A Robust Predictable L^{∞}-Martingale Representation Property for Marked Point Processes and Super-Additive Insurance Markets

Johannes Leitner
TU Vienna

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The Economic Model

- New information arrives at stopping times,
- **Super-additive** (insurance) market,
- No-arbitrage, completeness, replication of contingent claims.

Marked Point Process

- $(\Omega, \mathcal{F}, P)$ probability space. $(E, \mathcal{B})$ separable metric space, $\bar{E} := E \cup \{\Delta\}, \Delta \notin E$. E.g. $E = \mathbf{R}^d \setminus \{0\}$.

- Sequence $\mathbf{R}_+ \times \bar{E}$-valued random variables $(T_n, X_n)_{n \geq 1}$ with

  1. Points: $T_0 := 0 < T_n < T_{n+1}$ on $\{T_n < \infty\}, n \geq 1$,

  2. Marks: $\{X_n = \Delta\} = \{T_n = \infty\}, n \geq 1$.

- Interpretation: price, interest rate, etc., jumps by X_n at T_n.

Random Measure

Corresponding random measure:

$$\mu(\omega, dt, dx) = \sum_{n \geq 1} \epsilon_{(T_n(\omega), X_n(\omega))}(dt, dx) \mathbf{1}_{\{T_n(\omega) < \infty\}}, \quad \omega \in \Omega,$$

$\epsilon_{(t,x)}$ probability measure concentrated in $(t, x) \in \mathbf{R}_+ \times \bar{E}$.

μ is optional w.r.t.

$$\mathcal{F}_t := \mathcal{F}_0 \vee \sigma\Big(X_n \mathbf{1}_{[T_n, \infty)}(t), n \geq 1\Big), t \geq 0.$$

Stochastic Integrals w.r.t. Random Measures

H predictable (optional) process, define **pathwise** (if integral exists):

$$H * \mu_t(\omega) := \int_{[0,t]\times E} H(\omega, s, x)\mu(\omega; ds, dx), \quad \omega \in \Omega.$$

Marked point process: On $[0, \sup_{n\geq 1} T_n)$

$$H * \mu = \sum_{n\geq 1} H(T_n, X_n)\mathbf{1}_{[T_n,\infty)}.$$

Compensator of a Random Measures

Let μ be optional, σ-finite. There exists a unique **predictable** random measure $\widehat{\mu}$ such that

$H * \mu - H * \widehat{\mu}$ is a local **martingale**

for all predictable H such that $H * \mu$ is of locally integrable total variation.

$\widehat{\mu}$ is the predictable compensator of μ.

$\exists$ kernel K from $(\Omega \times \mathbf{R}_+, \mathcal{P})$ into $(E, \mathcal{B})$ and non-decreasing predictable process A such that:

$$\widehat{\mu}(\omega; dt, dx) = K(\omega, t, dx) dA_t(\omega).$$

Counting process $N := \sum_{n \geq 1} \mathbf{1}_{[T_n, \infty)}$, Compensator $\widehat{N} = A$

Assume throughout

- $\widehat{N}$ is continuous.
- $\widehat{N}_\infty$ is uniformly bounded.

Insurance Market

$K(t, dx)$ describes the law of X_n given $\mathcal{F}_{t-}$ and $T_n = t$.

$H(\omega, t, x) = x\mathbf{1}_{x>0}$, $\lambda \geq 0$ **security loading**:

$$H * \mu - H * \Big((1+\lambda) \cdot \widehat{\mu}\Big) = H * \mu - \Big(E_K[H.](1+\lambda)\Big) \cdot \hat{N}$$

is a **risk process**.

Spaces of predictable integrands

$H : \Omega \times \mathbf{R}_+ \times E \to \mathbf{R}$ measurable w.r.t. $\mathcal{P} \otimes \mathcal{B}$ (predictable)

Insurance claim equals $H(\omega, t, x)$ if $\Big(T_n(\omega), X_n(\omega)\Big) = (t, x)$

$G_\infty := \{H$ predictable$\} \cap L^\infty$, and for $p < \infty$

$$G_p := \{H \text{ predictable} | \int_E |H(\cdot; x)|^p K(\cdot; dx) \in L^\infty\}.$$

Change of measure

For q conjugate to p set

$$G_q^{++} := \{Y \in G_q | Y > 0\}.$$

For $Y \in G_1^{++}$ and $M^Y := (Y-1) * (\mu - \widehat{\mu})$ define Q^Y by

$$\frac{dQ^Y}{dP} = \mathcal{E}(M^Y)_\infty > 0.$$

W.r.t. Q^Y the compensator $\widehat{\mu}^{Q^Y}$ of μ is given as $Y \cdot \widehat{\mu}$.

