On Highly Efficient Methods for Pricing Options with and without Early Exercise

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Contents

- Brief overview of derivative pricing
- Our contribution: The COS method:
  - Efficient way to recover the density function;
  - Efficient alternative for FFT-based methods for calibration;
  - Focus on Lévy processes and Heston stochastic volatility
- COS method for European options
- Bermudan and discretely-monitored barrier options
- Credit Default Swaps

Multi-D asset prices

- Asset price, S_i, can be modeled by geometric Brownian motion:

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i dW_i(t),$$

with $W_i(t)$ Wiener process, μ_i drift, σ_i volatility.

$\Rightarrow$ Itô's Lemma: multi-D Black-Scholes equation: (for a European option)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{d}[\sigma_i\sigma_j\rho_{i,j}S_iS_j\frac{\partial^2 V}{\partial S_i \partial S_j}] + \sum_{i=1}^{d}[rS_i\frac{\partial V}{\partial S_i}] - rV = 0 \ .$$

- Correlation between a pair of assets, S_i and S_j, is $\rho_{i,j}$.

Pricing: Feynman-Kac Theorem

Given the system of stochastic differential equations:

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i dW_i(t)$$

with $\mathbb{E}\{dW_i(t)dW_j(t)\} = \rho_{ij}dt$ and an option, V, such that

$$V(\mathbf{S}, t) = e^{-r(T-t)}\mathbb{E}^Q\{V(\mathbf{S}(T), T)|\mathbf{S}(t)\}$$

with the sum of the first derivatives of the option square integrable.

Then the value, $V(\mathbf{S}(t), t)$, is the unique solution of the final condition problem

$$\begin{cases} \frac{\partial V}{\partial t} & + & \frac{1}{2}\sum_{i,j=1}^{d}[\sigma_i\sigma_j\rho_{i,j}S_iS_j\frac{\partial^2 V}{\partial S_i \partial S_j}] + \sum_{i=1}^{d}[rS_i\frac{\partial V}{\partial S_i}] - rV = 0, \\ V(\mathbf{S}, T) & = & \text{given} \end{cases}$$

Numerical Pricing Approach

- One can apply several numerical techniques to calculate the option price:
  - Numerical integration,
  - Monte Carlo simulation,
  - Numerical solution of the partial-(integro) differential equation (P(I)DE)
- Each of these methods has its merits and demerits.
- Numerical challenges:
  - The problem's dimensionality
  - Speed of solution methods
  - Early exercise feature (P(I)DE $\longrightarrow$ free boundary problem)

Lévy Processes

- Use Heston's model, or a Lévy process with jumps, to better fit market data, and allow for smile effects
- A Lévy process is a stochastic process that starts at 0 and has independent and stationary increments.
- The Lévy processes of our interest here include
  - The CGMY model (generalized VG model; driven by four parameters);
  - The Normal Inverse Gaussian (NIG) model (a variance-mean mixture of a Gaussian distribution with an inverse Gaussian; driven by four parameters).

Motivation

- Our motivation: To derive pricing methods that
  - are computationally fast
  - are not restricted to Gaussian-based models
  - should work as long as we have a characteristic function,

$$\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx; \quad f(x) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re}\,(\phi(\omega) e^{-i\omega x}) d\omega$$

  - Preferably faster than approaches based on the FFT
- The characteristic function of a Lévy process equals:

$$\phi(\omega) \quad = \quad \exp\left(t(i\mu\omega - \frac{1}{2}\sigma^2\omega^2 + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x \mathbf{1}_{[|x|<1]} \nu(dx))\right),$$

the celebrated Lévy-Khinchine formula.

Fourier-Cosine Expansion

- The COS method:
  - Exponential convergence;
  - Greeks are obtained at no additional cost.
  - For discretely-monitored barrier and Bermudan options as well;
- The basic idea:
  - Replace the density by its Fourier-cosine series expansion;
  - Series coefficients have simple relation with characteristic function.

Series Coefficients of the Density and the Ch.F.

- Fourier-Cosine expansion of density function on interval $[a, b]$:

$$f(x) = \sum{}'_{n=0}^{\infty} F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

- F_n has direct relation to ch.f., $\phi(\omega) := \int_{\mathbb{R}} f(x) e^{i\omega x} dx$ ($\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0$),

$$\begin{aligned} F_n \approx A_n \quad &:= \quad \frac{2}{b-a} \int_{\mathbb{R}} f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \quad \frac{2}{b-a} \mathrm{Re}\left\{ \phi\left(\frac{n\pi}{b-a}\right) \exp\left(-i \frac{ka\pi}{b-a}\right)\right\}. \end{aligned}$$

Recovering Densities

- Replace F_n by A_n, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} \sum{}'_{n=0}^{N-1} \mathrm{Re}\left\{ \phi\left(\frac{n\pi}{b-a}; x\right) \exp\left(in\pi \frac{-a}{b-a}\right)\right\} \cos\left(n\pi \frac{x-a}{b-a}\right),$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a,b] = [-10, 10]$ and $x = \{-5, -4, \cdots, 4, 5\}$.

N	4	8	16	32	64
error	0.2538	0.1075	0.0072	4.04e-07	3.33e-16
cpu time (sec.)	0.0025	0.0028	0.0025	0.0031	0.0032

Exponential error convergence in N.

Pricing European Options

- Start from the risk-neutral valuation formula:

$$v(x,t_0) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[v(y,T)|x\right] = e^{-r\Delta t}\int_{\mathbb{R}} v(y,T)f(y|x)dy.$$

- Truncate the integration range:

$$v(x,t_0) = e^{-r\Delta t}\int_{[a,b]} v(y,T)f(y|x)dy + \varepsilon.$$

- Replace the density by the COS approximation, and interchange summation and integration:

$$\hat{v}(x,t_0) = e^{-r\Delta t}\sum{}'_{n=0}^{N-1}\mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a};x\right)e^{-in\pi\frac{a}{b-a}}\right\}V_n,$$

where the series coefficients of the payoff, V_n, are analytic.

