On Multi-Period Risk Functionals

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Why to measure risk/acceptability

In longer term planning such as

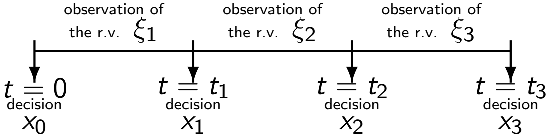
- portfolio planning,
- pension fund management,
- electricity portfolio management,
- gas portfolio management,

we have to find decision strategies in a random environment. A good decision aims at maximizing the expected return among all acceptable decisions.
What is the overall expected return is clear. The overall acceptability is measured by a functional $\mathcal{A}(Y_1, \ldots, Y_T)$ from which we require that is is larger than some threshold.

A typical multi-period problem

$\xi = (\xi_1, \ldots, \xi_T)$	a scenario process
$x = (x_0, \ldots, x_{T-1})$	the decision process
$Y_t = H_t(x_0, \ldots, x_{t-1}; \xi_1, \ldots, \xi_t)$	the generated income process

$$\begin{array}{lll} \text{Maximize} & \mathbb{E}[\sum_{t=1}^{T} Y_t] & \\ \text{subject to} & \mathcal{A}_t(Y_1, \ldots, Y_t) \geq q_t & t = 1, \ldots, T \\ & x \lhd \mathcal{F} & \\ & x \in \mathbb{X} & \end{array}$$



What is a single -period risk/acceptability measure?

A mapping $\mathcal{A} : L_p(\Omega, \mathcal{F}, \mathbb{P}) \to \overline{\mathbb{R}}$ is called *acceptability functional* if it satisfies the following conditions for all Y, $\tilde{Y} \in \mathcal{Y}$, $c \in \mathbb{R}$, $\lambda \in [0, 1]$:

(A1) $\mathcal{A}(Y + c) = \mathcal{A}(Y) + c$ (*translation-equivariance*),

(A1') There is a linear subspace $\mathcal{W} \subseteq L_p$ and a function $Z^* \in L_q(\mathcal{F})$ $(1/p + 1/q = 1)$ such that for $W \in \mathcal{W}$

$$\mathcal{A}(W) = \mathbb{E}(W\, Z^*).$$

It then follows that

$$\mathcal{A}(Y+W) = \mathcal{A}(Y)+\mathbb{E}(W\, Z^*), \quad (\textit{the } (\mathcal{W}, Z^*) \textit{ translation property}).$$

(A2) $\mathcal{A}(\lambda Y + (1 - \lambda)\tilde{Y}) \geq \lambda\mathcal{A}(Y) + (1 - \lambda)\mathcal{A}(\tilde{Y})$ (*concavity*),

(A3) $Y \leq \tilde{Y}$ implies $\mathcal{A}(Y) \leq \mathcal{A}(\tilde{Y})$ (*monotonicity*).

An acceptability functional $\mathcal{A}$ is called

- *version-independent (law-invariant)* if

  $\mathcal{A}(Y)$ depends only on the distribution function $G_Y(u) = \mathbb{P}\{Y \le u\}$

Given an acceptability functional $\mathcal{A}$, the mappings

$$\rho := -\mathcal{A} \quad \text{and} \quad \mathcal{D} := \mathbb{E} - \mathcal{A}$$

are called *risk capital* and *deviation risk functional*, respectively.

By the Fenchel-Moreau Theorem, every concave upper semicontinuous (u.s.c.) functional $\mathcal{A}$ on $\mathcal{Y}$ has a representation of the form

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) - \mathcal{A}^+(Z) : Z \in \mathcal{Z}\}, \tag{1}$$

where $\mathcal{A}^+(Z) = \inf\{\mathbb{E}(Y\,Z) - \mathcal{A}(Y) : Y \in \mathcal{Y}\}$. We call (1) a *dual representation*. Let $dom(\mathcal{A}^+) = \{Z : \mathcal{A}^+(Z) > -\infty\}$. Then

- $\mathcal{A}$ is monotonic, iff $dom(\mathcal{A}^+) \subseteq L_q^+$
- $\mathcal{A}$ has the $(\mathcal{W}, Z^*)$ translation property (A1'), iff $dom(\mathcal{A}^+) \subseteq \mathcal{W}^\perp + Z^*$
- $\mathcal{A}$ is positively homogeneous, iff $\mathcal{A}^+$ takes only the values 0 and $-\infty$

