Some properties of American option prices in exponential Lévy models

Damien Lamberton Mohammed Mikou

Université Paris-Est

Workshop on Optimization and Optimal Control

Linz, October 2008

Outline

Outline

1 Optimal stopping of Lévy processes

Outline

1 Optimal stopping of Lévy processes

2 The American put price in an exponential Lévy model

Outline

1 Optimal stopping of Lévy processes

2 The American put price in an exponential Lévy model

3 The exercise boundary

Outline

1 Optimal stopping of Lévy processes

2 The American put price in an exponential Lévy model

3 The exercise boundary

4 The smooth fit property

Optimal stopping of Lévy processes

Consider a d-dimensional Lévy process $X = (X_t)_{t \geq 0}$, with characteristic exponent ψ and generating triplet (A, ν, γ), which means

$$\mathbb{E}\left(e^{iz.X_t}\right) = \exp[t\psi(z)], \quad z \in \mathbb{R}^d,$$

where

$$\psi(z) = -\frac{1}{2}z.Az + i\gamma.z + \int \left(e^{iz.x} - 1 - iz.x\mathbf{1}_{\{|x|\leq 1\}}\right)\nu(dx),$$

the matrix $A = (A_{ij})$ is the covariance matrix of the Brownian part, the measure ν on $\mathbb{R}^d \setminus \{0\}$ is the Lévy measure of X, which satisfies $\int (|x|^2 \wedge 1)\nu(dx) < \infty$, and γ is a vector in $\mathbb{R}^d$.

- Given a bounded and continuous function f on $\mathbb{R}^d$, we introduce

$$u_f(t,x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}\left(f(x+X_\tau)\right), \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

where $\mathcal{T}_{0,t}$ is the set of all stopping times with values in $[0,t]$. We want to characterize u_f as the unique solution of a variational inequality.

- Given a bounded and continuous function f on $\mathbb{R}^d$, we introduce

$$u_f(t,x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}\left(f(x+X_\tau)\right), \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d,$$

  where $\mathcal{T}_{0,t}$ is the set of all stopping times with values in $[0,t]$. We want to characterize u_f as the unique solution of a variational inequality.
- Denote by $\mathcal{L}$ the infinitesimal generator of X. The operator $\mathcal{L}$ can be written as a sum $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is the *local* (differential) part and $\mathcal{B}$ is the *non-local* (integral) part.

- For $g \in \mathcal{C}_b^2(\mathbb{R}^d)$, we have

$$\mathcal{A}g(x) = \frac{1}{2} \sum_{i,j=1}^{d} A_{i,j} \frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d} \gamma_i \frac{\partial g}{\partial x_i}(x),$$

and

$$\mathcal{B}g(x) = \int \nu(dy) \left(g(x+y) - g(x) - y.\nabla g(x) \mathbf{1}_{\{|y| \leq 1\}} \right),$$

where ∇g denotes the gradient of g.

- For $g \in \mathcal{C}_b^2(\mathbb{R}^d)$, we have

$$\mathcal{A}g(x) = \frac{1}{2}\sum_{i,j=1}^{d} A_{i,j}\frac{\partial^2 g}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{d}\gamma_i \frac{\partial g}{\partial x_i}(x),$$

  and

$$\mathcal{B}g(x) = \int \nu(dy)\left(g(x+y) - g(x) - y.\nabla g(x)\mathbf{1}_{\{|y|\leq 1\}}\right),$$

  where ∇g denotes the gradient of g.
- The local part $\mathcal{A}g$ can be defined in the sense of distributions if g is a locally integrable function.

- We will show that $\mathcal{B}g$ can be defined in the sense of distributions if g is bounded and Borel measurable.

