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Optimal management of pension funds: a stochastic control approach

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Overview

- Motivating problem: ALM for a pension fund with minimum guarantee.

- The model: a stochastic control problem with state constraints and delay terms.

- The case with no surplus (i.e. no delay): regular solutions to HJB and feedback control strategies.

- The case with delay: some partial results on the infinite dimensional HJB.

- Further research and work in progress.

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Motivating problem: ALM for a pension fund with minimum guarantee

Starting point: an italian insurance company (INA) asking for a model of optimal management of a defined contribution pension fund with a minimum guarantee.

First outcame: a paper where a discrete time model containing all the features required by the company are present (number and type of assets, transaction costs, objective function, demographic variables, various constraints required by the law or by the company, etc.):

(2003) Sbaraglia, S.; Papi, M.; Briani, M.; Bernaschi, M.; Gozzi, F., A model for the optimal asset-liability management for insurance companies, *Int. J. Theor. Appl. Finance* , **6**, No. 3, 277-299.

Model unsolvable with the known techniques.

Paper above devoted to present the model and some simulation by scenario generation and static optimization methods.

Present Goal: formulate and study a continuous time stochastic model of optimal allocation for a defined contribution pension fund with a minimum guarantee that still contains some key features of the applied model above:

- The manager invests in a Black-Scholes market and maximizes the discounted utility from wealth over an infinite horizon.

- There is a stationary flow of contributions and benefits.

- The wealth process $x(\cdot)$ that must stay above a solvency level l (**state constraint**).

- The benefits depend on the past performance of the fund (**delay term in the state equation**).

THE LITERATURE

Some papers on defined contribution pension fund without minimum guarantee:

(2000) Gerber H.U., and Shiu E.S.W., Investing for retirement: optimal capital growth and dynamic asset allocation, *North American Actuarial Journal*, **4**, 1, 42–62

(2000) Cairns A.J G. , Blake D., and Dowd K., Optimal dynamic asset allocation for defined-contribution pension plans, *Proceedings of the 10th AFIR International Colloquium*, 131–154

(2001) Vigna E., and Haberman S., Optimal investment strategy for defined contribution pension schemes, *Insurance: Mathematics and Economics*, **28**, 233–262

(2002) Haberman S., and Vigna E., Optimal investment strategies and risk measures in defined contribution pension schemes, *Insurance: Mathematics and Economics*, **31**, 35–69

(2004) Battocchio P., and Menoncin F., Optimal pension management in a stochastic framework, *Insurance: Mathematics and Economics*, **34**, 79–95

Defined contribution pension fund with minimum guarantee:

(2001) Boulier J.F., Huang S.J., and Taillard G., Optimal management under stochastic interest rates: the case of a protected pension fund, *Insurance: Mathematics and Economics*, **28**, 173-179.

(2003) Deelstra G., Grasselli M., and Koehl P.F., Optimal investment strategies in the presence of a minimum guarantee, *Insurance: Mathematics and Economics*, **33**, 189–207.

- Others?

In particular [Boulier et al., 2001] and [Deelstra et al., 2003] study the optimal management over the accumulation phase:

- in a complete financial market
- in a continuous and finite time horizon
- assuming as terminal date the time of retirement of a representative agent (i.e. single cohort)
- by considering the guarantee as a contingent claim
- by applying a martingale and duality approach
- by using the CRRA utility function

They find explicit solutions by maximizing the expected utility function of the terminal wealth under the constraint that the terminal wealth must exceed the minimum guarantee

Moreover [Boulier et al., 2001] consider

- the contribution flow is a deterministic process
- the guarantee has a very specific form
- the Vasiček model for the term structure of interest rates

On the contrary [Deelstra et al., 2003] assume that

- the contribution flow is a stochastic process but generated by the market (since the market is complete)
- the guarantee is a general process
- the interest rates follow the affine dynamics in the one-dimensional version, which include as a special case the CIR model and the Vasiček model

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The model: a stochastic control problem with state constraints and delay terms

A - SECURITY MARKET

- The security market is a standard Black & Scholes market: one riskless asset and one risky asset

- Randomness is described by a one-dimensional standard Brownian motion $B(t)$, $t \geq 0$, defined on a filtered probability space $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t^B\}_{t\geq 0},\ \mathbb{P})$

- The interest rate is constant: this restriction is done for simplicity to focus on the other features of the model.

B - DYNAMICS OF WEALTH

Let $X(t)$, $t \geq 0$, be the process giving the amount of fund wealth (**State variable**).

Let $\theta(t)$, $t \geq 0$, be the $\{\mathcal{F}_t^B\}_{t\geq 0}$-adapted process giving the proportion of wealth invested in the risky asset (**Control variable**).

The fund starts at $t = 0$ but we may look at it when it is already working. So we are given initial data $t_0 \geq 0$, $x_0 \geq 0$ and we assume that the wealth process satisfies the equation:

$$\begin{cases} dX(t) = \Big\{ \left[\theta(t)\sigma\lambda + r\right] X(t) + c(t) - b(t)\Big\} dt + \theta(t)\sigma X(t) dB(t), \quad t \geq t_0 \\ X(t_0) = x_0 \end{cases}$$

This the standard wealth equation with the extra terms given by the flow of **contributions** $c(\cdot)$ and **benefits** $b(\cdot)$.

C - CONTRIBUTIONS AND BENEFITS

Assuming **demographic stationarity** we have:

Hypothesis 1 *The flow of aggregate contributions is given by:*

$$c(t) := \frac{t \wedge T}{T} \alpha \, N \, w, \quad 0 < \alpha < 1, \quad \forall t \geq 0,$$

where

- α *is the average contribution rate;*
- T *is the average time spent in the fund by members;*
- $N \in \mathbb{N}$ *is the average number of members after* T*.*
- $w > 0$ *is the average per capita wage bill earned by the fund members (see* [Boulier et al., 2001]*)*

Hypothesis 2 *The flow of aggregate benefits is given by:*

$$b(t) := \begin{cases} 0 & \textit{if} \quad 0 \leq t < T \\ g(t) + s\left(t, X(\cdot)\,|_{[t-T,t]}\right) & \textit{if} \quad t \geq T \end{cases}$$

where

- $g(\cdot)$ *is the flow of minimum guarantee;*
- $s(\cdot, \cdot)$ *is the 'surplus' function. At time* $t \geq T$ *it depends on the fund wealth level in the time period* $[t-T, t]$*.*

The minimum guarantee

Hypothesis 3 *The flow of minimum guarantee is, for $t \geq T$:*

$$g(t) := \int_{t-T}^{t} \bar{c}(t) e^{\delta(t-u)} du, \quad \eta \geq 0,$$

where $\delta \in [0, r]$ is the instantaneous guaranteed rate of return and $\bar{c}(t)$ is flow of contributions of new members per unit of time. By demographic stationarity $\bar{c}(t) = \frac{1}{T} \alpha N w$.

It follows:

$$g(t) = \alpha N w \, \frac{e^{\delta T} - 1}{\delta T} > \alpha N w, \quad \forall t \geq T$$

The surplus A reasonable choice of the surplus is a two variables function $f(X(t), X(t-T))$ or, more precisely,

$$f_1\left(\frac{X(t)}{X(t-T)}\right) \qquad \text{or} \qquad f_2(X(t) - kX(t-T))$$

for suitable functions $f_1, f_2 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ increasing and convex.

Taking a nonzero surplus function the state equation becomes a **delay differential equation** and the related problem becomes more complex as it requires techniques of **stochastic control in infinite dimension**.

We look at the stationary phase (the accumulation phase is studied in a forthcoming paper of S. Federico) i.e. $t_0 = T$. The equation for the wealth becomes

$$(1) \begin{cases} dX(t) = \ [(r+\sigma\lambda\theta(t))\, X(t) - A]\, dt - f\,(X(t), X(t-T))\, dt \\ \qquad\qquad\qquad\qquad\qquad\qquad +\sigma\theta(t)X(t)\, dB(t) \\ \\ X(T) = \eta_0,\ \ X(T+\zeta) = \eta_1(\zeta), \quad \zeta \in [-T, 0), \end{cases}$$

where $\eta = (\eta_0, \eta_1(\cdot)) \in \mathbb{R} \times L^2(-T, 0)$ is the **initial datum** and

$$A = \alpha N w \left[\frac{e^{\delta T} - 1}{\delta T} - 1 \right] > 0$$

is the balance between benefits and contributions flow.

D - SOLVENCY CONSTRAINT

Hypothesis 4 *The process X is subject to the following constraint:*

$$X(t) \geq l(t) \quad \mathbb{P} - a.s., \qquad \forall t \geq 0,$$

where l is nonnegative and constant after T.

Remark 1 The function l gives a solvency level set up by the authority to avoid "improper" behavior of the fund manager. It does not need to be constant after T. This is done for simplicity. ∎

A possible choice of l.

A startup level l_0 plus a "share" of the due minimum guarantee in a unit of time:

$$l(t) = l_0 + \zeta \int_{(t-T)\wedge 0}^{t} \bar{c}(u) e^{\delta(t-u)} du \qquad\qquad t \geq 0,$$

where $l_0 \geq 0$ and $0 \leq \zeta \leq T$.

Note that for $t \geq T$ $l(t)$ is constant and

$$l(t) = l(T) = \zeta \alpha N w \, \frac{e^{\delta T} - 1}{\delta T}$$

Another possible choice (treatable in our setting with some more work) is that $l(t) = l_0$ plus a share of the contributions of active workers, evaluated at the rate of return of minimum guarantee δ.

In this case we would put $c(u)$ instead of $\bar{c}(u)$ in the above integral.