Pricing European Options

- Log-asset prices: $x := \ln(S_0/K) \quad \text{and} \quad y := \ln(S_T/K)$,
- The payoff for European options reads

$$v(y,T) \equiv [\alpha \cdot K(e^y - 1)]^+.$$

- For a call option, we obtain

$$\begin{aligned} V_k^{call} &= \frac{2}{b-a}\int_0^b K(e^y-1)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}K\left(\chi_k(0,b)-\psi_k(0,b)\right), \end{aligned}$$

- For a vanilla put, we find

$$V_k^{put} = \frac{2}{b-a}K\left(-\chi_k(a,0)+\psi_k(a,0)\right).$$

Characteristic Functions Heston Model

- The characteristic function of the log-asset price for Heston's model:

$$\begin{aligned}\varphi_{hes}(\omega;u_0) &= \exp\left(i\omega\mu\Delta t + \frac{u_0}{\eta^2}\left(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}}\right)(\lambda - i\rho\eta\omega - D)\right)\cdot \\ &\quad \exp\left(\frac{\lambda\bar{u}}{\eta^2}\left(\Delta t(\lambda - i\rho\eta\omega - D) - 2\log(\frac{1-Ge^{-D\Delta t}}{1-G})\right)\right),\end{aligned}$$

with $D = \sqrt{(\lambda - i\rho\eta\omega)^2 + (\omega^2 + i\omega)\eta^2}$ and $G = \frac{\lambda - i\rho\eta\omega - D}{\lambda - i\rho\eta\omega + D}$.

- For Lévy and Heston models, the ChF can be represented by

$$\begin{aligned}\phi(\omega;\mathbf{x}) &= \varphi_{levy}(\omega)\cdot e^{i\omega\mathbf{x}} \quad \text{with} \quad \varphi_{levy}(\omega) := \phi(\omega;0), \\ \phi(\omega;\mathbf{x},u_0) &= \varphi_{hes}(\omega;u_0)\cdot e^{i\omega\mathbf{x}},\end{aligned}$$

Characteristic Functions Lévy Processes

- For the CGMY/KoBoI model:

$$\begin{aligned}\varphi_{levy}(\omega) &= \exp\left(i\omega(r-q)\Delta t - \frac{1}{2}\omega^2\sigma^2\Delta t\right)\cdot \\ &\quad \exp\left(\Delta t C\Gamma(-Y)[(M-i\omega)^Y - M^Y + (G+i\omega)^Y - G^Y]\right),\end{aligned}$$

  where $\Gamma(\cdot)$ represents the gamma function.
- The parameters should satisfy $C \geq 0, G \geq 0, M \geq 0$ and $Y < 2$.
- The characteristic function of the log-asset price for NIG:

$$\varphi_{NIG}(\omega) = \exp\left(i\omega\mu + \delta(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+i\omega)^2})\right)$$

  with $\alpha, \delta > 0, \beta \in (-\alpha, \alpha - 1)$

Heston Model

- We can present the V_k as $\mathbf{V}_k = U_k \mathbf{K}$, where

$$U_k = \begin{cases} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for a call} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for a put.} \end{cases}$$

- The pricing formula simplifies for Heston and Lévy processes:

$$v(\mathbf{x}, t_0) \approx \mathbf{K} e^{-r\Delta t} \cdot \mathrm{Re}\left\{ \sum\nolimits_{n=0}^{\prime N-1} \varphi\left(\frac{n\pi}{b-a}\right) U_n \cdot e^{in\pi \frac{\mathbf{x}-a}{b-a}} \right\},$$

where $\varphi(\omega) := \phi(\omega; 0)$

Numerical Results

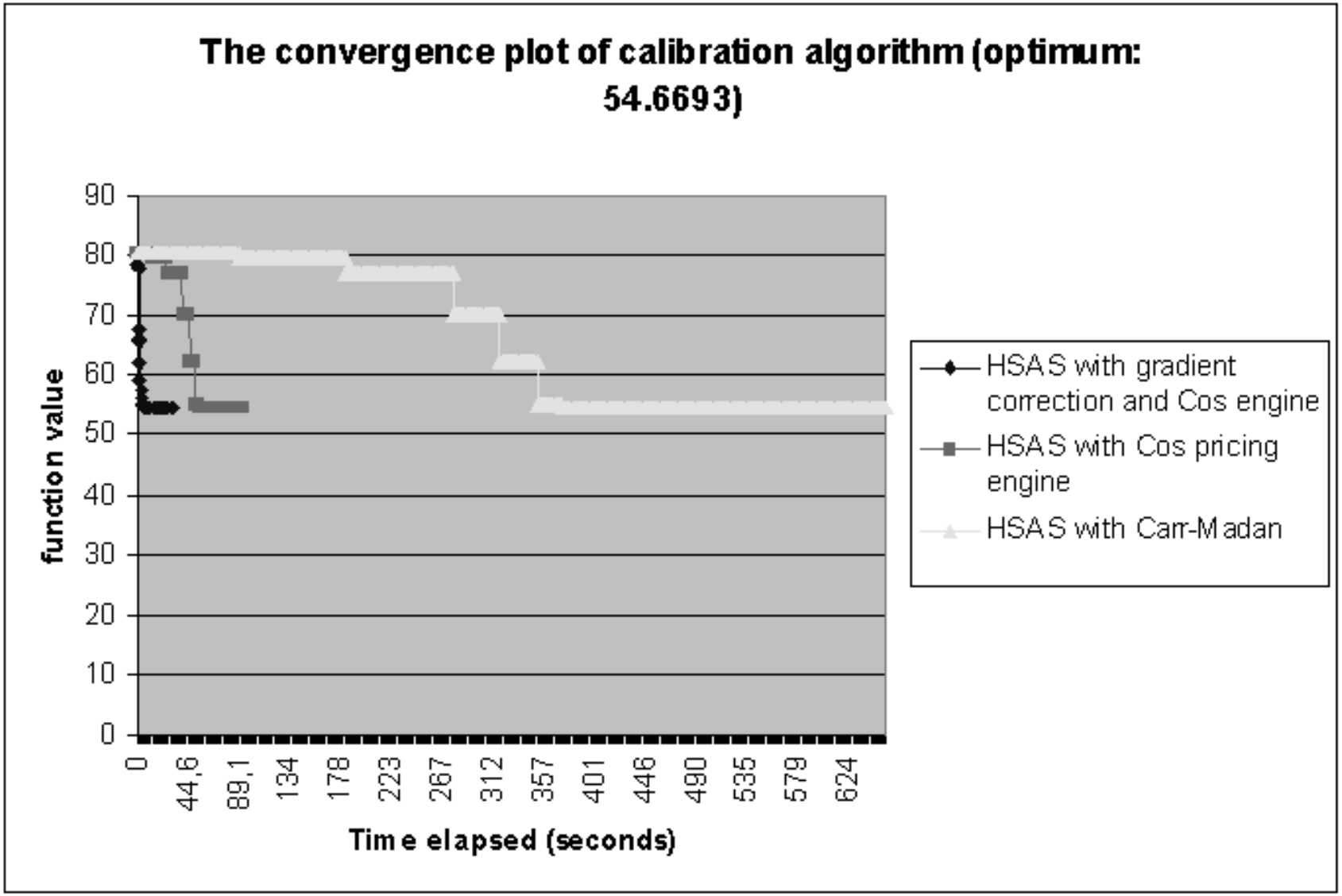
Pricing for 21 strikes $K = 50, 55, 60, \cdots, 150$ under Heston's model. Other parameters: $S_0 = 100, r = 0, q = 0, T = 1, \lambda = 1.5768, \eta = 0.5751, \bar{u} = 0.0398, u_0 = 0.0175, \rho = -0.5711$.