- We will show that $\mathcal{B}g$ can be defined in the sense of distributions if g is bounded and Borel measurable.
- If $\mathcal{O}$ is an open subset of $\mathbb{R}^d$, we denote by $\mathcal{D}(\mathcal{O})$ the set of all $\mathcal{C}^\infty$ functions with compact support in $\mathcal{O}$ and by $\mathcal{D}'(\mathcal{O})$ the space of distributions on $\mathcal{O}$. If $u \in \mathcal{D}'(\mathcal{O})$ and $\varphi \in \mathcal{D}(\mathcal{O})$, $\langle u, \varphi \rangle$ denotes the evaluation on the test function φ of the distribution u. Note that if u is a locally integrable function on $\mathcal{O}$,

$$\langle u, \varphi \rangle = \int_{\mathcal{O}} u(x)\varphi(x)dx.$$

- We will show that $\mathcal{B}g$ can be defined in the sense of distributions if g is bounded and Borel measurable.
- If $\mathcal{O}$ is an open subset of $\mathbb{R}^d$, we denote by $\mathcal{D}(\mathcal{O})$ the set of all $\mathcal{C}^\infty$ functions with compact support in $\mathcal{O}$ and by $\mathcal{D}'(\mathcal{O})$ the space of distributions on $\mathcal{O}$. If $u \in \mathcal{D}'(\mathcal{O})$ and $\varphi \in \mathcal{D}(\mathcal{O})$, $\langle u, \varphi \rangle$ denotes the evaluation on the test function φ of the distribution u. Note that if u is a locally integrable function on $\mathcal{O}$,

$$\langle u, \varphi \rangle = \int_{\mathcal{O}} u(x)\varphi(x)dx.$$

And the partial derivatives of u are defined by

$$\langle \frac{\partial u}{\partial x_j}, \varphi \rangle = -\int_{\mathcal{O}} u(x)\frac{\partial \varphi}{\partial x_j}(x)dx.$$

- Introduce the adjoint operator $\mathcal{B}^*$ of $\mathcal{B}$. For $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$, let

$$\mathcal{B}^*\varphi(x) = \int \left(\varphi(x-y) - \varphi(x) + y.\nabla\varphi(x)\mathbf{1}_{\{|y|\leq 1\}}\right)\nu(dy), \quad x \in \mathbb{R}^d.$$

- Introduce the adjoint operator $\mathcal{B}^*$ of $\mathcal{B}$. For $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d)$, let

$$\mathcal{B}^*\varphi(x) = \int \left(\varphi(x-y) - \varphi(x) + y.\nabla\varphi(x)\mathbf{1}_{\{|y|\le 1\}}\right)\nu(dy), \quad x \in \mathbb{R}^d.$$

- For the next Proposition, we will use the following notations.

$$||D^2\varphi||_\infty = \sup_{x\in\mathbb{R}^d}\sup_{|y|\le 1}\left|\sum_{i=1}^d\sum_{j=1}^d y_i y_j \frac{\partial^2\varphi}{\partial x_i \partial x_j}(x)\right|,$$

$$B_1 = \left\{y \in \mathbb{R}^d \mid |y| \le 1\right\}.$$

Proposition 1 *If $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the function $\mathcal{B}^*\varphi$ is continuous and integrable on $\mathbb{R}^d$, and we have*

$$||\mathcal{B}^*\varphi||_{L^1} \leq \frac{1}{2}||D^2\varphi||_\infty \lambda_d(K+B_1)\int_{B_1}|y|^2\nu(dy) + 2||\varphi||_{L^1}\nu(B_1^c),$$

where $K = \text{supp}\varphi$ and λ_d is the Lebesgue measure. Moreover, if $g \in \mathcal{C}_b^2(\mathbb{R}^d)$, we have

$$\langle \mathcal{B}g, \varphi\rangle = \int_{\mathbb{R}^d} g(x)\mathcal{B}^*\varphi(x)dx.$$

For $g \in L^\infty(\mathbb{R}^d)$, the distribution $\mathcal{B}g$ can be defined by setting

$$\langle \mathcal{B}g, \varphi\rangle = \int_{\mathbb{R}^d} g(x)\mathcal{B}^*\varphi(x)dx, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).$$

We can now characterize the value function u_f of an optimal stopping problem with reward function f as the solution of a variational inequality. Note that in the following statement $\partial_t v + \mathcal{L}v$ is to be understood as a distribution.

Theorem 2 *Fix $T > 0$ and let f be a continuous and bounded function on $\mathbb{R}^d$. The function v defined by $v(t,x) = u_f(T-t,x)$ is the only continuous and bounded function on $[0,T] \times \mathbb{R}^d$ satisfying the following conditions:*

1. $v(T,.) = f$,
2. $v \geq f$,
3. *On* $(0,T) \times \mathbb{R}^d$, $\partial_t v + \mathcal{L}v \leq 0$,
4. *On the open set* $\{(t,x) \in (0,T) \times \mathbb{R}^d \mid v(t,x) > f(x)\}$, $\partial_t v + \mathcal{L}v = 0$.