Examples for acceptability functionals

- The expectation. $\mathcal{A}(Y) = \mathbb{E}(Y)$.
- The Average Value-at-Risk. $\mathbb{A}\mathsf{V@R}_\alpha(Y) = \frac{1}{\alpha}\int_0^\alpha G_Y^{-1}(p)\,dp$.
- The distortion functional. $\mathcal{A}(Y) = \int_0^1 G_Y^{-1}(p)\,k(p)\,dp$.
- Risk corrected expectation.
  - $\mathbb{E}(Y) - \delta\mathbb{S}\mathrm{td}^-(Y)$
  - $\mathbb{E}(Y) - \delta\mathrm{Gini}(Y)$
  - $\mathbb{E}(Y) - \delta\mathrm{Mad}(Y)$

Examples for dual representations

Let h be a convex, nonnegative function satisfying $h(0) = 0$ and let $h^*(v) = \sup\{uv - h(u) : u \in \mathbb{R}\}$ be its Fenchel dual.

primal	dual
$\mathcal{A}(Y) = \mathbb{E}Y - \mathbb{E}[h(Y - \mathbb{E}Y)]$	$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) + D_{h*}(Z) : \mathbb{E}Z = 1\}$ $D_{h*}(Z) = \inf\{\mathbb{E}[h^*(Z - a)] : a \in \mathbb{R}\}$
$\mathcal{A}(Y) = \mathbb{E}Y - \inf\{\mathbb{E}[h(Y - a)] : a \in \mathbb{R}\}$	$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) + \mathbb{E}(h^*(1 - Z)) : \mathbb{E}(Z) = 1\}$
$\mathcal{A}(Y) = \mathbb{E}(Y) - M_h(Y - \mathbb{E}Y)$ $M_h(Y) = \inf\{a \geq 0 : \mathbb{E}[h(\frac{Y}{a})] \leq h(1)\}$	$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) : \mathbb{E}(Z) = 1, \inf_a\{D^*_{h*}(Z - a)\} \leq 1\}$ $D^*_{h*}(Z) = \sup\{\mathbb{E}(Z\,V) : \mathbb{E}[h^*(V)] \leq h^*(1)\}$.
$\mathcal{A}(Y) = \int_0^1 G_Y^{-1}(p)\,k(p)\,dp$ k nonnegative, monotonic, bounded	$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) : \mathbb{E}(\phi(Z)) \leq \int \phi(k(u))\,du, \phi \text{ convex }, \phi(0) = 0\}$,

A special case is the Average Value-at-Risk $\mathbb{A}\mathsf{V@R}_\alpha$

primal: $\mathbb{A}\mathsf{V@R}_\alpha(Y) = \frac{1}{\alpha}\int_0^1 G_Y^{-1}(p)\,dp = \max\{a - \mathbb{E}([Y-a]^-) : a \in \mathbb{R}\}$

dual: $\mathbb{A}\mathsf{V@R}_\alpha(Y) = \inf\{E(Y\,Z) : 0 \le Z \le 1/\alpha, \mathbb{E}(Z) = 1\}.$

Conditional acceptability functionals

Let $\mathcal{F}_1$ be a σ-field contained in $\mathcal{F}$. A mapping $\mathcal{A}_{\mathcal{F}_1} : L_p(\mathcal{F}) \to L_p(\mathcal{F}_1)$ is called *conditional acceptability mapping* if the following conditions are satisfied for all Y, $\lambda \in [0, 1]$:

(CA1) $\mathcal{A}_{\mathcal{F}_1}(Y + Y^{(1)}) = \mathcal{A}_{\mathcal{F}_1}(Y) + Y^{(1)}$
for $Y^{(1)} \lhd \mathcal{F}_1$ (*predictable translation-equivariance);*

(CA2) $\mathcal{A}_{\mathcal{F}_1}(\lambda Y + (1-\lambda)\tilde{Y}) \geq \lambda \mathcal{A}_{\mathcal{F}_1}(Y) + (1-\lambda)\mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$
*(*concavity*),*

(CA3) $Y \leq \tilde{Y}$ *implies* $\mathcal{A}_{\mathcal{F}_1}(Y) \leq \mathcal{A}_{\mathcal{F}_1}(\tilde{Y})$ *(*monotonicity*).*

Theorem. A mapping $\mathcal{A}_{\mathcal{F}_1}$ is a conditional acceptability mapping if and only if for all $B \in \mathcal{F}_1$ the functional $Y \mapsto \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B)$ is an acceptability functional, which has the $(L_p(\mathcal{F}_1), \mathbb{1}_B)$ translation property, that is

$$\mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y + Y^{(1)})\mathbb{1}_B) = \mathbb{E}(\mathcal{A}_{\mathcal{F}_1}(Y)\mathbb{1}_B) + \mathbb{E}(Y^{(1)}\mathbb{1}_B)$$

for all $Y^{(1)} \in L_p(\mathcal{F}_1)$.

