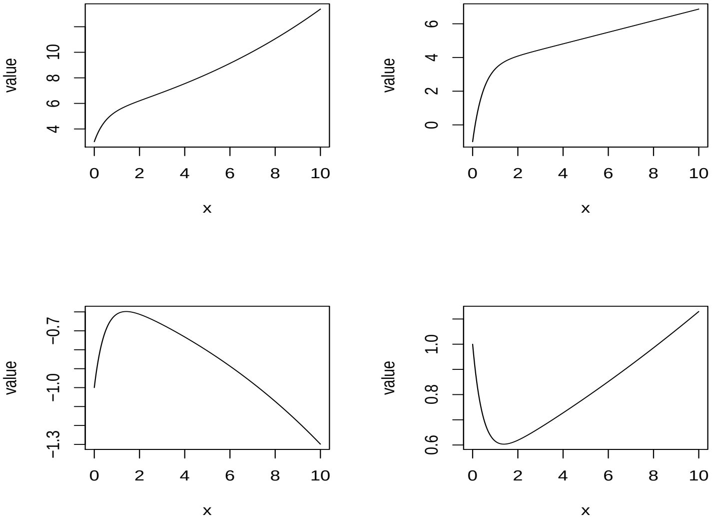
*Papers written for combinations of dividend payments, investment policies and reinsurance policy, but restricted to Brownian motion are*

**Cadenillas, Sarkar and Zapatero (2007),**
**Cadenillas, Choulli, Taksar and Zhang (2006).**

# General considerations

**Why is it possible to give a complete solution for such a general model?**
Consider again the equation $Lg(y) = 0$. Four (or five) basic solutions

# General considerations



# The variational problem

Consider the variational problem for unknown $V$, $y^*$, $\gamma^* \in (0, y^*)$ and $\delta^* \in (0, y^*)$,

# The variational problem

$$
\begin{aligned}
LV(y) &= 0, \quad 0 < y < y^*, \\
V(\gamma^*) &= V(0) + \frac{\gamma^* + c_0}{1 - c_1}, \\
V'(\gamma^*) &= \frac{1}{1 - c_1}, \\
V(y^*) &= V(y^* - \delta^*) + \frac{\delta^* - d_0}{1 + d_1}, \\
V'(y^* - \delta^*) &= \frac{1}{1 + d_1}, \\
V'(y^*) &= \frac{1}{1 + d_1}, \\
V(y) &= V(y^*) + \frac{y - y^*}{1 + d_1}, \quad y > y^*.
\end{aligned}
$$

# The variational problem

**a)** If this has a solution this solution is unique and

$$V(y) = V^*(y), \quad y \geq 0.$$

The optimal policy is to pay $\delta^*$ in dividends whenever $Y_{t-} = y^*$ and to reinvest $\gamma^*$ whenever $Y_{t-} = 0$.

**b)** If this has no solution there is no optimal policy, but

$$V^*(y) = \lim_{\bar{y}\to\infty} V_{\bar{y},\gamma(\bar{y}),\delta(\bar{y})}(y)$$

and this limit exists and is finite for every $y \geq 0$.

# The variational problem

**Proposition 1**

**a)** Assume there is no optimal solution. Then there exists a solution $g_2$ of $Lg = 0$ so that

$$\lim_{y\to\infty} g_2(y) = \lim_{y\to\infty} g_2'(y) = 0.$$

Furthermore, for any other independent solution $g_1$,

$$\lim_{y\to\infty} g_1'(y) = \lim_{y\to\infty} \frac{g_1(y)}{y} = \bar{g}_1$$

for some positive and finite $\bar{g}_1$.

# The variational problem

**b)** Assume that there are two solutions $g_1$ and $g_2$ of $Lg = 0$ so that

$$\begin{aligned} \lim_{y\to\infty} g_1'(y) &= \bar{g}_1, \\ \lim_{y\to\infty} g_2(y) &= 0, \end{aligned}$$

where $\bar{g}_1$ is finite and nonzero. Assume in addition that

$$\lim_{y\to\infty} \left( \frac{g_1(y)}{\bar{g}_1} - y \right) > \frac{\mu(0)}{r} - d_0.$$

Then there is no optimal solution.

# The variational problem

**c)** Assume there is a solution $g$ of $Lg = 0$ so that

$$\lim_{y\to\infty} \frac{g(y)}{y} = \infty$$

or equivalently

$$\lim_{y\to\infty} g'(y) = \infty.$$

Then there is an optimal solution.

