On Trades, Volume, and the Martingale Estimating Function Approach for Stochastic Volatility Models with Jumps

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Our papers

- Friedrich Hubalek and Petra Posedel, Joint analysis and estimation of stock prices and trading volume in Barndorff-Nielsen and Shephard stochastic volatility models, arXiv:0807.3464 (July 2008)
- Friedrich Hubalek and Petra Posedel, Asymptotic analysis for a simple explicit estimator in Barndorff-Nielsen and Shephard stochastic volatility models, arXiv:0807.3479 (July 2008)
- Friedrich Hubalek and Petra Posedel, Asymptotic analysis for an optimal estimating function for Barndorff-Nielsen-Shephard stochastic volatility models, Work in progress.

The Barndorff-Nielsen and Shephard stochastic volatility models with jumps

- Logarithmic returns (discounted)

$$dX(t) = (\mu + \beta V(t-))dt + \sqrt{V(t-)}dW(t) + \rho dZ_\lambda(t)$$

- Instantaneous variance

$$dV(t) = -\lambda V(t-)dt + dZ_\lambda(t)$$

  $W\ldots$ Brownian motion, $Z\ldots$ subordinator,
  $Z_\lambda(t) = Z(\lambda t)$ [...]

- Parameters: $\mu \in \mathbb{R}\ldots$ linear drift, $\beta \in \mathbb{R}\ldots$ Itô drift, $\rho \in \mathbb{R}\ldots$ leverage, $\lambda > 0\ldots$ acf parameter.

Analytical tractability

- $(X(t), V(t), t \geq 0)$... Markov, affine model (in continuous time)
- simple Riccati-type equations for characteristic resp. moment generating function
- general solution (up to one integral)
- Γ-OU and IG-OU completely explicitly in terms of elementary functions

Exploited in

- Option pricing (Nicolato and Venardos)
- Portfolio optimization (Benth et al.)
- Minimum entropy martingale measure (Benth et al., Rheinländer and Steiger)
- Semimartingal Esscher transform (Hubalek and Sgarra)
- ...

But statistical inference seems difficult! Bayesian, MCMC — computer intensive approaches!

- Barndorff-Nielsen O.E., Shephard N. (2001), Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics.
- Roberts G.O., Papaspiliopoulos O., Dellaportas P. (2004), Bayesian inference for non-Gaussian Ornstein-Uhlenbeck stochastic volatility processes,
- J.E. Griffin, M.F.J. Steel (2006), Inference with non-Gaussian Ornstein-Uhlenbeck processes for stochastic volatility
- Matthew P.S. Gandera and David A. Stephens (2007), Stochastic volatility modelling in continuous time with general marginal distributions: Inference, prediction and model selection
- Sylvia Frühwirth-Schnatter and Leopold Sögner (2007?), Bayesian estimation of stochastic volatility models based on OU processes with marginal Gamma laws.

Discrete observations

Grid $t_i = i\delta$, $i \geq 0$, fixed width $\Delta > 0$, discrete time observations

$$X_i = X(t_i) - X(t_{i-1}), \qquad V_i = V(t_i)$$

Discrete dynamics

$$X_i = \mu\Delta + \beta Y_i + \sqrt{Y_i} W_i + \rho Z_i, \qquad V_i = e^{\lambda\Delta} V_{i-1} + U_i$$

Auxiliary quantities (no discretization error!)

$$Z_i = Z_\lambda(t_i) - Z_\lambda(t_{i-1}), \quad U_i = \int_{t_{i-1}}^{t_i} e^{-\lambda(t_i - s)} dZ_\lambda(s)$$

and

$$Y_i = \int_{t_{i-1}}^{t_i} V(s-) ds, \qquad W_i = \frac{1}{\sqrt{Y_i}} \int_{t_{i-1}}^{t_i} \sqrt{V(s-)} dW(s).$$

$(X_i, V_i, i \in \mathbb{N})\ldots$ Markov affine model (in discrete time)