The conditional Average Value-at-Risk.

For $Y \in L_1$, $\mathbb{AV@R}_\alpha(Y|\mathcal{F}_1)$ is defined on $L_1(\mathcal{F})$ by the relation

$$\mathbb{E}(\mathbb{AV@R}_\alpha(Y|\mathcal{F}_1)\mathbb{1}_B) = \inf\{\mathbb{E}(Y\,Z\,\mathbb{1}_B) : 0 \le Z \le \frac{1}{\alpha}, \mathbb{E}(Z|\mathcal{F}_1) = 1\}.$$

$(B \in \mathcal{F}_1)$. There is a version such that $\alpha \mapsto \mathbb{AV@R}_\alpha(Y|\mathcal{F}_1)$ is monotonically increasing a.s. for $\alpha \in (0,1]$.
The L_1 space is an order complete Banach lattice, which implies that every set of elements from L_1, which is bounded from below has an infimum. Denote this infimum by **inf**. We may also write

$$\mathbb{AV@R}_\alpha(Y|\mathcal{F}_1) = \mathbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) : 0 \le Z \le \frac{1}{\alpha}, \mathbb{E}(Z|\mathcal{F}_1) = 1\}.$$

From unconditional to conditional functionals

By considering the trivial σ-algebra $\mathcal{F}_0 = (\emptyset, \Omega)$ one may specialize every conditional acceptability mapping to an ordinary acceptability measure.
Conversely, one may lift version-independent acceptability functionals to conditional acceptability mappings: Assume that $\mathcal{A}$ is defined by

$$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) - \mathcal{A}^{+}(Z) : \mathbb{E}(Z) = 1, Z \geq 0, Z \in \mathcal{Z}\},$$

where
$\mathcal{A}^{+}$ is the conjugate functional and
$\mathcal{Z}$ is the set of subgradients.

Conditional subgradient sets and conjugates

Assume that the subgradient set of $\mathcal{A}$ is defined by

$$\mathcal{Z} = \{Z : Z \geq 0, Z \in A \, a.s., \sup_{c \in C} \mathbb{E}(\phi_c(Z)) \leq 0, \inf_{d \in D} \mathbb{E}(\psi_d(Z)) \leq 0\}$$

where C and D are countable index sets. Then the conditional subgradient set is

$$\begin{aligned} \mathcal{Z}(\mathcal{F}_1) \quad = \quad & \{Z : \mathbb{E}(Z|\mathcal{F}_1) = 1, Z \geq 0, Z \in A \, a.s., \\ & \sup_{c \in C} \mathbb{E}(\phi_c(Z)|\mathcal{F}_1) \leq 0, a.s., \quad \inf_{d \in D} \mathbb{E}(\psi_d(Z)|\mathcal{F}_1) \leq 0, a.s.\}. \end{aligned}$$

In many cases the conditional conjugate $\mathcal{A}^+(\cdot|\mathcal{F}_1)$ may be found in a direct way.

Examples for conditional acceptability functionals

unconditional: $\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) + \inf\{\mathbb{E}[h^*(Z-a)] : a \in \mathbb{R}\} : \mathbb{E}Z = 1\}$
conditional: $\mathcal{A}(Y|\mathcal{F}_1) = \textbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) + \textbf{inf}\{\mathbb{E}[h^*(Z-a)|\mathcal{F}_1] : a \lhd \mathcal{F}_1\} : \mathbb{E}(Z|\mathcal{F}_1) = 1\}$

unconditional: $\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) + \mathbb{E}(h^*(1-Z)) : \mathbb{E}(Z) = 1\}$
conditional: $\mathcal{A}(Y|\mathcal{F}_1) = \textbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) + \mathbb{E}(h^*(1-Z)|\mathcal{F}_1) : \mathbb{E}(Z) = 1\}$

unconditional:$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) : \mathbb{E}(Z) = 1, \inf_a\{\sup\{\mathbb{E}[(Z-a)\,V] : \mathbb{E}[h(V)|\mathcal{F}_1] \le h(1)\} \le 1\}$
conditional:
$\mathcal{A}(Y|\mathcal{F}_1) = \textbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) : \mathbb{E}(Z|\mathcal{F}_1) = 1, \inf_a\{\sup\{\mathbb{E}[(Z-a)\,V|\mathcal{F}_1] : \mathbb{E}[h(V)|\mathcal{F}_1] \le h(1)\} \le 1\}$