COS	N	96	128	160
	(msec.)	2.039	2.641	3.220
	max. abs. err.	4.52e-04	2.61e-05	$4.40e-06$
Carr-Madan	N	2048	4096	8192
	(msec.)	20.36	37.69	76.02
	max. abs. error	2.61e-01	$2.15e-03$	2.08e-07

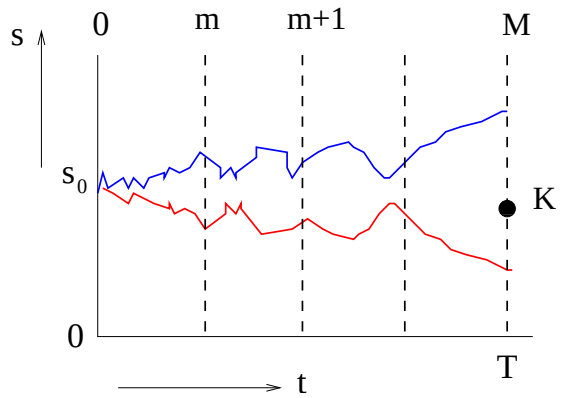
Error analysis for the COS method is provided in the paper.

Numerical Results within Calibration

- Calibration for Heston's model: Around 10 times faster than Carr-Madan.



Pricing Bermudan Options



- The pricing formulae

$$\begin{cases} c(x,t_m) &= e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_{m+1})f(y|x)dy \\ v(x,t_m) &= \max\left(g(x,t_m),c(x,t_m)\right) \end{cases}$$

and $v(x,t_0) = e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_1)f(y|x)dy$.

  - Use Newton's method to locate the early exercise point x_m^*, which is the root of $g(x,t_m) - c(x,t_m) = 0$.
  - Recover $V_n(t_1)$ recursively from $V_n(t_M)$, $V_n(t_{M-1}), \cdots, V_n(t_2)$.
  - Use the COS formula for $v(x,t_0)$.

V_k-Coefficients

- Once we have x_m^*, we split the integral, which defines $V_k(t_m)$:

$$V_k(t_m) = \begin{cases} C_k(a, x_m^*, t_m) + G_k(x_m^*, b), & \text{for a call,} \\ G_k(a, x_m^*) + C_k(x_m^*, b, t_m), & \text{for a put,} \end{cases}$$

for $m = M-1, M-2, \cdots, 1$. whereby

$$G_k(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

and

$$C_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} \hat{c}(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

Theorem

The $G_k(x_1, x_2)$ are known analytically and the $C_k(x_1, x_2, t_m)$ can be computed in $O(N \log_2(N))$ operations with the Fast Fourier Transform.

Bermudan Details

- Formula for the coefficients $C_k(x_1, x_2, t_m)$:

$$C_k(x_1, x_2, t_m) = e^{-r\Delta t}\mathrm{Re}\left\{\sum{}'_{j=0}^{N-1} \varphi_{levy}\left(\frac{j\pi}{b-a}\right) V_j(t_{m+1}) \cdot M_{k,j}(x_1, x_2)\right\},$$

where the coefficients $M_{k,j}(x_1, x_2)$ are given by

$$M_{k,j}(x_1, x_2) := \frac{2}{b-a}\int_{x_1}^{x_2} e^{ij\pi\frac{x-a}{b-a}} \cos\left(k\pi\frac{x-a}{b-a}\right) dx,$$

- With fundamental calculus, we can rewrite $M_{k,j}$ as

$$M_{k,j}(x_1, x_2) = -\frac{i}{\pi}\left(M^c_{k,j}(x_1, x_2) + M^s_{k,j}(x_1, x_2)\right),$$

Hankel and Toeplitz

- Matrices $M_c = \{M^c_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ and $M_s = \{M^s_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ have special structure for which the FFT can be employed: M_c is a Hankel matrix,

$$M_c = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{N-1} \\ m_1 & m_2 & \cdots & \cdots & m_N \\ \vdots & & & & \vdots \\ m_{N-2} & m_{N-1} & \cdots & & m_{2N-3} \\ m_{N-1} & \cdots & & m_{2N-3} & m_{2N-2} \end{bmatrix}_{N\times N}$$