Proof

- Continuity of $(t, x) \mapsto u_f(t, x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}\left(f(x + X_\tau)\right)$. The process $(U_t = v(t, x + X_t))_{0 \leq t \leq T}$ is the Snell envelope of the process $(Z_t = f(x + X_t))_{0 \leq t \leq T}$.

Proof

- Continuity of $(t, x) \mapsto u_f(t, x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}\left(f(x + X_\tau)\right)$. The process $(U_t = v(t, x + X_t))_{0 \leq t \leq T}$ is the Snell envelope of the process $(Z_t = f(x + X_t))_{0 \leq t \leq T}$.
- Therefore, $(U_t)_{0 \leq t \leq T}$ is a supermartingale, and, if

$$\tau^* = \inf\{t \in [0, T] \mid U_t = Z_t\},$$

the stopped process $(U_{t \wedge \tau^*})_{0 \leq t \leq T}$ is a martingale.

Proof

- Continuity of $(t, x) \mapsto u_f(t, x) = \sup_{\tau \in \mathcal{T}_{0,t}} \mathbb{E}\left(f(x + X_\tau)\right)$. The process $(U_t = v(t, x + X_t))_{0 \leq t \leq T}$ is the Snell envelope of the process $(Z_t = f(x + X_t))_{0 \leq t \leq T}$.
- Therefore, $(U_t)_{0 \leq t \leq T}$ is a supermartingale, and, if

$$\tau^* = \inf\{t \in [0, T] \mid U_t = Z_t\},$$

the stopped process $(U_{t \wedge \tau^*})_{0 \leq t \leq T}$ is a martingale.
- Note that τ^* is the exit time from the open set $\{v > f\}$ for the process $(t, x + X_t)_{0 \leq t \leq T}$.

- Given $x \in \mathbb{R}^d$ and an open subset U of $\mathbb{R}^d$, define

$$\tau_U^x = \inf\{t \geq 0 \mid x + X_t \notin U\}.$$

- Given $x \in \mathbb{R}^d$ and an open subset U of $\mathbb{R}^d$, define

$$\tau_U^x = \inf\{t \geq 0 \mid x + X_t \notin U\}.$$

- If g is a bounded continuous function on $\mathbb{R}^d$, the following conditions are equivalent
  1- For every $x \in \mathbb{R}^d$, the process $(g(x + X_{t\wedge\tau_U^x}))_{t\geq 0}$ is a supermartingale.
  2- The distribution $\mathcal{L}g$ is a nonpositive measure on U.

- Given $x \in \mathbb{R}^d$ and an open subset U of $\mathbb{R}^d$, define

$$\tau_U^x = \inf\{t \geq 0 \mid x + X_t \notin U\}.$$

- If g is a bounded continuous function on $\mathbb{R}^d$, the following conditions are equivalent
  1- For every $x \in \mathbb{R}^d$, the process $(g(x + X_{t \wedge \tau_U^x}))_{t \geq 0}$ is a supermartingale.
  2- The distribution $\mathcal{L}g$ is a nonpositive measure on U.
- For recent results on viscosity solutions, see Barles and Imbert (*Ann. IHP* 2008).

The American put price

The American put price

- In an exponential Lévy model, the price $(S_t)_{t\in[0,T]}$ of the risky asset is given, under the pricing measure, by

$$S_t = S_0 e^{(r-\delta)t+X_t},$$

where $r > 0$ is the interest rate, $\delta \geq 0$ the dividend rate, and $X = (X_t)_{0\leq t\leq T}$ is a real Lévy process, with generating triplet (σ^2, ν, γ), such that $(e^{X_t})_{0\leq t\leq T}$ is a martingale.

The American put price

- In an exponential Lévy model, the price $(S_t)_{t\in[0,T]}$ of the risky asset is given, under the pricing measure, by

$$S_t = S_0 e^{(r-\delta)t+X_t},$$

where $r > 0$ is the interest rate, $\delta \geq 0$ the dividend rate, and $X = (X_t)_{0\leq t\leq T}$ is a real Lévy process, with generating triplet (σ^2, ν, γ), such that $(e^{X_t})_{0\leq t\leq T}$ is a martingale.