E - MAXIMIZING THE OBJECTIVE

We want to maximize the objective

$$J(T,\eta;\theta(\cdot)) = \mathbb{E}\left[\int_T^{+\infty} e^{-\rho t} U(X(t;T,\eta,\theta))\,dt\right]$$

where

- $U : [l_T, +\infty) \longrightarrow \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, belongs to $\mathrm{C}^2((0,+\infty))$ and satisfies, for suitable $C > 0$ and $\beta \in [0,1)$,

  $$U(x) \le C(1+x^\beta), \qquad x \ge l_T;$$

- the discount rate ρ satisfies

  $$\rho > \beta r + \frac{\lambda^2}{2} \cdot \frac{\beta}{1-\beta}.$$

  This ensures finiteness of the value function.

Why infinite horizon?

- In the papers on this subject one maximizes the final wealth at time T (single cohort).

- Here we take the horizon of the manager that can be different, finite of infinite.

  See on this e.g. Starks 1997 and Goetzmann et al 2001 (contract design for managers to incentive them to undertake risky investments).

- We take infinite horizon as it simplify the mathematical treatment.

Admissible strategies

The set of admissible strategies is

$$\Theta_{ad}(T,\eta) := \Big\{\theta : [l_T, +\infty) \times \Omega \longrightarrow [0,1] \text{ adapted to } \{\mathcal{F}_t^B\}_{t\geq T} \text{ s.t. } X(t;T,\eta,\theta) \in [l_T, +\infty),\ t \geq T\Big\}$$

This set is nonempty for every η such that $\eta_0 \geq l_T$ **if and only if** the null strategy is admissible.

In the case when $f \equiv 0$ this is equivalent to:

$$rl_T \geq A.$$

We will assume this from now on.

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The case with no surplus (i.e. no delay): regular solutions to HJB and feedback control strategies (Di Giacinto, Federico, G.)

THE VALUE FUNCTION

Here $f \equiv 0$ and the initial datum is only the present $\eta_0 =: x$.

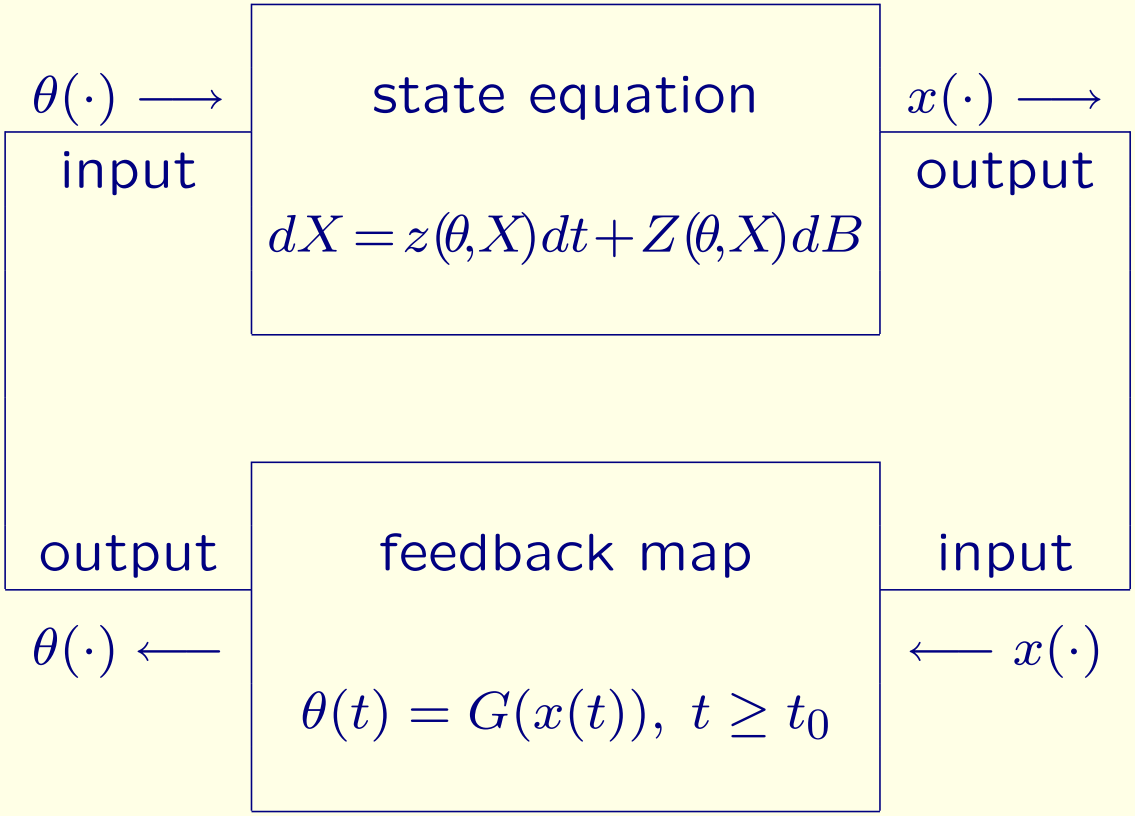
Since the problem is autonomous we define the value function independent of t:

$$V(x) := \sup_{\theta(\cdot) \in \Theta_{ad}(T,x)} J(T, x; \theta(\cdot)), \qquad x \geq l_T.$$

Dynamic Programming: main problems

A - Prove that V is a classical solution of the Hamilton-Jacobi-Bellman (HJB) equation.

B - Apply a verification theorem to get the optimal strategies in feedback form.



$\theta(\cdot) \longrightarrow$
input
state equation
$dX = z(\theta, X)dt + Z(\theta, X)dB$
$x(\cdot) \longrightarrow$
output
output
feedback map
$\theta(t) = G(x(t)),\ t \geq t_0$
input
$\theta(\cdot) \longleftarrow$
$\longleftarrow x(\cdot)$

The associated HJB equation is given by:

$$\rho v(x) - H\big(x, Dv(x), D^2 v(x)\big) = 0,\ \forall x \in [l_T, +\infty),$$

where

$$H\big(x, Dv(x), D^2 v(x)\big) := \sup_{\theta \in [0,1]} H_{cv}\big(x, Dv(x), D^2 v(x); \theta\big)$$

$$= \sup_{\theta \in [0,1]} \left\{ U(x) + \Big[(\theta\sigma\lambda + r)x - A \Big] Dv(x) + \frac{1}{2}\theta^2 \sigma^2 x^2 D^2 v(x) \right\}$$

$$= U(x) + (rx - A) Dv(x) + \sup_{\theta \in [0,1]} \left\{ \theta\sigma\lambda x Dv(x) + \frac{1}{2}\theta^2 \sigma^2 x^2 D^2 v(x) \right\}$$

It is similar to the equations for optimal portfolio studied in various papers (e.g. Zariphopolou, Duffie - Fleming - Soner - Zariphopolou, Choulli - Taksar - Zhou, Sethi - Taksar, etc.).

The main issue here is the **presence of the state constraint together with the degeneracy**.

PROPERTIES OF THE VALUE FUNCTION

We show that V is

- concave,
- strictly increasing,
- continuous on the interval $(l_T, +\infty)$ (also in l_T if it is finite in l_T).

Then, studying the HJB equation, we prove the

THEOREM 1

- V is the unique concave viscosity solution of the HJB equation,
- V belongs to $\mathrm{C}\big([l_T, +\infty); \mathbb{R}\big) \cap \mathrm{C}^2\big((l_T, +\infty); \mathbb{R}\big)$.

$\Longrightarrow$ *We can find optimal feedback control policies*

Remarks on the HJB equation.

- We use the concept of constrained viscosity solution (Soner, Katsoulakis): solution in the interior and subsolution up to the boundary. This provide the appropriate boundary conditions.

- The C^1 regularity is proven as in the paper of Choulli - Taksar - Zhou.

- The C^2 regularity is more difficult and we could not use the arguments of other papers. So we prove ad hoc estimates for the second derivative and get the regularity form them.

  The estimates are based on the idea that the optimal θ should be bounded away from 0 in the interior of the state region.

OPTIMAL FEEDBACK STRATEGIES

The candidate optimal feedback map in the interior of the state region is

$$G(x) := G_0\left(x, DV(x), DV^2(x)\right), \quad x > l_T,$$

where

$$\begin{aligned} G_0(x, DV(x), DV^2(x)) &= \arg\max_{\theta\in[0,1]} H_{cv}\big(x, DV(x), D^2V(x); \theta\big) \\ &= \min\left\{1, -\frac{\lambda}{\sigma}\frac{DV(x)}{xD^2V(x)}\right\} \end{aligned}$$

while at the boundary we must have $G(l_T) = 0$ (the only way to satisfy the constraint).

Problem: regularity of G up to the boundary.

Closed loop equation:

$$\begin{cases} dX(t) = \ [(r+\sigma\lambda G(X(t)))\,X(t) - A]\,dt + \sigma G(X(t))X(t)\,dB(t) \\ \\ x(T) = x, \end{cases}$$

To find a strong solution to it we need at least G to be continuous and 1/2 Hölder continuous up to the boundary.

The two cases $rl_T > A$ and $rl_T = A$ are structurally different:

- when $rl_T > A$ we expect to reach the boundary with positive probability and to leave it immediately applying the control zero;

- when $rl_T = A$ then l_T is an absorbing point and we expect that is never reached as in the standard portfolio problems.

Case $rl_T > A$ with $U(l_T), U'(l_T)$ both finite:

the boundary condition (V subsolution up to the boundary) implies

$$\lim_{x \longrightarrow l_T^+} V''(x) = -\infty \qquad \lim_{x \longrightarrow l_T^+} (x - l)[V''(x)]^2 = \frac{\lambda^2}{2} \frac{[V'(l_T^+)]^2}{rl_T - A}$$

This implies that G is 1/2 Hölder continuous up to the boundary.

THEOREM 2

Assume that $rl_T > A$ and $U(l_T), U'(l_T)$ be both finite.

Then there exists a **unique optimal strategy** given by the feedback map G above.