Construction and moments

Two starting points

- L ... infinitely divisible distribution on $\mathbb{R}_+ \Rightarrow$ subordinator Z with $Z(1) \stackrel{d}{=} L \Rightarrow$ (OU-L)
- D ... self-decomposable distribution on $\mathbb{R}_+ \Rightarrow$ stationary Ornstein-Uhlenbeck process V with $V(t) \stackrel{d}{=} D \Rightarrow$ (D-OU)

Moments of D resp. $L \rightarrow$ all (mixed, conditional, unconditional) integer moments by simple algebra (multivariate Faa di Bruno formula resp. Bell polynomials, practical calculations best by recursions!)

$$E[X_i^n], E[V_i^n], E[X_i^m V_i^n], E[X_i^\ell V_i^m V_{i-1}^n],$$
$$E[X_i^n | V_{i-1}], E[V_i^n | V_{i-1}], E[X_i^m V_i^n | V_{i-1}], \ldots$$

$\Rightarrow$ method of moments estimation

Various methods of moments

- Method of moments — MM (Pearson 1893)
- Generalized method of moments — GMM (Hansen 1982)
- Simulated method of moments — SMM (. . .)
- Efficient method of moments — EMM (Gallant and Tauchen 1996),
- . . .
- [Methods of moments for weak convergence]

Estimation: Setting and problems

Grid, fixed width, horizon (number of observations) going to Infinity for asymptotics! (Several other possibilities...)

- Rich, well-informed financial institutions and traders observe and trade in continous-time
- Poor, academic statisticians and econometers do inference with daily (or less frequent!) observations
- [But: High-frequence analyses ...]

Discrete time observations $\Rightarrow V_i$ not observed, BNS becomes non-Markovian, (a hidden Markov model)!

Remedies

- Substitute unobserved $V_i \mapsto$ model-implied $\hat{V}_i$ from option data, i.e., joint analysis of P and Q. Cf.
  - Jun Pan, The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study (2002).

  (GMM, realistic, complicated, many assumptions.)
  Also our long term goal!
- Ignore the problem. Purely theoretical study, exhibits methodology, provides an upper bound for the accuracy for this type of methods. See our first paper!
- NOW: Substitute unobserved V_i by an observable proxy, volume or number of trades.

Prices, volatility, trading intensity

Our incentive

- Carl Lindberg, The estimation of the Barndorff-Nielsen and Shephard model from daily data based on measures of trading intensity. Applied Stochastic Models in Business and Industry 24 (4), 2008.

Some earlier/classical references

- J. M. Karpoff, The relation between price changes and trading volume: a survey. JFQA 22, 1987.
- R.P.E. Gallant, A.R. and G. Tauchen, Stock prices and volume, Rev.Fin.Stud. 5:199–242, 1992.
- K.G. Jones, C. and M.L. Lipson, Transactions, volume and volatility. Rev.Fin.Stud. 7:631–651, 1994.
- G.E. Tauchen and M.Pitts, The Price Variability-Volume Relationship on Speculative Markets Econometrica 51,(1983).

The new variant/interpretation of the BNS models

Bold simplification/assumption: Instantaneous variance IS a (constant) multiple of the trading volume resp. number of trades. Introduce a proportionality parameter $\sigma > 0$. [...]

- Logarithmic returns

$$dX(t) = (\mu + \beta V(t-))dt + \sigma \sqrt{V(t-)} dW(t) + \rho dZ_\lambda(t)$$

- Trading volume (or number of trades)

$$dV(t) = -\lambda V(t-)dt + dZ_\lambda(t)$$

  W ... Brownian motion, Z ... subordinator,
  $Z_\lambda(t) = Z(\lambda t)$ [...]

- Parameters: $\mu \in \mathbb{R}$ ... linear drift, $\beta \in \mathbb{R}$ ... Itô drift, $\sigma > 0$ ... proportionality, $\rho \in \mathbb{R}$ ... leverage, $\lambda > 0$ ... acf parameter.

What about maximum likelihood ?