- The martingale property for e^{X_t} is equivalent to the following conditions: $\int_{\{|x|\geq 1\}} e^x \nu(dx) < \infty$ and

$$\frac{\sigma^2}{2} + \gamma + \int \left(e^x - 1 - x\mathbf{1}_{\{|x|\leq 1\}}\right) \nu(dx) = 0.$$

Using this condition, the infinitesimal generator $\mathcal{L}$ of the process $(\tilde{X}_t = \log(S_t/S_0))$ can be written as follows. For $g \in \mathcal{C}_b^2$, we have

$$\mathcal{L}g(x) = \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2}(x) + \left(r - \delta - \frac{\sigma^2}{2}\right)\frac{\partial g}{\partial x}(x) + \mathcal{B}g(x),$$

where

$$\mathcal{B}g(x) = \int \nu(dy)\left(g(x+y) - g(x) - (e^y - 1)\frac{\partial g}{\partial x}(x)\right).$$

Note that, as in the general setting, $\mathcal{B}g$ can be defined in the sense of distributions for $g \in L^\infty(\mathbb{R}^d)$.

The value at time t of an American put with maturity T and strike price K is given by

$$P_t = P(t, S_t),$$

with,

$$P(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}(e^{-r\tau} f(S_\tau^x)), \tag{1}$$

where $S_t^x = xe^{\tilde{X}_t} = xe^{(r-\delta)t + X_t}$ and $f(x) = (K - x)_+$.
It follows from (1) that $x \mapsto P(t,x)$ is convex and that $t \mapsto P(t,x)$ is non-increasing.

Define

$$\tilde{P}(t,x) = P(t,e^x), \quad (t,x) \in [0,T] \times \mathbb{R}.$$

We have

$$\tilde{P}(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}(e^{-r\tau}\tilde{f}(x+\tilde{X}_\tau)),$$

where $\tilde{f}(x) = f(e^x) = (K - e^x)_+$.

Theorem 3 *The distribution* $(\partial_t + \mathcal{L} - r)\tilde{P}$ *is a nonpositive measure on* $(0,T) \times \mathbb{R}$*, and, on the open set* $\{(t,x) \in (0,T) \times \mathbb{R} \mid \tilde{P}(t,x) > \tilde{f}(x)\}$*, we have* $(\partial_t + \mathcal{L} - r)\tilde{P} = 0$*.*

The exercise boundary

The exercise boundary

We assume that one of the following conditions holds

$$\sigma \neq 0, \quad \nu((-\infty, 0)) > 0 \quad \text{or} \quad \int_{(0,+\infty)} (x \wedge 1)\nu(dx) = +\infty.$$

The exercise boundary

We assume that one of the following conditions holds

$$\sigma \neq 0, \quad \nu((-\infty, 0)) > 0 \quad \text{or} \quad \int_{(0,+\infty)} (x \wedge 1)\nu(dx) = +\infty.$$

Under this assumption, we have

$$\forall t \in [0, T), \quad \forall x \in [0, +\infty), \quad P(t, x) > 0.$$

The exercise boundary

We assume that one of the following conditions holds

$$\sigma \neq 0, \quad \nu((-\infty, 0)) > 0 \quad \text{or} \quad \int_{(0,+\infty)} (x \wedge 1)\nu(dx) = +\infty.$$

Under this assumption, we have

$$\forall t \in [0, T), \quad \forall x \in [0, +\infty), \quad P(t, x) > 0.$$

We now define the *critical price* at time $t \in [0, T)$ by

$$b(t) = \inf\{x \geq 0 \mid P(t, x) > f(x)\}.$$

The exercise boundary

We assume that one of the following conditions holds

$$\sigma \neq 0, \quad \nu((-\infty, 0)) > 0 \quad \text{or} \quad \int_{(0,+\infty)} (x \wedge 1)\nu(dx) = +\infty.$$

Under this assumption, we have

$$\forall t \in [0, T), \quad \forall x \in [0, +\infty), \quad P(t, x) > 0.$$

We now define the *critical price* at time $t \in [0, T)$ by

$$b(t) = \inf\{x \geq 0 \mid P(t, x) > f(x)\}.$$

Note that $b(t) \in [0, K)$. It can be proved that $b(t) > 0$.