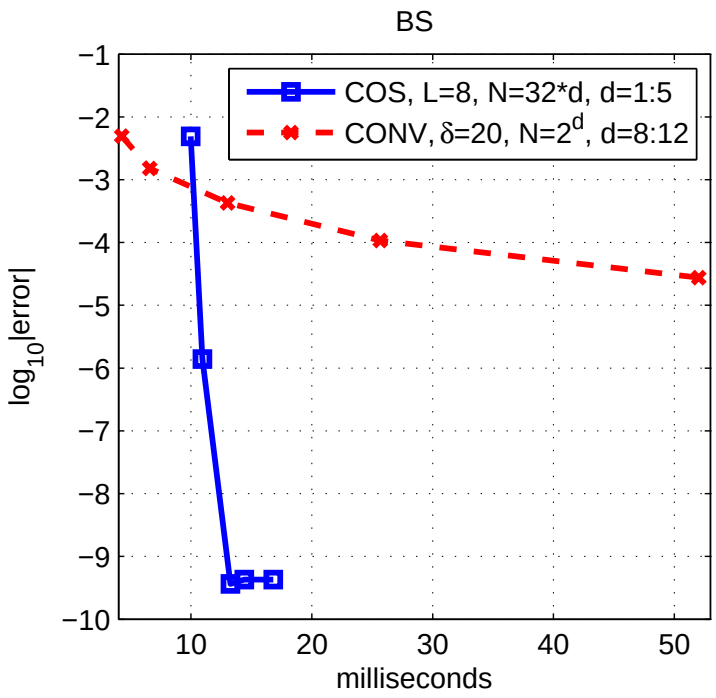
and M_s is a Toeplitz matrix,

$$M_s = \begin{bmatrix} m_0 & m_1 & \cdots & m_{N-2} & m_{N-1} \\ m_{-1} & m_0 & m_1 & \cdots & m_{N-2} \\ \vdots & & \ddots & & \vdots \\ m_{2-N} & \cdots & m_{-1} & m_0 & m_1 \\ m_{1-N} & m_{2-N} & \cdots & m_{-1} & m_0 \end{bmatrix}_{N\times N}$$

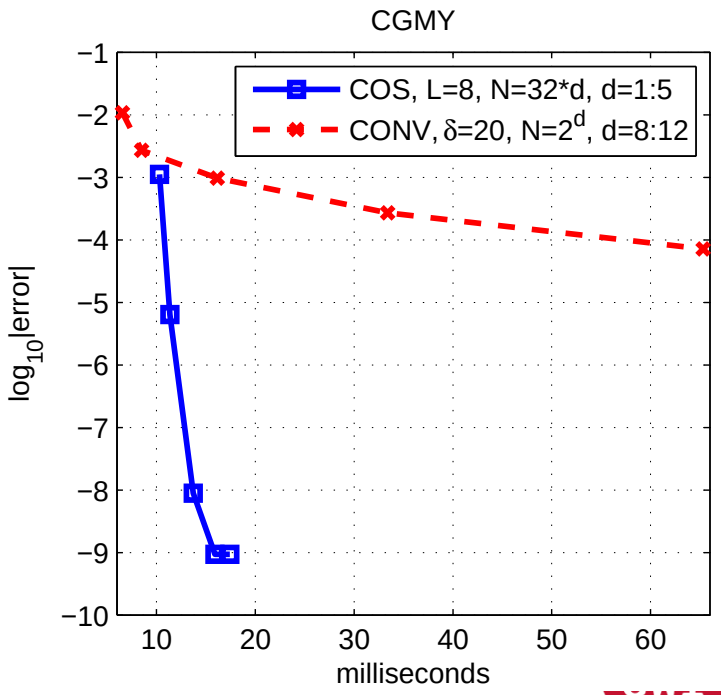
Bermudan puts with 10 early-exercise dates

Table: Test parameters for pricing Bermudan options

Test No.	Model	S_0	K	T	r	σ	Other Parameters
2	BS	100	110	1	0.1	0.2	–
3	CGMY	100	80	1	0.1	0	$C = 1, G = 5, M = 5, Y = 1.5$



(a) BS



(b) CGMY with $Y = 1.5$

Pricing Discrete Barrier Options

- The price of an M-times monitored up-and-out option satisfies

$$\begin{cases} c(x, t_{m-1}) &= e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} v(x, t_m) f(y|x) dy \\ v(x, t_{m-1}) &= \begin{cases} e^{-r(T - t_{m-1})} Rb, & x \geq h \\ c(x, t_{m-1}), & x < h \end{cases} \end{cases}$$

  where $h = \ln(H/K)$, and $v(x, t_0) = e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} v(x, t_1) f(y|x) dy$.
- The technique:
  - Recover $V_n(t_1)$ recursively, from $V_n(t_M)$, $V_n(t_{M-1}), \cdots, V_n(t_2)$ in $O((M-1)N \log_2(N))$ operations.
  - Split the integration range at the barrier level (no Newton required)
  - Insert $V_n(t_1)$ in the COS formula to get $v(x, t_0)$, in $O(N)$ operations.

Monthly-monitored Barrier Options

Table: Test parameters for pricing barrier options

Test No.	Model	S_0	K	T	r	q	Other Parameters
1	NIG	100	100	1	0.05	0.02	$\alpha = 15, \beta = -5, \delta = 0.5$

Option Type	Ref. Val.	N N	time (milli-sec.)	error
DOP	2.139931117	2^7	3.7	1.28e-3
		2^8	5.4	4.65e-5
		2^9	8.4	1.39e-7
		2^{10}	14.7	1.38e-12
DOC	8.983106036	2^7	3.7	1.09e-3
		2^8	5.3	3.99e-5
		2^9	8.3	9.47e-8
		2^{10}	14.8	5.61e-13

Credit Default Swaps (with W. Schoutens, H. Jönsson)

- Credit default swaps (CDSs), the basic building block of the credit risk market, offer investors the opportunity to either buy or sell default protection on a reference entity.
- The protection buyer pays a premium periodically for the possibility to get compensation if there is a credit event on the reference entity until maturity or the default time, which ever is first.
- If there is a credit event the protection seller covers the losses by returning the par value. The premium payments are based on the CDS spread.

CDS and COS

- CDS spreads are based on a series of default/survival probabilities, that can be very efficiently recovered using the COS method. It is also very flexible w.r.t. the underlying process as long as it is Lévy.
- The flexibility and the efficiency of the method are demonstrated via a calibration study of the iTraxx Series 7 and Series 8 quotes.

Lévy Default Model

- Definition of default: For a given recovery rate, R, default occurs the first time the firm's value is below the "reference value" RV_0.
- As a result, the survival probability in the time period $(0, t]$ is nothing but the price of a digital down-and-out barrier option without discounting.

$$
\begin{aligned}
\mathrm{P}_{surv}(t) &= \mathrm{P}_{\mathbb{Q}}\left(X_s > \ln R, \text{for all } 0 \leq s \leq t\right) \\
&= \mathrm{P}_{\mathbb{Q}}\left(\min_{0 \leq s \leq t} X_s > \ln R\right) \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\left(\min_{0 \leq s \leq t} X_s > \ln R\right)\right]
\end{aligned}
$$

Survival Probability

- Assume there are only a finite number of observing dates.

$$\mathrm{P}_{surv}(\tau) = \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\Big(X_{\tau_1} \in [\ln R, \infty)\Big) \cdot \mathbf{1}\Big(X_{\tau_2} \in [\ln R, \infty)\Big) \cdots \mathbf{1}\Big(X_{\tau_M} \in [\ln R, \infty)\Big)\right]$$

where $\tau_k = k\Delta\tau$ and $\Delta\tau := \tau/M$.