unconditional:$\mathcal{A}(Y) = \inf\{\mathbb{E}(Y\,Z) : \mathbb{E}(\phi(Z)) \le \int \phi(k(u))\,du, \phi$ convex $, \phi(0) = 0\}$
conditional: $\mathcal{A}(Y|\mathcal{F}_1) = \textbf{inf}\{\mathbb{E}(Y\,Z|\mathcal{F}_1) : \mathbb{E}(\phi(Z)|\mathcal{F}_1) \le \int \phi(k(u))\,du, \phi$ convex $, \phi(0) = 0\}$

Multi-period acceptability functionals

Let $Y = (Y_1, \ldots, Y_T)$ be an income process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\boldsymbol{\mathcal{F}} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$ denote a filtration which models the available information over time, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$, $\mathcal{F}_t \subseteq \mathcal{F}_{t+1} \subseteq \mathcal{F}$, and Y_t is $\mathcal{F}_t$ measurable for every $t = 1, \ldots, T$. Let $\mathcal{Y} \subseteq \times_{t=1}^{T} L_1(\Omega, \mathcal{F}, \mathbb{P})$ be a linear space of income processes $Y = (Y_1, \ldots, Y_T)$, which are all adapted to $\boldsymbol{\mathcal{F}}$.
Definition. A multi-period functional $\mathcal{A}$ with values $\mathcal{A}(Y; \boldsymbol{\mathcal{F}})$ is called *multi-period acceptability functional*, if satisfies

(MA0) **Information monotonicity.** If $Y \in \mathcal{Y}$ and $\mathcal{F}_t \subseteq \mathcal{F}'_t$, for all t, then

$$\mathcal{A}(Y; \mathcal{F}_0, \ldots, \mathcal{F}_{T-1}) \leq \mathcal{A}(Y; \mathcal{F}'_0, \ldots, \mathcal{F}'_{T-1}).$$

(MA1) **Predictable translation-equivariance.** If $W \in \mathcal{Y}$ such that W_t is $\mathcal{F}_{t-1}$ measurable for all t, then

$$\mathcal{A}(Y + W; \boldsymbol{\mathcal{F}}) = \sum_{t=1}^{T} \mathbb{E}(W_t) + \mathcal{A}(Y; \boldsymbol{\mathcal{F}}). \tag{2}$$

(MA2) **Concavity.** The mapping $Y \mapsto \mathcal{A}(Y; \boldsymbol{\mathcal{F}})$ is concave on $\mathcal{Y}$ for all filtrations $\boldsymbol{\mathcal{F}}$.

(MA3) **Monotonicity.** If $Y_t \leq \tilde{Y}_t$ holds a.s. for all t, then

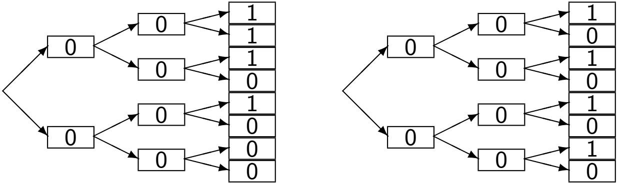
$$\mathcal{A}(Y; \boldsymbol{\mathcal{F}}) \leq \mathcal{A}(\tilde{Y}; \boldsymbol{\mathcal{F}}).$$

(MA1)* $(\pi, \mathcal{W})$-**translation-equivariance.** There exists a linear subspace $\mathcal{W}$ of $\times_{t=1}^{T} L_1(\Omega, \mathcal{F}_{t-1}, \mathbb{P})$ and a linear continuous functional $\pi : \mathcal{W} \to \mathbb{R}$ such that for all $W \in \mathcal{W}$, $Y \in \mathcal{Y}$

$$\mathcal{A}(Y + W; \boldsymbol{\mathcal{F}}) = \pi(W) + \mathcal{A}(Y; \boldsymbol{\mathcal{F}}).$$

When is a multi-period functional version-independent?

Recall that our functionals are defined on pairs of processes and filtrations. For illustration, we use a tree representation.



The two value processes are identical in distribution, but differ in the information.