Since $t \mapsto P(t,x)$ is nonincreasing, the function $t \mapsto b(t)$ is nondecreasing.
We obviously have $P(t,x) = f(x)$ for $x \in [0, b(t))$ and also for $x = b(t)$, due to the continuity of P and f. We also deduce from the convexity of $x \mapsto P(t,x)$ and the fact that $P(t,x) > 0$ that

$$\forall t \in [0,T), \quad \forall x > b(t), \quad P(t,x) > f(x).$$

Since $t \mapsto P(t,x)$ is nonincreasing, the function $t \mapsto b(t)$ is nondecreasing.
We obviously have $P(t,x) = f(x)$ for $x \in [0, b(t))$ and also for $x = b(t)$, due to the continuity of P and f. We also deduce from the convexity of $x \mapsto P(t,x)$ and the fact that $P(t,x) > 0$ that

$$\forall t \in [0,T), \quad \forall x > b(t), \quad P(t,x) > f(x).$$

In other words, the continuation region C can be written as

$$C = \{(t,x) \in [0,T) \times [0,+\infty) \mid x > b(t)\}.$$

The graph of b is called the *exercise boundary* or *free boundary*.

Theorem 4 *The function* $t \mapsto b(t)$ *is continuous on* $[0, T)$.

The following result characterizes the limit of the critical price $b(t)$ as t approaches T. This extends and clarifies recent results of Levendorski (2004). See also Yang, Jiang and Bian (2006), Bayraktar, Xing (2008).

Theorem 5 *If* $\int (e^x - 1)_+ \nu(dx) \leq r - \delta$, *we have*

$$\lim_{t \to T} b(t) = K.$$

If $\int (e^x - 1)_+ \nu(dx) > r - \delta$, *we have* $\lim_{t \to T} b(t) = \xi$, *where* ξ *is the unique real number in the interval* $(0, K)$ *such that* $\varphi(\xi) = rK$, *and* φ *is the function defined by*

$$\varphi(x) = \delta x + \int (xe^y - K)_+ \nu(dy), \quad x \in (0, K).$$

The smooth fit property

The continuity of the derivative (with respect to the underlying stock price) of the American put price is a well known property in the Black-scholes model, called the *smooth fit* property (see also Zhang (1994) and Bayraktar (2007) for jump-diffusions). In the context of exponential Lévy models, this property may no longer be true.

The smooth fit property

The continuity of the derivative (with respect to the underlying stock price) of the American put price is a well known property in the Black-scholes model, called the *smooth fit* property (see also Zhang (1994) and Bayraktar (2007) for jump-diffusions). In the context of exponential Lévy models, this property may no longer be true.

In the case of *perpetual* American options, Alili and Kyprianou (2004) proved that a necessary and sufficient condition for smooth fit is that the point 0 is *regular* with respect to the set $(-\infty, 0)$ for the process $\tilde{X}_t := (r-\delta)t + X_t$, which means that $\mathbb{P}(\tau_0 = 0) = 1$, where

$$\tau_0 = \inf\{t > 0 \mid \tilde{X}_t < 0\}.$$

In the case of finite horizon, it can be proved that regularity implies smooth fit (G. Peskir). It follows that the smooth fit property is satisfied by the American put in an exponential Lévy model, if the underlying Lévy process has infinite variation.
Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded and Lipschitz continuous function. Define

$$v(t,x) = \sup_{\tau \in \mathcal{T}_{0,T-t}} \mathbb{E}\left(e^{-r\tau} f(x + X_\tau)\right), \quad 0 \le t \le T,$$

where X is a real Lévy process. Assume there exists $b : [0,T) \mapsto \mathbb{R}$ such that for every $t \in [0,T)$, we have

$$v(t,x) > f(x) \Leftrightarrow x > b(t).$$

Proposition 6 *Assume that* 0 *is a regular point with respect to the set* $(-\infty, 0)$ *for the process* X *and that* f *is of class* C^1 *in a neighborhood of* $b(0)$. *Then, the function* $x \mapsto v(0, x)$ *is differentiable at* $x = b(0)$ *and* $\dfrac{\partial v}{\partial x}(0, b(0)) = f'(b(0))$.