- Practical issue: Bivariate Markov, known transition probability (in terms of characteristic resp. cumulant function) $\Rightarrow$ invert for each observation in each iterations [Possible remedies, approximate inversions, LeCam's trick,...]
- Theoretical issue: For infinite activity BDLP (e.g., IG-OU) fine, for finite activity (e.g., Γ-OU with exponential compound Poisson BDLP)

$$P_\lambda[V_1 = ve^{-\lambda\Delta} | V_0 = v] = e^{-\lambda\Delta} \quad \text{(no jump)}$$

  $\Rightarrow$ No dominating sigma-finite measure! $\Rightarrow$ Usual ML framework does not apply!
  - Generalized ML (Kiefer and Wolfowitz 1956) [...]
  - Much better than $\sqrt{n}$ by ad hoc (?) methods! [...]

Martingale estimating functions

E.g., $\Gamma(\nu,\alpha)$-OU: Parameter vector $(3+4=7)$

$$\theta = (\lambda, \nu, \alpha, \mu, \beta, \sigma, \rho)$$

Moments

$$\Xi_i = (V_i, V_i V_{i-1}, V_i^2, X_i, X_i V_{i-1}, X_i V_i, X_i^2), \qquad \Upsilon_i = (V_{i-1}, V_{i-1}^2)$$

Martingale estimating function

$$G_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} \left[\Xi_i - f(V_{i-1}, \theta)\right], \qquad f(v,\theta) = E_\theta[\Xi_1 | V_0 = v]$$

Estimator: Solve $G_n(\theta) = 0$! Sample moments

$$\xi_n = \frac{1}{n}\sum_{i=1}^{n} \Xi_i, \qquad \upsilon_n = \frac{1}{n}\sum_{i=1}^{n} \Upsilon_i,$$

The explicit estimator

Unique solution exists on $C_n = \left\{\xi_n^2 - \xi_n^1 \upsilon_n^1 > 0, \upsilon_n^2 - (\upsilon_n^1)^2 > 0\right\}$ and is given by

$$\gamma_n = (\xi_n^2 - \xi_n^1 \upsilon_n^1)/(\upsilon_n^2 - (\upsilon_n^1)^2); \quad \zeta_n = \frac{\gamma_n \upsilon_n^1 - \xi_n^1}{-1 + \gamma_n} \quad \lambda_n = -\log(\gamma_n)/\Delta;$$

$$\eta_n = -\frac{(-1+\gamma_n^2)(\upsilon_n^1)^2 - \gamma_n^2 \upsilon_n^2 + \xi_n^3}{-1+\gamma_n^2} \quad \epsilon_n = (1-\gamma_n)/\lambda_n;$$

$$\beta_n = \frac{(\xi_n^5 - \upsilon_n^1 \xi_n^4)}{\epsilon_n(\upsilon_n^2 - (\upsilon_n^1)^2)};$$

$$\rho_n = \big(-\beta_n \epsilon_n(-(\upsilon_n^1)^2 + \epsilon_n \lambda_n(\eta_n + (\upsilon_n^1)^2 - \upsilon_n^2) + \upsilon_n^2) - \xi_n^1 \xi_n^4 + \xi_n^6\big)/(2\epsilon_n \eta_n \lambda_n);$$

$$\mu_n = \big(-\Delta \lambda_n \rho_n \zeta_n - \beta_n(\Delta \zeta_n + \epsilon_n(-\zeta_n + \upsilon_n^1)) + \xi_n^4\big)/\Delta;$$

$$\sigma_n = \sqrt{a_n/b_n}; \quad b_n = \Delta \zeta_n + \epsilon_n(-\zeta_n + \upsilon_n^1);$$

$$a_n = \lambda_n^{-1}\Big[4\beta_n(-\Delta + \epsilon_n)\eta_n \lambda_n \rho_n + \beta_n^2(-2\Delta \eta_n + \epsilon_n(\eta_n(2 + \epsilon_n \lambda_n) + \epsilon_n \lambda_n((\upsilon_n^1)^2 - \upsilon_n^2))) + \lambda_n(-2\Delta \eta_n \lambda_n \rho_n^2 - (\xi_n^4)^2 + \xi_n^7)\Big];$$

Structure: First $\lambda_n, \nu_n, \alpha_n$ are simple AR(1) estimators, then μ_n, β_n, ρ_n from a simple linear system, finally σ_n from a quadratic equation.