The proof is nontrivial since the boundary is reached and left and since V'' is $-\infty$ at the boundary $\longrightarrow$ approximation procedure.

EXPLICIT SOLUTION WHEN $rl_T = A$

Case $rl_T = A$: we consider an explicit example with power utility:

$$U(x) = \frac{(x - l_T)^\gamma}{\gamma}, \qquad \gamma \in (-\infty, 0) \cup (0, 1)$$

Given suitable constraints on the solvency level l, our HJB equation is solved by

$$V(x) = \frac{\left(x - \frac{A}{r}\right)^\gamma}{\gamma\left[\rho - \gamma\left(r + \frac{\lambda^2}{2(1-\gamma)}\right)\right]}, \qquad \rho - \gamma\left(r + \frac{\gamma\lambda^2}{2(1-\gamma)}\right) > 0$$

The optimal feedback map becomes:

$$G(x) = \min\left\{1, \frac{\lambda}{\sigma(1-\gamma)x}\left(x - \frac{A}{r}\right)\right\}.$$

and 0 is an absorbing boundary that is never reached.

EXPLICIT SOLUTION WHEN $rl_T > A$?

In the case $rl > A$ take again, for $x \geq l > \frac{A}{r}$

$$U(x) = \frac{\left(x - \dfrac{A}{r}\right)^{\gamma}}{\gamma}, \qquad \gamma \in (0,1)$$

In this case, given suitable constraints on l, our HJB equation is solved, for $x > l$, by the function:

$$W(x) = \frac{\left(x - \frac{A}{r}\right)^{\gamma}}{\gamma\left[\rho - \gamma\left(r + \frac{\lambda^2}{2(1-\gamma)}\right)\right]}, \qquad \rho - \gamma\left(r + \frac{\gamma\lambda^2}{2(1-\gamma)}\right) > 0$$

However this function **is not the value function** as it does not satisfy the boundary condition at $x = l$. We have $V < W$.

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The case with delay: some partial results on the infinite dimensional HJB

DELAY EQUATIONS

A delay equation is a differential equation in which the knowledge of the future depends also on the past of the state:

$$x'(t) = f\left(x(t), x(t+\xi)|_{\xi \in [-T,0]}\right).$$

In general for stating the evolution of the system such an equation requires as initial datum the knowledge of the whole past trajectory

$$x(\cdot)|_{[-T,0]}.$$

Thus the problem is basically infinite-dimensional.

DELAY EQUATIONS: A SPECIAL CASE

There are some special case for which the evolution of the system can be reduced to a finite dimensional system. For example:

$$x'(t) = f\left(x(t), \int_{-\infty}^{0} e^{\lambda\xi} x(t+\xi) d\xi\right), \quad \lambda \geq 0.$$

In this case the variable

$$y(t) := \int_{-\infty}^{0} e^{\lambda\xi} x(t+\xi) d\xi$$

is like a "sufficient statistics" for the system, which could be rewritten as a bi-dimensional system

$$\begin{cases} x'(t) = f\left(x(t), y(t)\right), \\ y'(t) = -\lambda y(t) + x(t). \end{cases}$$

See e.g. papers of Elsanosi, Larssen, Risebro, Oksendal,... where this is exploited in various control problems.

DELAY EQUATIONS: THE INFINITE DIMENSIONAL REPRESENTATION

A classical approach to treat the delay equations, which applies quite in general, consists in rewriting them as evolution equations in a suitable Hilbert spaces.
The idea behind is to consider as state not only the present, but also the past, i.e. to define a new state variable representing the present and the past of the old state variable.

Formally in $H = \mathbb{R} \times L^2([-T, 0]; \mathbb{R})$:

$$X'(t) = AX(t) + F(X(t)),$$

where

$$X(\cdot) := (X_0(\cdot), X_1(\cdot)) = \Big(x(\cdot), x(\cdot + \xi)|_{\xi \in [-T,0]}\Big),$$

A is a first order operator and F "translate" f in the infinite dimensional setting.

PENSION FUNDS: THE STATE EQUATION

In a general pension fund model the state equation has to take in account two types of added cashflows:

- Contributions paid by the members who are adhering to the fund;
- Benefits which the fund has to pay to the members who have accrued the right to the pension and are leaving the fund.

$$\begin{aligned} dx(t) &= \left[\left(r + \sigma\lambda\theta(t) \right) x(t) \right] dt + \sigma\theta(t)x(t)dB(t) \\ &\quad + [\underbrace{c(t)}_{\text{Contributions}} - \underbrace{b(t)}_{\text{Benefits}}]\, dt. \end{aligned}$$

THE STATE EQUATION IN A PENSION FUND MODEL WITH SURPLUS

In our model the state equation is

$$\begin{cases} dx(t) \;=\; [(r+\sigma\lambda\theta(t))\,x(t)]\,dt+\sigma\theta(t)x(t)dB(t) \\ \qquad\qquad\qquad -f_0\,(x(t)-x(t-T))\,dt, \\ x(0)=\eta_0,\ x(\zeta)=\eta_1(\zeta),\quad \zeta\in[-T,0), \end{cases} \tag{1}$$

- $(\eta_0, \eta(\cdot))$ is the initial (functional) datum;
- f_0 is a suitable function containing the surplus term.

This is a stochastic delay differential equation and it is treated by the infinite-dimensional approach.
(e.g. Vinter & Kwong, 1981; Da Prato & Zabczyk, 1996; Gozzi & Marinelli, 2006)

CONSTRAINTS

We impose the following constraints for the variables:

- **Control constraint:** $\theta(\cdot)$ is a $[0,1]$-valued adapted process;

- **State Constraint:** $x(t) \geq l \geq 0$ (solvency level), for each $t \geq 0$.

THE OPTIMIZATION PROBLEM

We want to maximize the functional

$$E\left[\int_0^{+\infty} e^{-\rho t} U(x(t))\, dt\right],$$

over the set of the admissible strategies.

- $\rho > 0$ is the discount rate;
- $U : [l, +\infty) \longrightarrow \mathbb{R}$ is continuous, increasing and concave.

THE (FORMALLY) EQUIVALENT INFINITE DIMENSIONAL PROBLEM

We pass from the SDDE to an infinite dimensional SDE. We define the Hilbert space

$$H = \mathbb{R} \times L^2([-T,0];\mathbb{R}), \quad \langle \eta, \zeta \rangle = \eta_0 \zeta_0 + \int_{-T}^{0} \eta_1(\xi)\zeta_1(\xi) d\xi,$$

and the infinite dimensional SDE

$$\begin{cases} dX(t) = AX(t)dt + \sigma\lambda\theta(t)\Phi X(t)dt - F(X(t))dt + \sigma\theta(t)\Phi X(t)dB(t), \\ X(0) = \eta \in E, \end{cases} \tag{2}$$

where E is a suitable subset of H.

In the previous equation:

- $A : D(A) \subset H \rightarrow H$ is the closed unbounded defined by

$$(\eta_0, \eta_1(\cdot)) \mapsto (r\eta_0, \eta_1'(\cdot)),$$

  with

$$D(A) = \{(\eta_0, \eta_1(\cdot)) \in H \mid \eta_1(\cdot) \in W^{1,2}([-T,0];\mathbb{R}),\ \eta_0 = \eta_1(0)\};$$

  A is the generator of a C_0-semigroup $S(\cdot)$ on H.

- $F : E \rightarrow H$ is the nonlinear map

$$\begin{pmatrix} \eta_0 \\ \eta_1(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} f(\eta_0, \eta_1(\cdot)) \\ 0 \end{pmatrix} := \begin{pmatrix} f_0(\eta_0 - \eta_1(-T)) \\ 0 \end{pmatrix}.$$

- $\Phi : H \rightarrow H$ is the bounded linear operator defined by

$$(\eta_0, \eta_1(\cdot)) \mapsto (\eta_0, 0).$$

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Unfortunately the infinite dimensional equation obtained above is non standard.

$\Longrightarrow$ we had to prove everything almost from scratch.

First of all the well posedness of the state equation:

SPACE OF SOLUTIONS

We have to give sense to the term $F(X(\cdot))$ in the equation. Therefore we choose as space on which look for a solution the space

$$C_{\mathcal{P}}\big(\,[0,+\infty);\; L^2(\Omega;E)\,\big),$$

where E is the Banach space

$$E = \Big\{(\eta_0, \eta_1(\cdot)) \in H \;\Big|\; \eta_1 \in C([-T,0];\mathbb{R}),\;\; \eta_0 = \eta_1(0)\Big\}.$$

A mild solution for the SDE (2) is a process $X \in C_{\mathcal{P}}\left(\,[0,+\infty);\; L^2(\Omega;E)\,\right)$ which satisfies, for $t \geq 0$, the integral equation

$$\begin{aligned} X(t) \;&=\; S(t)x + \int_0^t \sigma\lambda\theta(\tau)S(t)\left[\Phi X(\tau)\right] d\tau - \int_0^t S(t)F(X(\tau))d\tau \\ &\quad + \int_0^t \sigma\theta(\tau)S(t)\left[\Phi X(\tau)\right] dB(\tau). \end{aligned}$$

THE EQUIVALENCE BETWEEN THE PROBLEMS

The equation (2) is not covered by the classical literature. Therefore S. Federico (2008, submitted) proved:

Theorem 1 *For each $\eta \in E$, the equation (2) admits a unique mild solution.*

To give sense to the infinite-dimensioal approach one has to prove an equivalence result:

Proposition 1 *[S. Federico] Let $x(\cdot)$ be the unique solution of the one-dimensional SDDE (1) and let $X(\cdot)$ be the unique mild solution of the infinite-dimensional SDE (2). Then*

$$X(t) = (x(t), x(t+\zeta)|_{\zeta \in [-T,0)}).$$

PROPERTIES OF THE VALUE FUNCTION

- The time dependence of the value function is
$$V(t,\eta) = e^{-\rho t} V(0,\eta).$$
Thus the problem reduces to study $V_0(\eta) := V(0,\eta)$.

