# Credit-rating based Lévy Libor model

(joint work with Ernst Eberlein)

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# Outline

Lévy Libor model without risk

Credit-rating based Lévy Libor model

- Rating-dependent Libor rates
- Credit migration process

Pricing problems

Concluding remarks

# Lévy Libor model - notation

- $T^* > 0$ fixed time horizon
- discrete tenor structure $0 = T_0 < T_1 < \ldots < T_n = T^*$, with $\delta_k = T_{k+1} - T_k$
- default-free zero coupon bonds $B(\cdot, T_1), \ldots, B(\cdot, T_n)$
- forward Libor rate at time $t \leq T_k$ for the accrual period $[T_k, T_{k+1}]$

$$L(t, T_k) = \frac{1}{\delta_k}\left(\frac{B(t, T_k)}{B(t, T_{k+1})} - 1\right),$$

# The driving process

Let $(\Omega, \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T^*}, \mathbb{P}_{T^*})$ be a complete stochastic basis

We work with a *time-inhomogeneous Lévy process* $X^{T^*}$ with values in $\mathbb{R}^d$, i.e an adapted, cádlág processs with $X_0^{T^*} = 0$ and such that

(1) $X^{T^*}$ has independent increments

(2) the law of $X_t^{T^*}$ is given by its characteristic function

$$\mathbb{E}[\exp(\mathrm{i}\langle u, X_t^{T^*}\rangle)] = \exp\left(\int_0^t \theta_s(\mathrm{i}u)\,\mathrm{d}s\right),$$

$$\theta_s(\mathrm{i}u) = \mathrm{i}\langle u, b_s\rangle - \frac{1}{2}\langle u, c_s u\rangle + \int_{\mathbb{R}^d}\left(e^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, x\rangle F_s(\mathrm{d}x)\right).$$

Here $b_s \in \mathbb{R}^d$, $c_s$ a symmetric, nonneg. definite $d \times d$-matrix and $F_s^{T^*}$ a Lévy measure ($s \in [0, T^*]$).
Assume the following integrability conditions:

$$\sup_{0 \le s \le T^*}\left(|b_s| + ||c_s|| + \int_{\mathbb{R}^d}(|x|^2 \wedge 1)F_s(\mathrm{d}x)\right) < \infty$$

and

$$\sup_{0 \le s \le T^*}\int_{|x|>1}\exp\langle u, x\rangle F_s(\mathrm{d}x) < \infty \quad (u \in [-(1+\varepsilon)M, (1+\varepsilon)M]^d).$$

$X^{T^*}$ is a special semimartingale with canonical decomposition

$$X_t^{T^*} = \int_0^t b_s \, \mathrm{d}s + \int_0^t \sqrt{c_s} \, \mathrm{d}W_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu^X - \nu^{T^*})(\mathrm{d}s, \mathrm{d}x),$$

where $W^{T^*}$ denotes standard Brownian motion and
$\mu^X$ is the random measure of jumps of $X^{T^*}$ with the compensator $\nu^{T^*}$.

The semimartingale characteristics of $X^{T^*}$ is given by

$$B_t = \int_0^t b_s \, \mathrm{d}s, \quad C_t = \int_0^t c_s \, \mathrm{d}s, \quad \nu^{T^*}(\mathrm{d}s, \mathrm{d}x) = F_s^{T^*}(\mathrm{d}x) \, \mathrm{d}s$$

We assume from now on that $b_s = 0$.

# Construction

The model is constructed by a *backward induction* procedure.

**Assumptions:**

(**L**.1) For every $T_k$ there is a deterministic function $\sigma(\cdot, T_k) : [0, T^*] \to \mathbb{R}^d_+$, which represents the volatility of the forward Libor rate $L(\cdot, T_k)$. We assume that

$$\sum_{k=1}^{n-1} \sigma^j(s, T_k) \leq M,$$

for all $s \in [0, T^*]$ and every coordinate $j \in \{1, \ldots, d\}$, where $M > 0$ is a constant from the integrability condition for $X^{T^*}$. If $s > T_k$, then $\sigma(s, T_k) = 0$.

(**L**.2) The initial term structure $B(0, T_k)$ is strictly positive and strictly decreasing in $k$.

Start by specifying the dynamics of the most distant Libor rate under $\mathbb{P}_{T^*}$ (forward measure associated with date $T^*$) as

$$L(t,T_{n-1}) = L(0,T_{n-1}) \exp\left(\int_0^t b^L(s,T_{n-1},T^*)\,\mathrm{d}s + \int_0^t \sigma(s,T_{n-1})\,\mathrm{d}X_s^{T^*}\right),$$

where the drift is chosen in such a way that $L(\cdot,T_{n-1})$ becomes a $\mathbb{P}_{T^*}$-martingale, namely

$$\begin{aligned} b^L(s,T_{n-1},T^*) &= -\frac{1}{2}\langle \sigma(s,T_{n-1}), c_s\sigma(s,T_{n-1})\rangle \\ &\quad - \int_{\mathbb{R}^d} \left(e^{\langle \sigma(s,T_{n-1}),x\rangle} - 1 - \langle \sigma(s,T_{n-1}),x\rangle\right) F_s^{T^*}(\mathrm{d}x). \end{aligned}$$

Next define a forward measure $\mathbb{P}_{T_{n-1}}$ associated with the date $T_{n-1}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_{n-1}}}{\mathrm{d}\mathbb{P}_{T^*}} = \frac{1+\delta_{n-1}L(T_{n-1},T_{n-1})}{1+\delta_{n-1}L(0,T_{n-1})}$$

and proceed with modeling of $L(\cdot,T_{n-2})$.