Consistency

The basic (and only!) assumption: V_0 self-decomposable rv on $\mathbb{R}_+$ with

$$E[V_0^n] < \infty \quad \forall n \in \mathbb{N}.$$

The basic convergence result

$$\frac{1}{n}\sum_{i=1}^{n} X_i^p V_i^q V_{i-1}^r \xrightarrow{a.s.} E[X_1^p V_1^q V_0^r] \quad \forall p, q, r \in \mathbb{N}.$$

Remark: Ergodicity vs. simple proof. Martingale differences $\Rightarrow$ uncorreclated $\Rightarrow$ elementary convergence result.

Theorem

We have $P(C_n) \to 1$ and the estimator $\hat{\theta}_n$ is consistent on C_n, namely

$$\hat{\theta}_n I_{C_n} \xrightarrow{a.s.} \theta_0$$

as $n \to \infty$.

Asymptotic normality — delta method

- Explicit estimator ⇒ Delta-Method applies: Sample moments

$$(\xi_n, \upsilon_n) \xrightarrow{\mathcal{D}} N(0, \Sigma)$$

  estimator

$$\theta_n = h(\xi_n, \upsilon_n)$$

  result

$$\sqrt{n}(\theta_n - \theta_0) \xrightarrow{\mathcal{D}} N(0, T) \qquad T = J\Sigma J^\top$$

  Jacobian $J = \nabla h$. Messy.
- Better: General framework (implicit function theorem)
  - Michael Sørensen, Statistical inference for discretely observed diffusions, Lecture Notes, Berlin, 1997.
  - Michael Sørensen, On asymptotics of estimating functions, Brazil. J. Prob. Stat. (1999).

  Also when estimating functions $G_n(\theta)$ explicit, but estimator θ_n is not [...optimal estimating functions].

Asymptotic normality — general framework

Basic result: asymptotic normality of estimating functions

$$\frac{1}{\sqrt{n}} G_n(\theta_0) \xrightarrow{\mathcal{D}} N(0, \Lambda), \qquad \Lambda = E[\mathrm{Var}[\Xi_1] | V_0]$$

Proof by multivariate martingale central limit theorem.

Theorem

The estimator $\theta_n I_{C_n}$ is asymptotically normal, namely

$$\sqrt{n}\big[\hat{\theta}_n - \theta_0\big] \xrightarrow{\mathcal{D}} N(0, T), \qquad T = A^{-1}\Lambda(A^{-1})^\top$$

as $n \to \infty$, with Jacobian $A = E[\nabla f(V_0, \theta_0)]$.

- Recall $f(v, \theta) = E_\theta[\Xi_1 | V_0 = v]$ and $E = E_{\theta_0}$.
- Matrices A and Λ simple, explicit, (slightly lengthy).

Finite sample performance — the controlled simulation experiment

Γ-OU: Volume $V(t) \sim \Gamma(\nu, \alpha)$ stationary, BDLP Z compound Poisson, intensity λ exponential jumps with mean $1/\alpha$.

- Parameter values (annual, 250 trading days)

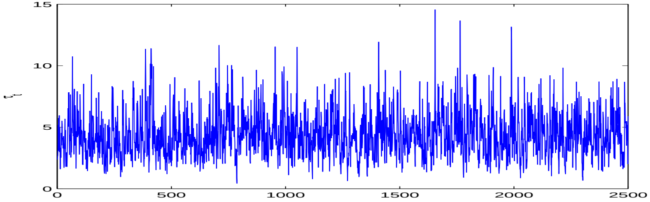
$$\nu = 6.17, \quad \alpha = 1.42, \quad \lambda = 177.95,$$

$$\beta = -0.015, \quad \rho = -0.00056, \quad \mu = 0.435, \quad \sigma = 0.087.$$

