A new approach to LIBOR modeling application of affine processes

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Interest rates – Notation

- $B(t, T)$: time-t price of a zero coupon bond for T, i.e. $B(T, T) = 1$;
- $L(t, T)$: time-t forward LIBOR for $[T, T + \delta]$:

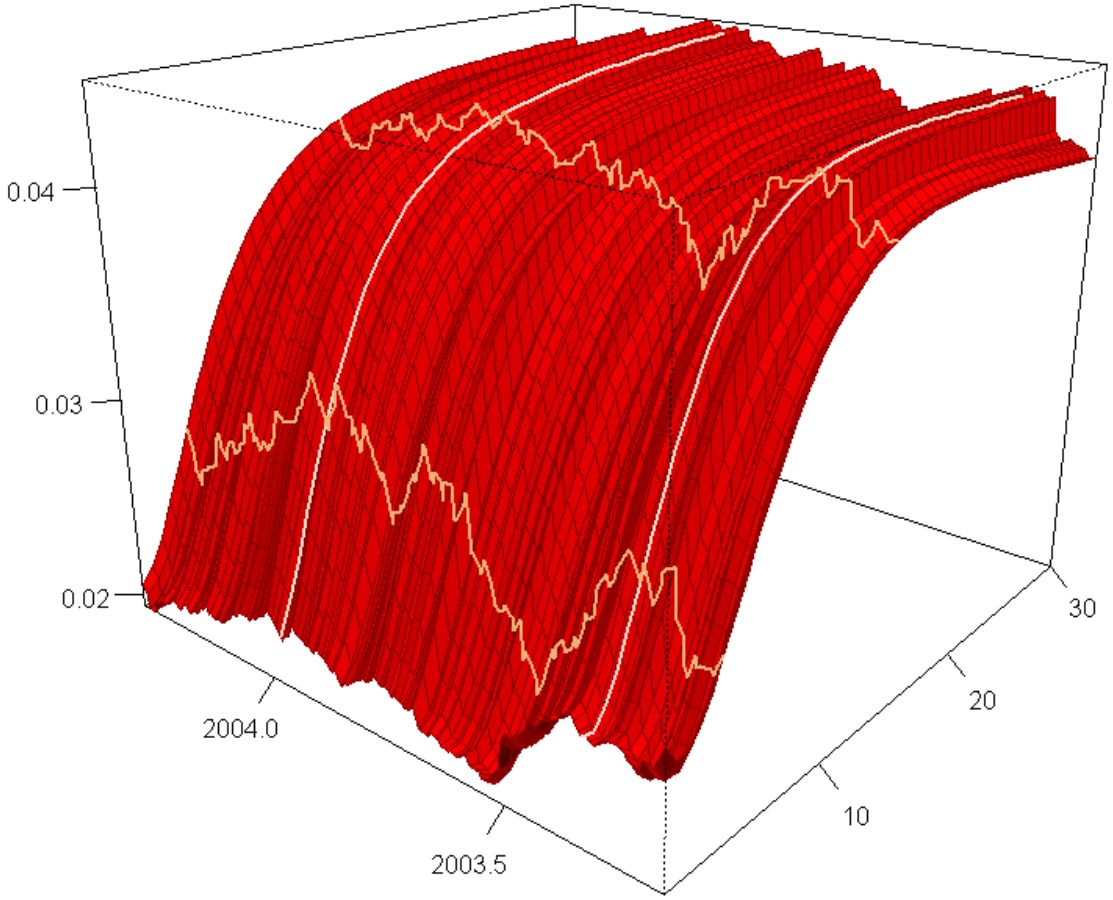
$$L(t, T) = \frac{1}{\delta}\left(\frac{B(t, T)}{B(t, T + \delta)} - 1\right)$$

- $F(t, T, U)$: time-t forward price for T and U: $F(t, T, U) = \frac{B(t,T)}{B(t,U)}$