**For each $k$:**

(1) define the forward measure $\mathbb{P}_{T_{k+1}}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}} = \prod_{l=k+1}^{n-1} \frac{1+\delta_l L(T_{k+1}, T_l)}{1+\delta_l L(0, T_l)}.$$

(2) the dynamics of the Libor rate $L(\cdot, T_k)$ under this measure

$$L(t, T_k) = L(0, T_k) \exp\left( \int_0^t b^L(s, T_k, T_{k+1})\, \mathrm{d}s + \int_0^t \sigma(s, T_k)\, \mathrm{d}X_s^{T_{k+1}} \right), \quad (1)$$

where

$$X_t^{T_{k+1}} = \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu^X - \nu^{T_{k+1}})(\mathrm{d}s, \mathrm{d}x)$$

with

$$\nu^{T_{k+1}}(\mathrm{d}s, \mathrm{d}x) = \prod_{l=k+1}^{n-1} \frac{\delta_l L(s-, T_l)}{1+\delta_l L(s-, T_l)} \left( (e^{\sigma(s, T_l)x} - 1) + 1 \right) \nu^{T^*}(\mathrm{d}s, \mathrm{d}x)$$

and the drift $b^L(s, T_k, T_{k+1})$ is chosen such that it becomes a $\mathbb{P}_{T_{k+1}}$-martingale.

This construction guarantees that processes

$$\left(\frac{B(\cdot, T_j)}{B(\cdot, T_k)}\right)$$

are martingales for all $j = 1, \ldots, n$ under the forward measure $\mathbb{P}_{T_k}$ associated with the date $T_k$ ($k = 1, \ldots, n$).

The $t$-time price of a contingent claim with payoff $X$ at maturity $T_k$ can be calculated as

$$\pi_t^X = B(t, T_k)\mathbb{E}_{\mathbb{P}_{T_k}}[X|\mathcal{F}_t].$$

# Credit ratings

- credit ratings identified with elements of a finite set $\mathcal{K} = \{1, 2, \ldots, K\}$, where 1 is the best possible rating and $K$ is the default event
- these ratings correspond to the states of a conditional Markov chain $C$ with the absorbing state $K$
- the default time $\tau$ = the first time when $C$ reaches state $K$

$$\tau = \inf\{t > 0 : C_t = K\}$$

# Defaultable bonds with credit ratings

- defaultable bonds whose credit migration process is denoted by $C$ and with zero recovery upon default: $B_C(\cdot, T_1), \ldots, B_C(\cdot, T_n)$

- time-$t$ price of such a defaultable bond can be expressed as

$$B_C(t, T_k) = \mathbf{1}_{\{C_t=1\}} B_1(t, T_k) + \cdots + \mathbf{1}_{\{C_t=K-1\}} B_{K-1}(t, T_k),$$

  where $B_i(t, T_k)$ represents the bond price at time $t$ provided that the bond has the rating $i$ during the time interval $[0, t]$.

- Payoff at maturity equals

$$B_C(T_k, T_k) = \sum_{i=1}^{K-1} \mathbf{1}_{\{C_{T_k}=i\}} = \mathbf{1}_{\{\tau > T_k\}}$$

  and it holds $B_i(T_k, T_k) = 1$, for all $i$.

# Libor rates for different credit ratings

- The *Libor rates for credit rating* class $i$

$$L_i(t,T_k) := \frac{1}{\delta_k}\left(\frac{B_i(t,T_k)}{B_i(t,T_{k+1})} - 1\right), \quad i = 1,2,\ldots,K-1.$$

  We put $L_0(t,T_k) := L(t,T_k)$ (default-free Libor rates)

- The *forward credit spreads* between two successive classes

$$S_i(t,T_k) := L_i(t,T_k) - L_{i-1}(t,T_k), \quad i = 1,2,\ldots,K-1.$$

- The corresponding *forward credit spread intensities*

$$H_i(t,T_k) := \frac{L_i(t,T_k) - L_{i-1}(t,T_k)}{1 + \delta_k L_{i-1}(t,T_k)}$$

Observe that Libor rates for the rating $i$ can be expressed as

$$\begin{aligned} 1 + \delta_k L_i(t, T_k) &= (1 + \delta_k L_{i-1}(t, T_k))(1 + \delta_k H_i(t, T_k)) \\ &= \underbrace{(1 + \delta_k L(t, T_k))}_{\text{default-free Libor}} \prod_{j=1}^{i} \underbrace{(1 + \delta_k H_j(t, T_k))}_{\text{intensity between } j \text{ and } j-1} \end{aligned}$$

