Approximative closed formulas for models with local/stochastic volatility

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Papers available on HAL.fr, ssrn.com, arXiv.org

Outline of the talk

1. Motivation and litterature review

2. Our approach using a proxy

3. Applications to local volatility model

4. Applications to local volatility model with jumps

5. Applications to Heston model

1) Motivations: pricing/calibration of financial products

- The standard example = call option = gives the right to its holder to buy an asset at a future time T and at a guaranteed price K.
- Cash flow: $\max(0, S_T - K) = (S_T - K)_+$.
- No arbitrage theory: the (forward) price of this contract at time t must be

$$\mathbb{E}_{\mathbb{Q}^T}((S_T - K)_+ | \mathcal{F}_t) = \mathrm{Call}_t^{Model}(T, K)$$

under a suitable probability measure $\mathbb{Q}^T$ (forward measure).

- How to choose a stochastic model for S to ensure that

the model price equals the market price?

Calibration

$\rightsquigarrow$ Find a model such that for a set of ≈ 30 maturities T_i and strikes K_i

$$\mathrm{Call}_0^{Model}(T_i, K_i) = \mathrm{Call}_0^{Market}(T_i, K_i), \quad \forall i.$$

Our purpose: fast numerical pricing using closed formulas!

Literature review on semi-analytical formulas

1. **stochastic volatility model** (**no jumps**)

$$\frac{dS_t}{S_t} = r_t dt + f(V_t)dW_t, \quad dV_t = \alpha(\beta - V_t)dt + \cdots .$$

- **Fouque, Papanicolaou, Sircar ...:** asymptotic **expansion w.r.t. the correlation time** $\frac{1}{\alpha} = \epsilon \to 0$ (*fast mean reverting volatility*).

  Expansion of the pricing infinitesimal operator

$$\mathcal{L}^{\epsilon} = \frac{1}{\epsilon}\mathcal{L}^{(0)} + \frac{1}{\sqrt{\epsilon}}\mathcal{L}^{(1)} + \mathcal{L}^{(2)},$$

$$\Longrightarrow \text{Call} = \text{Call}_{BS}^{(0)} + \sqrt{\epsilon}\ \text{Correction}^{(1)} + \cdots$$

  with $\text{Correction}^{(1)}$ solving a PDE depending on $\mathcal{L}^{(1)}$ and $\text{Call}_{BS}^{(0)}$...

  To get an explicit solution, requires **time homogeneous coefficients**.

  Accurate **for medium and long maturities**.

  Error bounds available (w.r.t. ϵ).

- **Antonelli-Scarlati:** **expansion w.r.t. the correlation** between the assets and its volatility.

  Time homogeneous coefficients.

  May require to solve other PDEs.

  Error bounds available.

2. **local volatility model** **(Hagan etal.)** (**no jumps**):

$$\frac{dS_t}{S_t} = r_t dt + \sigma(t, S_t) dW_t.$$

   Singular perturbations techniques on the PDE.

   To get explicit approached formulas, restriction to volatility of the form $\sigma(t, x) = \sigma_1(t)\sigma_2(x)$.

3. **SABR model** **(Hagan etal., Osajima):** **singular perturbations techniques** on the PDE (Hagan) or **expansion of transition density functions** using Malliavin calculus (Osajima).

   Time independent coefficients.

Summary:

- Nice explicit formulas for prices and implied volatilities
- Allows for fast calibration (directly on implied volatilities)
- Allows to understand the role of model paramaters
- Sometimes error analysis available (Osajima, Fouque etal...).
- Coefficients has to be time independent (no term structure).
- No jumps.

2) Our approach

1. We use a **model proxy**: $X \simeq X^P$.
2. We derive a **decomposition of the price** as follows:

$$\text{Price}(X) = \text{Price}(X^P) + \sum_{i=1}^{3 \text{ or } \dots} \alpha_i \text{Greek}_i(X^p) + \text{Error}.$$

3. We **analyse the error**: depends on payoff, on maturity and on model.

Model proxy and formal decomposition

- Given the dynamics of the underlying asset X (or log-asset), **find a proxy** X^P for this model. For instance
  - **log-normal proxy** (Black-Scholes price)
  - **normal proxy** (Bachelier price)
  - **Merton** (log-normal diffusion + log-normal jumps)
  - ...

  Constraints: price and Greeks in this model should be given in closed forms.