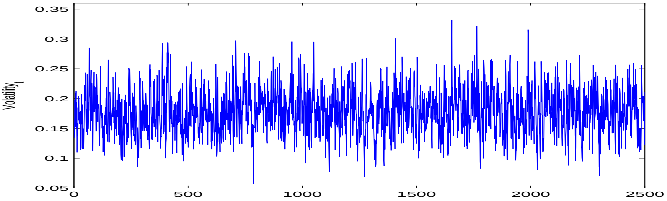
- BDLP: 4.4 jumps per day (interesting pieces of news arriving?), each jump with mean and stddev 0.704.
- Volume (in Mio): Stationary mean 4.35, variance 0.033 $\Rightarrow$ Volatility $\approx$ 18%. ACF half-life $\approx$1 day.
- Log returns: Mean -6.5%, volatility 18%.
- Experiments: n=2500 (10 years), $n = 8000$ (32 years, theoretical check).

Simulated paths 1

Volume

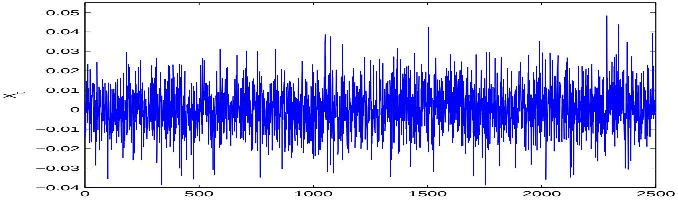


Volatility



Simulated paths 2

Returns



Asymptotic performance

- True values $\theta = (\nu, \alpha, \lambda, \beta, \rho, \mu, \sigma)$

$$\theta = (6.17, 1.42, 177.95, -0.015, -0.00056, 0.435, 0.087)$$

- Asymptotic stddev $s/\sqrt{n}$

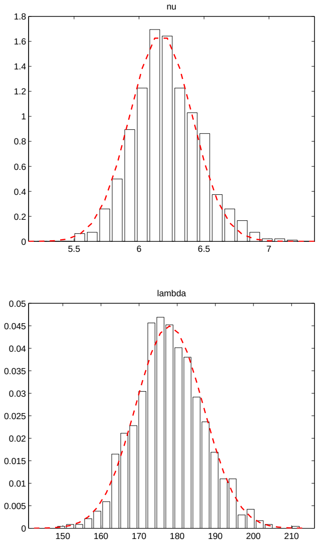
$$s = (12.0, 2.8, 440, 9.0, 2.6, 0.066, 0.007)$$

- Asymptotic correlation r

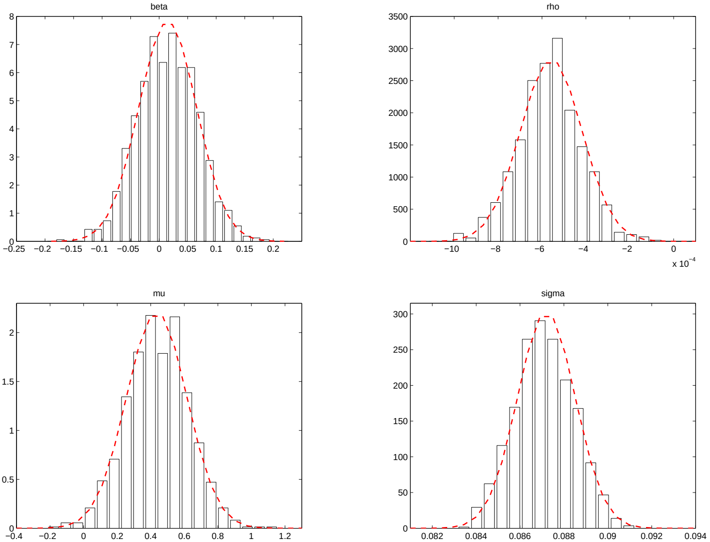
$$r = \begin{bmatrix} 1 & 0.9 & 0.6 & 0.007 & 0.05 & 0.006 & -0.003 \\ 0.9 & 1 & 0.6 & 0.007 & 0.05 & 0.01 & -0.004 \\ 0.6 & 0.6 & 1 & 0.01 & 0.09 & -0.0006 & 0.00 \\ 0.007 & 0.008 & 0.01 & 1 & -0.8 & -0.01 & 0.03 \\ 0.05 & 0.05 & 0.09 & -0.8 & 1 & 0.01 & -0.5 \\ 0.006 & 0.01 & -0.0006 & -0.01 & 0.01 & 1 & -0.005 \\ -0.003 & -0.004 & 0.00 & 0.03 & -0.5 & -0.005 & 1 \end{bmatrix}$$