Integrability Condition

$(\mathcal{H}, \mathcal{Y}) := (G_p, G_q^{++})$ satisfies the following condition

(INT) For all $(H, Y) \in \mathcal{H} \times \mathcal{Y}$, we have $Q^Y \sim P$ and $H * (\mu - \widehat{\mu}^{Q^Y}) = H * \mu - (HY) * \widehat{\mu}$ is a uniformly integrable Q^Y-martingale.

Super-Additive Markets

Linear space $\mathcal{V}_0 \subseteq L^1(\mathcal{F}_0)$ of initial capitals, $\mathcal{H}$ linear space (of actions in a market)

Functional $W : \mathcal{V}_0 \times \mathcal{H} \to L^0, (v, H) \mapsto W_v^H$ such that $W_v^0 = v$ and

$$W_{v+\widehat{v}}^{H+\widehat{H}} \geq W_v^H + W_{\widehat{v}}^{\widehat{H}}.$$

EMM Condition

$\mathcal{Z} \neq \emptyset$, $Z > 0$ for all $Z \in \mathcal{Z}$.

(EMM) For all $(v, H) \in \mathcal{V}_0 \times \mathcal{H}$ there exists a $Z \in \mathcal{Z}$ such that

(i) $ZW_v^H \in L^1$,

(ii) $E[ZW_v^H \,|\, \mathcal{F}_0] = v$.

Abstract No-Arbitrage

Let $(v, H), (\tilde{v}, \tilde{H}) \in \mathcal{V}_0 \times \mathcal{H}$.

Proposition. *Under Condition* (EMM) *we have*

$$W_v^H \leq W_{\tilde{v}}^{\tilde{H}} \text{ and } v \geq \tilde{v} \text{ imply } v = \tilde{v} \text{ and } W_v^H = W_{\tilde{v}}^{\tilde{H}}.$$

I.e. a no-domination *property holds:*

$$W_v^H \leq W_{\tilde{v}}^{\tilde{H}} \text{ and } P(W_v^H < W_{\tilde{v}}^{\tilde{H}}) > 0 \text{ imply } P(v < \tilde{v}) > 0.$$

In particular the no-arbitrage *property holds:*

$$W_v^H \geq v = W_v^0 \text{ implies } W_v^H = v.$$

Worst Case Scenario Condition

Let $\mathcal{H} \times \mathcal{Y} \subseteq G_p \times G_q^{++}$, (INT).

(WCS) For all $H \in \mathcal{H}$ there exists a $Y^H \in \mathcal{Y}$ such that $(HY^H) * \widehat{\mu}_\infty \leq (HY) * \widehat{\mu}_\infty$ for all $Y \in \mathcal{Y}$.

The Irreversible Insurance Market

Assume $(\mathcal{H}, \mathcal{Y})$ to satisfy property (WCS).

For $(v, H) \in L^p(\mathcal{F}_0) \times \mathcal{H}$, define

$$\begin{aligned} W_v^H &:= v + (H * \mu)_\infty - (HY^{-H}) * \widehat{\mu}_\infty \\ &= v + (H * \mu)_\infty - \left(\int_E HY^{-H} dK \right) \cdot \widehat{N}_\infty . \end{aligned}$$

Let $(\mathcal{H}, \mathcal{Y})$ satisfy Conditions (INT) and (WCS).

Set $\mathcal{Z}_{\mathcal{Y}} := \{\frac{dQ^Y}{dP} \mid Y \in \mathcal{Y}\}$:

Proposition. *$\mathcal{Z}_{\mathcal{Y}}$ satisfies property* (EMM)*. I.e. the insurance market described by $\{W_v^H \mid v \in L^p(\mathcal{F}_0), H \in \mathcal{H}\}$ satisfies the* **no-domination** *condition.*

Irreversible Contracts

Y^{-H} does in general not equal $Y^{-H\mathbf{1}_{[0,t]}}$ on $[0,t]$.

Change of contract not possible.

Pricing in general not compatible with starting and stopping.

Decomposability

Definition. *We say that $\mathcal{Y}$ is $\mathcal{P}$-***decomposable** *if $\mathcal{Y} \neq \emptyset$ and for all $A \in \mathcal{P}$ and $Y, \tilde{Y} \in \mathcal{Y}$, $\mathbf{1}_A Y + \mathbf{1}_{A^c} \tilde{Y} \in \mathcal{Y}$ holds.*

Set $\mathcal{Y}_H := \left\{ Y^H \in \mathcal{Y} \middle| \, \langle H, Y^H \rangle_K = \operatorname{essinf}_{Y \in \mathcal{Y}} \langle H, Y \rangle_K > -\infty \right\}$ for all $H \in \mathcal{H}$.

