Control for the Lundberg process

Reinsurance and investment

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Contents

Control in Insurance

Active research area with objectives: minimizing ruin probability, or maximizing dividend payment, or else; with control of

- investment,
- reinsurance,
- new business,
- premia,
- more than one of these.

a) in the classical Lundberg model or
b) in diffusion approximations.
Asmussen, Hoejgaard, Taksar with (b), Schmidli, Schachermeyer, H. and Plum, Vogt, Schmidli, with (a).
Good problems.

Simplest model for insurance

Lundberg's risk model (1905):

$$R(t) = s + ct - X_1 - \dots - X_{N(t)},$$

- s initial surplus,
- c constant premium rate,
- $N(t)$ homogeneous Poisson prozess for occurence of claims,
- $X_1, X_2, \dots$ iid claim sizes,
- $N(t), t \geq 0$, and $X_1, X_2, \dots$ independent.

Capital market

Logarithmic Brownian motion for stock, index or similar:

$$dZ(t) = \mu Z(t)dt + \sigma Z(t)dW(t),$$

independence between $Z(t)$ and $R(t), t \geq 0$, with $\mu, \sigma > 0$.
riskless asset (still existing?)

$$dB(t) = rB(t)dt, r \geq 0.$$

Simplifications

Simplifying assumptions are:

- short selling allowed;
- arbitrary sizes;
- equal interest rate for borrowing and lending;
- leverage possible;
- no transaction costs;
- no tax;
- ...

$r = 0$, leverage, investment in index

Minimize ruin probability by dynamic investment in index. If $\theta(t)Z(t) = A(t)$ is invested at time t then the total position of the insurer has the dynamics

$$dY(t) = cdt - dS(t) + dG(t) = (c + A(t)\mu)dt - dS(t) + A(t)\sigma dW(t),$$

with claims process

$$S(t) = X_1 + ... + X_{N(t)}, t \geq 0,$$

and investment gains (again existing?)

$$G(t) = \int_0^t \theta(u)dZ(u) = \int_0^t A(u)/Z(u)dZ(u).$$

General solution procedure

- HJB equation
- Existence of a smooth solution
- Verification argument
- numerical calculation of optimal strategy
- qualitative properties of optimal strategy

Different degrees of difficulty!

HJB equation

The HJB equation is

$$0 = \sup_A \{\lambda E[V(s - X) - V(s)] + (c + A\mu)V'(s) + \frac{1}{2}A^2\sigma^2 V''(s)\}.$$

Possible norming: $\mu = \sigma = 1$.
Maximizer

$$A(s) = -\frac{V'(s)}{V''(s)}$$

defines the optimal strategy in feedback form: invest $A(s)$ when you are in state s.

Equivalent system of equations

With $U(s) = (V'(s)/V''(s))^2 = A(s)^2$ equivalent to the following system of interacting differential equations:

$$V'(s) = \frac{\lambda(V(s) - g(s))}{c + \frac{1}{2}\sqrt{U(s)}} \quad (1)$$

$$\frac{1}{4}U'(s) = \sqrt{U(s)}\left[\lambda + 1/2 - \lambda\frac{g'(s)}{V'(s)}\right] + c, \quad (2)$$

where

$$g(s) = E[V(s - X)].$$

With boundary values $U(0) = 0$, $V(\infty) = 1$ this produces a stable and fast numerical algorithm for $A(s)$.

Exponentially distributed claim sizes

If $X \sim Exp(a)$ with density $f(x) = a\exp(-ax), x > 0, a > 0$, we have

$$g'(s) = a(V(s) - g(s))$$

and thus the equations separate with one equation being

$$\frac{1}{4}U'(s) = \sqrt{U(s)}\left[\lambda + 1/2 - ac - \frac{a}{2}\sqrt{U(s)}\right] + c.$$