"Master" relationship

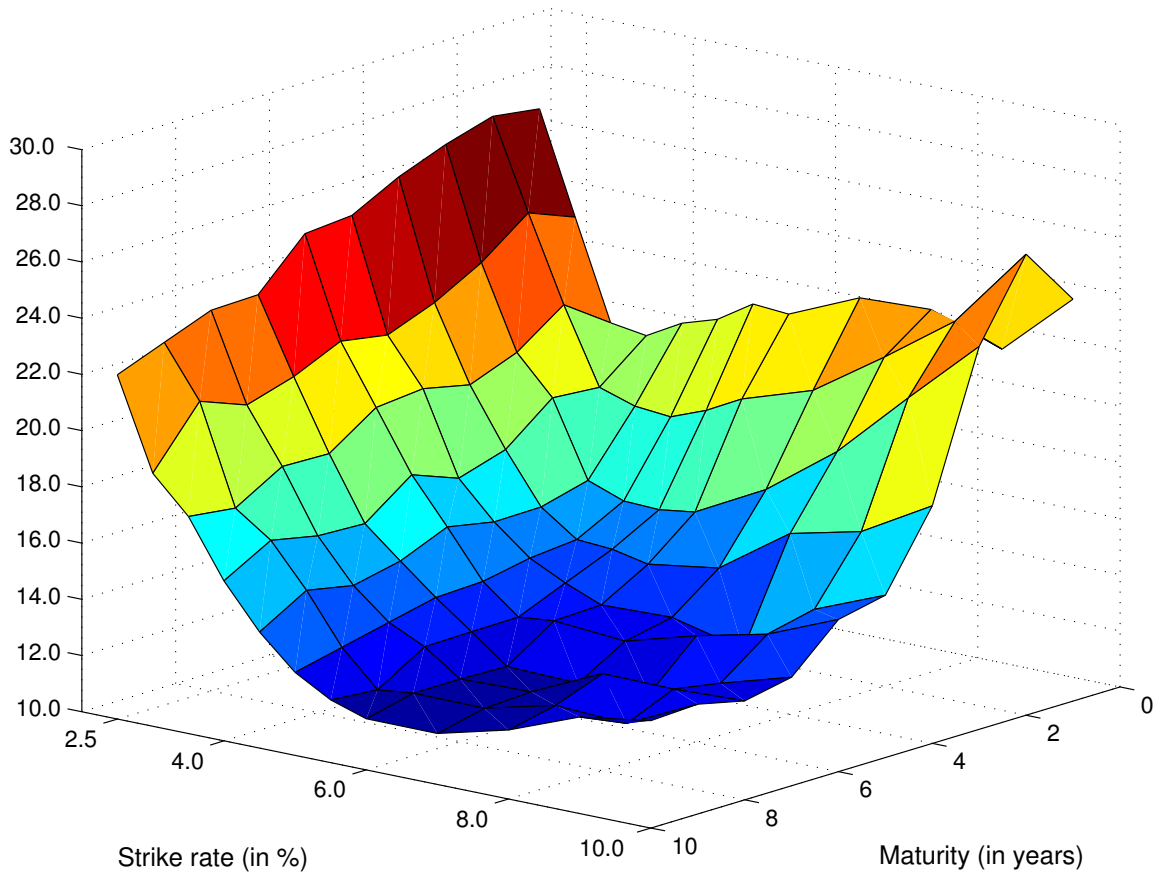
$$F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T) \tag{1}$$

Interest rates evolution



- Evolution of interest rate term structure, 2003 – 2004 (picture: Th. Steiner)

Calibration problems



1. Implied volatilities are constant neither across strike nor across maturity
2. Variance scales non-linearly over time (see e.g. Skovmand)

LIBOR model: Axioms

Economic thought dictates that LIBOR rates should satisfy:

Axiom 1

The LIBOR rate should be positive, i.e. $L(t, T) > 0$ *for all* t.

Axiom 2

The LIBOR rate process should be a martingale under the (corresponding) forward measure, i.e. $L(\cdot, T) \in \mathcal{M}(P_{T+\delta})$.

Practical applications require:

Models should describe the empirical evidence adequately.

Models should be calibrated to liquid products (caps, atm swaptions).

- What axioms do the existing models satisfy?

