On Highly Efficient Methods for Pricing Options with and without Early Exercise

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Linz, Semester on Finance, November 2008
Brief overview of derivative pricing

Our contribution: The COS method:
  ▶ Efficient way to recover the density function;
  ▶ Efficient alternative for FFT-based methods for calibration;
  ▶ Focus on Lévy processes and Heston stochastic volatility

COS method for European options

Bermudan and discretely-monitored barrier options

Credit Default Swaps
Multi-D asset prices

- Asset price, $S_i$, can be modeled by geometric Brownian motion:

$$dS_i(t) = \mu_i S_i(t) dt + \sigma_i S_i dW_i(t),$$

with $W_i(t)$ Wiener process, $\mu_i$ drift, $\sigma_i$ volatility.

⇒ Itô’s Lemma: multi-D Black-Scholes equation: (for a European option)

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} \left[ \sigma_i \sigma_j \rho_{i,j} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} \right] + \sum_{i=1}^{d} \left[ r S_i \frac{\partial V}{\partial S_i} \right] - r V = 0.$$ 

- Correlation between a pair of assets, $S_i$ and $S_j$, is $\rho_{i,j}$.
Pricing: Feynman-Kac Theorem

Given the system of stochastic differential equations:

$$dS_i(t) = rS_i(t)dt + \sigma_i S_i dW_i(t)$$

with $\mathbb{E}\{dW_i(t)dW_j(t)\} = \rho_{ij} dt$ and an option, $V$, such that

$$V(S, t) = e^{-r(T-t)}\mathbb{E}^Q\{V(S(T), T)|S(t)\}$$

with the sum of the first derivatives of the option square integrable.

Then the value, $V(S(t), t)$, is the unique solution of the final condition problem

$$\begin{cases}
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_i \sigma_j \rho_{i,j} S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{d} [rS_i \frac{\partial V}{\partial S_i}] - rV = 0, \\
V(S, T) = \text{given}
\end{cases}$$
Numerical Pricing Approach

- One can apply several numerical techniques to calculate the option price:
  - Numerical integration,
  - Monte Carlo simulation,
  - Numerical solution of the partial-(integro) differential equation (P(I)DE)

- Each of these methods has its merits and demerits.

Numerical challenges:
- The problem’s dimensionality
- Speed of solution methods
- Early exercise feature (P(I)DE $\rightarrow$ free boundary problem)
Lévy Processes

- Use Heston’s model, or a Lévy process with jumps, to better fit market data, and allow for smile effects.
- A Lévy process is a stochastic process that starts at 0 and has independent and stationary increments.
- The Lévy processes of our interest here include:
  - The CGMY model (generalized VG model; driven by four parameters);
  - The Normal Inverse Gaussian (NIG) model (a variance-mean mixture of a Gaussian distribution with an inverse Gaussian; driven by four parameters).
Motivation

- **Our motivation:** To derive pricing methods that
  - are computationally fast
  - are not restricted to Gaussian-based models
  - should work as long as we have a characteristic function,

\[
\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x) dx; \quad f(x) = \frac{1}{\pi} \int_{0}^{\infty} \text{Re} (\phi(\omega) e^{-i\omega x}) d\omega
\]

- Preferably faster than approaches based on the FFT

- **The characteristic function of a Lévy process equals:**

\[
\phi(\omega) = \exp \left( t(i\mu \omega - \frac{1}{2} \sigma^2 \omega^2 + \int_{\mathbb{R}} (e^{i\omega x} - 1 - i\omega x \mathbf{1}_{|x|<1} \nu(dx))) \right),
\]

the celebrated Lévy-Khinchine formula.
Fourier-Cosine Expansion

The **COS method**:

- Exponential convergence;
- Greeks are obtained at no additional cost.
- For discretely-monitored barrier and Bermudan options as well;

The basic idea:

- Replace the density by its *Fourier-cosine series expansion*;
- Series coefficients have simple relation with characteristic function.
Series Coefficients of the Density and the Ch.F.