Proof (G. Peskir): For simplicity, denote $x_0 = b(0)$, $x_h = x_0 + h$, and $v(x) = v(0, x)$. We need only prove that v has a right-hand derivative at x_0, since $v = f$ on $(\infty, x_0]$. For $h > 0$, we have

$$
\begin{aligned}
\frac{v(x_0 + h) - v(x_0)}{h} &= \frac{v(x_0 + h) - f(x_0)}{h} \\
&\geq \frac{f(x_0 + h) - f(x_0)}{h}.
\end{aligned}
$$

Hence

$$\liminf_{h\to 0} \frac{v(x_0+h)-v(x_0)}{h} \geq f'(x_0).$$

Hence

$$\liminf_{h\to 0} \frac{v(x_0+h)-v(x_0)}{h} \geq f'(x_0).$$

For the upper bound, introduce the optimal stopping time with initial point x_h:

$$\begin{aligned} \tau_h &= \inf\{t \geq 0 \mid v(t, x_h + X_t) = f(x_h + X_t)\} \\ &= \inf\{t \geq 0 \mid t = T \text{ or } x_h + X_t \leq b(t)\}. \end{aligned}$$

Hence

$$\liminf_{h\to 0} \frac{v(x_0+h)-v(x_0)}{h} \geq f'(x_0).$$

For the upper bound, introduce the optimal stopping time with initial point x_h:

$$\begin{aligned}\tau_h &= \inf\{t \geq 0 \mid v(t, x_h + X_t) = f(x_h + X_t)\} \\ &= \inf\{t \geq 0 \mid t = T \text{ or } x_h + X_t \leq b(t)\}.\end{aligned}$$

Note that

$$\begin{aligned}\tau_h &= \inf\{t \geq 0 \mid t = T \text{ or } X_t \leq b(t) - b(0) - h\} \\ &\leq \inf\{t \geq 0 \mid t = T \text{ or } X_t \leq -h\} \\ &= T \wedge \hat{\tau}_h, \quad \text{where } \hat{\tau}_h = \inf\{t \geq 0 \mid X_t \leq -h\}.\end{aligned}$$

It can be proved that

$$\lim_{h \to 0} \hat{\tau}_h = \tau_0 = \inf\{t \geq 0 \mid X_t < 0\},$$

so that, if 0 is regular for $(-\infty, 0)$, $\lim_{h \to 0} \tau_h = 0$, almost surely.

It can be proved that

$$\lim_{h \to 0} \hat{\tau}_h = \tau_0 = \inf\{t \geq 0 \mid X_t < 0\},$$

so that, if 0 is regular for $(-\infty, 0)$, $\lim_{h \to 0} \tau_h = 0$, almost surely.

On the other hand, we have

$$\begin{aligned}
v(x_h) - v(x_0) &= \mathbb{E}\left(e^{-r\tau_h} f(x_h + X_{\tau_h})\right) - v(x_0) \\
&\leq \mathbb{E}\left(e^{-r\tau_h} f(x_h + X_{\tau_h})\right) - \mathbb{E}\left(e^{-r\tau_h} f(x_0 + X_{\tau_h})\right. \\
&= \mathbb{E}\left[e^{-r\tau_h}\left(f(x_0 + h + X_{\tau_h}) - f(x_0 + X_{\tau_h})\right)\right].
\end{aligned}$$

It can be proved that

$$\lim_{h\to 0} \hat{\tau}_h = \tau_0 = \inf\{t \geq 0 \mid X_t < 0\},$$

so that, if 0 is regular for $(-\infty, 0)$, $\lim_{h\to 0} \tau_h = 0$, almost surely.

On the other hand, we have

$$\begin{aligned} v(x_h) - v(x_0) &= \mathbb{E}\left(e^{-r\tau_h} f(x_h + X_{\tau_h})\right) - v(x_0) \\ &\leq \mathbb{E}\left(e^{-r\tau_h} f(x_h + X_{\tau_h})\right) - \mathbb{E}\left(e^{-r\tau_h} f(x_0 + X_{\tau_h})\right) \\ &= \mathbb{E}\left[e^{-r\tau_h}\left(f(x_0 + h + X_{\tau_h}) - f(x_0 + X_{\tau_h})\right)\right]. \end{aligned}$$

Hence

$$\limsup_{h\to 0} \frac{v(x_0 + h) - v(x_0)}{h} \leq f'(x_0).$$

The regularity property is satisfied for Lévy processes which have paths with infinite variation.
Using the variational inequality, one can derive the following statement.