LIBOR models I (Sandmann et al, Brace et al, ..., Eberlein & Özkan)

Ansatz: model the LIBOR rate as the exponential of a semimartingale H:

$$L(t, T_k) = L(0, T_k) \exp\left(\int_0^t b(s, T_k) ds + \int_0^t \lambda(s, T_k) dH_s^{T_{k+1}} \right), \qquad (2)$$

where $b(s, T_k)$ ensures that $L(\cdot, T_k) \in \mathcal{M}(P_{T_{k+1}})$.
H has the $P_{T_{k+1}}$-canonical decomposition

$$H_t^{T_{k+1}} = \int_0^t \sqrt{c_s} dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{k+1}})(ds, dx), \qquad (3)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left(\sum_{l=k+1}^{N} \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \lambda(t, T_l) \right) \sqrt{c_s} ds, \qquad (4)$$

LIBOR models II

and the $P_{T_{k+1}}$-compensator of μ^H is

$$\nu^{T_{k+1}}(ds, dx) = \left(\prod_{l=k+1}^{N} \frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \left(e^{\lambda(t, T_l)x} - 1 \right) + 1 \right) \nu^{T_*}(ds, dx).$$

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Consequences for continuous semimartingales:

1. caplets can be priced in closed form;
2. swaptions and multi-LIBOR products cannot be priced in closed form;
3. Monte-Carlo pricing is very time consuming $\rightsquigarrow$ coupled high dimensional SDEs!

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Consequences for general semimartingales:

1. even caplets cannot be priced in closed form!
2. ditto for Monte-Carlo pricing.

LIBOR models III: Remedies

1. **"Frozen drift" approximation**
   - Brace et al, Schlögl, Glassermann et al ...
   - replace the random terms by their deterministic initial values:

$$\frac{\delta_l L(t-, T_l)}{1 + \delta_l L(t-, T_l)} \approx \frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)} \tag{5}$$

   - **(+)** deterministic characteristics $\rightsquigarrow$ closed form pricing
   - **(−)** "ad hoc" approximation, no error estimates, compounded error ...
2. **Strong Taylor approximation**
   - approximate the LIBOR rates in the drift/compensator by

$$L(t, T_l) \approx L(0, T_l) + Y(t, T_l)_+ \tag{6}$$

     where Y is the (scaled) exponential transform of H ($Y = \mathcal{L}oge^H$)
   - theoretical foundation, error estimates, simpler equations for MC
   - Siopacha and Teichmann; Hubalek, Papapantoleon & Siopacha

Forward price model I (Eberlein & Özkan, Kluge)

Ansatz: model the forward price as the exponential of a semimartingale H:

$$F(t, T_k) = F(0, T_k) \exp\left(\int_0^t b(s, T_k) ds + \int_0^t \lambda(s, T_k) dH_s^{T_{k+1}} \right), \quad (7)$$

where $b(s, T_k)$ ensures that $F(\cdot, T_k) = 1 + \delta L(\cdot, T_k) \in \mathcal{M}(P_{T_{k+1}})$.
H has the $P_{T_{k+1}}$-canonical decomposition

$$H_t^{T_{k+1}} = \int_0^t \sqrt{c_s} dW_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{k+1}})(ds, dx), \quad (8)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left(\sum_{l=k+1}^{N} \lambda(t, T_l) \right) \sqrt{c_s} ds, \quad (9)$$

Forward price model II

and the $P_{T_{k+1}}$-compensator of μ^H is

$$\nu^{T_{k+1}}(ds, dx) = \exp\left(x \sum_{l=k+1}^{N} \lambda(t, T_l)\right) \nu^{T_*}(ds, dx).$$

Consequences:

1. the model structure is preserved;
2. caps, swaptions and multi-LIBOR products priced in closed form.

So what is wrong?

Forward price model II

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Negative LIBOR rates can occur!

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Negative LIBOR rates can occur!

Aim: *design a model where the model structure is preserved and LIBOR rates are positive.*

Tool: *Affine processes*

Affine processes I

An affine process in the spirit of Duffie et al (henceforth DFS) is a time-homogeneous, stochastically continuous, Markov process X with state space $D = \mathbb{R}^m_{\geqslant 0} \times \mathbb{R}^n \subseteq \mathbb{R}^d$, with $X_0 = x \in D$, such that the moment generating function has the form

$$E_x[\exp\langle u, X_t\rangle] = \exp\left(\phi_t(u) + \langle \psi_t(u), x\rangle\right); \tag{10}$$

for all $u \in \mathbb{C}^d$ or $\mathbb{R}^d$ where the expectation is finite.
We assume that X is regular (and conservative).