- Fourier-Cosine expansion of density function on interval \([a, b]\):

\[
f(x) = \sum_{n=0}^{\infty} F_n \cos \left( n\pi \frac{x - a}{b - a} \right),
\]

with \(x \in [a, b] \subset \mathbb{R}\) and the coefficients defined as

\[
F_n := \frac{2}{b - a} \int_a^b f(x) \cos \left( n\pi \frac{x - a}{b - a} \right) dx.
\]

- \(F_n\) has direct relation to ch.f., \(\phi(\omega) := \int_{\mathbb{R}} f(x) e^{i\omega x} dx\) (\(\int_{\mathbb{R}\setminus[a,b]} f(x) \approx 0\)),

\[
F_n \approx A_n := \frac{2}{b - a} \int_{\mathbb{R}} f(x) \cos \left( n\pi \frac{x - a}{b - a} \right) dx
= \frac{2}{b - a} \text{Re} \left\{ \phi \left( \frac{n\pi}{b - a} \right) \exp \left( -i \frac{ka\pi}{b - a} \right) \right\}.
\]
Recovering Densities

- Replace $F_n$ by $A_n$, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} \sum_{n=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{n\pi}{b-a}; x \right) \exp \left( in\pi \frac{-a}{b-a} \right) \right\} \cos \left( n\pi \frac{x-a}{b-a} \right),$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a, b] = [-10, 10]$ and $x = \{-5, -4, \cdots, 4, 5\}$.

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<th>16</th>
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<td>0.0028</td>
<td>0.0025</td>
<td>0.0031</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

Exponential error convergence in $N$. 

C.W.Oosterlee (CWI) 

The COS Method 

CWI
Pricing European Options

- Start from the risk-neutral valuation formula:

\[ v(x, t_0) = e^{-r\Delta t} \mathbb{E}^Q \left[ v(y, T) | x \right] = e^{-r\Delta t} \int_{\mathbb{R}} v(y, T) f(y|x) dy. \]

- Truncate the integration range:

\[ v(x, t_0) = e^{-r\Delta t} \int_{[a,b]} v(y, T) f(y|x) dy + \varepsilon. \]

- Replace the density by the COS approximation, and interchange summation and integration:

\[ \hat{v}(x, t_0) = e^{-r\Delta t} \sum_{n=0}^{N-1} \text{Re} \left\{ \phi \left( \frac{n\pi}{b-a}, x \right) e^{-in\pi \frac{a}{b-a}} \right\} V_n, \]

where the series coefficients of the payoff, \( V_n \), are analytic.
Pricing European Options

- Log-asset prices: \( x := \ln(S_0/K) \) and \( y := \ln(S_T/K) \),
- The payoff for European options reads
  \[
  v(y, T) \equiv [\alpha \cdot K(e^y - 1)]^+.
  \]

For a call option, we obtain

\[
V_{k}^{\text{call}} = \frac{2}{b - a} \int_{0}^{b} K(e^y - 1) \cos \left( k\pi \frac{y - a}{b - a} \right) dy
\]

\[
= \frac{2}{b - a} K (\chi_k(0, b) - \psi_k(0, b)),
\]

For a vanilla put, we find

\[
V_{k}^{\text{put}} = \frac{2}{b - a} K (-\chi_k(a, 0) + \psi_k(a, 0)).
\]
The characteristic function of the log-asset price for Heston’s model:

\[
\varphi_{\text{hes}}(\omega; u_0) = \exp \left( i\omega \mu \Delta t + \frac{u_0}{\eta^2} \left( \frac{1 - e^{-D\Delta t}}{1 - Ge^{-D\Delta t}} \right) (\lambda - i\rho\eta\omega - D) \right) \cdot \\
\exp \left( \frac{\lambda \bar{u}}{\eta^2} \left( \Delta t(\lambda - i\rho\eta\omega - D) - 2 \log\left( \frac{1 - Ge^{-D\Delta t}}{1 - G} \right) \right) \right),
\]

with \( D = \sqrt{(\lambda - i\rho\eta\omega)^2 + (\omega^2 + i\omega)\eta^2} \) and \( G = \frac{\lambda - i\rho\eta\omega - D}{\lambda - i\rho\eta\omega + D} \).

For Lévy and Heston models, the ChF can be represented by

\[
\phi(\omega; x) = \varphi_{\text{levy}}(\omega) \cdot e^{i\omega x} \quad \text{with} \quad \varphi_{\text{levy}}(\omega) := \phi(\omega; 0),
\]

\[
\phi(\omega; x, u_0) = \varphi_{\text{hes}}(\omega; u_0) \cdot e^{i\omega x},
\]
Characteristic Functions Lévy Processes

- For the CGMY/KoBoL model:

\[
\varphi_{\text{levy}}(\omega) = \exp\left(i\omega(r - q)\Delta t - \frac{1}{2}\omega^2\sigma^2 \Delta t\right) \cdot \\
\exp(\Delta t C \Gamma(-Y)[(M - i\omega)^Y - M^Y + (G + i\omega)^Y - G^Y]),
\]

where \(\Gamma(\cdot)\) represents the gamma function.