- The value function V_0 is concave.

- Under good financial assumptions, the $(E, \|\cdot\|_H)$-interior part $\mathcal{V}$ of the effective domain of the value function V_0 is not empty. In particular it contains the points with financial meaning.

Proposition 2 (S. Federico) *The value function V_0 is $\|\cdot\|_H$-continuous on $\mathcal{V}$. Moreover, if some condition on the paramaters of the model are satisfied, then V_0 is continuous up to the boundary.*

THE HJB EQUATION

The Hamilton-Jacobi-Bellman equation associated with the value function V_0 in the space H is

$$\rho v(\eta) = \langle \eta, A^* \nabla v(\eta) \rangle + U(\eta_0) - f(\eta) v_{\eta_0}(\eta) + \mathcal{H}\big(\eta_0, v_{\eta_0}(\eta), v_{\eta_0 \eta_0}(\eta)\big),$$

where, for $p_0, q_0 \in \mathbb{R}$,

$$\mathcal{H}(\eta_0, p_0, q_0) := \sup_{\theta \in [0,1]} \left(\frac{1}{2} \sigma^2 \, \eta_0^2 \; q_0 \; \theta^2 + \sigma \, \lambda \, \eta_0 \, p_0 \; \theta \right).$$

State of the art

- Strong solution approach: initiated first by Barbu and Da Prato (1981) and then developped by various authors (Cannarsa - Da Prato, G., Goldys - Maslowski, G. - Rouy, Cerrai, Da Prato - Debussche, G. - Goldys, Chow - Menaldi, etc).
  - Uses regularisation properties of the Ornstein Uhlenbeck transition semigroup associated to the uncontrolled problem.
  - Finds regular solutions (at least C^1 or W^1 in space) so the optimal sinthesis is "possible".

Not applicable here since

– No regularizing properties of O-U semigroup (since A is of first order and the equation is fully nonlinear).

– Without these difficulties G. - Goldys (SPA 07) works but needs no delay in the control.

- Backward SDE approach: series of papers by Fuhrman, Tessitore, Masiero, etc.

  - Represents the solution of HJB using a suitable forward - backward system and finds regular solutions.

  Not applicable here since

  - it needs semilinear HJB equations

- Viscosity solutions approach: initiated by Crandall and Lions in '80 for finite dimensional PDE's. Infinite dimensional second order HJB first studied by Lions '88 and then by various authors (Swiech, G., Rouy, Sritharan, Kelome, etc).

  - More general theory of existence and uniqueness.

  - No regularity results (the solutions are continuous but no more: no space derivatives so sinthesis is much more complicated).

  Not devloped for our case but seems applicable here.

THE HJB EQUATION: SPECIFIC FEATURES

- It is a fully nonlinear equation.
- It is defined on the points of E, due to the presence of f.
- The linear term is unbounded.
- The term $f(\cdot)$ is not continuous with respect to $\|\cdot\|_H$.
- The nonlinear term involves only the derivatives with respect to the real component.

THE VALUE FUNCTION AS VISCOSITY SOLUTION OF THE HJB EQUATION

Theorem 2 (S. Federico) *The value function is a viscosity solution of the equation HJB on $\mathcal{V}$. Moreover, if it is continuous up to the boundary, then it is a viscocosity subsolution also at the boundary.*

- The subsolution viscosity property of the value function at the boundary plays the role of a boundary condition.

- When this happens in a finite-dimensional framework, the value function is said a constrained viscosity solution of the HJB equation.

- In the finite-dimensional framework very often this boundary condition is strong enough to guarantee a uniqueness result for the solution.

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Further research and work in progress

NATURAL FUTURE TARGETS FOR THE INFINITE DIMENSIONAL HJB

- Proving a uniqueness result for viscosity solutions which would give a full characterization for the value function.

- Proving the existence of the directional (along the "present" component) first and second derivatives for the value function.

- Proving a verification theorem in order to be able to find optimal feedback control strategies for the problem.

All these are very difficult: we then started to look at simpler problems to extend the existing theory. (Federico, Goldys, Gozzi work in progress).

OPTIMAL CONSUMPTION WITH DELAY IN THE STATE

State equation:

$$\begin{cases} x'(t) = rx(t) + f_0\left(x(t),\ \int_{-T}^{0} a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) = \eta_0,\ x(s) = \eta_1(s),\ s \in [-T, 0). \end{cases}$$

- a is a weighting function satisfying $a \in W^{1,2}([-T,0];\mathbb{R})$, $a(\cdot) > 0$ and $a(-T) = 0$;
- $f_0 : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is concave, Lipschitz, increasing on both the variables and $f_0(0,0) \geq 0$.
- state constraint: $x(\cdot) > 0$;
- control constraint: $c(\cdot) \geq 0$.

On the delay term:

We can imagine this kind of contract: the bank provides for the customer an interest spot rate r smaller than the market spot rate r^M; nevertheless, as a compensation, it provides a premium on the past of the wealth. For example we could have the following dynamics:

$$\begin{cases} x'(t) = rx(t) + g\left(\int_{-T}^{0} a(\xi)x(t+\xi)d\xi\right) - c(t), \\ x(0) = \eta_0, \ x(s) = \eta_1(s), \ s \in [-T, 0), \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$. Such a contract should incentive the customer to keep for longer periods his money within the bank account in order to perform the benefits coming by the term g_0.

Problem:

Maximize over the set of the admissible strategies $c(\cdot)$

$$\int_0^{+\infty} e^{-\rho t}\big[U_1(c(t)) + U_2(x(t))\big]\, dt.$$

- $\rho > 0$.
- $U_1,\ U_2$ are utility functions satisfying suitable conditions. (For many results it can be $U_2 \equiv 0$.)

THE INFINITE-DIMENSIONAL REPRESENTATION

We pass from the one-dimensional DDE to an infinite-dimensional DE (without delay): we define the Hilbert space

$$H = \mathbb{R} \times L^2([-T, 0]; \mathbb{R}).$$

The new state variable in this space is

$$X(t) = (X_0(t), X_1(t)) \in H.$$

Formally we want

$$X_0(t) = x(t); \quad X_1(t)(\xi) = x(t+\xi), \text{ for a.e. } \xi \in [-T, 0].$$

Define:

- the closed unbounded operator

$$A : D(A) \subset H \to H,$$

where

$$D(A) = \{(\eta_0, \eta_1(\cdot)) \in H \mid \eta_1(\cdot) \in W^{1,2}([-T,0];\mathbb{R}),\ \eta_0 = \eta_1(0)\};$$

and

$$D(A) \ni (\eta_0, \eta_1(\cdot)) \mapsto (r\eta_0, \eta_1'(\cdot)).$$

A is the generator of a C_0-semigroup $S(\cdot)$ on H.

- the nonlinear map $F : H \to H$, by

$$\begin{pmatrix} \eta_0 \\ \eta_1(\cdot) \end{pmatrix} \mapsto \begin{pmatrix} f(\eta_0, \eta_1(\cdot)) \\ 0 \end{pmatrix} := \begin{pmatrix} f_0\left(\eta_0, \int_{-T}^{0} a(\xi)\eta_1(\xi)d\xi\right) \\ 0 \end{pmatrix}.$$

Define the infinite-dimensional DE in the space H

$$\begin{cases} X'(t) = AX(t) + F(X(t)) - c(t)\hat{n}, \\ X(0) = \eta = (\eta_0, \eta_1(\cdot)) \in H, \end{cases}$$

where $\hat{n} = (1, 0) \in H$.

<u>The role of A:</u>

$$A(X_0(t), X_1(t)) = (rX_0(t), X_1(t)'(\cdot)).$$

On the first component A gives the linear evolution of the present; On the second component A moves the past as a shift.

<u>The role of the boundary condition in $D(A)$:</u>

$$D(A) = \{(\eta_0, \eta_1(\cdot)) \in H \mid \eta_1(\cdot) \in W^{1,2}([-T, 0]; \mathbb{R}),\ \eta_0 = \eta_1(0)\}.$$

This boundary condition forces the past to follow the present, i.e. the last point of the past has to follow the same evolution of the present.

MILD SOLUTIONS AND EQUIVALENCE

Proposition 3 *For any* $\eta \in H$ *and* $c(\cdot) \in L^1_{loc}([0, +\infty); \mathbb{R})$, *the equation*

$$\begin{cases} X'(t) = AX(t) + F(X(t)) - c(t)\hat{n}, \\ X(0) = \eta = (\eta_0, \eta_1(\cdot)) \in H, \end{cases}$$

admits a unique mild solution $X(\cdot)$, *i.e.*

$$X(t) = S(t)\eta + \int_0^t S(t-\tau)F(X(\tau))d\tau + \int_0^t c(\tau)S(t-\tau)\hat{n}\, d\tau.$$

Moreover

$$X(t) = \left(X_0(t), X_1(t)(\xi)|_{\xi \in [-T,0]} \right) = \left(x(t); x(t+\xi)|_{\xi \in [-T,0]} \right),$$

where $x(\cdot)$ *is the unique solution of the one-dimensional delay equation.*

HJB EQUATION

Formally the HJB equation for the problem is

$$\rho v(\eta) = \langle A\eta, \nabla v(\eta)\rangle + f(\eta) v_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(v_{\eta_0}(\eta));$$

this requires in particular $\eta \in D(A)$.

In order to allow $\eta \in H$ we rewrite it as

$$\rho v(\eta) = \langle \eta, A^* \nabla v(\eta)\rangle + f(\eta) v_{\eta_0}(\eta) + U_2(\eta_0) + \mathcal{H}(v_{\eta_0}(\eta)),$$

where $\mathcal{H}$ is the Legendre transform of U_1, i.e.

$$\mathcal{H}(p) := \sup_{c \geq 0} \left(U_1(c) - cp \right), \quad p > 0.$$

THE HJB EQUATION IN GENERAL: WHICH KIND OF SOLUTIONS?