# Linear Brownian Motion

Let the income process without dividends follow

$$dX_t = \mu dt + \sigma dW_t,$$

It is easy to verify that $Lg(y) = 0$ has the independent solutions

$$g_i(y) = e^{\theta_i y}, \quad i = 1,2,$$

where

$$\begin{aligned} \theta_1 &= \frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2r\sigma^2} - \mu\right) \\ \theta_2 &= -\frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2r\sigma^2} + \mu\right). \end{aligned}$$

Clearly $\theta_1 > 0$, hence an optimal solution exists by Proposition 1.c. This is the main result of Harrison & al. (1983).

# A useful comparison result

**Lemma**

Assume A2 and A3. Let $f_i(y)$, $i = 1{,}2$ solve

$$\frac{1}{2}\sigma^2(y)f_i''(y) + \mu_i(y)f_i'(y) - rf_i(y) = 0, \quad y \geq 0,$$

where $\mu_1(y) > \mu_2(y)$ for all $y \geq 0$ and

$$f_i(0) = f_0 \quad \text{and} \quad f_i'(0) = f_1 \geq 0, \quad i = 1{,}2.$$

Then $f_1'(y) < f_2'(y)$ for all $y > 0$, which in turn implies that $f_1(y) < f_2(y)$ for all $y > 0$.

# A useful comparison result

**Proposition 2**

Assume there is no optimal policy, and let $V$ be the value function. Consider the equation (in $\bar{\gamma}$).

$$\begin{aligned} V'(\bar{\gamma}) &= \frac{1}{1-c_1}, && (1) \\ V(\bar{\gamma}) &= V(0) + \frac{\bar{\gamma}+c_0}{1-c_1}. && (2) \end{aligned}$$

Furthermore, with $g_1$ and $g_2$ as in Proposition 1, write

$$V(y) = a_1 g_1(y) + a_2 g_2(y).$$

**a)** We have

$$\lim_{y\to\infty} V'(y) = \frac{1}{1+d_1}.$$

# A useful comparison result

**b)** If $c_1 + d_1 > 0$ then (1) has a unique solution. Furthermore

$$\begin{aligned} a_1 &= \frac{1}{1+d_1}\frac{1}{\bar{g}_1}, \\ a_2 &= \frac{1}{1-c_1}\frac{1}{g_2'(\bar{\gamma})} - \frac{1}{1+d_1}\frac{1}{\bar{g}_1}\frac{g_1'(\bar{\gamma})}{g_2'(\bar{\gamma})}. \end{aligned}$$

Here $\bar{g}_1 = \lim_{y\to\infty} g_1'(y)$ and $\bar{\gamma}$ is the solution of

$$\begin{aligned} c_0 &= \frac{1-c_1}{1+d_1}\frac{1}{\bar{g}_1}(g_1(y) - g_1(0)) \\ &\quad + \left(\frac{1}{g_2'(y)} - \frac{1-c_1}{1+d_1}\frac{1}{\bar{g}_1}\frac{g_1'(y)}{g_2'(y)}\right)(g_2(y) - g_2(0)) - y. \end{aligned}$$

# A useful comparison result

**c)** If $c_1 = d_1 = 0$ there are two possibilities.

  **(i)** The equation (1) has a unique solution and then $a_1$, $a_2$ and $\bar{\gamma}$ are is in part b above.

  **(ii)** The equation (1) has no solution, but

$$\begin{aligned} a_1 &= \frac{1}{\bar{g}_1}, \\ a_2 &= \frac{\lim_{y\to\infty}\left(\frac{g_1(y)}{g_1} - y\right) - \frac{g_1'(0)}{\bar{g}_1} - c_0}{g_2(0)}. \end{aligned}$$

# A financial example

Income process without dividends assumed to be a linear Brownian motion with drift $\mu$ and diffusion $\sigma$, but money can be invested in risk free assets with return $r$.
Investment costs are incurred with rate $\alpha(Y_t)$ so that total investment costs have intensity $\alpha(Y_t)Y_t$.
Assume that this consists of a fixed part $\alpha_0$ and a part that is proportional with the amount invested $\alpha_1$, i.e.

$$\alpha(y)y = \alpha_0 + \alpha_1 y.$$

This gives

$$dX_t = (\mu_0 + (r - \alpha_1)X_t)dt + \sigma dW_t,$$

where $\mu_0 = \mu - \alpha_0$. Assume that $\mu_0 > 0$ and $0 \leq \alpha_1 < r$. When $\alpha_0 = 0$ and $\alpha_1 = r$, this is Brownian motion.
The generator is

$$Lg(y) = \frac{1}{2}\sigma^2 g''(y) + (\mu_0 + (r - \alpha_1)y)g'(y) - rg(y) = 0.$$