- The parameters should satisfy \(C \geq 0, G \geq 0, M \geq 0\) and \(Y < 2\).

- The characteristic function of the log-asset price for NIG:

\[
\varphi_{\text{NIG}}(\omega) = \exp\left(i\omega \mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + i\omega)^2})\right)
\]

with \(\alpha, \delta > 0, \beta \in (-\alpha, \alpha - 1)\)
Heston Model

- We can present the $V_k$ as $V_k = U_k K$, where

$$U_k = \begin{cases} \frac{2}{b-a} (\chi_k(0, b) - \psi_k(0, b)) & \text{for a call} \\ \frac{2}{b-a} (-\chi_k(a, 0) + \psi_k(a, 0)) & \text{for a put} \end{cases}$$

- The pricing formula simplifies for Heston and Lévy processes:

$$v(x, t_0) \approx K e^{-r \Delta t} \cdot \text{Re} \left\{ \sum_{n=0}^{N-1} \varphi \left( \frac{n\pi}{b-a} \right) U_n \cdot e^{i n \pi \frac{x-a}{b-a}} \right\},$$

where $\varphi(\omega) := \phi(\omega; 0)$
Numerical Results

Pricing for 21 strikes $K = 50, 55, 60, \ldots, 150$ under Heston’s model. Other parameters: $S_0 = 100, r = 0, q = 0, T = 1, \lambda = 1.5768, \eta = 0.5751, \bar{u} = 0.0398, u_0 = 0.0175, \rho = -0.5711$.

<table>
<thead>
<tr>
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<th>$N$</th>
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<th>128</th>
<th>160</th>
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<tr>
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<td>(msec.)</td>
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<td>2.641</td>
<td>3.220</td>
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<td>max. abs. err.</td>
<td>4.52e-04</td>
<td>2.61e-05</td>
<td>4.40e-06</td>
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<td>Carr-Madan</td>
<td>$N$</td>
<td>2048</td>
<td>4096</td>
<td>8192</td>
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<tr>
<td></td>
<td>(msec.)</td>
<td>20.36</td>
<td>37.69</td>
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<tr>
<td></td>
<td>max. abs. error</td>
<td>2.61e-01</td>
<td>2.15e-03</td>
<td>2.08e-07</td>
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</table>

Error analysis for the COS method is provided in the paper.
Numerical Results within Calibration

- Calibration for Heston’s model: Around 10 times faster than Carr-Madan.

![Convergence Plot of Calibration Algorithm](image)
Pricing Bermudan Options

The pricing formulae

\[
\begin{align*}
\{ & = e^{-r \Delta t} \int_{\mathbb{R}} v(y, t_{m+1}) f(y|x) dy \\
\} & = \max(g(x, t_m), c(x, t_m))
\end{align*}
\]

and \( v(x, t_0) = e^{-r \Delta t} \int_{\mathbb{R}} v(y, t_1) f(y|x) dy. \)

- Use Newton’s method to locate the early exercise point \( x_m^\ast \), which is the root of \( g(x, t_m) - c(x, t_m) = 0. \)
- Recover \( V_n(t_1) \) recursively from \( V_n(t_M), V_n(t_{M-1}), \ldots, V_n(t_2). \)
- Use the COS formula for \( v(x, t_0). \)
Once we have $x_m^*$, we split the integral, which defines $V_k(t_m)$:

$$V_k(t_m) = \begin{cases} 
  C_k(a, x_m^*, t_m) + G_k(x_m^*, b), & \text{for a call,} \\
  G_k(a, x_m^*) + C_k(x_m^*, b, t_m), & \text{for a put,}
\end{cases}$$

for $m = M - 1, M - 2, \ldots, 1$. whereby

$$G_k(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} g(x, t_m) \cos \left( k\pi \frac{x - a}{b - a} \right) dx.$$

and

$$C_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} \hat{c}(x, t_m) \cos \left( k\pi \frac{x - a}{b - a} \right) dx.$$
**Bermudan Details**

- Formula for the coefficients $C_k(x_1, x_2, t_m)$:

  $$C_k(x_1, x_2, t_m) = e^{-r\Delta t} \text{Re} \left\{ \sum_{j=0}^{N-1} \varphi_{\text{levy}} \left( \frac{j\pi}{b-a} \right) V_j(t_{m+1}) \cdot M_{k,j}(x_1, x_2) \right\},$$

  where the coefficients $M_{k,j}(x_1, x_2)$ are given by

  $$M_{k,j}(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} e^{ij\pi \frac{x-a}{b-a}} \cos \left( k\pi \frac{x-a}{b-a} \right) dx,$$

