A new approach to LIBOR modeling
application of affine processes

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Interest rates – Notation

- \( B(t, T) \): time-\( t \) price of a zero coupon bond for \( T \), i.e. \( B(T, T) = 1 \);
- \( L(t, T) \): time-\( t \) forward LIBOR for \([T, T + \delta] \):
  \[
  L(t, T) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right)
  \]
- \( F(t, T, U) \): time-\( t \) forward price for \( T \) and \( U \):
  \[
  F(t, T, U) = \frac{B(t, T)}{B(t, U)}
  \]

“Master” relationship

\[
F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T) \quad (1)
\]
Interest rates evolution

- Evolution of interest rate term structure, 2003 – 2004 (picture: Th. Steiner)
Implied volatilities are constant neither across strike nor across maturity.

Variance scales non-linearly over time (see e.g. Skovmand)
Economic thought dictates that LIBOR rates should satisfy:

**Axiom 1**

*The LIBOR rate should be positive, i.e. \( L(t, T) > 0 \) for all \( t \).*

**Axiom 2**

*The LIBOR rate process should be a martingale under the (corresponding) forward measure, i.e. \( L(\cdot, T) \in \mathcal{M}(P_{T+\delta}) \).*

Practical applications require:

**Models should describe the empirical evidence adequately.**

**Models should be calibrated to liquid products (caps, atm swaptions).**

- What axioms do the existing models satisfy?
LIBOR models I (Sandmann et al, Brace et al, . . ., Eberlein & Özkan)

Ansatz: model the LIBOR rate as the exponential of a semimartingale $H$:

$$L(t, T_k) = L(0, T_k) \exp \left( \int_0^t b(s, T_k) \, ds + \int_0^t \lambda(s, T_k) \, dH_{T_k+1}^T \right), \quad (2)$$

where $b(s, T_k)$ ensures that $L(\cdot, T_k) \in \mathcal{M}(P_{T_{k+1}})$. $H$ has the $P_{T_{k+1}}$-canonical decomposition

$$H_{T_{k+1}}^t = \int_0^t \sqrt{c_s} \, dW_{T_{k+1}}^T + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu_{T_{k+1}})(ds, dx), \quad (3)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_{T_{k+1}}^t = W_{T_{k+1}}^T - \int_0^t \left( \sum_{l=k+1}^N \delta_l L(t-, T_l) \right) \frac{\lambda(t, T_l)}{1 + \delta_l L(t-, T_l)} \sqrt{c_s} \, ds, \quad (4)$$
LIBOR models II

and the $P_{T_{k+1}}$-compensator of $\mu^H$ is

$$\nu^{T_{k+1}}(ds, dx) = \left( \prod_{l=k+1}^{N} \frac{\delta_l L(t, T_l)}{1 + \delta_l L(t, T_l)} (e^{\lambda(t, T_l)x} - 1) + 1 \right) \nu^{T_*}(ds, dx).$$
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**Consequences for continuous semimartingales:**

1. Caplets can be priced in closed form;
2. Swaptions and multi-LIBOR products cannot be priced in closed form;
3. Monte-Carlo pricing is very time consuming $\Rightarrow$ coupled high dimensional SDEs!
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$$
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$$

**Consequences for continuous semimartingales:**

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**Consequences for general semimartingales:**

1. even caplets cannot be priced in closed form!
2. ditto for Monte-Carlo pricing.
“Frozen drift” approximation

- Brace et al, Schlögl, Glassermann et al ...
- replace the random terms by their deterministic initial values:

\[
\frac{\delta_t L(t-, T_l)}{1 + \delta_t L(t-, T_l)} \approx \frac{\delta_t L(0, T_l)}{1 + \delta_t L(0, T_l)}
\]  

\(\delta_t L(t-, T_l)\) deterministic characteristics \(\Rightarrow\) closed form pricing

\((-\) \) “ad hoc” approximation, no error estimates, compounded error ...