Proposition. *If $\mathcal{H} \subseteq L^p$ and $\mathcal{Y}$ is $\mathcal{P}$-decomposable and weakly compact in L^q, then it satisfies Condition (WCS) and $\mathcal{Y}_H \neq \emptyset$ for all $H \in \mathcal{H}$. Furthermore, under Condition (INT) no-arbitrage holds.*

Dynamic Insurance Markets

Assume $\mathcal{Y}$ to be $\mathcal{P}$-decomposable weakly compact in L^q.

For $(v, H) \in L^p(\mathcal{F}_0) \times L^p$ choose $Y^{-H} \in \mathcal{Y}_{-H}$ and define the semimartingale value process

$$
\begin{aligned}
V^{v,H} &:= v + H * \mu - (HY^{-H}) * \widehat{\mu} \\
&= v + H * \mu - \left(\int_E HY^{-H} dK \right) \cdot \widehat{N}.
\end{aligned}
$$

Starting and Stopping

Proposition. *Assume $\mathcal{Y}$ to be $\mathcal{P}$-decomposable, weakly compact in L^q.*
Then for all $(v, H) \in L^p(\mathcal{F}_0) \times L^p$ and all stopping times $\tau_0 \leq \tau_1$,

$V^{v,H} - V^{v,H}_{\tau_0}$ *equals* $V^{0,H\mathbf{1}_{(\tau_0,\tau_1]}}$ *on* $[\tau_0, \tau_1]$.

Robustness and Uniqueness

Assume $(\mathcal{H}, \mathcal{Y})$ to satisfy (INT), and $\mathcal{Y}$ to be $\mathcal{P}$-decomposable, weakly compact in L^q.

Theorem. *For all* $(v, H) \in L^p(\mathcal{F}_0) \times \mathcal{H}$, $V^{v,H}$ *is a local* Q^Y*-supermartingale for all* $Y \in \mathcal{Y}$ *and there exists a* $Y \in \mathcal{Y}$ *such that* $V^{v,H}$ *is a uniformly integrable* Q^Y*-martingale. No-arbitrage holds and uniqueness:* $V_\infty^{v,H} = V_\infty^{\tilde{v},\tilde{H}}$ *for* $(v, H), (\tilde{v}, \tilde{H}) \in L^p(\mathcal{F}_0) \times \mathcal{H}$ *implies* $V^{v,H} = V^{\tilde{v},\tilde{H}}$.

Robust Compensator

Define a *time-additive/spatially super-additive* random measure $\mathcal{Y} \cdot \widehat{\mu}$ by

$$H * (\mathcal{Y} \cdot \widehat{\mu}) := \textit{essinf}_{Y \in \mathcal{Y}} \langle H, Y \rangle_K \cdot \widehat{N}, \quad H \in \mathcal{H}.$$

$\mathcal{Y} \cdot \widehat{\mu}$ can be interpreted as a **robust** compensator for μ w.r.t. the probability measures in the closed convex hull of $\{Q^Y \,|\, Y \in \mathcal{Y}\}$.

$V^{v,H} = v + H * \mu - H * (\mathcal{Y} \cdot \widehat{\mu})$ is a local Q^Y-**super-martingale** for all $Y \in \mathcal{Y}$, resp. a uniformly integrable Q^Y-martingale for all $Y \in \mathcal{Y}_H$.

Example

Assume insurance contracts, described by $H^i \in \mathcal{H}, 1 \leq i \leq N$, to be given. Consider a market where trading in $V^i := V^{0,H^i}$ is possible under a **short-sale restriction**:
We assume for all $W^i \in L^\infty_+$, that $V := \sum_{i=1}^N W^i \cdot V^i$ is an attainable value process. Since $V = \sum_{i=1}^N V^{0,W^i H^i} \leq V^{0,H}$ for $H := \sum_{i=1}^N W^i H^i$, the resulting market is still arbitrage free, an investor never loses and possibly gains, buying the insurance H instead of trading in the single contracts V^i.

Main Result

Assume $\mathcal{Y} \subseteq G_1^{++}$ to be $\mathcal{P}$-decomposable and $\mathcal{Z}_{\mathcal{Y}}$ (or $\mathcal{Q}^{\mathcal{Y}} := \{Q^Y \,|\, Y \in \mathcal{Y}\}$) to be weakly compact in $L^1(\mathbf{\Omega})$, and

$(\mathcal{F}_t)_{t\in\mathbf{R}_+}$ to equal the internal filtration generated by μ and $\mathcal{F}_0$:

Theorem. *μ has the robust predictable martingale representation property for $L^\infty(\mathcal{F}_\infty)$ with respect to the closed convex hull of $\mathcal{Q}^{\mathcal{Y}}$.*

Hedging in Dynamic Insurance Markets.