- The survival probability then has the following recursive expression:

$$\begin{cases} \mathrm{P}_{surv}(\tau) & := & p(x=0,\tau_0) \\ p(x,\tau_m) & := & \int_{\ln R}^{\infty} f_{X_{\tau_{m+1}}|X_{\tau_m}}(y|x)p(y,\tau_{m+1})\,dy, \quad m = M-1,\cdots,2,1,0, \\ p(x,\tau_M) & := & 1 \end{cases}$$

$f_{X_{\tau_{m+1}}|X_{\tau_m}}(\cdot|\cdot)$ denotes the conditional probability density of $X_{\tau_{m+1}}$ given X_{τ_m}.

The Fair Spread of a Credit Default Swap

- The *fair spread*, C, of a CDS at the initialization date is the spread that equalizes the present value of the premium leg and the present value of the protection leg, i.e.

$$C = \frac{(1-R)\left(\int_0^T \exp(-r(s)s)\mathrm{d}\mathrm{P}_{def}(s)\right)}{\int_0^T \exp(-r(s)s)\mathrm{P}_{surv}(s)\mathrm{d}s},$$

- It is actually based on a series of survival probabilities on different time intervals:

$$C = \frac{(1-R)\sum_{j=0}^{J}\frac{1}{2}\left[\exp(-r_j t_j)+\exp(-r_{j+1}t_{j+1})\right]\left[\mathrm{P}_{surv}(t_j)-\mathrm{P}_{surv}(t_{j+1})\right]}{\sum_{j=0}^{J}\frac{1}{2}\left[\exp(-r_j t_j)\mathrm{P}_{surv}(t_j)+\exp(-r_{j+1}t_{j+1})\mathrm{P}_{surv}(t_{j+1})\right]\Delta t}+\epsilon,$$

The COS Formula for Survival Probabilities

- Replace the conditional density by the COS (semi-analytical) expression, the survival probability then satisfies

$$\begin{cases} \mathrm{P}_{surv}(\tau) & = & p(x=0,\tau_0). \\ p(x,\tau_0) & = & \sum\nolimits'_{n=0}^{N-1} \phi_n(x) \cdot P_n(\tau_1), \end{cases}$$

- The only thing one needs is $\{P_n(\tau_1)\}_{n=0}^{N-1}$, which can be recovered from $\{P_n(\tau_M)\}_{n=0}^{N-1}$ via backwards induction.

Backwards Induction

- Starting from the definition of $P_n(\tau_m)$, we apply the COS reconstruction of $p(y, \tau_m)$ to get
$$\mathbf{P}(\tau_m) = \mathrm{Re}\left\{\Omega\ \Lambda\right\}\ \mathbf{P}(\tau_{m+1}),$$

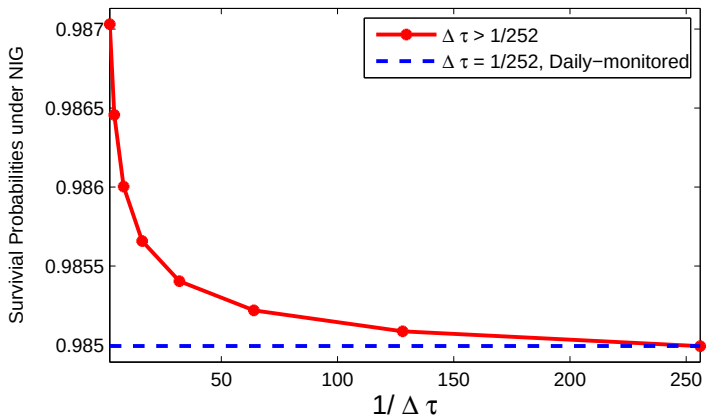
- Applying this recursively backwards in time, we get
$$\mathbf{P}(\tau_1) = \left(\mathrm{Re}\left\{\Omega\ \Lambda\right\}\right)^{M-1} \mathbf{P}(\tau_M)$$

- For this recursive matrix-vector-product, there exists a fast algorithm, e.g.
$$\mathbf{P}(\tau_1) = \mathrm{Re}\left\{\Omega\ \left[\Lambda\quad \mathrm{Re}\left\{\Omega\ \left[\Lambda\quad \mathrm{Re}\left\{\Omega\ \left[\Lambda\ \mathbf{P}(t_3)\right]\right\}\ \right]\right\}\ \right]\right\}.$$

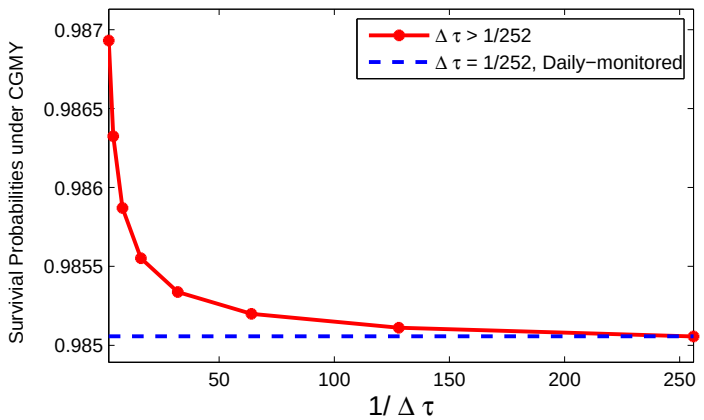
- The FFT algorithm can be applied because $\Omega = H + T$, where H is a Hankel matrix and T is a Toeplitz matrix.

Convergence of Survival Probabilities

- Ideally, the survival probabilities should be monitored daily, i.e. $\Delta\tau = 1/252$. That is, $M = 252T$, which is a bit too much for $T = 5, 7, 10$ years.
- For Black-Scholes' model, there exist rigorous proof of the convergence of discrete barrier options to otherwise identical continuous options [Kou,2003].
- We observe similar convergence under NIG, CGMY:



(c)



(d)

- Convergence of the 1-year survival probability w.r.t. $\Delta\tau$.