Exponentially distributed claim sizes, $\lambda + 1/2 = ac$

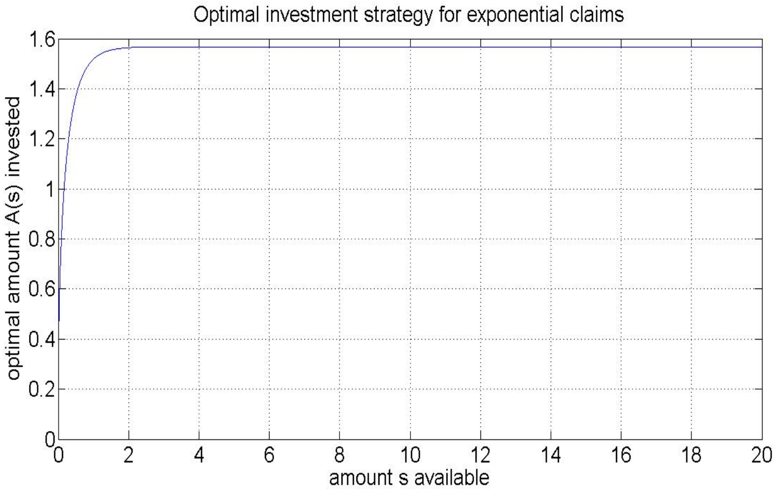
In this special case

$$U'(s) = -2aU(s) + 4c.$$

$$U(s) = \frac{2c}{a}(1 - \exp(-2as)),$$

or

$$A(s) = \sqrt{2c/a}\sqrt{1 - \exp(-2as)}.$$


Optimal investment strategy for exponential claims
optimal amount A(s) invested
amount s available
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10
12
14
16
18
20

Typical behaviour for small claims distributions

- $A(s) \sim C\sqrt{s}, s \to 0$;
- $V(s) \geq 1 - \exp(-Rs)$, where $R =$ adjustment coefficient;
- $R > 0$ unique positive solution of

$$\lambda + rc + \frac{\mu^2}{2\sigma^2} = E[exp(-rX)];$$

- fast convergence of $A(s) \to 1/R$;
- $V(s) \sim 1 - K\exp(-Rs), s \to \infty$.

Leverage

Leverage $A(s)/s$ is unbounded for $s \to 0$:

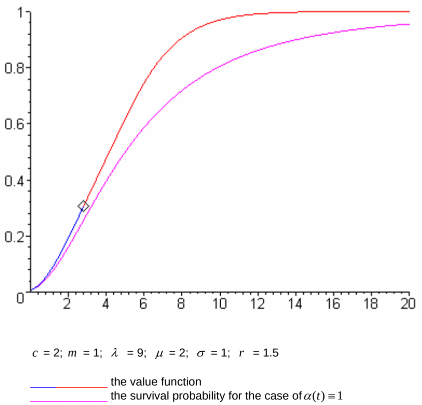
$$\lim_{s \to 0} A(s)/s = \lim_{s \to 0} K\sqrt{s}/s = \infty.$$

The HJB for the case without leverage and short selling: (normed)

$$0 = \sup_{0 \le A \le s} \{\lambda E[V(s-X) - V(s)] + (c+A)V'(s) + \frac{1}{2}A^2 V''(s)\}.$$

attained at $A = 0$ or $A = s$ or at $A = -V'(s)/V''(s)$.
Here, $V(s)$ need not be concave, $V''(s) = 0$ possible, even for exponential claims.

Investment without leverage



Tatjana Belkina, Moscow (2008): exponential claims, with $r > 0$.

Just after my presentation I learned from Stefan Thonhauser that the leverage problem has been solved completely by Pablo Azcue and Nora Muler in a paper which is accepted for publication in Insurance: Mathematics and Economics with title *Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constraints*.

General setup

Risk sharing between insurer and reinsurer: X is divided as

$$X = g(X) \;\; + \;\; X - g(X),$$

with $0 \leq g(x) \leq x$ the payment of the first insurer. Reinsurer charges a premium h.

Optimization problem: find the optimal dynamic reinsurance cover, given a set $g(x, a), a \in A$, of possible reinsurance contracts with prices $h(a), a \in A$, to minimize ruin probability.

HJB equation

for maximal survival probability $V(s)$

$$V'(s) = \min_{h(a)<c} \frac{\lambda E[V(s) - V(s - g(X,a))]}{c - h(a)}, s \geq 0.$$

Existence and uniqueness of a solution $V(s)$ satisfying $V(\infty) = 1$ and $cV'(s) = \lambda V(s)$ is easy. Numerical computation cumbersome.