Definition. ν is a tree process, iff the σ-fields generated by ν_t form a filtration (an increasing sequence of σ-fields)

Equivalence and Invariance

We assume that the filtration $\mathcal{F}$ is generated by a tree process ν with values in a Polish space and that the income process Y is adapted to it. We call

$$(Y, \mathcal{F}) \quad \text{resp.} \quad (Y, \nu)$$

a *process-and-information pair*. Notice that there are functions f_t such that $Y_t = f_t(\nu_t) \qquad$ a.e.

Definition. Two process-and-information pairs (Y, ν) and $(\bar{Y}, \bar{\nu})$ (which are defined on possibly different probability spaces) are *equivalent*, if there are bijective measurable functions k_t such that

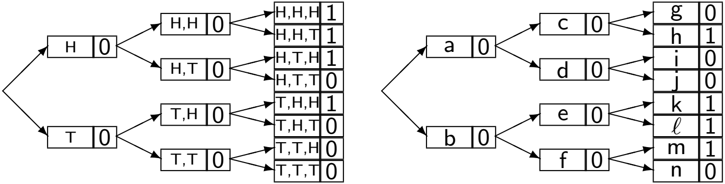
(i) $\bar{\nu}_t$ has the same distribution as $k_t(\nu_t)$.

(ii) $Y_t = f_t(\nu_t)$ and $\bar{Y}_t = f_t(k_t(\bar{\nu}_t))$.

Important observation. The solutions of stochastic optimization problems with version-independent objective are invariant w.r.t. the choice of equivalent process-and-information pairs.

An Example for Equivalence

$$
\begin{tikzcd}
\nu_t \arrow[r, "k_t"] \arrow[d, dash] \arrow[dr, dash] & \bar{\nu}_t \arrow[d, dash] \\
Y_t = f_t(\nu_t) & \bar{Y}_t = f_t(k_t^{-1}(\bar{\nu}_t))
\end{tikzcd}
$$



Equivalent process-and-information pairs.

Nested distributions

Let (Ξ, d) be a Polish space, i.e. complete separable metric space and let $\mathcal{P}_1(\Xi, d)$ be the family of all Borel probability measures P on (Ξ, d) such that

$$\int d(u, u_0)\, dP(u) < \infty$$

for some $u_0 \in \Xi$.
For two Borel probabilities, P and Q in $\mathcal{P}_1(\Xi, d)$, let $d(P, Q)$ denote the Kantorovich distance

$$d(P, Q) = \sup\{\int h(u)\, dP(u) - \int h(u)\, dQ(u) : |h(u) - h(v)| \leq d(u, v)\}$$

d metrizises the weak topology on $\mathcal{P}_1$.
$\mathcal{P}_1$ is a complete separable metric space (Polish space) under d.
Iterate the argument: $\mathcal{P}_1(\mathcal{P}_1(\Xi, d), d)$ is a Polish space, a space of distributions over distributions (i.e. what Bayesians would call a hyperdistribution).

If (Ξ_1, d_1) and (Ξ_2, d_2) are Polish spaces then so is the Cartesian product $(\Xi_1 \times \Xi_2)$ with metric

$$d^2((u_1, u_2), (v_1, v_2)) = d_1(u_1, v_1) + d_2(u_2, v_2).$$

Consider some metric d on $\mathbb{R}^m$, which makes it Polish (it needs not to be the Euclidean one). Then we define the following spaces

$$\begin{aligned}
\Xi_1 &= (\mathbb{R}^m, d) \\
\Xi_2 &= (\mathbb{R}^m \times \mathcal{P}_1(\Xi_1, d), d^2) = (\mathbb{R}^m \times \mathcal{P}_1(\mathbb{R}^m, d), d^2) \\
\Xi_3 &= (\mathbb{R}^m \times \mathcal{P}_1(\Xi_2, d), d^2) = (\mathbb{R}^m \times \mathcal{P}_1(\mathbb{R}^m \times \mathcal{P}_1(\mathbb{R}^m, d), d^2), d^2) \\
&\vdots \\
\Xi_T &= (\mathbb{R}^m \times \mathcal{P}_1(\Xi_{T-1}, d), d^2)
\end{aligned}$$

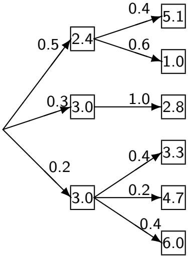
All spaces $\Xi_1, \ldots, \Xi_T$ are Polish spaces and they may carry probability distributions.