Proposition 7 *Consider an exponential Lévy model, in which the generating triplet of the Lévy process is given by* (σ^2, ν, γ)*. Suppose* $\sigma^2 = 0$*, and* $\int (|x| \wedge 1)\nu(dx) < \infty$*.*

- *If* $r - \delta - \int (e^y - 1)\nu(dy) < 0$*, smooth fit holds for American put options with finite maturity.*
- *If* $r - \delta - \int (e^y - 1)_+\nu(dy) > 0$*, smooth fit does not hold.*

Note that $\gamma_0 := r - \delta - \int (e^y - 1)\nu(dy)$ is the drift of the Lévy process $\tilde{X}_t = \log(S_T/S_0)$.

Proof of the second part of the Proposition

It follows from the variational inequality that, for $x > b(t)$,

$$\gamma_0 x \frac{\partial P}{\partial x}(t,x) + \int (P(t,xe^y) - P(t,x))\,\nu(dy) - rP(t,x) = -\frac{\partial P}{\partial t}(t,x).$$

Therefore, at $x = b(t)$,

$$\begin{aligned} \gamma_0 x \frac{\partial P}{\partial x}(t,x) &\geq r(K-x) - \int (P(t,xe^y) - P(t,x))\,\nu(dy) \\ &= r(K-x) - \int_{(-\infty,0)} x(1-e^y)\nu(dy) \\ &\quad - \int_{(0,+\infty)} (P(t,xe^y) - P(t,x))\,\nu(dy). \end{aligned}$$

Note: $P(t,\cdot)$ is nonincreasing and $\gamma_0 = r - \delta - \int (e^y - 1)\nu(dy)$

Hence (for $x = b(t)$)

$$\left(r - \delta - \int (e^y - 1)\nu(dy)\right) \frac{\partial P}{\partial x}(t,x) + \int_{(-\infty,0)} (1 - e^y)\nu(dy) \quad \geq \quad 0,$$

so that

$$\begin{aligned} \frac{\partial P}{\partial x}(t,x) \quad &\geq \quad -\frac{\int_{(-\infty,0)}(1-e^y)\nu(dy)}{r - \delta - \int (e^y-1)_+\nu(dy) + \int_{(-\infty,0)}(1-e^y)\nu(dy)} \\ &> \quad -1, \end{aligned}$$

if $r - \delta - \int (e^y - 1)_+ \nu(dy) > 0$.

Continuity of the free boundary

The right continuity can be deduced easily from the continuity of P and its monotonicity.
For the proof of the left continuity, we use the variational inequality. Define

$$\tilde{E} = \{(t, x) \in (0, T) \times \mathbb{R} \mid x < \tilde{b}(t)\},$$

with $\tilde{b}(t) = \ln b(t)$.
On the open set $\tilde{E}$, we have $\tilde{P} = \tilde{f}$, and

$$(\mathcal{L} - r)\tilde{P} = (\partial \tilde{P}/\partial t) + (\mathcal{L} - r)\tilde{P} \leq 0.$$

When computing $(\mathcal{L} - r)\tilde{P}$ in $\tilde{E}$, we have to be careful with the non local part of the operator.

We have

$$(\mathcal{L} - r)\tilde{P}(t,x) = \phi_t(e^x) + \delta e^x - rK, \quad (t,x) \in \tilde{E},$$

where

$$\phi_t(x) = \int (P(t, xe^y) + xe^y - K)\, \nu(dy).$$

For each $t \in [0, T)$, the function ϕ_t is nonnegative, convex and continuous on the interval $[0, b(t))$.
We now prove that b is left continuous. Equivalently, we will prove that $t \mapsto \tilde{b}(t) = \ln b(t)$ is left continuous.