Strong Taylor approximation

- approximate the LIBOR rates in the drift/compensator by

\[
L(t, T_l) \approx L(0, T_l) + Y(t, T_l)_+
\]

where \(Y\) is the (scaled) exponential transform of \(H\) \((Y = \log e^H)\)

- theoretical foundation, error estimates, simpler equations for MC
- Siopacha and Teichmann; Hubalek, Papapantoleon & Siopacha
Forward price model I (Eberlein & Özkan, Kluge)

Ansatz: model the forward price as the exponential of a semimartingale $H$:

$$F(t, T_k) = F(0, T_k) \exp \left( \int_0^t b(s, T_k) \, ds + \int_0^t \lambda(s, T_k) \, dH_s^{T_{k+1}} \right), \quad (7)$$

where $b(s, T_k)$ ensures that $F(\cdot, T_k) = 1 + \delta L(\cdot, T_k) \in \mathcal{M}(P_{T_{k+1}})$. $H$ has the $P_{T_{k+1}}$-canonical decomposition

$$H_t^{T_{k+1}} = \int_0^t \sqrt{c_s} \, dW_{s}^{T_{k+1}} + \int_0^t \int_{\mathbb{R}} x(\mu^H - \nu^{T_{k+1}})(ds, dx), \quad (8)$$

where the $P_{T_{k+1}}$-Brownian motion is

$$W_t^{T_{k+1}} = W_t^{T_*} - \int_0^t \left( \sum_{l=k+1}^N \lambda(t, T_l) \right) \sqrt{c_s} \, ds, \quad (9)$$
and the $P_{T_{k+1}}$-compensator of $\mu^H$ is

$$\nu^{T_{k+1}}(ds, dx) = \exp \left( x \sum_{l=k+1}^{N} \lambda(t, T_l) \right) \nu^{T*}(ds, dx).$$

**Consequences:**

1. the model structure is preserved;
2. caps, swaptions and multi-LIBOR products priced in closed form.

**So what is wrong?**
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Negative LIBOR rates can occur!
Forward price model II

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So what is wrong?

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Aim: design a model where the model structure is preserved and LIBOR rates are positive.

Tool: Affine processes
An affine process in the spirit of Duffie et al (henceforth DFS) is a time-homogeneous, stochastically continuous, Markov process $X$ with state space $D = \mathbb{R}_{\geq 0}^m \times \mathbb{R}^n \subseteq \mathbb{R}^d$, with $X_0 = x \in D$, such that the moment generating function has the form

$$E_x[\exp\langle u, X_t \rangle] = \exp \left( \phi_t(u) + \langle \psi_t(u), x \rangle \right);$$

for all $u \in \mathbb{C}^d$ or $\mathbb{R}^d$ where the expectation is finite. We assume that $X$ is regular (and conservative).

Lemma (Flow property)

The functions $\phi$ and $\psi$ satisfy the flow equations:

$$\phi_{t+s}(u) = \phi_t(u) + \phi_s(\psi_t(u))$$
$$\psi_{t+s}(u) = \psi_s(\psi_t(u))$$

for all suitable $u$ and $t, s > 0$. 


Theorem (DFS, Main characterization)

*If* $X$ *is a regular affine process, then* $\phi$ *and* $\psi$ *solve the generalized Riccati equations*

\[
\frac{\partial}{\partial t} \phi_t(u) = F(\psi_t(u)), \quad \phi_0(u) = 0 \quad (12)
\]

\[
\frac{\partial}{\partial t} \psi_t(u) = R(\psi_t(u)), \quad \psi_0(u) = u, \quad (13)
\]

*where* $F$ *and* $R$ *are defined via*

\[
F(u) = \frac{\partial}{\partial t} \bigg|_{t=0} \phi_t(u) \quad (14)
\]

\[
R(u) = \frac{\partial}{\partial t} \bigg|_{t=0} \psi_t(u) \quad (15)
\]