Definition. A Borel probability distribution $\mathbb{P}$ with finite first moment on Ξ_T is called a *nested distribution of depth* T.
For any nested distribution $\mathbb{P}$, there is an embedded multivariate distribution P. The projection from the nested distribution to the embedded distribution is not injective. Notation for discrete distributions:

$$\begin{array}{l} \text{probabilities:} \\ \text{values:} \end{array} \left[\begin{array}{ccc} 0.3 & 0.4 & 0.3 \\ \hline 3.0 & 1.0 & 5.0 \end{array}\right] \qquad \left[\begin{array}{ccc} 0.4 & 0.3 & 0.3 \\ \hline 1.0 & 5.0 & 3.0 \end{array}\right] \qquad \left[\begin{array}{cccc} 0.1 & 0.2 & 0.4 & 0.3 \\ \hline 3.0 & 3.0 & 1.0 & 5.0 \end{array}\right]$$

Left: A valid distribution. Middle: the same distribution. Right: Not a valid distribution

Examples for nested distributions



$$\left[\begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ \hline 3.0 & 3.0 & 2.4 \\ \left[\begin{array}{ccc}0.4 & 0.2 & 0.4 \\ \hline 6.0 & 4.7 & 3.3\end{array}\right] & \left[\begin{array}{c}1.0 \\ \hline 2.8\end{array}\right] & \left[\begin{array}{cc}0.6 & 0.4 \\ \hline 1.0 & 5.1\end{array}\right]\end{array}\right]$$

The embedded multivariate, but non-nested distribution of the scenario process can be gotten from it:

$$\left[\begin{array}{cccccc} 0.08 & 0.04 & 0.08 & 0.3 & 0.3 & 0.2 \\ \hline 3.0 & 3.0 & 3.0 & 3.0 & 2.4 & 2.4 \\ 6.0 & 4.7 & 3.3 & 2.8 & 1.0 & 5.1 \end{array}\right]$$

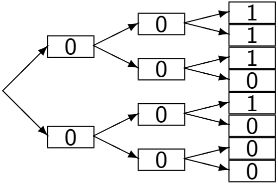
Minimal filtrations

$$\left[\begin{array}{cc} 0.5 & 0.5 \\ \hline \begin{array}{c} 0 \\ \left[\begin{array}{cc} 0.5 & 0.5 \\ \hline 0.0 & 1.0 \end{array}\right] \end{array} & \begin{array}{c} 0 \\ \left[\begin{array}{cc} 0.5 & 0.5 \\ \hline 0.0 & 1.0 \end{array}\right] \end{array} \end{array}\right] \qquad \left[\begin{array}{c} 1.0 \\ \hline \begin{array}{c} 1.0 \\ \left[\begin{array}{cc} 0.5 & 0.5 \\ \hline 0.0 & 1.0 \end{array}\right] \end{array} \end{array}\right]$$

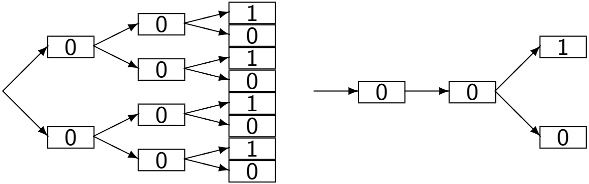
Left: Not a valid nested distribution. Right: A valid one

This fact leads to the concept of minimal filtrations.

Example



This process-and-information pair is already minimal.



Left: the original process-and-information pair, Right: the pertaining minimal pair.

Theorems

Theorem. Two minimal process-and-information pairs are equivalent, if and only if they induce the same nested distribution.
Theorem. If a multiperiod stochastic optimization problem is based on compound convex acceptability functionals, then the optimal solution can be chosen as measurable w.r.t the minimal filtration.

Version-independent acceptability functionals

If a process $Y = (Y_1, \ldots, Y_T)$ is defined on a probability space $(\Omega, \boldsymbol{\mathcal{F}}, P)$ with filtration $\boldsymbol{\mathcal{F}} = (\mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_T)$, it generates a nested distribution.

Definition. An acceptability functional $\mathcal{A}(Y_1, \ldots, Y_T, \mathcal{F}_0, \ldots, \mathcal{F}_{T-1})$ is called version-independent (law-invariant), if it depends only on the nested distribution of the process-and-information pair.

All functionals in this talk are version-independent. I do not know of any reasonable functional, which is not version-independent in this sense.

Construction of multi-period risk functionals

(a) Separable multi-period acceptability functionals:

$$\mathcal{A}(Y;\boldsymbol{\mathcal{F}}) := \sum_{t=1}^{T} \mathcal{A}_t(Y_t),$$

where $\mathcal{A}_t$ are single-period acceptability functionals, satisfy (MA1)', (MA2) and (MA3), but do not depend on $\boldsymbol{\mathcal{F}}$.