Definition. *We say that $\mathcal{Y}$ is $\mathcal{P}$-***additive** *if $\mathcal{Y}$ is $\mathcal{P}$-decomposable and if the predictable process $\Lambda := \langle \mathbf{1}, Y \rangle_K$ does not depend on $Y \in \mathcal{Y}$.*

$\mathcal{P}$-additive $\mathcal{Y} \subseteq G_1^+$, Q^Y-compensator of N:

$$\hat{N}^{Q^Y} = \Lambda \cdot \hat{N}, \quad Y \in \mathcal{Y}.$$

The law of N, resp. $(T_i)_{i\geq 1}$, under Q^Y does not depend on $Y \in \mathcal{Y}$.

Coherent Risk Measures

$\mathcal{Y}$ $\mathcal{P}$-additive, weakly compact in G_q^{++}, $H \in G_p$:

$$\rho.(H) := -\text{essinf}_{Y \in \mathcal{Y}} \langle H, Y \rangle_K \Lambda^{-1} \in L^1(\widehat{\mathbf{\Omega}}),$$

$(v, H) \in L^p(\mathcal{F}_0) \times G_p$:

$$V^{v,H} = v + H * \mu - \rho.(-H) \cdot (\Lambda \cdot \widehat{N}).$$

Random Set Theory:

$H_\cdot(x) := H(\cdot; x), x \in E$, $\tilde{\rho}_t : L^p(dK_t) \to \mathbf{R}$:

$$\tilde{\rho}_\cdot(H_\cdot) = \rho_\cdot(H).$$

E.g. law invariant **risk measure**:

$$\tilde{\rho}_\cdot(H_\cdot) := -\int_0^1 F_\cdot^{\leftarrow}(u) g_\cdot(1-u) du,$$

Representation: Finite Jump Case

$\mathcal{H} \times \mathcal{Y} \subseteq L^{\infty}(\tilde{\mathbf{\Omega}}) \times G_1^{++}$. $v \in L^{\infty}(\mathcal{F}_0)$ and $H \in L^{\infty}(\tilde{\mathbf{\Omega}})$

Consider the following SDE:

$$V = v + (H - V_-) * \mu + \operatorname{essinf}_{Y \in \mathcal{Y}} \langle V_- - H, Y \rangle_K \cdot \hat{N}. \qquad (1)$$

with terminal condition $V_\infty = Z \in L^\infty$.

Translation invariance of ρ implies $Y^{-H} = Y^{V_- - H}$!

Linear Inhomogeneous ODE

On $[T_i, T_{i+1}]$, wlog $i = 0$, $(T, X) := (T_1, X_1)$:

$$\begin{aligned} V &= v + (H - V_-) * \mu - \langle H, Y^{-H} \rangle_K \cdot \widehat{N} + V_- \cdot (\Lambda \cdot \widehat{N}) \\ &= v + (H - V_-) * \mu - \Big(\rho.(-H) - V_-\Big) \cdot (\Lambda \cdot \widehat{N}). \end{aligned}$$

$V_T = V_{T-} + \Delta V_T = H(T, X)$ on $\{T < \infty\}$.

We can try to choose v such that $V_\infty = Z$ on $\{T = \infty\}$ too !

Internal Filtration

Condition $(\mathcal{F}_t)_{t\geq 0}$ generated by $\mathcal{F}_0$ and μ: $Z \in L^\infty(\mathcal{F}_T)$

$$Z = \tilde{H}(T,X)\mathbf{1}_{\{T<\infty\}} + \tilde{H}_\infty \mathbf{1}_{\{T=\infty\}},$$

for $\tilde{H} : \tilde{\Omega} \to \mathbf{R}$ is uniformly bounded and $\mathcal{F}_0 \otimes \mathcal{B}_+ \otimes \mathcal{B}$-measurable and $H_\infty : \Omega \to \mathbf{R}$ is uniformly bounded and $\mathcal{F}_0$-measurable.

Explicit Solution

$$v := \frac{\tilde{H}_\infty}{\mathcal{E}(\tilde{\Lambda} \cdot \tilde{N})_\infty} + \frac{\langle \tilde{H}, \tilde{Y}^{-H} \rangle_{\tilde{K}}}{\mathcal{E}(\tilde{\Lambda} \cdot \tilde{N})} \cdot \tilde{N}_\infty. \tag{2}$$

Define

$$R := v - \frac{\langle \tilde{H}, \tilde{Y}^{-H} \rangle_{\tilde{K}}}{\mathcal{E}(\tilde{\Lambda} \cdot \tilde{N})} \cdot \tilde{N}. \tag{3}$$

For $\tilde{V} := \mathcal{E}(\tilde{\Lambda} \cdot \tilde{N})R$, we have on $[0, T)$:

$$V = \tilde{V}.$$

Applications

- **Term structure** of (defaultable) zero bonds,

- Time-Discretized versions can be applied to CDS-pricing, **risk transfer** problems.