- Big r in AR(1)-part! $\Rightarrow$ Optimal estimating function.

Histograms: $m = 1000$ replications, each $n = 2500$ observations, volume parameters



Histograms : $m = 1000$ replications, each $n = 2500$ observations, return parameters



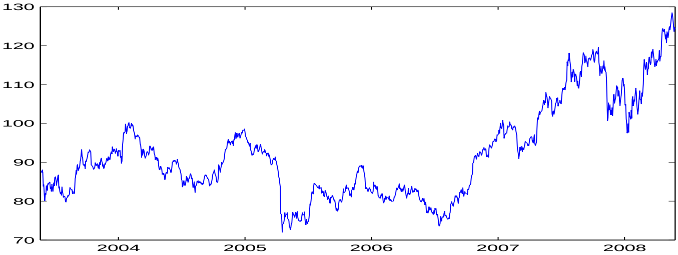
A first empirical analysis — data

Closing price and volume

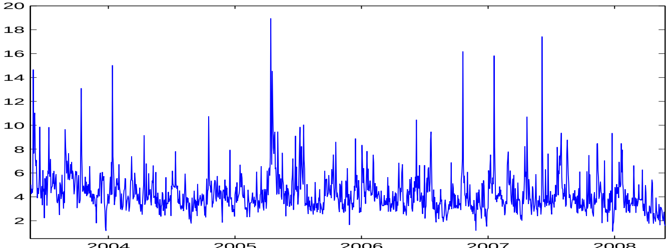
- IBM: March 23, 2003 – March 23, 2008 [NYSE], 1259 observations
- MSFT: April 11, 2003 – Feb 4, 2008 [Nasdaq], 1212 observations

IBM data

Price

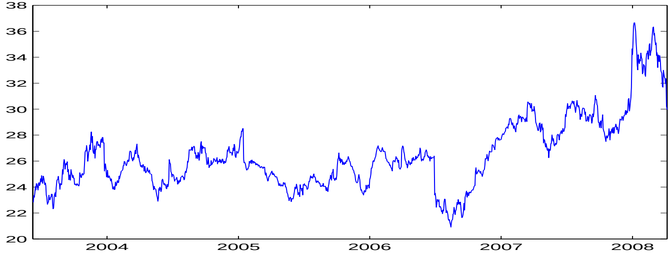


Volume

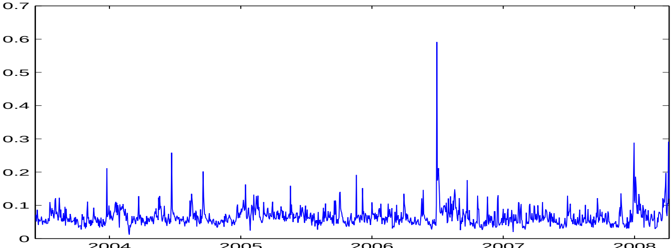


MSFT data

Price



Volume



Estimation results

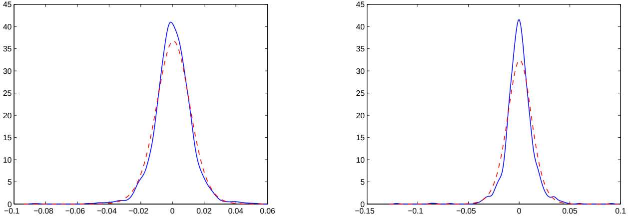
	IBM	stddev
$\hat{\nu}$	6.17	0.339
$\hat{\alpha}$	1.42	0.079
$\hat{\lambda}$	177.95	12.509
$\hat{\mu}$	0.435	0.254
$\hat{\beta}$	-0.015	0.072
$\hat{\sigma}$	0.087	0.002
$\hat{\rho}$	-0.00056	0.0002