**Idea:** model $H_j(\cdot, T_k)$ as exponential processes and therefore ensure automatically the *monotonicity* of Libor rates w.r.t the credit rating:

$$L(t, T_k) < L_1(t, T_k) < \cdots < L_{K-1}(t, T_k)$$

$\Longrightarrow$ worse credit rating, higher interest rate

**Assumptions:**

**(RBL.1)** For every $i \in \{1, \dots, K-1\}$ and every maturity $T_k$ there is a deterministic function $\gamma_i(\cdot, T_k) : [0, T^*] \to \mathbb{R}^d_+$, which represents the volatility of the forward spread intensity $H_i(\cdot, T_k)$ for the rating $i$. We assume that $\gamma_i(s, T_k) = 0$ for $s > T_k$ and that

$$\sum_{k=1}^{n-1} (\sigma^j(s, T_k) + \gamma_1^j(s, T_k) + \cdots + \gamma_{K-1}^j(s, T_k)) \le M,$$

for all $s \in [0, T^*]$ and every coordinate $j \in \{1, \dots, d\}$.

**(RBL.2)** The initial term structure $L_i(0, T_k)$ of Libor rates satisfies

$$L_i(0, T_k) > L_{i-1}(0, T_k),$$

for all $k = 1, \dots, n-1$, i.e.

$$\frac{B_i(0, T_k)}{B_i(0, T_{k+1})} > \frac{B_{i-1}(0, T_k)}{B_{i-1}(0, T_{k+1})}.$$

# Construction of rating-based Libor rates

For rating 1 and all settlement dates $T_k$ we model $H_1(\cdot, T_k)$ as

$$H_1(t, T_k) = H_1(0, T_k) \exp\left( \int_0^t b^{H_1}(s, T_k, T_{k+1}) \, \mathrm{d}s + \int_0^t \gamma_1(s, T_k) \, \mathrm{d}X_s^{T_{k+1}} \right)$$

with initial condition

$$H_1(0, T_k) = \frac{1}{\delta_k} \left( \frac{B_1(0, T_k) B(0, T_{k+1})}{B(0, T_k) B_1(0, T_{k+1})} - 1 \right).$$

$X^{T_{k+1}}$ is defined as earlier and $b^{H_1}(s, T_k, T_{k+1})$ is the drift term (we assume $b^{H_1}(s, T_k, T_{k+1}) = 0$, for $s \geq T_k \Rightarrow H_1(t, T_k) = H_1(T_k, T_k)$, for $t \geq T_k$).

Choose drift term in such a way that the process $\left( \prod_{l=0}^{k} \frac{1}{1+\delta_l H_1(t, T_l)} \right)_{t \leq T_k}$ becomes a $\mathbb{P}_{T_{k+1}}$-martingale (Kluge (2005)).

Hence we can introduce the following change of measure

$$\frac{d\mathbb{P}_{1,T_{k+1}}}{d\mathbb{P}_{T_{k+1}}} := \frac{\prod_{l=0}^{k} \frac{1}{1+\delta_l H_1(T_l,T_l)}}{\prod_{l=0}^{k} \frac{1}{1+\delta_l H_1(0,T_l)}} = \frac{B(0,T_{k+1})}{B_1(0,T_{k+1})} \prod_{l=0}^{k} \frac{1}{1+\delta_l H_1(T_l,T_l)}.$$

$\mathbb{P}_{1,T_{k+1}}$ - *forward measure associated with the rating 1 and the settlement date* $T_{k+1}$

Use Girsanov's theorem for semimartingales. Denote

$$h_1(s,T_l,T_{l+1}) := \frac{\delta_l H_1(s,T_l)}{1+\delta_l H_1(s,T_l)}, \quad s \le T_l.$$

Then

$$W_t^{1,T_{k+1}} := W_t^{T_{k+1}} + \int_0^t \sum_{l=1}^{k} h_1(s-,T_l,T_{l+1}) \gamma_1(s,T_l) \sqrt{c_s}\, ds$$

is a $\mathbb{P}_{1,T_{k+1}}$- Brownian motion and

$$\begin{aligned} \nu_{1,T_{k+1}}(dt,dx) &:= \prod_{l=1}^{k} \left(1 + h_1(s-,T_l,T_{l+1}) \left(e^{\langle \gamma_1(s,T_l), x\rangle} - 1\right)\right)^{-1} \nu^{T_{k+1}}(dt,dx) \\ &=: F_t^{1,T_{k+1}}(dx)\, dt \end{aligned}$$

is the $\mathbb{P}_{1,T_{k+1}}$-compensator of $\mu^X$.