Fix $t \in (0,T)$ and denote by $\tilde{b}(t^-)$ the left limit of $\tilde{b}$ at t. Recall that $\tilde{b}$ is nondecreasing, so that the limit exists and $\tilde{b}(t^-) \leq \tilde{b}(t)$.
Suppose $\tilde{b}(t^-) < \tilde{b}(t)$, and let $(s,x) \in (0,t) \times (\tilde{b}(t^-), \tilde{b}(t))$. We have $x > \tilde{b}(t^-) \geq \tilde{b}(s)$, so that $\tilde{P}(s,x) > \tilde{f}(x)$. Therefore, on the open set $(0,t) \times (\tilde{b}(t^-), \tilde{b}(t))$, we have $(\partial_t + \mathcal{L} - r)\tilde{P} = 0$. Hence

$$(\mathcal{L} - r)\tilde{P} = -\partial_t \tilde{P} \geq 0, \quad \text{on } (0,t) \times (\tilde{b}(t^-), \tilde{b}(t)),$$

Using the continuity of $\tilde{P}$, we deduce that for every $s \in (0,t)$, we have $(\mathcal{L} - r)\tilde{P}(s,.) \geq 0$ on the open interval $(\tilde{b}(t^-), \tilde{b}(t))$.

By passing to the limit as $s \to t$, we get

$(\mathcal{L} - r)\tilde{P}(t,.) \geq 0$ on the open interval $(\tilde{b}(t^-), \tilde{b}(t))$.

By passing to the limit as $s \to t$, we get

$$(\mathcal{L} - r)\tilde{P}(t, .) \geq 0 \text{ on the open interval } (\tilde{b}(t^-), \tilde{b}(t)).$$

On the other hand, we have $(t, T) \times (-\infty, \tilde{b}(t)) \subset \tilde{E}$, so that, on this set, $(\mathcal{L} - r)\tilde{P} \leq 0$. Hence

$$(\mathcal{L} - r)\tilde{P}(t, .) = 0 \quad \text{ on } (\tilde{b}(t^-), \tilde{b}(t)).$$

By passing to the limit as $s \to t$, we get

$$(\mathcal{L} - r)\tilde{P}(t, .) \geq 0 \text{ on the open interval } (\tilde{b}(t^-), \tilde{b}(t)).$$

On the other hand, we have $(t, T) \times (-\infty, \tilde{b}(t)) \subset \tilde{E}$, so that, on this set, $(\mathcal{L} - r)\tilde{P} \leq 0$. Hence

$$(\mathcal{L} - r)\tilde{P}(t, .) = 0 \quad \text{ on } (\tilde{b}(t^-), \tilde{b}(t)).$$

Recall $(\mathcal{L} - r)\tilde{P}(t, x) = \phi_t(e^x) + \delta e^x - rK, \quad (t, x) \in \tilde{E}$

By passing to the limit as $s \to t$, we get

$$(\mathcal{L} - r)\tilde{P}(t,.) \geq 0 \text{ on the open interval } (\tilde{b}(t^-), \tilde{b}(t)).$$

On the other hand, we have $(t,T) \times (-\infty, \tilde{b}(t)) \subset \tilde{E}$, so that, on this set, $(\mathcal{L} - r)\tilde{P} \leq 0$. Hence

$$(\mathcal{L} - r)\tilde{P}(t,.) = 0 \quad \text{ on } (\tilde{b}(t^-), \tilde{b}(t)).$$

Recall $(\mathcal{L} - r)\tilde{P}(t,x) = \phi_t(e^x) + \delta e^x - rK, \quad (t,x) \in \tilde{E}$ We then have $\phi_t(x) + \delta x = rK, \quad$ for $x \in (b(t^-), b(t))$.

By passing to the limit as $s \to t$, we get

$$(\mathcal{L} - r)\tilde{P}(t, .) \geq 0 \text{ on the open interval } (\tilde{b}(t^-), \tilde{b}(t)).$$

On the other hand, we have $(t, T) \times (-\infty, \tilde{b}(t)) \subset \tilde{E}$, so that, on this set, $(\mathcal{L} - r)\tilde{P} \leq 0$. Hence

$$(\mathcal{L} - r)\tilde{P}(t, .) = 0 \quad \text{ on } (\tilde{b}(t^-), \tilde{b}(t)).$$

Recall $(\mathcal{L} - r)\tilde{P}(t, x) = \phi_t(e^x) + \delta e^x - rK, \quad (t, x) \in \tilde{E}$ We then have $\phi_t(x) + \delta x = rK, \quad$ for $x \in (b(t^-), b(t))$. Now, let $\hat{\phi}_t(x) = \phi_t(x) + \delta x$. Note that $\hat{\phi}_t$ is continuous, convex on $[0, b(t))$, nonnegative, and that $\hat{\phi}_t(0) = 0$. Therefore, $\hat{\phi}_t$ cannot be equal to a positive constant on an open interval.