Unlimited XL reinsurance

Unlimited XL (excess of loss) reinsurance with $A = [0, \infty]$ and $g(x, a) = \min(x, a)$. First: reinsurance price according to expectation principle:

$$h(a) = \rho\lambda E[(X - a)^+], a \geq 0,$$

with $\rho\lambda E[X] > c$ (expensive reinsurance). Optimal strategy in feedback form: If we are in state s, then choose $a^*(s)$, where $a^*(s)$ is the minimizer in HJB equation.

Three cases

HJB:

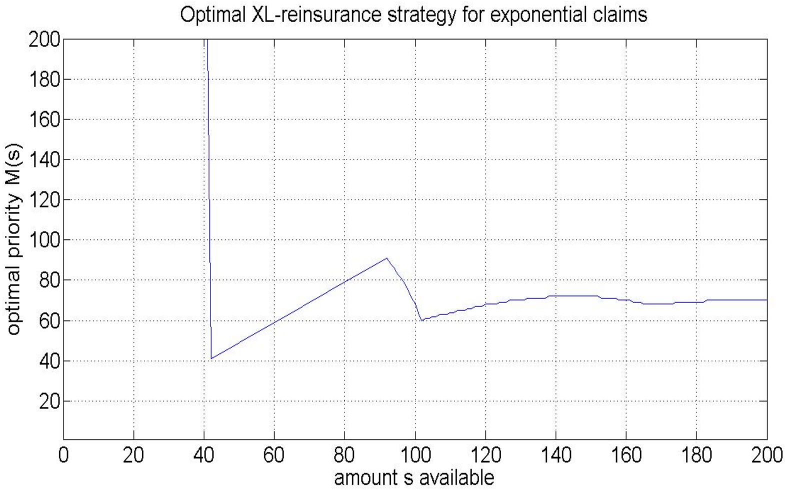
$$V'(s) = \min_{h(a)<c} \frac{\lambda E[V(s) - V(s - \min(X,a))]}{c - h(a)}, s \geq 0.$$

For $a > s$ we obtain from $V(x) = 0, x < 0$, the equation

$$E[V(s - \min(X,a))] = E[V(s - X)]$$

which does not depend on a, so the minimum is at a minimizing $h(a)$ which is at $h(a) = 0$ (no reinsurance) or $a = \infty$. So the optimal value for a is either $a = \infty$, or $a = s$ or $a < s$.

Exponential distribution



Expectation principle

On the market only limited XL contracts are liquid or affordable:

$$g(x,a) = \min(x,M) + (x-M-L)^+,$$

or

$$x - g(x,a) = \min\{(x-M)^+, L\}.$$

Here $a = (M,L) \in [0,\infty] \times [0,\infty]$.
$L = 0$ is *no reinsurance*.
For an expectation pricing formula for reinsurance premia we obtain the strategies from above: $L = \infty$ is always optimal.

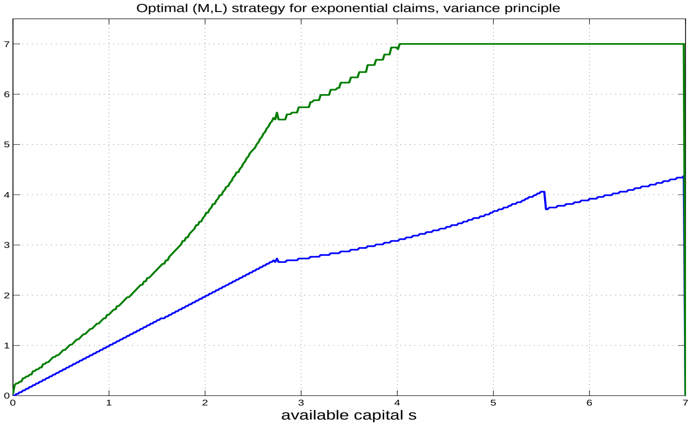
Variance principle

Under the variance principle the tail of the distribution gets more weight and so the first insurer will accept a limit and will take the tail risk himself. In this case the reinsurer's pricing formula will be:

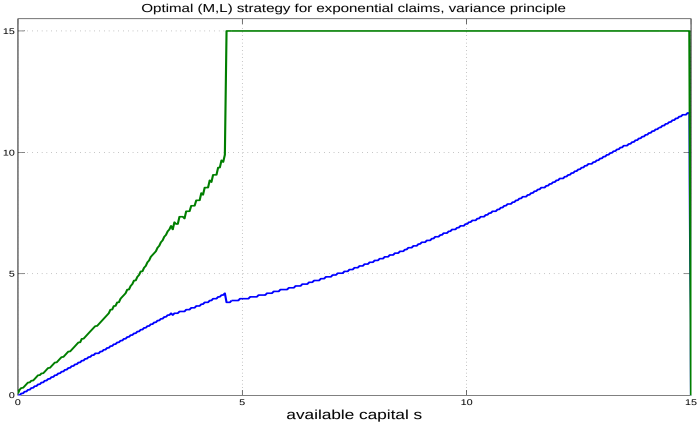
$$h(a) = \lambda E[X - g(X,a)] + \alpha E[(X - g(X,a))^2],$$

with $\lambda E[X] + \alpha E[X^2] > c$ (expensive reinsurance).

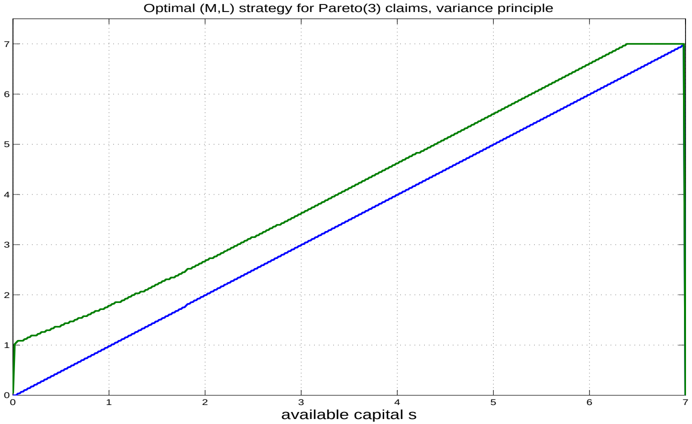
Exponential distribution



Exponential distribution



Pareto distribution



Crude discretisation

Only crude discretization because of computational complexity

- M and L discretized with 200 points;
- s discretized with 500 points;
- in each of 40.000 tests an integral is computed numerical, yielding a sum with at most 500 terms;
- computation for all the 500 s: 10^{10} multiplications.

Efficient algorithm in MatLab via matrices:
Form a matrix H with all point probabilities of discretized $g(X, a)$;
The vector of all needed integrals computed by the command

```
V(k:-1,1)*H(1:k,:)';
```

Simplification

$$
\begin{aligned}
g(s,M,L) &= E[V(s - X \wedge M - (X - M - L)^+)] \\
g_s(s,M,L) &= g_s(s,M,\infty) \\
&\quad + \frac{1 - F(M+L)}{F(M+L) - F(M)} g_M(s,M,L) \\
&\quad - g_L(s,M,L).
\end{aligned}
$$

Simplification

Notation:

$$
\begin{aligned}
v(i) &= V(i\Delta), M = m\Delta, L = l\Delta, s = k\Delta, \\
p(i) &= \mathbb{P}\{(i-1)\Delta < X < i\Delta\}
\end{aligned}
$$

$$
\begin{aligned}
g(s, M, L) &= \int_0^M V(s-x)f(x)dx \\
&+ V(s-M)P\{M \leq X \leq M+L\} \\
&+ \int_M^s V(s-x))f(x+L)dx,
\end{aligned}
$$

Simplification

approximated by

$$\sum_{i=1}^{m} v(k-i)p(i) \quad + \quad v(k-m)(pp(m+l)-pp(m)) \\ + \quad \sum_{i=m+1}^{k} v(k-i)p(i+l),$$

with $pp(i) = \mathbb{P}\{X > i\Delta\}$.
Can be represented with the quantities

$$c(k,l,m) = \sum_{i=0}^{m} v(k-i)p(i+l).$$

These can be computed recursively (in k)

Recursive computation of $c(k,l,m)$:

$l = 0, ..., L-1, m = 1, ..., k$:

$$c(k+1,l,m) = c(k,l+1,m-1) + p(l)v(k+1);$$

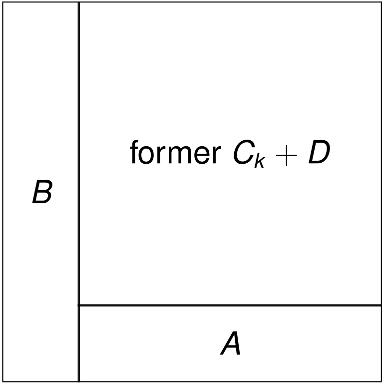
$$c(k+1,l,0) = v(k+1)p(l);$$

$$c(k+1,L,m) = v(k+1)p(L) + \sum_{i=0}^{m} v(k-i)p(i+M+1).$$

initialize with:

$$c(0,l,0) = v(0)p(l).$$

Recursion



MatLab command: $C_{k+1} = [B, [C_k + D * e'; A]]$.