## Proposition

(a) *The Libor rate* $L_1(\cdot, T_k)$ *is a* $\mathbb{P}_{1,T_{k+1}}$*-martingale for all* $T_k$*.*

(b) *The connection between two subsequent forward measures is given by*

$$\frac{d\mathbb{P}_{1,T_k}}{d\mathbb{P}_{1,T_{k+1}}} = \frac{B_1(0,T_{k+1})}{B_1(0,T_k)}(1+\delta_k L_1(T_k,T_k)).$$

*Proceeding backwards, the connection to the terminal forward measure* $\mathbb{P}_{1,T^*}$ *is given by*

$$\frac{d\mathbb{P}_{1,T_k}}{d\mathbb{P}_{1,T^*}} = \frac{B_1(0,T^*)}{B_1(0,T_k)}\prod_{l=k}^{n-1}(1+\delta_l L_1(T_k,T_l)).$$

**For each $i \in \mathcal{K} \backslash K$:**

(1) Remember that the forward Libor rate for the rating class $i$

$$1 + \delta_k L_i(t, T_k) = (1 + \delta_k L(t, T_k)) \prod_{j=1}^{i-1} (1 + \delta_k H_j(t, T_k))(1 + \delta_k H_i(t, T_k)),$$

where $H_1(\cdot, T_k), \ldots, H_{i-1}(\cdot, T_k)$ are already modelled.

(2) It remains to model $H_i(\cdot, T_k)$ as an exponential process of the form

$$H_i(t, T_k) = H_i(0, T_k) \exp\left( \int_0^t b^{H_i}(s, T_k, T_{k+1})\, \mathrm{d}s + \int_0^t \gamma_i(s, T_k)\, \mathrm{d}X_s^{i-1, T_{k+1}} \right), \tag{2}$$

where

$$X_t^{i-1, T_{k+1}} = \int_0^t \sqrt{c_s}\, \mathrm{d}W_s^{i-1, T_{k+1}} + \int_0^t \int_{\mathbb{R}^d} x(\mu^X - \nu^{i-1, T_{k+1}})(\mathrm{d}s, \mathrm{d}x)$$

and the drift term $b^{H_i}(s, T_k, T_{k+1}) = 0$, for $s \geq T_k$, i.e. $H_i(t, T_k) = H_i(T_k, T_k)$ for $t \geq T_k$.

(3) Choose the drift term in such a way that the process

$$\left( \prod_{l=0}^{k} \frac{1}{1 + \delta_l H_i(t, T_l)} \right)_{t \leq T_k}$$

becomes a $\mathbb{P}_{i-1, T_{k+1}}$-martingale.

We can define a new measure $\mathbb{P}_{i,T_{k+1}}$ called *forward measure associated with the rating i and date $T_{k+1}$* on $(\Omega, \mathcal{F}_{T_{k+1}})$ via

$$\frac{\mathrm{d}\mathbb{P}_{i,T_{k+1}}}{\mathrm{d}\mathbb{P}_{i-1,T_{k+1}}} := \frac{\prod_{l=0}^{k} \frac{1}{1+\delta_l H_i(T_l,T_l)}}{\prod_{l=0}^{k} \frac{1}{1+\delta_l H_i(0,T_l)}} = \frac{B_{i-1}(0,T_{k+1})}{B_i(0,T_{k+1})} \prod_{l=0}^{k} \frac{1}{1+\delta_l H_i(T_l,T_l)}.$$

Denoting

$$h_i(s,T_l,T_{l+1}) := \frac{\delta_l H_i(s,T_l)}{1+\delta_l H_i(s,T_l)}, \quad s \leq T_l,$$

it follows that

$$W_t^{i,T_{k+1}} := W_t^{i-1,T_{k+1}} + \int_0^t \sum_{l=1}^{k} h_i(s-,T_l,T_{l+1})\gamma_i(s,T_l)\sqrt{c_s}\,\mathrm{d}s$$

is a $\mathbb{P}_{i,T_{k+1}}$- standard Brownian motion and

$$\begin{aligned} \nu^{i,T_{k+1}}(\mathrm{d}t,\mathrm{d}x) &:= \prod_{l=1}^{k} \left(1 + h_i(s-,T_l,T_{l+1})\left(e^{\langle \gamma_i(s,T_l),x\rangle} - 1\right)\right)^{-1} \nu^{i-1,T_{k+1}}(\mathrm{d}t,\mathrm{d}x) \\ &=: F_t^{i,T_{k+1}}(\mathrm{d}x)\,\mathrm{d}t \end{aligned}$$

is the $\mathbb{P}_{i,T_{k+1}}$-compensator of $\mu^X$.

Observe that we can express the measure $\mathbb{P}_{i,T_{k+1}}$ with respect to the (default-free) forward measure $\mathbb{P}_{T_{k+1}}$