(b) Scalarization:

$$\mathcal{A}(Y;\boldsymbol{\mathcal{F}}) := \mathcal{A}_0(s(Y))$$

where $\mathcal{A}_0$ is a (single-period) acceptability functional and $s : \mathcal{Y} \to L_1(\Omega, \mathcal{F}, \mathbb{P})$ a mapping satisfying concavity, monotonicity and $s(Y_1 + r, Y_2, \ldots, Y_T) := s(Y_1, \ldots, Y_T) + r$ for all $Y \in \mathcal{Y}$ and $r \in \mathbb{R}$. Examples:
(i) $s(Y) = \sum_{t=1}^{T} Y_t$.
(ii) $s(Y) := \min_{t=1,\ldots,T} \sum_{\tau=1}^{t} Y_\tau$.

(c) Separable expected conditional (SEC) multi-period acceptability functionals:

$$\mathcal{A}(Y;\boldsymbol{\mathcal{F}}) := \sum_{t=1}^{T} \mathbb{E}(\mathcal{A}_t(Y_t|\mathcal{F}_{t-1}))$$

where $\mathcal{A}_t(\cdot\,|\mathcal{F}_{t-1})$, $t = 1, \ldots, T$, are conditional (single-period) acceptability functionals, satisfy (MA0)–(MA3).

Remark. If $\mathcal{A}(Y;\mathcal{F})$ is SEC functional, then also its conjugate of $\mathcal{A}^+(\cdot;\mathcal{F})$ is SEC.
Example. (Multi-period Average Value-at-Risk)

$$m\mathbb{A}\mathrm{V@R}_\alpha(Y;\mathcal{F}) := \sum_{t=1}^{T} \mathbb{E}(\mathbb{A}\mathrm{V@R}_\alpha(Y_t|\mathcal{F}_{t-1}))$$

$$= \inf\left\{\sum_{t=1}^{T} \mathbb{E}(Y_t Z_t) : Z_t \in [0,\frac{1}{\alpha}], \mathbb{E}(Z_t|\mathcal{F}_{t-1}) = 1, t = 1,\dots,T\right\}$$

The multi-period average value-at-risk is Lipschitz w.r.t. the nested distance.

Composition of conditional acceptability mappings

Let a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\boldsymbol{\mathcal{F}} = (\mathcal{F}_0, \ldots, \mathcal{F}_T)$ of σ-fields $\mathcal{F}_t$, $t = 0, ..., T$, with $\mathcal{F}_T = \mathcal{F}$ be given.
Let, for each $t = 1, \ldots, T$, conditional acceptability mappings $\mathcal{A}_{t-1} := \mathcal{A}(\cdot\,|\mathcal{F}_{t-1})$ from $\mathcal{Y}_T$ to $\mathcal{Y}_{t-1}$ be given. We introduce a multi-period probability functional $\mathcal{A}$ on $\mathcal{Y} := \times_{t=1}^{T} \mathcal{Y}_t$ by compositions of the conditional acceptability mappings $\mathcal{A}_{t-1}$, $t = 1, \ldots, T$, namely,

$$\begin{aligned}
\mathcal{A}(Y; \boldsymbol{\mathcal{F}}) &:= \mathcal{A}_0[Y_1 + \cdots + \mathcal{A}_{T-2}[Y_{T-1} + \mathcal{A}_{T-1}(Y_T)]\cdot] \\
&= \mathcal{A}_0 \circ \mathcal{A}_1 \circ \cdots \circ \mathcal{A}_{T-1}(\sum_{t=1}^{T} Y_t)
\end{aligned}$$

for every $Y_t \in \mathcal{Y}_t$. (Ruszczynski and Shapiro)

Example. We consider the conditional Average Value-at-Risk (of level $\alpha \in (0,1]$) as conditional acceptability mapping

$$\mathcal{A}_{t-1}(Y_t) := \mathbb{A}\mathsf{V@R}_\alpha(\cdot\,|\mathcal{F}_{t-1})$$

for every $t = 1, \ldots, T$. Then the multi-period probability functional

$$n\mathbb{A}\mathsf{V@R}_\alpha(Y;\boldsymbol{\mathcal{F}}) = \mathbb{A}\mathsf{V@R}_\alpha(\cdot\,|\mathcal{F}_0) \circ \cdots \circ \mathbb{A}\mathsf{V@R}_\alpha(\cdot\,|\mathcal{F}_{T-1})\left(\sum\nolimits_{t=1}^{T} Y_t\right)$$

satisfies (MA0), (MA1'), (MA2), (MA3) according to the Proposition. It is called the *nested Average Value-at-Risk*.