Lemma (Flow property)

The functions ϕ and ψ satisfy the **flow equations***:*

$$\begin{aligned} \phi_{t+s}(u) &= \phi_t(u) + \phi_s(\psi_t(u)) \\ \psi_{t+s}(u) &= \psi_s(\psi_t(u)) \end{aligned} \tag{11}$$

for all suitable u and $t, s > 0$.

Affine processes II

Theorem (DFS, Main characterization)

If X is a regular affine process, then ϕ and ψ solve the generalized Riccati equations

$$\frac{\partial}{\partial t}\phi_t(u) = F(\psi_t(u)), \quad \phi_0(u) = 0 \tag{12}$$

$$\frac{\partial}{\partial t}\psi_t(u) = R(\psi_t(u)), \quad \psi_0(u) = u, \tag{13}$$

where F and R are defined via

$$F(u) = \frac{\partial}{\partial t}\Big|_{t=0}\phi_t(u) \tag{14}$$

$$R(u) = \frac{\partial}{\partial t}\Big|_{t=0}\psi_t(u) \tag{15}$$

Moreover, ...

Affine processes III

Theorem (continued)

. . . the functions F and R are of Lévy–Khintchine form:

$$F(u) = \langle b, u\rangle - c + \left\langle \frac{a}{2}u, u\right\rangle + \int_D \left(e^{\langle z,u\rangle} - 1 - \langle u, h(z)\rangle\right) m(dz)$$

$$R_i(u) = \langle \beta_i, u\rangle - \gamma_i + \left\langle \frac{\alpha_i}{2}u, u\right\rangle + \int_D \left(e^{\langle z,u\rangle} - 1 - \langle u, h(z)\rangle\right) \mu_i(dz)$$

where $(a, b, c, m, \alpha_i, \beta_i, \gamma_i, \mu_i)_{1\le i\le d}$ are admissible parameters.
X is a Feller process, the generator $\mathcal{A}$ has affine form, the semimartingale characteristics (conservative case) have affine form, etc.

Conversely, to each set of admissible parameters corresponds a regular affine process on D with generator $\mathcal{A}$.

Affine processes IV

1. **Affine processes on $\mathbb{R}$:** the admissibility conditions yield

$$F(u) = -c + bu + \frac{a}{2}u^2 + \int_{\mathbb{R}} \left(e^{zu} - 1 - uh(z)\right) m(dz)$$
$$R(u) = \beta u,$$

for $a, c \in \mathbb{R}_{\geqslant 0}$ and $b, \beta \in \mathbb{R}$.

   - Every affine process on $\mathbb{R}$ is an Ornstein–Uhlenbeck (OU) process.

2. **Affine processes on $\mathbb{R}_{\geqslant 0}$:** the admissibility conditions yield

$$F(u) = -c + bu + \int_D \left(e^{zu} - 1\right) m(dz)$$
$$R(u) = -\gamma + \beta u + \frac{\alpha}{2}u^2 + \int_D \left(e^{zu} - 1 - uh(z)\right) \mu(dz),$$

for $b, c, \alpha, \gamma \in \mathbb{R}_{\geqslant 0}$ and $\beta \in \mathbb{R}$.

   - There exist affine process on $\mathbb{R}_{\geqslant 0}$ which are not OU process.

Affine LIBOR model: martingales $\geqslant 1$

1. **Idea:** consider an affine process; insert it in its moment generating function with inverted time. The resulting process is a martingale.
2. If the affine process is positive, the martingale is greater than one.

Theorem

The process $M^u = (M_t^u)_{0 \leq t \leq T}$ defined by

$$M_t^u = \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle\right), \tag{16}$$

is a martingale. Moreover, if $D = \mathbb{R}^d_{\geqslant 0}$, $u \in \mathcal{I} \subseteq \mathbb{R}^d_{\geqslant 0}$ then $M_t \geq 1$ a.s. for all $t \in [0, T]$, for any $X_0 \in \mathbb{R}^d_{\geqslant 0}$.