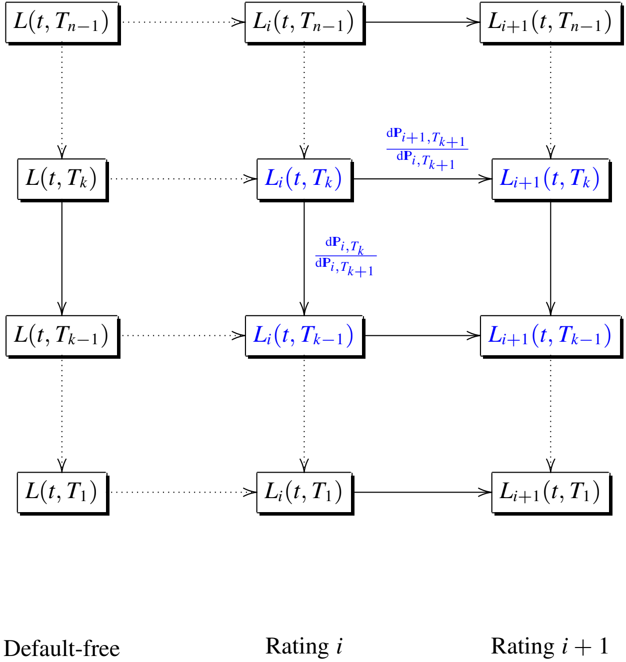
$$\frac{d\mathbb{P}_{i,T_{k+1}}}{d\mathbb{P}_{T_{k+1}}} = \frac{\prod_{j=1}^{i}\prod_{l=0}^{k}\frac{1}{(1+\delta_l H_j(T_l,T_l))}}{\prod_{j=1}^{i}\prod_{l=0}^{k}\frac{1}{(1+\delta_l H_j(0,T_l))}} = \frac{B(0,T_{k+1})}{B_i(0,T_{k+1})}\prod_{j=1}^{i}\prod_{l=0}^{k}\frac{1}{(1+\delta_l H_j(T_l,T_l))}$$

and we have

$$W_t^{i,T_{k+1}} = W_t^{i-1,T_{k+1}} + \int_0^t \sum_{l=1}^{k} h_i(s-,T_l,T_{l+1})\gamma_i(s,T_l)\sqrt{c_s}\,ds = W_t^{T_{k+1}} + \int_0^t \sum_{j=1}^{i}\sum_{l=1}^{k} h_j(s-,T_l,T_{l+1})\gamma_j(s,T_l)\sqrt{c_s}\,ds$$

and

$$\nu^{i,T_{k+1}}(dt,dx) = \prod_{l=1}^{k}\left(1+h_i(s-,T_l,T_{l+1})\left(e^{\langle\gamma_i(s,T_l),x\rangle}-1\right)\right)^{-1}\nu^{i-1,T_{k+1}}(dt,dx) = \prod_{j=1}^{i}\prod_{l=1}^{k}\left(1+h_j(s-,T_l,T_{l+1})\left(e^{\langle\gamma_j(s,T_l),x\rangle}-1\right)\right)^{-1}\nu^{T_{k+1}}(dt,dx).$$


$L(t, T_{n-1})$
$L_i(t, T_{n-1})$
$L_{i+1}(t, T_{n-1})$
$\frac{d\mathbf{P}_{i+1,T_{k+1}}}{d\mathbf{P}_{i,T_{k+1}}}$
$L(t, T_k)$
$L_i(t, T_k)$
$L_{i+1}(t, T_k)$
$\frac{d\mathbf{P}_{i,T_k}}{d\mathbf{P}_{i,T_{k+1}}}$
$L(t, T_{k-1})$
$L_i(t, T_{k-1})$
$L_{i+1}(t, T_{k-1})$
$L(t, T_1)$
$L_i(t, T_1)$
$L_{i+1}(t, T_1)$
Default-free
Rating $i$
Rating $i + 1$

## Theorem

*Assume that* $(\mathbf{L}.1)$, $(\mathbf{L}.2)$, $(\mathbf{RBL}.1)$ *and* $(\mathbf{RBL}.2)$ *are in force and that* $L(\cdot, T_k)$ *and* $H_i(\cdot, T_k)$ *are given by (1) and (2). The credit-rating based Libor model has the following properties:*

(a) *For every* $T_k$ *and* $t \le T_k$ *it holds*

$$L(t, T_k) < L_1(t, T_k) < \cdots < L_{K-1}(t, T_k),$$

*i.e. Libor rates are monotone with respect to the credit rating.*

(b) *The forward measure* $\mathbb{P}_{i,T_k}$ *associated with the rating* $i$ *and the date* $T_k$ *is defined via*

$$\frac{\mathrm{d}\mathbb{P}_{i,T_k}}{\mathrm{d}\mathbb{P}_{T_k}} = \frac{B(0, T_k)}{B_i(0, T_k)} \prod_{j=1}^{i} \prod_{l=0}^{k-1} \frac{1}{(1 + \delta_l H_j(T_l, T_l))}.$$

(c) *For each rating* $i$ *and each date* $T_k$ *the Libor rate* $L_i(\cdot, T_k)$ *is a martingale with respect to the forward measure* $\mathbb{P}_{i,T_{k+1}}$.