Some considerations:

- If the value function is smooth, then it solves the HJB equation. However this argument is in general only formal. Indeed in general the value function is not smooth.

- Even if the value function is smooth, it is difficult to prove a priori regularity results for the value function going beyond the continuity.
- The usual theory of classical or generalized solutions for PDE does not adapt to PDE of HJB type in general.

- The best concept of solution in the context of HJB equations seems to be the concept of **viscosity solution**, which does not require regularity (classical or generalized) for the definition of solution. It was developed in the early '80s by Crandall and Lions.

Bad case (viscosity approach): We do not know whether there exists a classical solution of the HJB equation or not. Then:

- We can prove that the value function is a viscosity solution (possibly unique) of the HJB equation (this is quite standard).

- We can try to prove, by using this viscosity property, that the value function is indeed smooth and so it is a classical solution.

  - This is a **regularity result**, as well as when in the classical PDE's theory it is proved that a generalized solution is indeed a classical solution.

  - Such a result does not hold in general, because, as said, the value function is not smooth in general.

- We can use the fact that the value function is a classical solution of the HJB equation to prove a verification theorem giving an optimal strategy for the problem.

BACK TO THE PROBLEM: RESULTS

- Concavity:

  The value function is concave. The domain $\mathcal{D}(V)$, i.e. the set where $V > -\infty$, is an open set of H with respect to the norm

  $$\|\eta\|_{A^{-1}} := \|A^{-1}\eta\|.$$

- Continuity:

  The value function is continuous on $\mathcal{D}(V)$ with respect to $\|\cdot\|_{A^{-1}}$.

- Properties of superdifferentials:

  Some properties for superdifferentials of concave and $\|\cdot\|_{A^{-1}}$-continuous functions are proved. These properties are very important to prove the regularity result.

- Viscosity:

  The value function is a viscosity solution of HJB in the following sense:

  Definition 1 *Define the set of test functions*

  $$\tau := \left\{ \varphi \in C^1(H) \;\middle|\; \nabla\varphi(\cdot) \in \mathcal{D}(A^*), \eta_n \to \eta \Rightarrow A^*\nabla\varphi(\eta_n) \rightharpoonup A^*\nabla\varphi(\eta) \right\}.$$

  A continuous function $v : \mathcal{D}(V) \to \mathbb{R}$ is called a viscosity subsolution of HJB on $\mathcal{D}(V)$ if, for any $\varphi \in \tau$ and any $\eta_M \in \mathcal{D}(V)$ such that $v - \varphi$ has a local maximum at η_M, we have

  $$\rho v(\eta_M) \leq \langle \eta_M, A^*\nabla\varphi(\eta_M)\rangle + f(\eta_M)\varphi_{\eta_0}(\eta_M) + U_2(\eta_0) + \mathcal{H}(\varphi_{\eta_0}(\eta_M)).$$

  Analogous definition for viscosity supersolution.

- Regularity **(main result)**:

  The value function is continuously differentiable on $\mathcal{D}(V)$ along the "present" direction, i.e.

$$\exists\ V_{\eta_0}(\eta), \qquad \forall \eta \in \mathcal{D}(V),$$

and

$$\eta \longmapsto V_{\eta_0}(\eta)$$

is continuous.

- <u>On the (formal) optimal feedback strategy:</u>

  Thanks to the regularity result we can write the feedback map, defined by

  $$C(\eta) := \mathrm{argmax}_{c \geq 0} \left(U_1(c) - cV_{\eta_0}(\eta) \right), \quad \eta \in \mathcal{D}(V).$$

  If U_2 is not integrable at 0^+, then the formal optimal feedback strategy exists (in the sense that the closed loop equation associated with the feedbak map C has a global solution) and is admissible.

OTHER FUTURE TARGETS

Research project also with

Elena Vigna (University of Torino, Italy)

Martino Grasselli (University of Verona, Italy)

- To take stochastic interest rates
- To release the hypotheses of demographic stationarity
- To introduce a stochastic wage
- To analyze the decumulation phase

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THIS IS THE END, THANKS

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A stochastic advertising model with delay

Monopolistic firm preparing the market introduction of a new product at some time T in the future.

Nerlove-Arrow (1962) framework: the state is the "goodwill stock" $y(t)$, $0 \leq t \leq T$.

The control is the rate of advertising spending $z(t)$.

The state equation is linear and allows for delay effects both in the state and in the control.

Literature: Buratto, Grosset, Viscolani, Marinelli, etc.:

stochastic problems with no delay.

State equation

$$
\begin{cases}
dy(t) = \left[a_0 y(t) + \int_{-r}^{0} a_1(\xi) y(t+\xi)\, d\xi + b_0 z(t) + \int_{-r}^{0} b_1(\xi) z(t+\xi)\, d\xi\right] dt \\
\qquad\qquad +\sigma\, dW_0(t), \quad s \le t \le T \le +\infty \\
y(s) = x_0; \quad y(s+\xi) = x_1(\xi),\ z(s+\xi) = \delta(\xi),\ \ \xi \in [-r, 0],
\end{cases}
\tag{3}
$$

where:

- the Brownian motion W_0 is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \ge 0}, \mathbb{P})$, with $\mathbb{F}$ being the completion of the filtration generated by W_0. The volatility $\sigma > 0$ is constant.

- the advertising spending rate $z(t)$ is constrained to remain in the set $\mathcal{U} := L^2_{\mathbb{F}}([0, T], U)$, the space of square integrable processes adapted to $\mathbb{F}$ taking values in a closed convex set $U \subseteq \mathbb{R}_+$, such as $U = [0, R]$, with R a positive constant, finite or infinite.

- a_0 and $a_1(\cdot)$ describe the process of goodwill deterioration when the advertising stops,

- b_0 and $b_1(\cdot)$ provide the characterization of the effect of the current and the past advertising rates on the goodwill level.

- the values of x_0, $x_1(\cdot)$ and $\delta(\cdot)$ reflect the “initial” goodwill and advertising trajectories.

Note that we recover the model of Nerlove and Arrow (1962) from (3) in the deterministic setting ($\sigma = 0$) in the absence of delay effects ($a_1(\cdot) = b_1(\cdot) = 0$).

In addition, we assume that the following conditions hold:

(i) $a_0 \leq 0$;

(ii) $a_1(\cdot) \in L^2([-r, 0], \mathbb{R})$;

(iii) $b_0 \geq 0$;

(iv) $b_1(\cdot) \in L^2([-r, 0], \mathbb{R}_+)$;

(v) $x_0 \geq 0$;

(vi) $x_1(\cdot) \geq 0$, with $x_1(0) = x_0$;

(vii) $\delta(\cdot) \geq 0$.

Setting $x := (x_0, x_1(\cdot))$ and denoting by $y^{s,x,z}(t)$, $t \in [0,T]$, "the" solution of (3), we define the objective functional

$$J(s,x;z) = \left[\varphi_0(y^{s,x,z}(T)) - \int_s^T h_0(z(t))\,dt\right], \tag{4}$$

where $\varphi_0 : \mathbb{R} \longrightarrow \mathbb{R}$ and $h_0 : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are measurable utility and cost functions, respectively, satisfying the growth condition

$$|f(x)| \leq K(1+|x|)^m,\ K > 0,\ m \geq 0, \tag{5}$$

for $f = \varphi_0,\ h_0$. If $T = +\infty$ we remove the final reward φ_0 and put it inside the integral with a discount factor

$$J(s,x;z) = \left[\int_s^{+\infty} e^{-\rho t}[\varphi_0(y^{s,x,z}(t)) - h_0(z(t))]\,dt\right].$$

Let us also define the value function V for this problem as follows:

$$V(s,x) = \sup_{z \in \mathcal{U}} J(s,x;z).$$

We shall say that $z^* \in \mathcal{U}$ is an optimal strategy if it is such that

$$V(s,x) = J(s,x;z^*).$$

Problem: maximization of the objective functional J over all admissible strategies

$$\mathcal{U} = L^2_{\mathbb{F}}([0,T],U).$$

.

An equivalent infinite dimensional setting

(for the advertising problem)

The state space is infinite dimensional

- In the previously quoted literature (e.g. the case when a_1 and b_1 are 0) the state of the system at time t is described only by a (real) number: the stock of goodwill at time t: the state space is one dimensional.

- In models with delay (to have a Markovian state equation) it is useful to consider a bigger state. In this case the state at time t is given by the history of the goodwill in the whole period $[t-r,t]$. So the system is described by a function

$$[-r,0] \longrightarrow \mathbb{R}$$

which is not a finite dimensional vector.

The choice of the state space

- There is not a unique possible choice of the infinite dimensional state space and state variable (see e.g. Ichikawa (1982), Vinter - Kwong (1981), Bensoussan - Da Prato - Delfour - Mitter, (2006)).

- Here we choose (following Vinter - Kwong ('81)) of the Hilbert-state space
$$X := \mathbb{R} \times L^2(-r, 0).$$

- We call the state $x(t) \in X$.

The new state equation

Let us define an operator $A : D(A) \subset X \longrightarrow X$ as follows:

$$A : (x_0, x_1(\cdot)) \;\mapsto\; \Big(a_0 x_0 + x_1(0), a_1(\cdot)x_0 - x_1'\Big) \quad \text{a.e.} \in [-r, 0],$$

$$D(A) \;=\; \Big\{x \in X : x_1 \in W^{1,2}([-r,0];\mathbb{R}),\; x_1(-r) = 0\Big\}.$$

Moreover, setting $U := \mathbb{R}_+$, we define the bounded linear control operator $B : U \longrightarrow X$ as

$$B : u \mapsto \Big(b_0 u, b_1(\cdot)u\Big), \tag{6}$$

and finally the operator $G : \mathbb{R} \longrightarrow X$ as $G : x_0 \mapsto (\sigma x_0, 0)$.