*Moreover, . . .*
Theorem (continued)

...the functions $F$ and $R$ are of Lévy–Khintchine form:

$$F(u) = \langle b, u \rangle - c + \left\langle \frac{a}{2} u, u \right\rangle + \int_D \left( e^{\langle z, u \rangle} - 1 - \langle u, h(z) \rangle \right) m(dz)$$

$$R_i(u) = \langle \beta_i, u \rangle - \gamma_i + \left\langle \frac{\alpha_i}{2} u, u \right\rangle + \int_D \left( e^{\langle z, u \rangle} - 1 - \langle u, h(z) \rangle \right) \mu_i(dz)$$

where $(a, b, c, m, \alpha_i, \beta_i, \gamma_i, \mu_i)_{1 \leq i \leq d}$ are admissible parameters.

$X$ is a Feller process, the generator $A$ has affine form, the semimartingale characteristics (conservative case) have affine form, etc.

Conversely, to each set of admissible parameters corresponds a regular affine process on $D$ with generator $A$. 
Affine processes IV

1. **Affine processes on** $\mathbb{R}$: the admissibility conditions yield

$$F(u) = -c + bu + \frac{a}{2}u^2 + \int_{\mathbb{R}} (e^{zu} - 1 - uh(z)) m(dz)$$

$$R(u) = \beta u,$$

for $a, c \in \mathbb{R}_{\geq 0}$ and $b, \beta \in \mathbb{R}$.

- Every affine process on $\mathbb{R}$ is an Ornstein–Uhlenbeck (OU) process.

2. **Affine processes on** $\mathbb{R}_{\geq 0}$: the admissibility conditions yield

$$F(u) = -c + bu + \int_{D} (e^{zu} - 1) m(dz)$$

$$R(u) = -\gamma + \beta u + \frac{\alpha}{2}u^2 + \int_{D} (e^{zu} - 1 - uh(z)) \mu(dz),$$

for $b, c, \alpha, \gamma \in \mathbb{R}_{\geq 0}$ and $\beta \in \mathbb{R}$.

- There exist affine process on $\mathbb{R}_{\geq 0}$ which are not OU process.
**Idea:** consider an affine process; insert it in its moment generating function with inverted time. The resulting process is a martingale.

If the affine process is positive, the martingale is greater than one.

### Theorem

The process \( M^u = (M^u_t)_{0 \leq t \leq T} \) defined by

\[
M^u_t = \exp \left( \phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle \right),
\]

is a martingale. Moreover, if \( D = \mathbb{R}^d_{\geq 0}, u \in \mathcal{I} \subseteq \mathbb{R}^d_{\geq 0} \) then \( M_t \geq 1 \) a.s. for all \( t \in [0, T] \), for any \( X_0 \in \mathbb{R}^d_{\geq 0} \).
Affine LIBOR model: martingales $\geq 1$

Proof (martingality).

\[
E[M^u_t | \mathcal{F}_s] = E[\exp (\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle) | \mathcal{F}_s]
= \exp(\phi_{T-t}(u))E[\exp (\langle \psi_{T-t}(u), X_t \rangle) | \mathcal{F}_s]
= \exp (\phi_{T-t}(u) + \phi_{t-s}(\psi_{T-t}(u)) + \langle \psi_{t-s}(\psi_{T-t}(u)), X_s \rangle).
\]

Now, using the flow properties (11), we get

\[
\phi_{T-t}(u) + \phi_{t-s}(\psi_{T-t}(u)) = \phi_{T-s}(u)
\psi_{t-s}(\psi_{T-t}(u)) = \psi_{T-s}(u).
\]

Hence, the statement is proved, since

\[
E[M^u_t | \mathcal{F}_s] = \exp (\phi_{T-s}(u) + \langle \psi_{T-s}(u), X_s \rangle) = M_s.
\]
Proof (positivity).