(d) *For every* $i$ *it holds*

$$\frac{\mathrm{d}\mathbb{P}_{i,T_k}}{\mathrm{d}\mathbb{P}_{i,T_{k+1}}} = \frac{B_i(0, T_{k+1})}{B_i(0, T_k)} (1 + \delta_k L_i(T_k, T_k)).$$

# Credit migration process

Bielecki and Rutkowski (2002)

- Let $\Lambda = (\Lambda_t)_{0 \le t \le T^*}$ be a matrix-valued $\mathbb{F}$-adapted stochastic process on $(\Omega, \mathcal{F}_{T^*}, \mathbb{P}_{T^*})$

$$\Lambda(t) = \begin{bmatrix} \lambda_{11}(t) & \lambda_{12}(t) & \dots & \lambda_{1K}(t) \\ \lambda_{21}(t) & \lambda_{22}(t) & \dots & \lambda_{2K}(t) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

  where $\lambda_{ij}$ are nonnegative processes, integrable on every $[0, t]$ and $\lambda_{ii}(t) = -\sum_{j \in \mathcal{K} \setminus \{i\}} \lambda_{ij}(t)$.
- Let $\mu = (\delta_{ij}, j \in \mathcal{K})$ be a probability distribution on $\overline{\Omega} = \mathcal{K}$.
- Define

$$(\tilde{\Omega}, \mathcal{G}, \mathbb{Q}_{T^*}) = (\Omega \times \Omega^U \times \overline{\Omega}, \mathcal{F}_{T^*} \otimes \mathcal{F}^U \otimes 2^{\overline{\Omega}}, \mathbb{P}_{T^*} \otimes \mathbb{P}^U \otimes \mu),$$

- On $(\Omega^U, \mathcal{F}^U, \mathbb{P}^U)$ a sequence $(U_{i,j})$, $i, j \in \mathbb{N}$, of mutually independent random variables, uniformly distributed on $[0, 1]$.

# Construction of the migration process

- The jump times $\tau_k$ are constructed recursively as

$$\tau_k := \tau_{k-1} + \inf\left\{ t \geq 0 : \exp\left( \int_{\tau_{k-1}}^{\tau_{k-1}+t} \lambda_{\overline{C}_{k-1},\overline{C}_{k-1}}(u)\,\mathrm{d}u \right) \leq U_{1,k} \right\},$$

  where we set $\tau_0 := 0$.

- The new state at the jump time $\tau_k$ is defined as

$$\overline{C}_k := \mathbb{C}(U_{2,k}, \overline{C}_{k-1}, \tau_k),$$

  where we set $\overline{C}_0(\omega, \omega^U, \overline{\omega}) = \overline{\omega}$ and where $\mathbb{C} : [0,1] \times \mathcal{K} \times \mathbb{R}_+ \times \Omega \to \mathcal{K}$ is any mapping such that for any $i, j \in \mathcal{K}$, $i \neq j$, it holds

$$\mathrm{Leb}\left(\{u \in [0,1] : \mathbb{C}(u,i,t) = j\}\right) = -\frac{\lambda_{ij}(t)}{\lambda_{ii}(t)}$$

  if $\lambda_{ii}(t) < 0$ and $0$ if $\lambda_{ii}(t) = 0$.

- Finally, we define for every $t \geq 0$

$$C_t := \overline{C}_{k-1}, \quad \text{for } t \in [\tau_{k-1}, \tau_k), k \geq 1.$$

Process $C$ is a *conditionally Markov chain* relative to $\mathbb{F}$, i.e. for every $0 \leq t \leq s$ and any function $h : \mathcal{K} \to \mathbb{R}$ it holds

$$\mathbb{E}[h(C_s)|\mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}[h(C_s)|\mathcal{F}_t \vee \sigma(C_t)],$$

where $\mathbb{F}^C = (\mathcal{F}_t^C)$ denotes the filtration generated by $C$ and $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_t^C$.

## Proposition

*The conditional expectations with respect to enlarged $\sigma$-algebras can be expressed in terms of $\mathcal{F}_t$-conditional expectations. It holds*

$$\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t \vee \sigma(C_t)] = \sum_{i=1}^{K} \mathbf{1}_{\{C_t=i\}} \frac{\mathbb{E}_{\mathbb{Q}_{T^*}}[Y\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T^*}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]},$$

*for any $\mathcal{G}$-measurable random variable $Y$.*

# Properties of $C$

(a) for every $t \leq s \leq u$ and any function $h : \mathcal{K} \to \mathbb{R}$ a stronger version of conditional Markov property holds:

$$\mathbb{E}_{\mathbb{Q}_{T*}}\left[h(C_s)|\mathcal{F}_u \vee \mathcal{F}_t^C\right] = \mathbb{E}_{\mathbb{Q}_{T*}}\left[h(C_s)|\mathcal{F}_u \vee \sigma(C_t)\right]$$

(b) for every $t \leq s$ and $B \in \mathcal{F}_t^C$:

$$\mathbb{E}_{\mathbb{Q}_{T*}}\left[\mathbf{1}_B|\mathcal{F}_s\right] = \mathbb{E}_{\mathbb{Q}_{T*}}\left[\mathbf{1}_B|\mathcal{F}_t\right]$$

(c) *conditional* Chapman-Kolmogorov equality

$$P(t,s) = P(t,u)P(u,s),$$

where $P(t,s) = [p_{ij}(t,s)]_{i,j\in\mathcal{K}}$ and

$$p_{ij}(t,s) := \frac{\mathbb{Q}_{T*}\left(C_s = j, C_t = i|\mathcal{F}_s\right)}{\mathbb{Q}_{T*}\left(C_t = i|\mathcal{F}_s\right)}$$

(d) *conditional* forward Kolmogorov equation

$$\frac{\mathrm{d}P(t,s)}{\mathrm{d}s} = P(t,s)\Lambda(s)$$

Note that the $(\mathcal{H})$-hypothesis

$(\mathcal{H})$ Every local $\mathbf{F}$-martingale is a local $\mathbf{G}$-martingale

holds in this conditional Markov setting.