- With fundamental calculus, we can rewrite $M_{k,j}$ as

  $$M_{k,j}(x_1, x_2) = -\frac{i}{\pi} \left( M^c_{k,j}(x_1, x_2) + M^s_{k,j}(x_1, x_2) \right),$$
Hankel and Toeplitz

Matrices $M_c = \{M^c_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ and $M_s = \{M^s_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ have special structure for which the FFT can be employed: $M_c$ is a Hankel matrix,

$$
M_c = \begin{bmatrix}
    m_0 & m_1 & m_2 & \cdots & m_{N-1} \\
    m_1 & m_2 & \cdots & \cdots & m_N \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    m_{N-2} & m_{N-1} & \cdots & m_{2N-3} & m_{2N-2} \\
    m_{N-1} & \cdots & m_{2N-3} & m_{2N-2} & \cdots \\
\end{bmatrix}_{N \times N}
$$

and $M_s$ is a Toeplitz matrix,

$$
M_s = \begin{bmatrix}
    m_0 & m_1 & \cdots & m_{N-2} & m_{N-1} \\
    m_{-1} & m_0 & m_1 & \cdots & m_{N-2} \\
    \vdots & \vdots & \ddots & \ddots & \vdots \\
    m_{2-N} & \cdots & m_{-1} & m_0 & m_1 \\
    m_{1-N} & m_{2-N} & \cdots & m_{-1} & m_0 \\
\end{bmatrix}_{N \times N}
$$
Bermudan puts with 10 early-exercise dates

Table: Test parameters for pricing Bermudan options

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Model</th>
<th>$S_0$</th>
<th>$K$</th>
<th>$T$</th>
<th>$r$</th>
<th>$\sigma$</th>
<th>Other Parameters</th>
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<td>110</td>
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<td>0.1</td>
<td>0.2</td>
<td></td>
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<tr>
<td>3</td>
<td>CGMY</td>
<td>100</td>
<td>80</td>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>$C = 1, G = 5, M = 5, Y = 1.5$</td>
</tr>
</tbody>
</table>

(a) BS

(b) CGMY with $Y = 1.5$
The price of an $M$-times monitored up-and-out option satisfies

\[
\begin{align*}
    c(x, t_{m-1}) &= e^{-r(t_m-t_{m-1})} \int_{\mathbb{R}} v(x, t_m)f(y|x)dy \\
    v(x, t_{m-1}) &= \begin{cases} 
        e^{-r(T-t_{m-1})} Rb, & x \geq h \\
        c(x, t_{m-1}), & x < h 
    \end{cases}
\end{align*}
\]

where $h = \ln(H/K)$, and $v(x, t_0) = e^{-r(t_m-t_{m-1})} \int_{\mathbb{R}} v(x, t_1)f(y|x)dy$.

The technique:

- Recover $V_n(t_1)$ recursively, from $V_n(t_M)$, $V_n(t_{M-1})$, · · · , $V_n(t_2)$ in $O((M-1)N \log_2(N))$ operations.
- Split the integration range at the barrier level (no Newton required)
- Insert $V_n(t_1)$ in the COS formula to get $v(x, t_0)$, in $O(N)$ operations.
### Monthly-monitored Barrier Options

#### Table: Test parameters for pricing barrier options

<table>
<thead>
<tr>
<th>Test No.</th>
<th>Model</th>
<th>$S_0$</th>
<th>$K$</th>
<th>$T$</th>
<th>$r$</th>
<th>$q$</th>
<th>Other Parameters</th>
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<tr>
<td>1</td>
<td>NIG</td>
<td>100</td>
<td>100</td>
<td>1</td>
<td>0.05</td>
<td>0.02</td>
<td>$\alpha = 15, \beta = -5, \delta = 0.5$</td>
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<table>
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<th>Option Type</th>
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<th>$N \backslash N$</th>
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<td>$2^{10}$</td>
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Credit Default Swaps (with W. Schoutens, H. Jönsson)

- Credit default swaps (CDSs), the basic building block of the credit risk market, offer investors the opportunity to either buy or sell default protection on a reference entity.
- The protection buyer pays a premium periodically for the possibility to get compensation if there is a credit event on the reference entity until maturity or the default time, whichever is first.
- If there is a credit event the protection seller covers the losses by returning the par value. The premium payments are based on the CDS spread.
CDS and COS