The verification theorem

A smooth solution to HJB solves the optimization problem needs that for arbitrary admissible control, the reserve process either goes to ruin, or it takes arbitrarily large values. A simple proof for this which is due to Freddy Delbaen, here for the case of a diffusion process:

Theorem: $dX(t) = a(t)dt - b(t)dW(t), X(0) = x_0,$
with predictable processes a, b satisfying $|a| + |b| < M$.
Assume that there exist ε, δ for which

$$a < -\delta \text{ whenever } |b| < \varepsilon.$$

Then for all $N > 0$ with $\tau = \inf\{t : X(t) \notin [0, N]\}$ $\mathbb{P}\{\tau < \infty\} = 1$.

Proof: For large enough $K > 0$ consider $Y(t) = \exp(-KX(t))$. Then

$$dY(t) = KY(t)[-a(t) + \frac{1}{2}Kb(t)^2)dt - b(t)dW(t),$$

$$1 \geq E\left[\int_0^{\tau} K\exp(-KX(s))[\frac{1}{2}Kb^2(s) - a(s)]ds\right].$$

Using

$$\frac{1}{2}Kb^2(s) - a(s) > \delta$$

we obtain that $X(t)$ is unbounded on $\{\tau = \infty\}$.

Optimal investment and XL reinsurance

This problem has been solved completely – using ideas of Schmidli – by Ming Fang and Fei Wang. HJB after norming:

$$0 = \sup_{A,M}\{\lambda E[V(s-X\wedge M)-V(s)]-(c-h(M)+A)V'(s)+\frac{1}{2}A^2V''(s)\}$$

with $h(M) = \rho E[(X-M)^+]$.

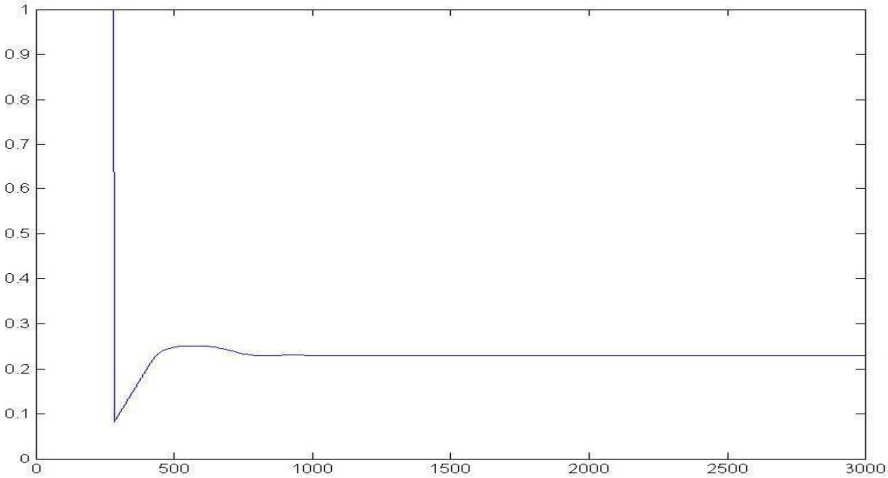
Optimal investment and XL reinsurance

is equivalent for $M < s$ to:

$$\begin{aligned} V'(s) &= \inf_M \frac{\lambda V(s) - \lambda E[V(s - X \wedge M)]}{\sqrt{U(s)}/2 + c - h(M)}, \\ \frac{1}{4} U'(s) &= \sqrt{U(s)} \left(\lambda + \frac{1}{2} - h(a) - \frac{G_s(s, M)}{V'(s)} \right) \\ &\quad + c - h(M) + h(M)\sqrt{U(s - M)}, \end{aligned}$$

where $G(s, M) = E[V(s - X \wedge M)]$ and M is the minimizer in the first equation.

Optimal reinsurance strategy Pareto claims



Optimal investment strategy Pareto claims