It is equivalent to any of the following three equalities:

$(\mathcal{H}1)$ $\mathbb{E}_{\mathbb{Q}_{T^*}}[XY|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_t]\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t]$,

i.e. $\mathcal{F}_{T^*}$ and $\mathcal{F}_t^C$ are conditionally independent given $\mathcal{F}_t$

$(\mathcal{H}2)$ $\mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_{T^*}] = \mathbb{E}_{\mathbb{Q}_{T^*}}[Y|\mathcal{F}_t]$,

for $Y$ bounded $\mathcal{F}_t^C$-measurable random variable

$(\mathcal{H}3)$ $\mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_t \vee \mathcal{F}_t^C] = \mathbb{E}_{\mathbb{Q}_{T^*}}[X|\mathcal{F}_t]$,

for $X$ bounded $\mathcal{F}_{T^*}$-measurable random variable

# No-arbitrage condition for the model with migration

Remember the defaultable bond price process with zero recovery

$$B_C(t, T_k) = \sum_{i=1}^{K-1} B_i(t, T_k) \mathbf{1}_{\{C_t = i\}}.$$

Then we can write it as

$$\frac{B_C(t, T_k)}{B(t, T_k)} = \sum_{i=1}^{K-1} \underbrace{\prod_{j=1}^{i} \prod_{l=0}^{k-1} \frac{1}{1 + \delta_k H_j(t, T_l)}}_{:= \mathbb{H}(i,t,T_k)} \frac{B_i(t, 0)}{B(t, 0)} \mathbf{1}_{\{C_t = i\}}, \tag{3}$$

where we put

$$\frac{B_i(t, 0)}{B(t, 0)} := \exp\left( \int_0^t \lambda_i(s)\, \mathrm{d}s \right),$$

for some $\mathbb{F}$-adapted process $(\lambda_i(\cdot))$ that is integrable on $[0, T^*]$ ($i \in \mathcal{K} \setminus K$).

## Proposition

*Assume that the processes $(H_j(\cdot, T_k))$ are given by (2) and satisfy the drift condition, i.e. the processes $(\mathbb{H}(i, \cdot, T_k))$ are $\mathbb{P}_{T_k}$-martingales. The process $\left(\frac{B_C(\cdot,T_k)}{B(\cdot,T_k)}\right)$ defined in (3) is a local martingale with respect to the forward measure $\mathbb{P}_{T_k}$ iff for every $i \in \mathcal{K} \setminus K$ and $t \leq T_k$ it holds*

$$\mathbb{H}(i,t,T_k)\lambda_{i,K}(t) \;+\; \sum_{j=1}^{K-1} \left( \mathbb{H}(i,t,T_k) - \mathbb{H}(j,t,T_k) \exp\Big( \int_0^t (\lambda_j(s) - \lambda_i(s))\, \mathrm{d}s \Big) \right) \lambda_{ij}(t)$$
$$= \mathbb{H}(i,t-,T_k)\lambda_i(t).$$

**Remark:** In case when the recovery is not zero, a similar condition can be derived. The bond price process has the additional term

$$B_C(t,T_k) = \sum_{i=1}^{K-1} B_i(t,T_k)\mathbf{1}_{\{C_t=i\}} + \delta_{C_{\tau-}} B(t,T_k)\mathbf{1}_{\{C_t=K\}}$$

(fractional recovery of Treasury value)

# Defaultable bond price

The price of a $T_k$-maturity defaultable bond with zero recovery at time $t \leq T_k$ equals

$$B_C(t,T_k) = B(t,T_k)\sum_{i=1}^{K-1}\mathbb{E}_{\mathbb{Q}_{T_k}}[1 - p_{iK}(t,T_k)|\mathcal{F}_t]\mathbf{1}_{\{C_t=i\}}.$$

**Proof:** The promised payoff of such a bond at maturity time $T_k$ is $\mathbf{1}_{\{C_{T_k}\neq K\}}$. Using properties of the process $C$, we get

$$\begin{aligned}
B_C(t,T_k) &= B(t,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}|\mathcal{F}_t \vee \mathcal{F}_t^C] \\
&= B(t,T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}|\mathcal{F}_t \vee \sigma(C_t)] \\
&= B(t,T_k)\sum_{i=1}^{K-1}\frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_{T_k}\neq K\}}\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}\mathbf{1}_{\{C_t=i\}} \\
&= B(t,T_k)\sum_{i=1}^{K-1}\mathbb{E}_{\mathbb{Q}_{T_k}}[1 - p_{iK}(t,T_k)|\mathcal{F}_t]\mathbf{1}_{\{C_t=i\}}.
\end{aligned}$$