Since $u$, $x$, and $X$ take values in $\mathbb{R}^d_{\geq 0}$, we simply have for any $t \geq 0$

$$E_x[\exp\langle u, X_t \rangle] = \exp \left( \phi_t(u) + \langle \psi_t(u), x \rangle \right) \geq 1.$$  \hfill (17)

Example (Lévy process)

Consider a Lévy subordinator, then

$$M^u_t = \exp \left( \phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle \right)$$
$$= \exp \left( (T-t)\kappa(u) + u \cdot X_t \right)$$
$$= \exp(T\kappa(u)) \exp(u \cdot X_t - t\kappa(u)),$$  \hfill (18)

which is obviously a positive martingale.
Affine LIBOR model: Ansatz

Consider a discrete tenor structure $0 = T_0 < T_1 < T_2 < \cdots < T_N$; discounted traded assets (bonds) are martingales with respect to the terminal martingale measure, i.e.

$$
\frac{B(\cdot, T_k)}{B(\cdot, T_N)} \in \mathcal{M}(P_{T_N}), \quad \text{for all } k \in \{1, \ldots, N-1\}. \quad (19)
$$

We model quotients of bond prices using the martingales $M$ as follows:

$$
\frac{B(t, T_1)}{B(t, T_N)} = M_{t}^{u_1} \quad (20)
$$

$$
\vdots
$$

$$
\frac{B(t, T_{N-1})}{B(t, T_N)} = M_{t}^{u_{N-1}}. \quad (21)
$$
Affine LIBOR model: forward prices

**Consequence:** forward prices have the following form

\[
\frac{B(t, T_k)}{B(t, T_{k+1})} = \frac{B(t, T_k)}{B(t, T_N)} \cdot \frac{B(t, T_N)}{B(t, T_{k+1})} = \frac{M_{t}^{u_k}}{M_{t}^{u_{k+1}}}
\]

\[
= \exp \left( \phi_{T_N-t}(u_k) - \phi_{T_N-t}(u_{k+1}) \right)
\]

\[
+ \langle \psi_{T_N-t}(u_k) - \psi_{T_N-t}(u_{k+1}), X_t \rangle. \tag{22}
\]

Forward measures are related via:

\[
\left. \frac{dP_{T_{k-1}}}{dP_{T_k}} \right|_{\mathcal{F}_t} = \frac{F(t, T_{k-1}, T_k)}{F(0, T_{k-1}, T_k)} = \frac{B(0, T_k)}{B(0, T_{k-1})} \times \frac{M_{t}^{u_{k-1}}}{M_{t}^{u_{k}}} \tag{23}
\]

or equivalently:

\[
\left. \frac{dP_{T_{k-1}}}{dP_{T_N}} \right|_{\mathcal{F}_t} = \frac{B(0, T_N)}{B(0, T_{k-1})} \times \frac{B(t, T_{k-1})}{B(t, T_N)} = \frac{B(0, T_N)}{B(0, T_{k-1})} \times M_{t}^{u_{k-1}}. \tag{24}
\]
The moment generating function of $X_t$ under a forward measure is

$$E_{P_{T_{k+1}}} [e^{vX_t}] = E_{P_{T_N}} [M_t^{u_{k+1}} e^{vX_t}]$$

$$= \exp \left( \phi_{T-t}(u_{k+1}) + \phi_t (\psi_{T-t}(u_{k+1}) + v) 
+ \langle \psi_t (\psi_{T-t}(u_{k+1}) + v), x \rangle \right).$$

(25)

Let us denote by $\frac{B(t, T_k)}{B(t, T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}} = e^{A_k + B_k \cdot X_t}$; then the moment generating function is

$$E_{P_{T_{k+1}}} [e^{v(A_k + B_k \cdot X_t)}] = \exp \left( v \phi_{T-N-t}(u_k) + (1 - v) \phi_{T-N-t}(u_{k+1}) 
+ \phi_t (v \psi_{T-N-t}(u_k) + (1 - v) \psi_{T-N-t}(u_{k+1})) 
+ \langle \psi_t (v \psi_{T-N-t}(u_k) + (1 - v) \psi_{T-N-t}(u_{k+1})), x \rangle \right).$$