Error Convergence

- The error convergence of the COS method is usually exponential in N

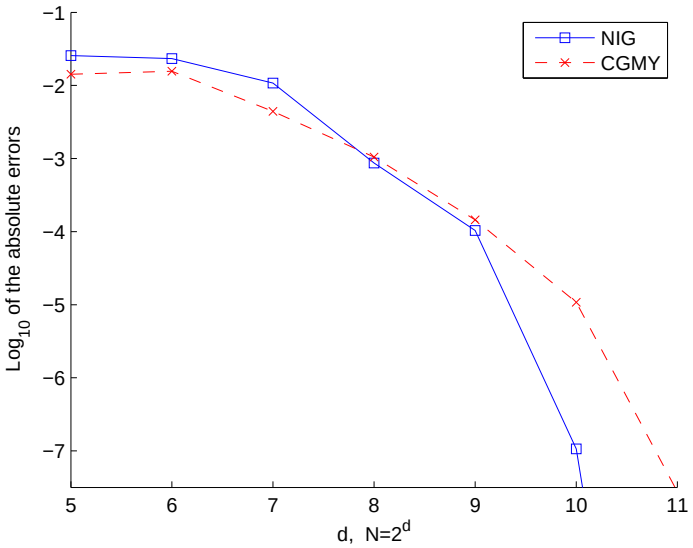


Figure: Convergence of $\mathrm{P}_{surv}(\Delta\tau = 1/48)$ w.r.t. N for NIG and CGMY

Calibration Setting

- The data sets: weekly quotes from iTraxx Series 7 (S7) and 8 (S8). After cleaning the data we were left with 119 firms from Series 7 and 123 firms from Series 8. Out of these firms 106 are common to both Series.
- The interest rates: EURIBOR swap rates.
- We have chosen to calibrate the models to CDSs spreads with maturities 1, 3, 5, 7, and 10 years.

The Objective Function

- To avoid the ill-posedness of the inverse problem we defined here, the objective function is set to

$$F_{obj} = \text{rmse} + \gamma \cdot ||\mathbf{X}_2 - \mathbf{X}_1||_2,$$

where

$$\text{rmse} = \sqrt{\sum_{\text{CDS}} \frac{(\text{market CDS spread} - \text{model CDS spread})^2}{\text{number of CDSs on each day}}},$$

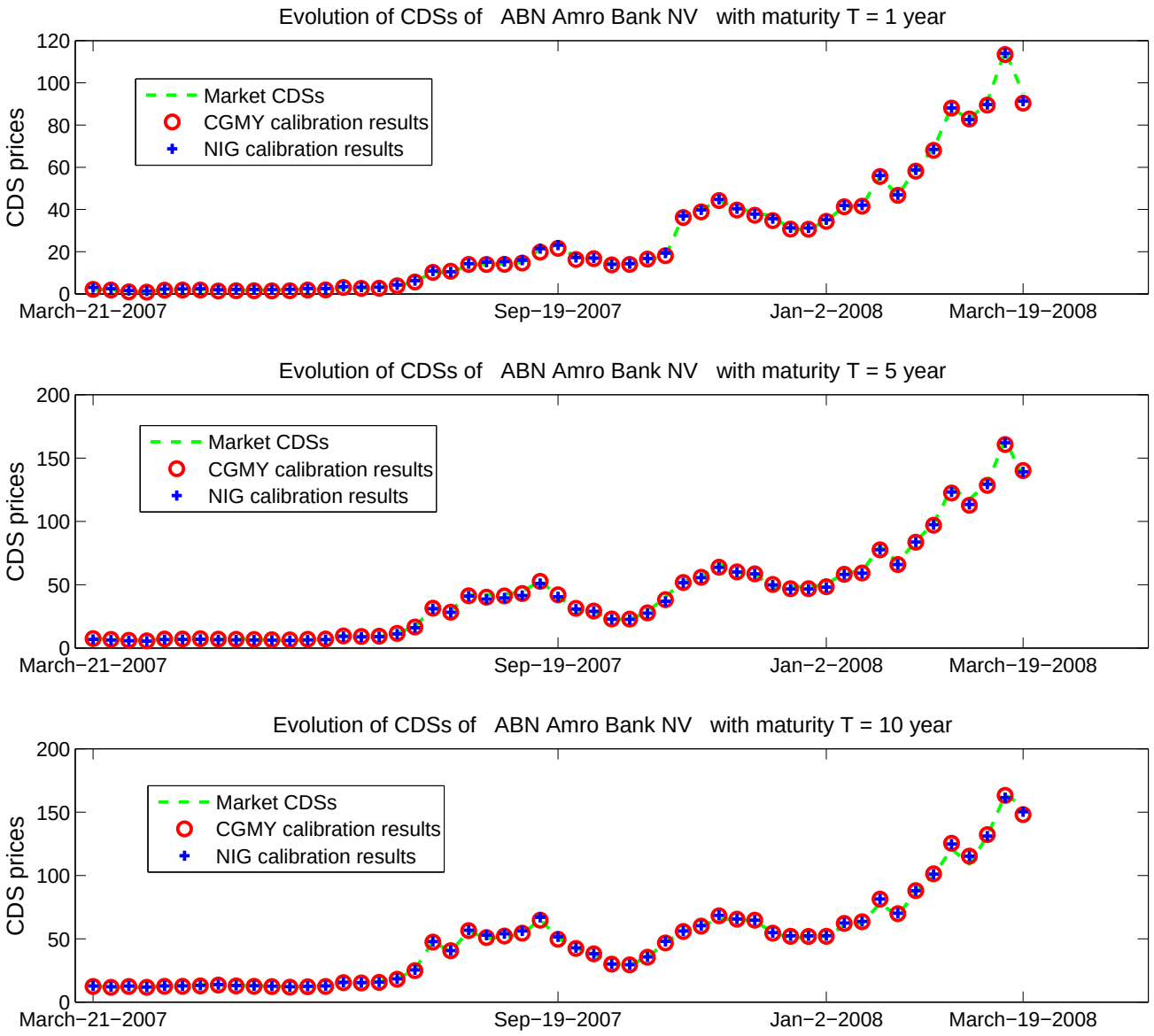
$|| \cdot ||_2$ denotes the L_2–norm operator, and $\mathbf{X}_2$ and $\mathbf{X}_1$ denote the parameter vectors of two neighbor data sets.