More generally, the price of a contingent claim with a promised $\mathcal{F}_{T_k}$ - measurable payoff $Y$ at maturity $T_k$ and zero recovery upon default equals

$$\pi_t(Y) = B(t, T_k) \sum_{i=1}^{K-1} \mathbb{E}_{\mathbb{Q}_{T_k}} [Y(1 - p_{iK}(t, T_k))|\mathcal{F}_t] \mathbf{1}_{\{C_t = i\}}.$$

- In a special case when no migration between time $t$ and $T_k$ is allowed, the pricing formula can be simplified considerably. It can be shown that

$$\pi_t(Y) = \sum_{i=1}^{K-1} B_i(t, T_k) \mathbb{E}_{\mathbb{P}_{i,T_k}} [Y|\mathcal{F}_t] \mathbf{1}_{\{C_t = i\}},$$

  where $\mathbb{P}_{i,T_k}$ is the forward measure associated with rating $i$ and date $T_k$.

  (see analogous result in Eberlein, Kluge and Schönbucher (2006))

- Unfortunately if this is not the case, one has to work with the full transition matrix of $C$ and the forward measures cannot be used this way.

# A rating-dependent Libor rate forward cap

- provides insurance against increasing interest rates
- cap is a series of *caplets*, i.e. call options on subsequent Libor rates

The payoff of a caplet with strike $K$ and maturity $T_k$ on the Libor rate for rating $i$ is

$$\delta_k (L_i(T_k, T_k) - K)^+.$$

.

Applying the previous result, time-$t$ price of the caplet equals

$$Caplet^i(t, T_k) = \delta_k B_i(t, T_{k+1}) \mathbb{E}_{\mathbf{P}_{i,T_{k+1}}} [(L_i(T_k, T_k) - K)^+ | \mathcal{F}_t].$$

Hence,

$$Cap^i(t) = \sum_{j=1}^{n} \delta_{j-1} B_i(t, T_j) \mathbb{E}_{\mathbf{P}_{i,T_j}} [(L_i(T_{j-1}, T_{j-1}) - K)^+ | \mathcal{F}_t].$$

- by using Laplace transformation techniques we can calculate the price at time 0

# Credit default swap

- provides protection against default of an underlying asset
- consider a maturity date $T_m$ and a defaultable bond with fractional recovery of Treasury value as the underlying asset
- protection buyer pays a fixed amount $s$ periodically at dates $T_1, \ldots, T_{m-1}$ until default
- protection seller promises to make a payment that covers the loss if default happens:

$$1 - \delta_{C_{\tau-}}$$

is received at $T_{k+1}$ if default occurs in $(T_k, T_{k+1}]$

The value of the premium leg at $t \leq T_1$:

$$\sum_{k=1}^{m-1} sB(t, T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{\tau > T_k\}}|\mathcal{G}_t]$$

The value of the default leg at $t \leq T_1$:

$$\sum_{k=2}^{m} B(t, T_k)\mathbb{E}_{\mathbb{Q}_{T_k}}[(1 - \delta_{C_{\tau-}})\mathbf{1}_{\{T_{k-1} < \tau \leq T_k\}}|\mathcal{G}_t]$$

The premium leg can be written as

$$s \sum_{k=1}^{m-1} \sum_{i=1}^{K-1} B_i(t, T_k) \mathbf{1}_{\{C_t=i\}}$$

and the default leg

$$\sum_{k=2}^{m} B(t, T_k) \sum_{i=1}^{K-1} \mathbf{1}_{\{C_t=i\}} \sum_{j=1}^{K-1} \frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-\delta_j)\mathbf{1}_{\{T_{k-1}<\tau\leq T_k, C_{\tau-}=j\}}\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_t=i\}}|\mathcal{F}_t]}.$$

Hence, the swap rate $s$ at time 0 is equal to

$$s = \sum_{i=1}^{K-1} \mathbf{1}_{\{C_0=i\}} \frac{\sum_{k=2}^{m} B(t, T_k) \sum_{j=1}^{K-1} \frac{\mathbb{E}_{\mathbb{Q}_{T_k}}[(1-\delta_j)\mathbf{1}_{\{T_{k-1}<\tau\leq T_k, C_{\tau-}=j\}}\mathbf{1}_{\{C_0=i\}}]}{\mathbb{E}_{\mathbb{Q}_{T_k}}[\mathbf{1}_{\{C_0=i\}}]}}{\sum_{k=1}^{m-1} B_i(0, T_k)}.$$

# Concluding remarks

- rating-dependent Libor rates are introduced
- the whole term structure of default-free and rating-dependent Libor rates is modelled
- credit rating migrations are added and no-arbitrage conditions derived
- rating-dependent forward measures can be used for pricing *only* in special cases where no migration between classes is allowed
  $\Rightarrow$ pricing formulae for caps, floors, swaps on rating-dependent Libor rates
- in general, explicit pricing formulae for derivatives depending on ratings (e.g. options on defaultable bonds, credit default swaps, credit default swaptions) require further assumptions on the migration process

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