Affine LIBOR model: martingales $\geqslant 1$

Proof (martingality).

$$\begin{aligned} E[M_t^u|\mathcal{F}_s] &= E[\exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t\rangle\right)|\mathcal{F}_s] \\ &= \exp(\phi_{T-t}(u))E[\exp\left(\langle \psi_{T-t}(u), X_t\rangle\right)|\mathcal{F}_s] \\ &= \exp\left(\phi_{T-t}(u) + \phi_{t-s}(\psi_{T-t}(u)) + \langle \psi_{t-s}(\psi_{T-t}(u)), X_s\rangle\right). \end{aligned}$$

Now, using the flow properties (11), we get

$$\begin{aligned} \phi_{T-t}(u) + \phi_{t-s}(\psi_{T-t}(u)) &= \phi_{T-s}(u) \\ \psi_{t-s}(\psi_{T-t}(u)) &= \psi_{T-s}(u). \end{aligned}$$

Hence, the statement is proved, since

$$E[M_t^u|\mathcal{F}_s] = \exp\left(\phi_{T-s}(u) + \langle \psi_{T-s}(u), X_s\rangle\right) = M_s.$$

□

Affine LIBOR model: martingales $\geqslant 1$

Proof (positivity).

Since u, x and X take values in $\mathbb{R}^d_{\geqslant 0}$, we simply have for any $t \geq 0$

$$E_x[\exp\langle u, X_t\rangle] = \exp\left(\phi_t(u) + \langle \psi_t(u), x\rangle\right) \geq 1. \tag{17}$$

□

Example (Lévy process)

Consider a Lévy subordinator, then

$$\begin{aligned} M_t^u &= \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t\rangle\right) \\ &= \exp\left((T-t)\kappa(u) + u \cdot X_t\right) \\ &= \exp(T\kappa(u)) \exp\left(u \cdot X_t - t\kappa(u)\right), \end{aligned} \tag{18}$$

which is obviously a positive martingale.

Affine LIBOR model: Ansatz

Consider a discrete tenor structure $0 = T_0 < T_1 < T_2 < \cdots < T_N$; discounted traded assets (bonds) are martingales with respect to the terminal martingale measure, i.e.

$$\frac{B(\cdot, T_k)}{B(\cdot, T_N)} \in \mathcal{M}(P_{T_N}), \qquad \text{for all } k \in \{1, \ldots, N-1\}. \tag{19}$$

We model quotients of bond prices using the martingales M as follows:

$$\frac{B(t, T_1)}{B(t, T_N)} = M_t^{u_1} \tag{20}$$

$$\vdots$$

$$\frac{B(t, T_{N-1})}{B(t, T_N)} = M_t^{u_{N-1}}. \tag{21}$$

Affine LIBOR model: forward prices

Consequence: forward prices have the following form

$$\frac{B(t,T_k)}{B(t,T_{k+1})} = \frac{B(t,T_k)}{B(t,T_N)}\frac{B(t,T_N)}{B(t,T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}}$$
$$= \exp\Big(\phi_{T_N-t}(u_k) - \phi_{T_N-t}(u_{k+1})$$
$$+ \big\langle \psi_{T_N-t}(u_k) - \psi_{T_N-t}(u_{k+1}), X_t \big\rangle\Big). \tag{22}$$

Forward measures are related via:

$$\frac{dP_{T_{k-1}}}{dP_{T_k}}\Big|_{\mathcal{F}_t} = \frac{F(t,T_{k-1},T_k)}{F(0,T_{k-1},T_k)} = \frac{B(0,T_k)}{B(0,T_{k-1})} \times \frac{M_t^{u_{k-1}}}{M_t^{u_k}} \tag{23}$$

or equivalently:

$$\frac{dP_{T_{k-1}}}{dP_{T_N}}\Big|_{\mathcal{F}_t} = \frac{B(0,T_N)}{B(0,T_{k-1})} \times \frac{B(t,T_{k-1})}{B(t,T_N)} = \frac{B(0,T_N)}{B(0,T_{k-1})} \times M_t^{u_{k-1}}. \tag{24}$$