Good Fit to Market Data

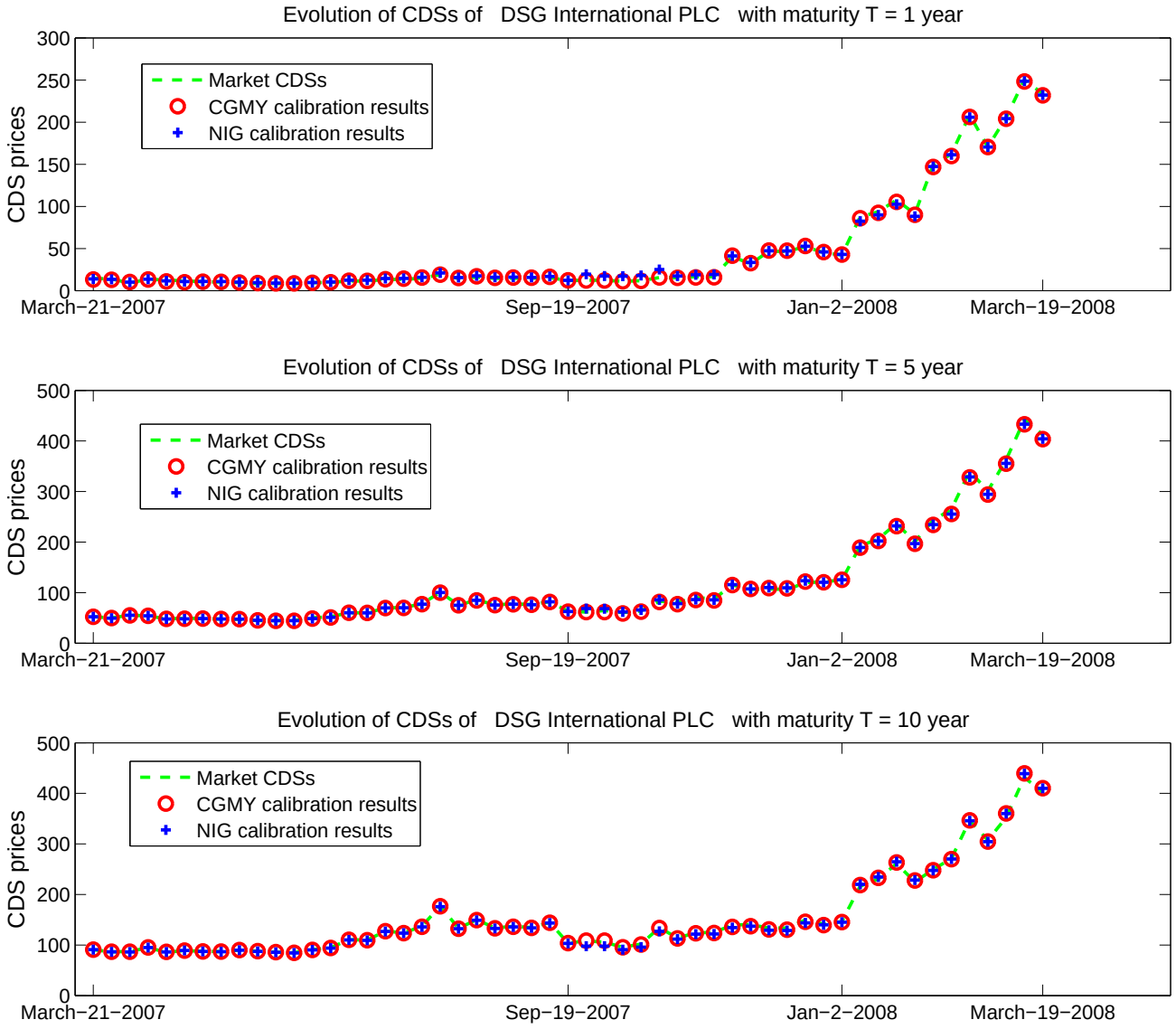
Table: Summary of calibration results of all 106 firms in both S7 and S8 of iTraxx quotes

RMSEs	NIG in S7	CGMY in S7	NIG in S8	CGMY in S8
Average (bp.)	0.89	0.79	1.65	1.54
Min. (bp.)	0.22	0.29	0.27	0.46
Max. (bp.)	2.29	1.97	4.27	3.52

A Typical Example



An Extreme Case



NIG Parameters for "ABN AMRO Bank"