Note that $b_1 \neq 0$ implies that $ImB \not\subset ImG$.

The new state equation is then the abstract evolution equation

$$\begin{cases} dY(t) = (AY(t) + Bz(t))\,dt + G\,dW_0(t) \\ Y(s) = \bar{x} \in X, \end{cases} \tag{7}$$

with arbitrary initial datum $\bar{x} \in X$ and control $z \in \mathcal{U}$.

(Controlled Ornstein Uhlenbeck process)

We have the following equivalence result (see Gozzi - Marinelli, 2006)

Proposition. For $t \geq r$, one has, $\mathbb{P}$-a.s.,

$$Y(t) = M(Y_0(t), Y_0(t+\cdot), z(t+\cdot)),$$

where

$$\begin{aligned} M : X \times L^2([-r,0],\mathbb{R}) & \longrightarrow X \\ (x_0, x_1(\cdot), v(\cdot)) & \mapsto (x_0, m(\cdot)), \end{aligned}$$

$$m(\xi) := \int_{-r}^{\xi} a_1(\zeta) x_1(\zeta - \xi)\, d\zeta + \int_{-r}^{\xi} b_1(\zeta) v(\zeta - \xi)\, d\zeta.$$

Moreover, let $\{y(t),\ t \geq -r\}$ be a continuous solution of the stochastic delay differential equation (3), and $Y(\cdot)$ be the weak solution of the abstract evolution equation (7) with initial condition

$$\bar{x} = M(x_0, x_1, \delta(\cdot)).$$

Then, for $t \geq 0$, one has, $\mathbb{P}$-a.s.,

$$Y(t) = M(y(t), y(t+\cdot), z(t+\cdot)),$$

hence $y(t) = Y_0(t)$, $\mathbb{P}$-a.s., for all $t \geq 0$.

Using this equivalence result, we can now give a Markovian reformulation on the Hilbert space X of the problem of maximizing (4). In particular, denoting by $Y^{s,\bar{x},z}(\cdot)$ a mild solution of (7), (4) is equivalent to

$$J(s,x;z) = \left[\varphi(Y^{s,\bar{x},z}(T)) + \int_s^T h(z(t))\,dt\right], \tag{8}$$

with the functions $h : U \longrightarrow \mathbb{R}$ and $\varphi : X \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} h(z) &= -h_0(z) \\ \varphi(x_0, x_1) &= \varphi_0(x_0). \end{aligned}$$

Hence also $V(s,x) = \sup_{z \in \mathcal{U}} J(s,x;z)$.

The same for the infinite horizon problem.

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The Dynamic Programming (DP) and the Hamilton-Jacobi-Bellman (HJB) equation

We consider now the infinite dimensional problem ($0 \leq s \leq t \leq T$).

State equation:

$$\begin{cases} dY(t) = (AY(t) + Bz(t))\, dt + G\, dW_0(t) \\ Y(s) = x \in X, \end{cases} \tag{9}$$

Objective functional (to maximize):

$$J(s, x; z) = \left[\varphi(Y(T)) + \int_s^T h(z(t))\, dt \right], \tag{10}$$

and similarly for $T = +\infty$.

Value function:

$$V(s, x) = \sup_{z \in \mathcal{U}} J(s, x; z)$$

.

We apply the DP approach.

A (naive) scheme of the DP approach is the following

DP-1 Write an equation for the value function: the so-called Dynamic Programming Principle and its infinitesimal version, the Hamilton-Jacobi-Bellman (HJB) equation:

(DPP): for every $(s,x) \in [0,T] \times X$ and $t_1 \in (s,T)$

$$V(s,x) = \sup_{z \in \mathcal{U}} \left[\int_s^T h(z(t))\, dt + V(t_1, Y(t_1)) \right],$$

This is a standard result but the proof is nontrivial (see e.g. Fleming - Soner (2005), Yong - Zhou (1999) in finite dimension; in the infinite dimensional case e.g. Lions, G. -Swiech - Sritharan)

(HJB): for every $(t,x) \in [0,T] \times D(A)$

$$\begin{cases} v_t(t,x) + \frac{1}{2} Tr(GG^* v_{xx}(t,x)) + \langle Ax, v_x(t,x) \rangle + H_0(v_x(t,x)) = 0, \\ v(T,x) = \varphi(x), \end{cases} \tag{11}$$

where $H_0(p) = \sup_{z \in U} (\langle Bz, p \rangle + h(z))$ is the so-called Hamiltonian.

DP-2 Find a solution of the HJB equation and prove that it is the value function. If not possible prove weaker results on the HJB equation (existence, uniqueness, regularity, etc.);

DP-3 (Verification Theorem). Prove that an optimal feedback formula (i.e. a formula expressing the optimal control as function of the optimal state) is given by

$$z^*(t) = F(v_x(t, Y^*(t))) \tag{12}$$

where F is the function giving the arg max of the Hamiltonian H_0.

DP-4 Plug such feedback formula into the state equation (obtaining the so-called Closed Loop Equation) to find the optimal trajectories of the state and of the control.

$$\begin{cases} dY(t) = (AY(t) + BF(v_x(t, Y(t))))\, dt + G\, dW_0(t) \\ Y(s) = x \in X, \end{cases} \tag{13}$$

The main issue is the study of the HJB equation.

.

The Main features of the HJB equation

State of the art

- Strong solution approach: initiated first by Barbu and Da Prato (1981) and then developped by various authors (Cannarsa - Da Prato, G., Goldys - Maslowski, G. - Rouy, Cerrai, Da Prato - Debussche, G. - Goldys, Chow - Menaldi, etc).
  - Uses regularisation properties of the Ornstein Uhlenbeck transition semigroup associated to the uncontrolled problem.
  - Finds regular solutions (at least C^1 or W^1 in space) so the optimal sinthesis is "possible".

Not applicable here since

- No regularizing properties of O-U semigroup (since A is of first order and G degenerate).
- Without this G. - Goldys still works but needs $ImB \subset ImG$ which is not true here due to the presence of the delay in the control ("carryover" effect).

- Backward SDE approach: series of papers by Fuhrman, Tessitore, Masiero, etc.

  - Represents the solution of HJB using a suitable forward - backward system and finds regular solutions (see the talk of Fuhrman for more on this).

  Not applicable here since

  - it needs $ImB \subset ImG$ which is not true here.

- Viscosity solutions approach: initiated by Crandall and Lions in '80 for finite dimensional PDE's. Infinite dimensional second order HJB first studied by Lions '88 and then by various authors (Swiech, G., Rouy, Sritharan, Kelome, etc).
  - More general theory of existence and uniqueness.
  - No regularity results (the solutions are continuous but no more: no space derivatives so sinthesis is much more complicated).

  Not devloped for our case but seems applicable here.

(And we did not consider the state constraints!!!!!)

.

Some results on existence, regularity and optimal synthesis, (under construction)

We use the viscosity solution approach.

Let us now consider the following Bellman equation on X

$$\rho v + \frac{1}{2}Tr(GG^* v_{xx}) + \langle Ax, v_x\rangle + H_0(v_x) = 0, \quad x \in X, \tag{14}$$

where $H_0(p) = \sup_{z\in U}(\langle Bz, p\rangle + h(z))$.

We introduce the following two classes of **test functions**:

Definition 2 *(i) We call $\mathcal{T}_1$ the set of functions $\psi \in C^2(X)$ such that, $\psi_x(x) \in D(A^*)$ for any $x \in X$ and $\psi, \psi_x, A^*\psi_x, \psi_{xx}$ are uniformly continuous.*

(ii) We call $\mathcal{T}_2$ the set of functions $g \in C_b^2(X)$ which are of the form

$$g(t, x) = g_0(\|x\|), \quad g_0 \in C^2([0, +\infty); \mathbb{R}), \quad g_0' \geq 0,$$

and g, g_x, g_{xx} are uniformly continuous.

Next we give the following definition of **viscosity solution**:

Definition 3 *(i) A continuous function $v : X \longrightarrow \mathbb{R}$ is called a* **viscosity subsolution (supersolution)** *of the HJB equation (12) on X if, for any triple $(x_M, \psi, g) \in X \times \mathcal{T}_1 \times \mathcal{T}_2$ such that x_M is a local maximum (minimum) point of $v - \psi - g$, we have*

$$\begin{aligned} \rho v(x_M) \;\; + \;\; & \frac{1}{2} Tr\left(GG^* \psi_{xx}(x_M) + g_{xx}(x_M)\right) + \langle x, A^* \psi_x(x_M) \rangle \\ & + H_0(\psi_{x_0}(x_M) + g_{x_0}(x_M)) \le (\ge) 0 \end{aligned}$$

(ii) v called a **viscosity solution** *of the HJB equation (9) if it is both a viscosity subsolution and a viscosity supersolution.*

- Existence of viscosity solutions.

  Theorem The value function is a viscosity solution of the HJB equation (9).

  This is ok also in the pension fund problem (see S. Federico 2008)

- Uniqueness of viscosity solution.

  Work in progress. The definition used to prove existence seems "compatible" with a uniqueness theorem.

- Regularity of viscosity solutions.

  First we remark that the feedback formula (10) in our case contains only the derivative of the value function with respect to the first (real) component of the state.

  So to write (10) we need only differentiability of V in the first component.

  To prove this we extend to the infinite dimensional case a method developped in finite dimension for the case when V is semiconcave (here is concave) and H_0 is strictly convex (see e.g. Bardi - Capuzzo Dolcetta).

  Ok for some deterministic case, work in progress for the advertising case.