- CDS spreads are based on a series of default/survival probabilities, that can be very efficiently recovered using the COS method. It is also very flexible w.r.t. the underlying process as long as it is Lévy.
- The flexibility and the efficiency of the method are demonstrated via a calibration study of the iTraxx Series 7 and Series 8 quotes.
Lévy Default Model

- Definition of default: For a given recovery rate, $R$, default occurs the first time the firm’s value is below the “reference value” $RV_0$.
- As a result, the survival probability in the time period $(0, t]$ is nothing but the price of a digital down-and-out barrier option without discounting.

\[
P_{\text{surv}}(t) = P_Q\left(X_s > \ln R, \text{for all } 0 \leq s \leq t\right) \\
= P_Q\left(\min_{0 \leq s \leq t} X_s > \ln R\right) \\
= E_Q\left[1\left(\min_{0 \leq s \leq t} X_s > \ln R\right)\right]
\]
Survival Probability

- Assume there are only a finite number of observing dates.

\[ P_{\text{surv}}(\tau) = \mathbb{E}_Q \left[ 1\left(X_{\tau_1} \in [\ln R, \infty)\right) \cdot 1\left(X_{\tau_2} \in [\ln R, \infty)\right) \cdots 1\left(X_{\tau_M} \in [\ln R, \infty)\right) \right] \]

where \( \tau_k = k\Delta \tau \) and \( \Delta \tau := \tau / M \).

- The survival probability then has the following recursive expression:

\[
\begin{cases}
    P_{\text{surv}}(\tau) &:= p(x = 0, \tau_0) \\
    p(x, \tau_m) &:= \int_{\ln R}^{\infty} f_{X_{\tau_{m+1}|X_{\tau_m}}}(y|x)p(y, \tau_{m+1}) \, dy, \quad m = M - 1, \cdots, 2, 1, 0, \\
    p(x, \tau_M) &:= 1
\end{cases}
\]

\( f_{X_{\tau_{m+1}|X_{\tau_m}}}(\cdot|\cdot) \) denotes the conditional probability density of \( X_{\tau_{m+1}} \) given \( X_{\tau_m} \).
The Fair Spread of a Credit Default Swap

- The *fair spread*, $C$, of a CDS at the initialization date is the spread that equalizes the present value of the premium leg and the present value of the protection leg, i.e.

$$C = \frac{(1 - R) \left( \int_0^T \exp(-r(s)s)dP_{\text{def}}(s) \right)}{\int_0^T \exp(-r(s)s)P_{\text{surv}}(s)ds},$$

- It is actually based on a series of survival probabilities on different time intervals:

$$C = \frac{(1 - R) \sum_{j=0}^J \frac{1}{2} \left[ \exp(-r_j t_j) + \exp(-r_{j+1} t_{j+1}) \right] \left[ P_{\text{surv}}(t_j) - P_{\text{surv}}(t_{j+1}) \right]}{\sum_{j=0}^J \frac{1}{2} \left[ \exp(-r_j t_j)P_{\text{surv}}(t_j) + \exp(-r_{j+1} t_{j+1})P_{\text{surv}}(t_{j+1}) \right] \Delta t} + \epsilon,$$
The COS Formula for Survival Probabilities

- Replace the conditional density by the COS (semi-analytical) expression, the survival probability then satisfies

\[
\begin{align*}
    P_{\text{surv}}(\tau) &= p(x = 0, \tau_0), \\
    p(x, \tau_0) &= \sum_{n=0}^{N-1} \phi_n(x) \cdot P_n(\tau_1),
\end{align*}
\]

- The only thing one needs is \( \{P_n(\tau_1)\}_{n=0}^{N-1} \), which can be recovered from \( \{P_n(\tau_M)\}_{n=0}^{N-1} \) via backwards induction.
Backwards Induction

- Starting from the definition of $P_n(\tau_m)$, we apply the COS reconstruction of $p(y, \tau_m)$ to get

$$P(\tau_m) = \text{Re} \{ \Omega \Lambda \} P(\tau_{m+1}),$$

- Applying this recursively backwards in time, we get

$$P(\tau_1) = (\text{Re} \{ \Omega \Lambda \})^{M-1} P(\tau_M)$$

- For this recursive matrix-vector-product, there exists a fast algorithm, e.g.