Affine LIBOR model: dynamics under forward measures

The moment generating function of X_t under a forward measure is

$$\begin{aligned} E_{P_{T_{k+1}}}[e^{vX_t}] &= E_{P_{T_N}}[M_t^{u_{k+1}} e^{vX_t}] \\ &= \exp\Big(\phi_{T-t}(u_{k+1}) + \phi_t\big(\psi_{T-t}(u_{k+1}) + v\big) \\ &\qquad + \big\langle \psi_t\big(\psi_{T-t}(u_{k+1}) + v\big), x\big\rangle\Big). \end{aligned} \tag{25}$$

Let us denote by $\frac{B(t,T_k)}{B(t,T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}} = e^{A_k + B_k \cdot X_t}$; then the moment generating function is

$$\begin{aligned} E_{P_{T_{k+1}}}[e^{v(A_k+B_k\cdot X_t)}] = \exp\Big(&v\phi_{T_N-t}(u_k) + (1-v)\phi_{T_N-t}(u_{k+1}) \\ &+ \phi_t\big(v\psi_{T_N-t}(u_k) + (1-v)\psi_{T_N-t}(u_{k+1})\big) \\ &+ \big\langle \psi_t\big(v\psi_{T_N-t}(u_k) + (1-v)\psi_{T_N-t}(u_{k+1})\big), x\big\rangle\Big). \end{aligned} \tag{26}$$

Affine LIBOR model: consequences

1. **The model structure is preserved!**
2. More precisely, with respect to any forward measure all LIBOR rates are of exponential-affine form.
3. Caps, swaptions and other multi-LIBOR products can be priced in closed form.
4. Pricing (always) takes place under the terminal measure!

Affine LIBOR model: caplet pricing

We can re-write the payoff of a caplet as follows (here $\mathcal{K} := 1 + \delta K$):

$$\begin{aligned}\delta(L(T_k, T_k) - K)^+ &= (1 + \delta L(T_k, T_k) - 1 + \delta K)^+ \\ &= \left(\frac{M_{T_k}^{u_k}}{M_{T_k}^{u_{k+1}}} - \mathcal{K}\right)^+ \\ &= \left(e^{A_k + B_k \cdot X_{T_k}} - \mathcal{K}\right)^+. \end{aligned} \tag{27}$$

Then we can price caplets by Fourier-transform methods:

$$\begin{aligned}\mathbb{C}(T_k, K) &= B(0, T_{k+1}) E_{P_{T_{k+1}}}\left[\delta(L(T_k, T_k) - K)^+\right] \\ &= \frac{\mathcal{K} B(0, T_{k+1})}{2\pi} \int_{\mathbb{R}} \mathcal{K}^{iv - R} \frac{\varphi_{A_k + B_k X_{T_k}}(R - iv)}{(R - iv)(R - 1 - iv)} dv \end{aligned} \tag{28}$$

where $\varphi_{A_k + B_k X_{T_k}}$ is given by (26).

Example: Γ-OU martingales I

Let H be a Gamma subordinator, i.e. $H_1 \sim \Gamma(\alpha, \beta)$, $\alpha, \beta > 0$. The cumulant is

$$\kappa_\Gamma(v) = -\beta \log\left(1 - \frac{v}{\alpha}\right). \tag{29}$$

The Gamma-OU process is an affine process X with state space $D = \mathbb{R}_{\geqslant 0}$, that satisfies the SDE

$$dX_t = -\lambda(X_t - \theta)dt + dH_t, \quad X_0 = x \in \mathbb{R}_{\geqslant 0}, \tag{30}$$

where $\lambda, \theta > 0$. The moment generating function of the Gamma-OU process is

$$E_x[e^{vX_t}] = \exp\left(\underbrace{\theta(1 - e^{-\lambda t})v + \int_0^t \kappa_\Gamma(e^{-\lambda s}v)ds}_{\phi_t(v)} + x \cdot \underbrace{e^{-\lambda t}v}_{\psi_t(v)}\right). \tag{31}$$