- Verification Theorem and sinthesis Under construction.

  Idea:

  approximate the solution with classical solutions of approximating equations (see e.g. G., '94) and use Ito formula

  or

  use weak version of Ito formula (see e.g. G.- Russo, 2006).

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THIS IS THE END, THANKS

Let us now consider the following Bellman equation on X

$$\rho v + \frac{1}{2} Tr(GG^* v_{xx}) + \langle Ax, v_x \rangle + H_0(v_x) = 0, \quad x \in X, \tag{15}$$

The main problem with (15) is that it is not solvable with any of the techniques currently available.

In particular, as of now, one cannot characterize the value function as the (unique) solution, in a suitable sense, of equation (15). This is work in progress using the theory of viscosity solutions.

Nevertheless, if we know a priori that a smooth solution exists, then we can apply a verification theorem (proved Gozzi - Marinelli).

This will be done in a special case, for which there exists a smooth solution in closed form, and hence we can fully characterize the optimal strategy.

DP-1

Like in the one dimensional model we write the HJB related to the optimal control problem:

$$\rho v(x) = \sup_{i \in [0, Ax^0]} \mathcal{H}_{CV}\left(x, Dv(x); i\right)$$

$$= \sup_{i \in [0, Ax^0]} \Big\{ \langle (x^0, x^1), G DV(x^0, x^1) \rangle_{M^2} + \\ + \langle i, \delta_0 (DV(x^0, x^1))^1 - \delta_{-T} (DV(x^0, x^1))^1 \rangle_{\mathbb{R}} + \\ + \frac{(Ax^0 - i)^{1-\sigma}}{(1-\sigma)} \Big\}$$

DP-2

To solve the HJB we require that:

- The solution of HJB is defined on a open set O of M^2 and C^1 on such set.

- On a closed subset Γ, where the trajectories interesting from the economic point of view remain, the solution has differential in $D(G)$ (on $D(G)$ also the Dirac δ makes sense).

- The solution satisfies on Γ the (HJB).

With some nontrivial work it is possible to find an explicit solution v of the HJB

$$v(x) = \nu \left(\int_{-T}^{0} e^{\xi s} x^1(s) ds + x^0 \right)^{1-\sigma}$$

which, in terms of the historical investment $\bar{\iota}$: $[-T, 0] \longrightarrow \mathbb{R}$ is written as

$$v(\bar{\iota}) = \nu \left(\int_{-T}^{0} (1 - e^{\xi(T+s)}) \bar{\iota}(s) ds \right)^{1-\sigma}$$

where ξ is the only positive root of the equation $z = A(1 - e^{-Tz})$ and

$$\nu = \frac{1}{(1-\sigma)\xi/A} \left(\frac{\rho - \xi(1-\sigma)}{\sigma \xi / A} \right)^{-\sigma}$$

Compare with the standard AK case.

DP-3

The explicit expression for the feedback ϕ is

$$\phi(x) = Ax^0 - \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi/A}\right)^{-\sigma}\left(\int_{-T}^{0} e^{\xi s}x^1(s)ds + x^0\right)$$

Again by a non trivial work we prove that it is optimal and that $v = V$.

DP-4

Putting the feedback into the state equation we get the following.

- The optimal investment path is the unique solution of the delay differential equation

$$i(t) = A \int_{-T}^{0} i(t+s)ds - b_2 \int_{-T}^{0} (1 - e^{\xi s}) i(-T+s+t) ds$$

- The optimal capital path is the unique solution of the delay differential equation

$$k^*(t) = \int_{(t-T)\wedge 0}^{0} \bar{\iota}(s)ds + \int_{(t-T)\vee 0}^{t} \left[ak(s) - \Lambda e^{gs}\right] ds$$

We cannot solve explicitly but we can get various informations from them.

OPTIMAL PATHS

The optimal consumption path

Along optimal trajectories we have

$$c(t) = Ak(t) - i(t) = \Lambda e^{gt}$$

where

$$\Lambda = \left(\frac{\rho - \xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{-T}^{0}(1-e^{\xi s})\bar{\iota}(T-s)\mathrm{d}s\right)$$

$$g = \frac{\xi - \rho}{\sigma}$$

Compare with the standard AK case.

Note that the optimal investment and the related capital are not exponential: they oscillate.

Long run behavior of capital and investment paths

Using the equation for the optimal investment and capital obtained above we get

We have

$$\lim_{t \longrightarrow +\infty} e^{-gt} k(t) = \frac{\Lambda}{a - \frac{g}{1 - e^{-gT}}} > 0$$

and

$$\lim_{t \longrightarrow +\infty} e^{-gt} i(t) = \frac{\Lambda}{\frac{a}{g}(1 - e^{-gT}) - 1} > 0$$

Moreover we can write the optimal paths in Fourier series and give an expression for the principal part of the oscillations.

This give a basis for an estimation of the model.

Balanced Growth Paths

The balanced growth path (BGP) are of the form

$$i(s) = a_0 e^{gs} \ for \, s \in [-T, +\infty)$$

$$k(s) = b_0 e^{gs} for \, s \in [0, +\infty)$$

where a_0 and b_0 are connected by the relation:

$$b_0 = a_0 \int_{-T}^{0} e^{gs} s$$

Equation for optimal co-state

The optimal co-state is exponential and satisfies

$$\lambda(t) = \frac{A}{\sigma g + \rho}(\Lambda^{-\sigma} e^{-gt\sigma})$$

Comparison with previous results

The main improvements obtained using the DP approach is the explicit expression of:

- The value function

- The optimal feedback

- The DDE for optimal investment and capital paths

- The constants like Λ, $\lim_{t\longrightarrow+\infty} e^{-gt}k(t)$, $\lim_{t\longrightarrow+\infty} e^{-gt}i(t)$.

Moreover we get the following.

- The use of lighter assumptions on the parameters of the model.

- The theoretical justification of the absence of corner solutions.

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THIS IS THE END

We can see that the model for $T \longrightarrow \infty$ tends to the one dimensional AK model. In particular:

- $\xi \longrightarrow A$
- $\nu \longrightarrow \frac{r^{-\sigma}}{1-\sigma}$

If $\bar{\iota} \in L^2(-\infty, 0)$

- The term

$$\left(\int_{-T}^{0} (1 - e^{\xi s}) \bar{\iota}(T - s) \mathrm{d}s \right) \longrightarrow k$$

Moreover

- $g \longrightarrow A - r$
- $\Lambda \longrightarrow rk$
- The value function tends to the one dimensional one
- The optimal trajectories tend to the one dimensional ones

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MAIN FEATURES OF THE PROBLEM

1

Both problems can be suitably rewritten as optimal controls of Hilbert space systems

- In the first case the Hilbert-state space is given by $H = \mathbb{R} \times L^2(-T, 0)$;
- In the first case the Hilbert-state space is given by $H = L^2(0, \overline{s})$.

2

In both cases the state equation in the state space is of the kind

$$\begin{cases} k'(\tau) = Ak(\tau) + Bu(\tau), \quad \tau \in]t, +\infty[\\ k(t) = x \in H, \end{cases}$$

where

- A generates a strongly continuous semigroup which is not analytic;
- the control operator B is unbounded.

3

In both cases the objective functional is not bounded neither from above nor from below

4

State or state - control constraints are present.

Back to the delay model We can come back to the economic model:

- The value function for an historical positive investment $i \colon [-T, 0]$ is V(i)=$\nu\left(\int_{-T}^{0}(1-e^{\xi s})i(-T-s)s\right)^{1-\sigma} where \xi$ is the only positive root of the equation $z = A(1-e^{-Tz})$ and $\nu = \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/A}\right)^{-\sigma}\frac{1}{(1-\sigma)\xi/A}$

- The optimal control solves the delay differential equation $i(t) = \mathrm{A}\int_{-T}^{0} i(t+s)s - b_2 \int_{-T}^{0} (1-e^{\xi s}) i(-T+s+t)s Theoptimalcapitalsolvesthedelaydifferentiale$ $\int_{(t-T)\wedge 0}^{0} \bar{\iota}(s)ds + \int_{(t-T)\vee 0}^{t} \left[ak(s) - \Lambda e^{gs}\right] ds$

- Along optimal trajectories we have c(t) =Ak(t)-i(t)=$\Lambda e^{gt} where \Lambda = \left(\frac{\rho-\xi(1-\sigma)}{\sigma\xi/a}\right)\left(\int_{-T}^{0}(1-e^{\xi s})\bar{\iota}(T-s)\mathrm{d}s\right) g = \frac{\xi-\rho}{\sigma} Note that the optimal investment an$

Other results: long run behavior Using the equation for the optimal investment obtained using DP approach we can find other information of the economic system:

We have $\lim_{t \longrightarrow +\infty} e^{-gt} k(t) = \frac{\Lambda}{a - \frac{g}{1-e^{-gT}}} > 0 and \lim_{t \longrightarrow +\infty} e^{-gt} i(t) = \frac{\Lambda}{\frac{a}{g}(1-e^{-gT})-1} > 0$

Other results: BGPs The balanced growth path (BGP) are of the form i(s)=a$_0 e^{gs}\ for\, s \in [-T,+\infty) k(s) = b_0 e^{gs} for\, s \in [0,+\infty) where$a$_0$ and b_0 are connected by the relation: b$_0 = a_0 \int_{-T}^{0} e^{gs} s$

Other results: Equation for optimal co-state The optimal co-state is exponential and satisfies $\lambda(t) = \frac{A}{\sigma g + \rho}(\Lambda^{-\sigma} e^{-gt\sigma})$

Comparison with one dimensional AK model We can see that the model for $T \longrightarrow \infty$ tends to the one dimensional AK model. In particular:

- $\xi \longrightarrow A$
- $\nu \longrightarrow \frac{r^{-\sigma}}{1-\sigma}$

If $\bar{\iota} \in L^2(-\infty, 0)$

- The term $\left(\int_{-T}^{0}(1-e^{\xi s})\bar{\iota}(T-s)\mathrm{d}s\right) \longrightarrow k$

Moreover

- $g \longrightarrow A - r$
- $\Lambda \longrightarrow rk$
- The value function tends to the one dimensional one
- The optimal trajectories tend to the one dimensional ones

Comparison with previous results The main improvements obtained using the DP approach in M^2 are:

- The value function of the problem

- The optimal feedback

- Explicit DDE for optimal investment and optimal capital

- Explicit expression for constants like Λ, $\lim_{t \longrightarrow +\infty} e^{-gt} k(t)$, $\lim_{t \longrightarrow +\infty} e$

- The use of lighter assumptions on the constants (we haven't see them in the discussion)

1. Natural abstract setting:

$$H := L^2(0, \bar{s}),\ U := \mathbb{R} \times L^2(0, \bar{s})$$

H space state, U control space.

$$\begin{cases} k'(\tau) = A_0 k(\tau) + Bu(\tau), & \tau \in]t, +\infty[\\ k(t) = x \in L^2(0, \bar{s}), \end{cases}$$

where

$$A_0 f(s) = -\frac{\partial}{\partial s} f(s) - \mu f(s);\ D(A) = \{f \in H^1(0, \bar{s})\ :\ f(0) = 0\}$$

$$Bu \equiv B(u_0, u_1) = u_1 + \delta_0 u_0$$

The control on the boundary yields

$$B \notin L(U, H)$$

Minimize

$$J_\infty(t,x,u) = \int_t^{+\infty} e^{-\lambda\tau}[g_0(k(\tau)) + h_0(u(\tau))]d\tau,$$

over the set

$$L^p_\lambda(t,+\infty;U) = \{u \in L^1_{loc}(t,+\infty;U)\ ;\ t \mapsto u(t)e^{-\frac{\lambda t}{p}} \in L^p(t,+\infty;U)\}$$

Value Function

$$Z(t,x) = \inf_{L^p_\lambda(t,+\infty;U)} J_\infty(t,x,u),$$

$Z(t,x) = e^{-\lambda t} Z(0,x)$, where $Z(0,x)$ is the candidate solution of the stationary HJB equation

$$-\lambda z(x) + \langle A^* z'(x), x\rangle_H - h_0^*(-B^* z'(x)) + g(x) = 0,$$

(where $h_0^*(u) = \sup_{v\in U}\{(u|v)_U + h_0(v)\}$);

The terms

$$\langle A^* z'(x), x\rangle_H, \quad and \quad - h_0^*(-B^* z'(x))$$

in HJB are not well defined.

2. Extended abstract setting

We set $V := D(A^*)$ and choose $V' = D(A^*)'$ as state space.

The state equation and HJB make sense, $B \in L(U, V'), B^* \in L(V, U)$, so that HJB reads as

$$-\lambda z(x) + \langle z'(x), Ax\rangle_{V' \times V} - h_0^*(-B^* z'(x)) + g(x) = 0,$$

General assumptions

1. $A : D(A) \subset V' \longrightarrow V'$ generates a s.c. semigroup $\{e^{\tau A}\}_{\tau \geq 0}$ on V';

2. $B \in L(U, V')$;

3. there exists $\omega \geq 0$ such that $|e^{\tau A}x|_{V'} \leq e^{\omega\tau}|x|_{V'}$, $\forall \tau \geq 0$;

4. $g_0, \phi_0 : V' \longrightarrow \mathbb{R}$, convex, C^1 with Lipschitz gradient.

5. h_0 is convex, lower semi–continuous, $\partial_u h_0$ is injective; $\exists a > 0$, $\exists b \in \mathbb{R}$, $\exists p > 1$: $h_0(u) \geq a|u|_U^p + b$, $\forall u \in U$;

   Moreover, either $p > 2$, $\lambda > 2\omega$, or $\lambda > \omega$, and g_0, ϕ_0 sublinear.

6. $h_0^*(0) = 0$, h_0^* is C^1 with Lipschitz gradient.

Crucial assumption:

g_0, that is naturally defined on H, can be extended on V' to a C^1 function with respect to the topology of V'.

Then:

Unboundedness of B is compensated by the regularity of g_0.

(Faggian, '04, for finite horizon)

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ESSENTIAL LITERATURE

1. Regular and strong solutions for general convex problem and distributed control, with and without constraints: Barbu-Da Prato, Cannarsa-Di Blasio.

2. Linear-quadratic problem: Lasiecka-Triggiani, Bensoussan-Delfour -Da Prato-Mitter, Acquistapace-Flandoli-Terreni (non autonomous systems).

3. Viscosity solutions: Crandall-Lions (distributed control), Cannarsa-Gozzi-Soner, Cannarsa-Tessitore (some boundary control, no regularity of the value function)

4. Regular and strong solutions for general convex problem and boundary control, on FINITE horizon:

- Unconstrained problem (Faggian, '04,'05)
- Constraints on the control (Faggian & Gozzi, '05)
- Constraints on the state (Faggian, '06)

THE RESULTS

All the results for infinite horizon are obtained by passing to limits as $T \longrightarrow \infty$ on the finite horizon case, with horizon T.

Finite horizon:

$$J_T(t,x,u) = \int_t^T \left[g_0\left(y(\tau)\right) + h_0\left(u(\tau)\right)\right] e^{-\lambda\tau} d\tau + \varphi_0(y(T))e^{-\lambda T}.$$

$$\phi_T(T-t,x) := \inf_{L^p_\lambda(t,T;U)} J_T(t,x,u)$$

then ϕ_T is proved to solve in strong sense the following evolutionary HJB

$$\begin{cases} \partial_t \phi(t,x) + e^{-\lambda t} h_0^*(-B^*[e^{\lambda t}\phi_x(t,x)]) - \langle Ax, \phi_x(t,x)\rangle = e^{-\lambda(T-t)} g_0(x), \\ \phi(0,x) = \varphi_0(x)e^{-\lambda T}. \end{cases}$$

Strong solutions of the evolutionary HJB are, roughly speaking, limits of classical solutions of equations approximating HJB associated to the finite horizon problem.

Strong solutions are proved to be Lipschits in t and C^1 in x, having Lipschitz spatial gradient.

Dynamic Programming is completely performed for finite horizon, yielding a feedback formula for optimal strategies by means of the spatial gradient of the value function ϕ_T.

Theorem 1: passing to limits.

Let $\phi_T(t,x)$ be the unique strong solution to evolutionary HJB. Then the function

$$\Psi(t,x) := e^{\lambda(T-t)}\phi_T(t,x)$$

is independent of T and there exists the following limit

$$\Psi_\infty(x) := \lim_{t \longrightarrow +\infty} \Psi(t,x).$$

uniformly on bdd subsets of V'. Moreover, if

$$\lambda > \omega \max\{2, \frac{p}{p-1}\},$$

then Ψ_∞ is C^1 and has Lipschitz gradient, and

$$\Psi_x(t,x) \longrightarrow \Psi'_\infty(x), \text{ weakly in } V, \text{ as } \; t \longrightarrow +\infty.$$

Theorem 2: the value function solves HJB.

(i)

$$\Psi_\infty(x) = Z_\infty(0,x) = \inf_{u \in L^p_\lambda(0,+\infty;U)} J_\infty(0,x,u).$$

(that is Ψ_∞ is the value function of the infinite horizon problem with initial time $t = 0$). Moreover $Z_\infty(t,x) = e^{-\lambda t}\Psi_\infty(x)$.

(ii) Ψ_∞ is the unique classical solution of the stationary HJB equation:

$$-\lambda\Psi_\infty(x) + \langle \Psi'_\infty(x), Ax\rangle - h_0^*(-B^*\Psi'_\infty(x)) + g(x) = 0.$$

Theorem 3: Existence of a unique optimal pair.

$\forall t \geq 0$ and $x \in V'$, $\exists$! (u^*, y^*) optimal pair.

The optimal state y^* is the unique solution of the CLE

$$y(\tau) = e^{(\tau-t)A}x + \int_t^\tau e^{(\tau-\sigma)A}B(h_0^*)'(-B^*\Psi'_\infty(y(s)))d\sigma, \quad \tau \in [t, +\infty[,$$

while the optimal control u^* is given by the feedback formula

$$u^*(s) = (h_0^*)'(-B^*\Psi'_\infty(y^*(s))).$$

Theorem 4: Verification Theorem. $\forall t \geq 0$ and $x \in V'$

$$e^{-\lambda t}\Psi_\infty(x) = J_\infty(t,x,u)$$
$$- \int_t^T e^{-\lambda s}\big[h_0^*(-B^*\Psi'_\infty(y(s))) + (B^*\Psi'_\infty(y(s)) \mid u(s))_U + h_0(u(s))\big]ds.$$

As a consequence, an admissible pair (u,y) at (t,x) is optimal if and only if

$$\sup_{u\in U}\left\{\big(u| - B^*\Psi'_\infty(y(s))\big)_U - h_0(u)\right\} = \big(u(s)| - B^*\Psi'_\infty(y(s))\big)_U - h_0(u(s))$$

for a.e. $s \geq 0$, which is equivalent to

$$u(s) = (h_0^*)'[-B^*\Psi'_\infty(y(s))]$$

for a.e. $s \geq 0$.

Comments:

The general model applies to a broad class of problems (not only to hyperbolic-type). The value function is C^1, so that a meaningful feedback formula is provided in terms of its spatial gradient.

No comparable results exist, as far as we know, within viscosity solution theory for the general problem: existence may be easy, but uniqueness is not (at all). Regularity?

[Fabbri, '06] gives an existence and uniqueness result for viscosity solution for optimal investment with vintage capital, that does not extend to the general case.