$$P(\tau_1) = \text{Re} \{ \Omega [\Lambda \text{Re} \{ \Omega [\Lambda \text{Re} \{ \Omega [\Lambda P(t_3)] \} \} \} \} \} \}.$$ 

- The FFT algorithm can be applied because $\Omega = H + T$, where $H$ is a Hankel matrix and $T$ is a Toeplitz matrix.
Convergence of Survival Probabilities

- Ideally, the survival probabilities should be monitored daily, i.e. $\Delta \tau = 1/252$. That is, $M = 252T$, which is a bit too much for $T = 5, 7, 10$ years.
- For Black-Scholes' model, there exist rigorous proof of the convergence of discrete barrier options to otherwise identical continuous options [Kou,2003].
- We observe similar convergence under NIG, CGMY:

![Convergence of Survival Probabilities](c)

![Convergence of Survival Probabilities](d)

- Convergence of the 1-year survival probability w.r.t. $\Delta \tau$. 
The error convergence of the COS method is usually exponential in $N$.

Figure: Convergence of $P_{\text{surv}}(\Delta \tau = 1/48)$ w.r.t. $N$ for NIG and CGMY
The data sets: weekly quotes from iTraxx Series 7 (S7) and 8 (S8). After cleaning the data we were left with 119 firms from Series 7 and 123 firms from Series 8. Out of these firms 106 are common to both Series.

The interest rates: EURIBOR swap rates.

We have chosen to calibrate the models to CDSs spreads with maturities 1, 3, 5, 7, and 10 years.
To avoid the ill-posedness of the inverse problem we defined here, the objective function is set to

\[
F_{obj} = \text{rmse} + \gamma \cdot ||X_2 - X_1||_2,
\]

where

\[
\text{rmse} = \sqrt{\sum_{\text{CDS}} \left( \frac{\text{market CDS spread} - \text{model CDS spread}}{\text{number of CDSs on each day}} \right)^2},
\]

\( || \cdot ||_2 \) denotes the \( L_2 \)-norm operator, and \( X_2 \) and \( X_1 \) denote the parameter vectors of two neighbor data sets.
Good Fit to Market Data

Table: Summary of calibration results of all 106 firms in both S7 and S8 of iTraxx quotes

<table>
<thead>
<tr>
<th>RMSEs</th>
<th>NIG in S7</th>
<th>CGMY in S7</th>
<th>NIG in S8</th>
<th>CGMY in S8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average (bp.)</td>
<td>0.89</td>
<td>0.79</td>
<td>1.65</td>
<td>1.54</td>
</tr>
<tr>
<td>Min. (bp.)</td>
<td>0.22</td>
<td>0.29</td>
<td>0.27</td>
<td>0.46</td>
</tr>
<tr>
<td>Max. (bp.)</td>
<td>2.29</td>
<td>1.97</td>
<td>4.27</td>
<td>3.52</td>
</tr>
</tbody>
</table>
A Typical Example

Evolution of CDSs of ABN Amro Bank NV with maturity $T = 1$ year

Evolution of CDSs of ABN Amro Bank NV with maturity $T = 5$ year

Evolution of CDSs of ABN Amro Bank NV with maturity $T = 10$ year
An Extreme Case

Evolution of CDSs of DSG International PLC with maturity T = 1 year

Evolution of CDSs of DSG International PLC with maturity T = 5 year

Evolution of CDSs of DSG International PLC with maturity T = 10 year
NIG Parameters for “ABN AMRO Bank”

Figure: Evolution of the NIG parameters and densities of “ABN AMRO Bank”
Figure: Evolution of the NIG parameters and densities of “DSG International PLC”
Both Lévy processes gave good fits, but

- The NIG model returns more consistent measures from time to time and from one company to another.
- From a numerical point of view, the NIG model is also more preferable.
  - Small $N$ (e.g. $N = 2^{10}$) can be applied.
  - The NIG model is much less sensitive to the initial guess of the optimum-searching procedure.
  - Fast convergence to the optimal parameters are observed (usually within 200 function evaluations). However, averagely 500 to 600 evaluations for the CGMY model are needed.
Conclusions

- The COS method is highly efficient for density recovery, for pricing European, Bermudan and discretely-monitored barrier options.
- Convergence is exponential, usually with small $N$.
- We relate the credit default spreads to a series survival/default probabilities with different maturities, and generalize the COS method to value these survival probabilities efficiently.
- Calibration results are also discussed. Both the NIG and the CGMY models give very good fits to the market CDSs, but the NIG model turns out to be more advantageous.