Proposition. Suppose that for every t the conditional acceptability functional $\mathcal{A}_t(\cdot|\mathcal{F}_t)$ maps $L_p(\mathcal{F}_t)$ to $L_p(\mathcal{F}_{t-1})$ and is defined by

$$\begin{aligned}\mathcal{A}_t(Y|\mathcal{F}_t) \;&=\; \mathbf{inf}\{\mathbb{E}(Y\ Z|\mathcal{F}_t) - \mathcal{A}^+(Z|\mathcal{F}_t) : Z \geq 0,\\ &\qquad \mathbb{E}(Z|\mathcal{F}_t) = 1, Z \in \mathcal{Z}_t(\mathcal{F}_t)\}.\end{aligned}$$

Then the nested acceptability functional
$\mathcal{A}(Y;\boldsymbol{\mathcal{F}}) = \mathcal{A}^{(T)}(Y_1 + \cdots + Y_T)$ has the dual representation

$$\mathcal{A}(Y;\boldsymbol{\mathcal{F}}) = \inf\{\mathbb{E}[(Y_1 + \cdots + Y_T)M_T] - \sum_{t=1}^{T} \mathbb{E}[\mathcal{A}_t^+(Z_t|\mathcal{F}_t)M_{t-1}] :$$
$$\mathbb{E}(Z_t|\mathcal{F}_t) = 1, Z_t \geq 0, Z_t \in \mathcal{Z}_t(\mathcal{F}_t)\}$$

where $M_t = \prod_{s=1}^{t} Z_t$ and $M_0 = 1$. Notice that (M_t) is a martingale w.r.t. $\boldsymbol{\mathcal{F}}$ with $\mathbb{E}(|M_t|^q) < \infty$.

Example. The nested $\mathbb{A}$V@R has the following dual representation:

$$n\mathbb{A}\mathrm{V@R}_\alpha(Y;\mathcal{F}) = \inf\{\mathbb{E}[(Y_1 + \cdots + Y_T)M_T] : 0 \le M_t \le \frac{1}{\alpha}M_{t-1},$$
$$\mathbb{E}(M_t|\mathcal{F}_{t-1}) = M_{t-1}, M_0 = 1, t = 1, \ldots, T\}.$$

The nested average value-at-risk $n\mathbb{A}$V@R is given by a linear stochastic optimization problem containing functional constraints. The nested average value-at-risk $n\mathbb{A}$V@R is Lipschitz w.r.t. the nested distance.

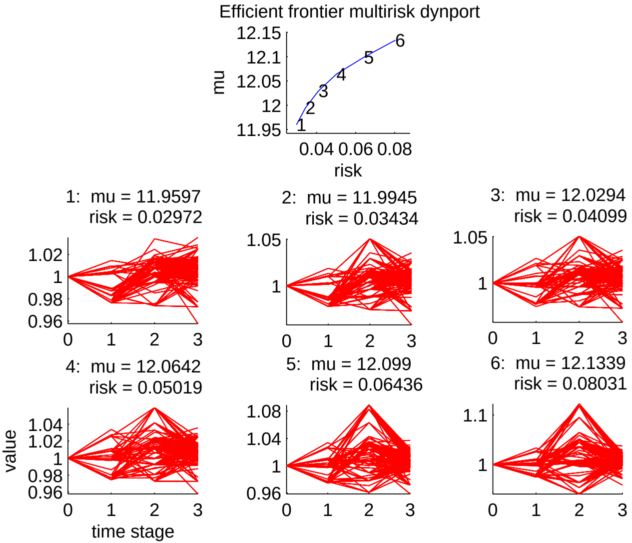
Dynamic programming solution?

Not every problem of the form

$$\begin{array}{ll} \text{Maximize} & \mathbb{E}[\sum_{t=1}^{T} Y_t] - \sum_{t=1}^{T} \mathcal{A}_t(Y_1, \ldots, Y_t) \\ \text{subject to} & x \lhd \mathcal{F} \\ & x \in \mathbb{X} \end{array}$$

allows a dynamic programming solution in the sense that one may solve subproblems on subtrees from right to left until the root is reached. Some authors call this property *time consistency*. However, all nested acceptability functionals and all SEC functionals are time consistent in this sense.

Example: dynamic portfolio management



An efficient frontier using the (negative) multi-period $\mathbb{A}$V@R as risk