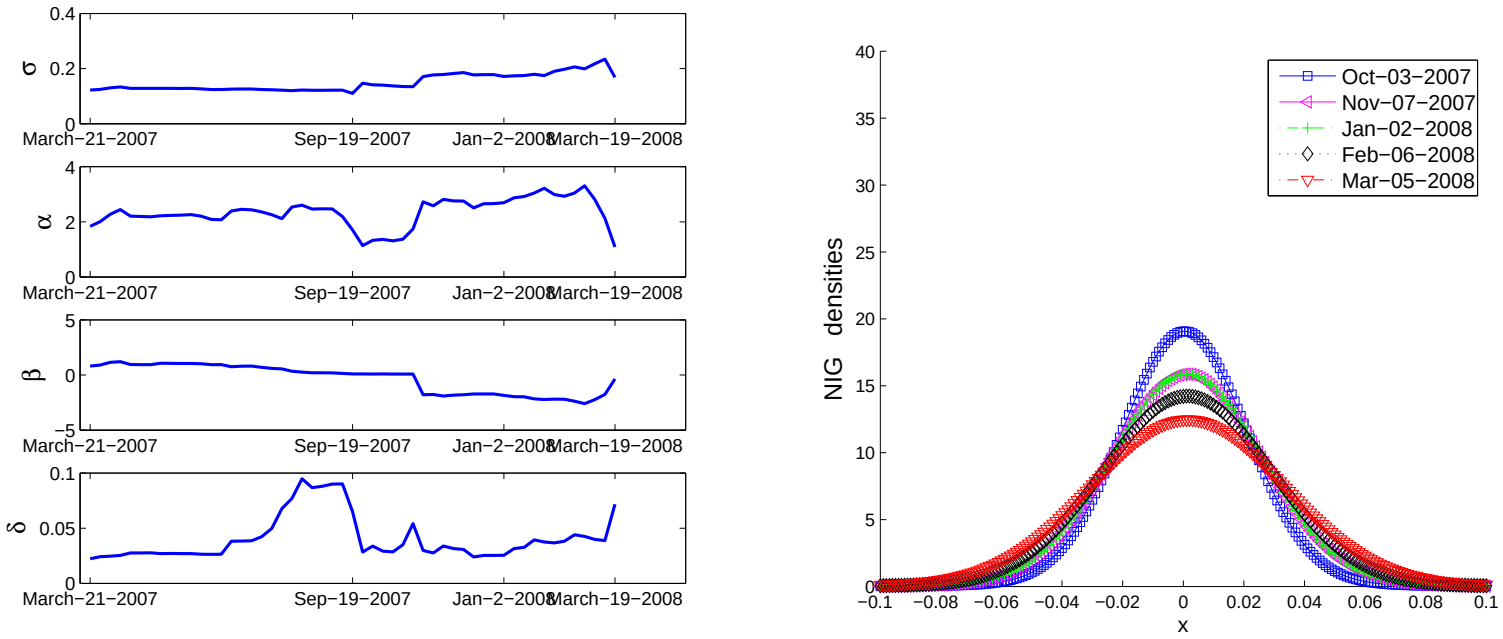


Figure: Evolution of the NIG parameters and densities of "ABN AMRO Bank"

NIG Parameters for "DSG International PLC"

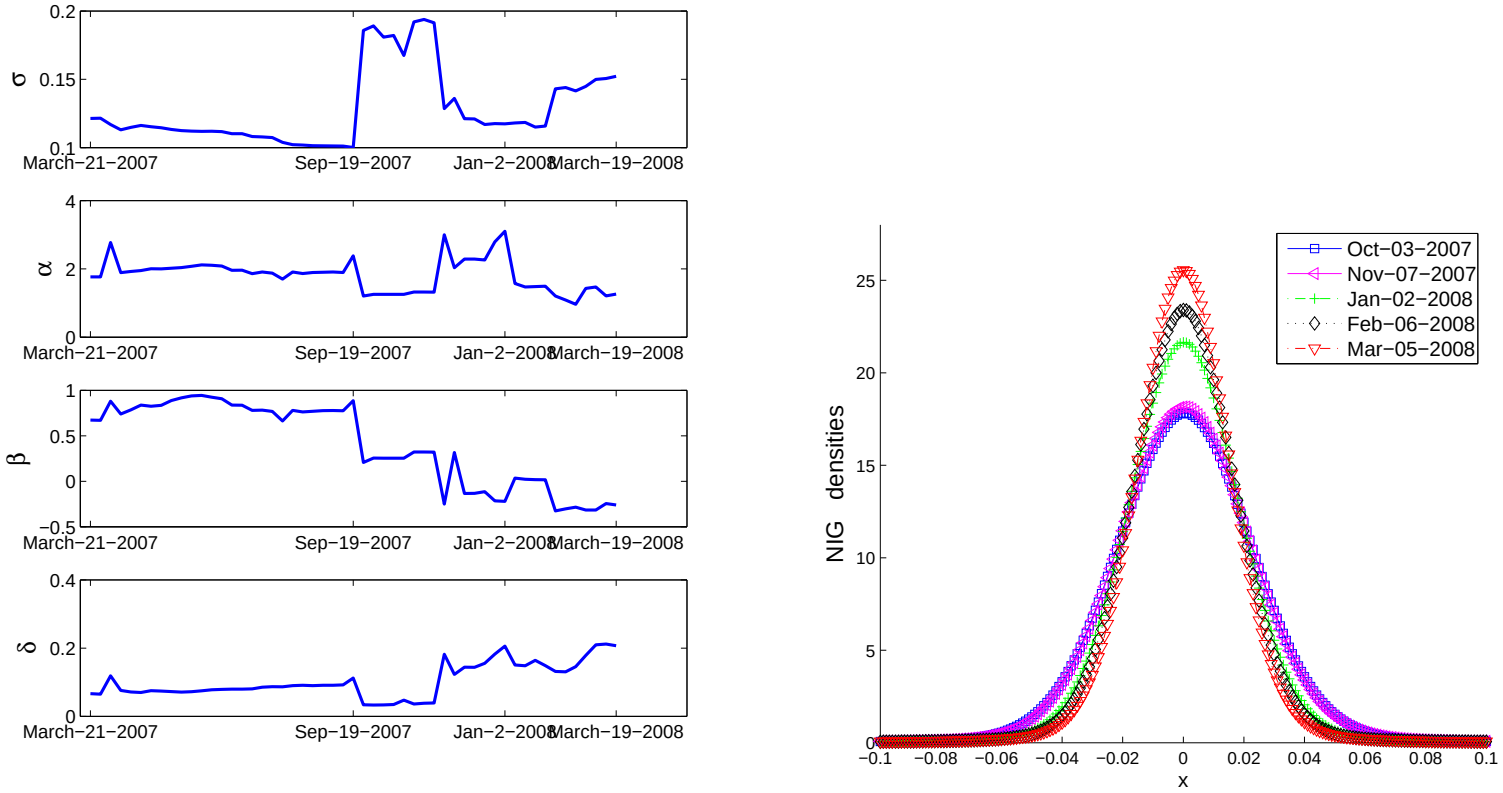


Figure: Evolution of the NIG parameters and densities of "DSG International PLC"

NIG vs. CGMY

Both Lévy processes gave good fits, but

- The NIG model returns more consistent measures from time to time and from one company to another.
- From a numerical point of view, the NIG model is also more preferable.
  - Small N (e.g. $N = 2^{10}$) can be applied.
  - The NIG model is much less sensitive to the initial guess of the optimum-searching procedure.
  - Fast convergence to the optimal parameters are observed (usually within 200 function evaluations). However, averagely 500 to 600 evaluations for the CGMY model are needed.

Conclusions

- The COS method is highly efficient for density recovery, for pricing European, Bermudan and discretely -monitored barrier options
- Convergence is exponential, usually with small N
- We relate the credit default spreads to a series survival/default probabilities with different maturities, and generalize the COS method to value these survival probabilities efficiently.
- Calibration results are also discussed. Both the NIG and the CGMY models give very good fits to the market CDSs, but the NIG model turns out to be more advantageous.

Black, F., and J. Cox (1976) Valuing corporate securities: some effects on bond indenture provisions, *J. Finance*, **31**, pp. 351–367.