Example: Γ-OU martingales II

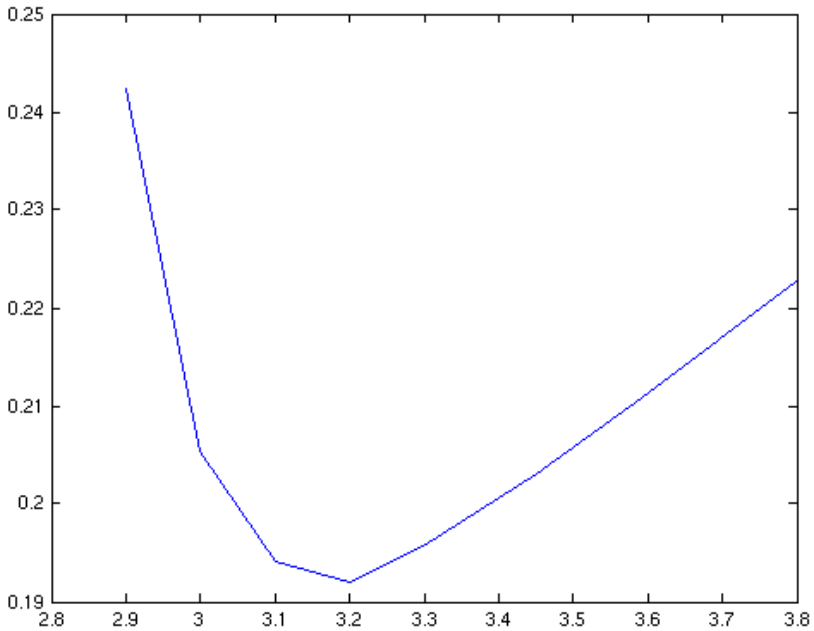
The Γ-OU martingales have the form

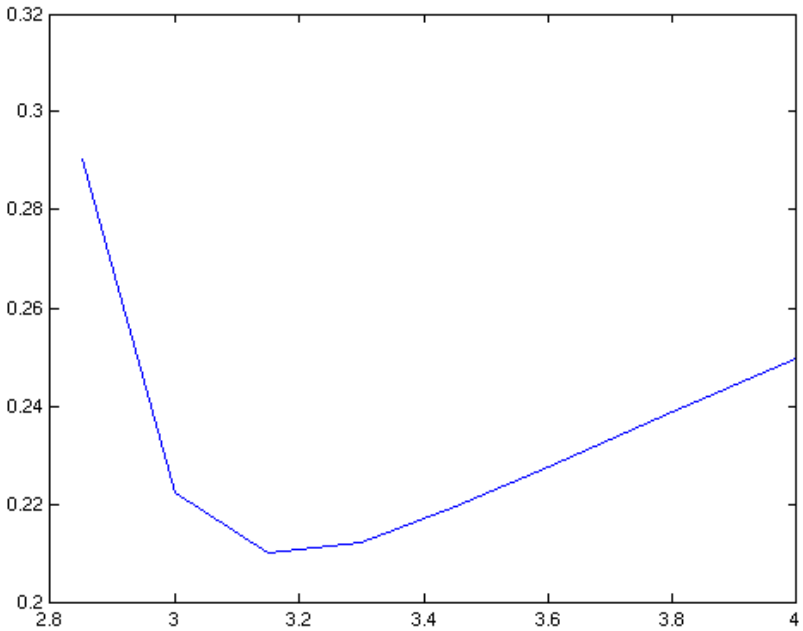
$$M_t^u = \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle\right)$$
$$= \exp\left(\underbrace{\theta(1-e^{-\lambda(T-t)})u + \int_0^{T-t} \kappa_\Gamma(e^{-\lambda s}u)ds}_{=:A} + \underbrace{e^{-\lambda(T-t)}u}_{=:B} \cdot X_t\right).$$

The moment generating function is

$$\varphi_{A+BX_{T_{N-1}}}(v) = e^{vA} E_x[e^{vBX_{T_{N-1}}}]$$
$$= \exp\left(vu\theta(1-e^{-\lambda T_N}) + v\int_0^\delta \kappa_\Gamma(e^{-\lambda s}u)ds \right.$$
$$\left. + \int_\delta^{T_N} \kappa_\Gamma(vue^{-\lambda s}) + vue^{-\lambda T_N} X_0\right).$$

Numerical illustration





Examples of caplet implied volatilities for the Γ–OU martingales

Summary and Outlook

1. We have presented a LIBOR model that
   - is very simple (Axiom 0 !), and yet ...
   - captures all the important features ...
   - especially positivity and analytical tractability.
2. Future work:
   - thorough empirical analysis
   - extensions: multiple currencies, default risk
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Thank you for your attention!