(26)
Affine LIBOR model: consequences

1. **The model structure is preserved!**
2. More precisely, with respect to any forward measure all LIBOR rates are of exponential-affine form.
3. Caps, swaptions and other multi-LIBOR products can be priced in closed form.
4. Pricing (always) takes place under the terminal measure!
Affine LIBOR model: caplet pricing

We can re-write the payoff of a caplet as follows (here $\mathcal{K} := 1 + \delta K$):

$$
\delta(L(T_k, T_k) - K)^+ = (1 + \delta L(T_k, T_k) - 1 + \delta K)^+
$$

$$
= \left( \frac{M_{T_k}^{u_k}}{M_{T_k}^{u_{k+1}}} - \mathcal{K} \right)^+
$$

$$
= \left( e^{A_k + B_k \cdot X_{T_k}} - \mathcal{K} \right)^+. \quad (27)
$$

Then we can price caplets by Fourier-transform methods:

$$
C(T_k, K) = B(0, T_{k+1}) E_{\mathcal{P}_{T_{k+1}}} \left[ \delta(L(T_k, T_k) - K)^+ \right]
$$

$$
= \frac{\mathcal{K} B(0, T_{k+1})}{2\pi} \int_{\mathbb{R}} \mathcal{K}^{iv-R} \frac{\varphi_{A_k+B_k X_{T_k}}(R - iv)}{(R - iv)(R - 1 - iv)} dv \quad (28)
$$

where $\varphi_{A_k+B_k X_{T_k}}$ is given by (26).
Example: Γ-OU martingales

Let $H$ be a Gamma subordinator, i.e. $H_t \sim \Gamma(\alpha, \beta)$, $\alpha, \beta > 0$. The cumulant is

$$\kappa_{\Gamma}(v) = -\beta \log \left(1 - \frac{v}{\alpha}\right). \quad (29)$$

The Gamma-OU process is an affine process $X$ with state space $D = \mathbb{R}_{\geq 0}$, that satisfies the SDE

$$dX_t = -\lambda(X_t - \theta)dt + dH_t, \quad X_0 = x \in \mathbb{R}_{\geq 0}, \quad (30)$$

where $\lambda, \theta > 0$. The moment generating function of the Gamma-OU process is

$$E_x[e^{vX_t}] = \exp \left(\theta (1 - e^{-\lambda t})v + \int_0^t \kappa_{\Gamma}(e^{-\lambda s}v)ds + x \cdot e^{-\lambda t} \phi_t(v)\right). \quad (31)$$
The \(\Gamma\)-OU martingales have the form

\[
M_t^u = \exp \left( \phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle \right)
\]

\[
= \exp \left( \theta(1 - e^{-\lambda(T-t)})u + \int_0^{T-t} \kappa_{\Gamma}(e^{-\lambda s} u)\,ds + e^{-\lambda(T-t)}u \cdot X_t \right).
\]

\[
=: A
\]

The moment generating function is

\[
\varphi_{A+BX_{T_{N-1}}}(v) = e^{vA} E_x [e^{vBX_{T_{N-1}}}] = \exp \left( vu\theta(1 - e^{-\lambda T_N}) + v \int_0^{\delta} \kappa_{\Gamma}(e^{-\lambda s} u)\,ds + \int_0^{T_N} \kappa_{\Gamma}(v e^{-\lambda s}) + v u e^{-\lambda T_N} X_0 \right).
\]
Numerical illustration

Examples of caplet implied volatilities for the Γ–OU martingales
We have presented a LIBOR model that
- is very simple (Axiom 0!), and yet . . .
- captures all the important features . . .
- especially positivity and analytical tractability.

Future work:
- thorough empirical analysis
- extensions: multiple currencies, default risk

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Thank you for your attention!