	MSFT	stddev
$\hat{\nu}$	4.496	0.247
$\hat{\alpha}$	67.895	3.773
$\hat{\lambda}$	201.99	14.420
$\hat{\mu}$	0.4162	0.265
$\hat{\beta}$	-0.464	5.059
$\hat{\sigma}$	0.81	0.018
$\hat{\rho}$	-0.025	0.013

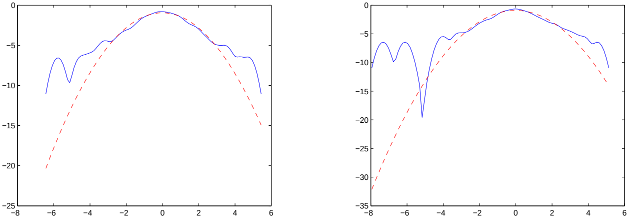
Interpretation: [. . .]

Unconditional return distributions

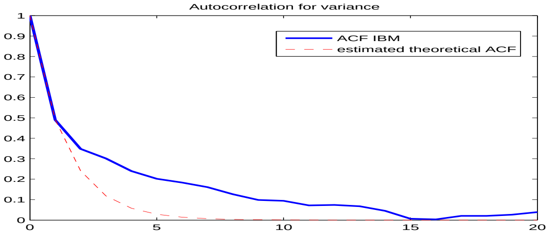
Theoretical BNS (dashed) versus kernel estimates (solid)

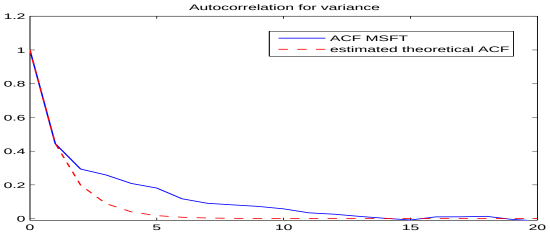


Log densities



Autocorrelation function (volume)





$\Rightarrow$ BNS with Superposition of OU-processes [...]

Model fit — residual analysis

- Volume: Usual (and exact) AR(1) analysis, though with funny innovations (U_i) iid,

$$V_i - e^{-\lambda\Delta} = U_i, \qquad U_i = \int_{t_{i-1}}^{t_i} e^{-\lambda(t_i - s)} dZ_\lambda(s)$$

- Returns: Not exact (?), Euler approximation

$$\dots$$

Further developments and directions 1

Superposition

$$V(t) = w_1 V_1(t) + \cdots + w_m V_m(t), \quad dV_i(t) = -\lambda_i V_i(t-)dt + dZ_i(\lambda_i t)$$

$(X, V_1, \ldots, V_m)$ Markov affine $\Rightarrow$ Observations?

- $V_1 \ldots$ common factor (market volume,...)
- $V_2 \ldots$ idiosyncratic factor (asset volume,...)
- $V_3 \ldots$? (similar asset? ...?)

Further developments and directions 2

- Number of trades (Lindberg!)
- Optimal martingale estimating functions

$$G_n(\theta) = \sum_{i=1}^{n} B(\theta, V_{i-1}) \left[\Xi_i - f(\theta, V_{i-1})\right) \quad f(\theta, v) = E_\theta[\Xi_i | V_{i-1} = v]$$

- Comparison with ML and related methods (for infinite activity)
- Comparison with GMM
- Hybrid approaches
- Other moments (trigonometric, c.f., Singleton, ...)
- Other time-scales (!!!)
- Integrated analysis for asset and derivatives