# A financial example

Assume first that $\alpha_1 = 0$. Two solutions are

$$\begin{aligned} g_1(y) &= ry + \mu_0, \\ g_2(y) &= e^{-k(y)} U(1, \tfrac{1}{2}, k(y)), \end{aligned}$$

where

$$\begin{aligned} k(y) &= \frac{r}{\sigma^2}\left(y + \frac{\mu_0}{r}\right)^2, \\ U(a,b,x) &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1} dt, \quad a > 0. \end{aligned}$$

In this case there is no optimal solution, but if $c_1 = d_1 = 0$,

$$V^*(y) = y + \frac{\mu_0}{r} - \frac{c_0}{U(1, \frac{1}{2}, k(0))} e^{-(k(y)-k(0))} U(1, \tfrac{1}{2}, k(y)).$$

The first two terms are the value if there were no costs when reaching zero, i.e. when $c_0 = 0$.

# A financial example

When $\alpha_1 > 0$, we have the solutions

$$\begin{aligned} g_1(y) &= e^{-k(y)} F(1, \tfrac{1}{2}, k(y)), \\ g_2(y) &= e^{-k(y)} U(1, \tfrac{1}{2}, k(y)). \end{aligned}$$

Also

$$e^{-k(y)} F(a, b, k(y)) \sim \left( y + \frac{\mu_0}{r - \alpha_1} \right)^{\frac{r}{r-\alpha_1}},$$

hence there is always a solution.

# A financial example

In all tables fixed values are $\sigma^2 = \mu_0 = 1$, $c_0 = d_0 = 0.1$, $c_1 = d_1 = 0.05$, $r = 0.1$ and $\alpha = 0.02$.
Solutions were obtained by using Runge-Kutta for $g_1(0) = 0,\ g_1'(0) = 1$ and $g_2(0) = 1,\ g_2'(0) = 0$, together with the MATLAB function fsolve.

# A financial example

| $c_0$ | 0 | 0.1 | 1 | 3 | 5 | 7.76 | 10 |
|---|---|---|---|---|---|---|---|
| $y^*$ | 4.50 | 5.14 | 5.89 | 6.33 | 6.54 | 6.73 | 6.84 |
| $\gamma^*$ | 0 | 0.61 | 1.31 | 1.72 | 1.92 | 2.10 | 2.20 |
| $y^* - \delta^*$ | 0.47 | 1.06 | 1.75 | 2.15 | 2.35 | 2.52 | 2.62 |
| $V^*(0)$ | 8.81 | 8.52 | 7.36 | 5.13 | 2.96 | 0 | -2.39 |
| $V^*(1)$ | 9.77 | 9.66 | 9.44 | 9.15 | 8.90 | 8.56 | 8.29 |
| $V^*(5)$ | 13.50 | 13.28 | 13.23 | 13.16 | 13.11 | 13.08 | 13.07 |

# A financial example

| $d_0$ | 0 | 0.1 | 1 | 3 | 5 | 10 |
|---|---|---|---|---|---|---|
| $y^*$ | 1.94 | 5.14 | 14.83 | 29.28 | 41.80 | 70.53 |
| $\gamma^*$ | 0.67 | 0.61 | 0.50 | 0.45 | 0.43 | 0.40 |
| $y^* - \delta^*$ | 1.94 | 1.06 | 0.73 | 0.61 | 0.57 | 0.52 |
| $V^*(0)$ | 8.95 | 8.52 | 7.53 | 6.67 | 6.19 | 5.48 |
| $V^*(1)$ | 10.10 | 9.66 | 8.64 | 7.75 | 7.26 | 6.51 |
| $V^*(5)$ | 13.92 | 13.38 | 11.98 | 10.76 | 10.08 | 9.06 |

# A financial example

| $c_0 = d_0$ | 0 | 0.1 | 1 | 3 | 5 | 5.42 | 10 |
|---|---|---|---|---|---|---|---|
| $y^*$ | 1.30 | 5.14 | 15.68 | 30.84 | 43.80 | 46.71 | 73.28 |
| $\gamma^*$ | 0 | 0.61 | 1.15 | 1.45 | 1.60 | 1.62 | 1.80 |
| $y^* - \delta^*$ | 1.30 | 1.06 | 1.37 | 1.60 | 1.73 | 1.75 | 1.91 |
| $V^*(0)$ | 9.24 | 8.52 | 6.35 | 3.22 | 0.53 | 0 | -5.61 |
| $V^*(1)$ | 10.22 | 9.66 | 8.45 | 7.32 | 6.59 | 6.46 | 5.28 |
| $V^*(5)$ | 14.04 | 13.38 | 11.88 | 10.66 | 10.00 | 9.88 | 8.99 |