- **Expansion around the proxy:**

  $\mathbb{E}(\mathbf{h}(\mathbf{X_T})) = \mathbb{E}(\mathbf{h}(\mathbf{X_T^P})) + \mathbb{E}(\mathbf{h}'(\mathbf{X_T^P})(\mathbf{X_T} - \mathbf{X_T^P})) + \frac{\mathbf{1}}{\mathbf{2}}\mathbb{E}(\mathbf{h}''(\mathbf{X_T^P})(\mathbf{X_T} - \mathbf{X_T^P})^{\mathbf{2}}) + \cdots$

  ? Closed approximation forms for terms with derivatives (Malliavin calculus)

- **Final formula:** $\mathbb{E}(\mathbf{h}(\mathbf{X_T})) = \mathbb{E}(\mathbf{h}(\mathbf{X_T^P})) + \sum_{\mathbf{i}} \alpha_{\mathbf{i}} \mathrm{Greek}_{\mathbf{i}}^{\mathbf{h}}(\mathbf{X_T^P}) + \mathrm{Error}.$

  ? Error analysis.

2) Application to local volatility models $\sigma(t,x) = \sigma_t(x)$

Example: time-dependent CEV $dS_t = \nu_t S_t^{\beta_t} dW_t$.

For constant parameters, see Schröder'89 for closed formulas.

Model on the log-asset

$X_t = \log(S_t)$.

Given x_0, $dX_t = -\sigma_t^2(X_t)dt + \sigma_t(X_t)dW_t$

Call payoff: $(e^{X_T} - K)_+$.

More Generally: payoff $h(X_T)$.

Which proxy?

Log-normal proxy: $X_T^{BS} = x_0 - \int_0^T \sigma_t^2(x_0)dt + \int_0^T \sigma_t(x_0)dW_t$

Approximation $\sigma_t(X_t) \approx \sigma_t(x_0)$ justified by one the following reasons:

i) the **function $\sigma_t(.)$ has slow variations** (spatial derivatives of σ are small).
In CEV model $(\sigma_t(x) = \nu_t e^{x(\beta_t - 1)})$, this means β **close to 1**.

ii) X_T close to X_0 because the **drift and volatility terms are small**.
In CEV, it means ν **close to 0**.

iii) X_T close to X_0 because T **is small** (**short maturity approximation**).

Good candidates because price and Greeks are explicit in the proxy:
Black-Scholes formula with volatility input $\sqrt{\frac{1}{T}\int_0^T \sigma_t^2(x_0)dt}$.

Model on the asset

Given x_0, $dX_t = \sigma_t(X_t)dW_t$

For CEV model, $\sigma_t(x) = \nu_t x^{\beta_t}$.

Call payoff: $(X_T - K)_+$.

Which proxy?

Normal proxy: $X_T^N = x_0 + \int_0^T \sigma_t(x_0)dW_t$

Approximation $\sigma_t(X_t) \approx \sigma_t(x_0)$ justified by one the following reasons (in CEV)

i) the **function $\sigma_t(.)$ has slow variations** (β **close to 0**).

ii) X_T close to X_0 because of **small volatility** (ν **close to 0**).

iii) X_T close to X_0 because **T is small**.

Pricing formulas and Greeks also available in closed forms (Bachelier formula with normal model).

Theorem (2nd order formula) with log-normal proxy (Black-Scholes)

Theorem (2nd order formula). For vanilla payoff h,

$$\mathbb{E}[\mathbf{h}(\mathbf{X_T})] = \mathbb{E}[\mathbf{h}(\mathbf{X_T^{BS}})] + \mathbf{C_{1,T}}\left(\frac{\mathbf{1}}{\mathbf{2}}\mathrm{Greek}_{\mathbf{1}}^{\mathbf{h}}(\mathbf{X_T^{BS}}) - \frac{\mathbf{3}}{\mathbf{2}}\mathrm{Greek}_{\mathbf{2}}^{\mathbf{h}}(\mathbf{X_T^{BS}}) + \mathrm{Greek}_{\mathbf{3}}^{\mathbf{h}}(\mathbf{X_T^{BS}})\right) + \mathrm{Error}_{\mathbf{2}}$$

with

$$C_{1,T} = \int_0^T \sigma_t^2 \left(\int_t^T \sigma_s \sigma_s^{(1)} ds\right) dt, \qquad M_0 = \max_{0\le i\le 4}(|\sigma^{(i)}|_\infty),$$

$$|\mathrm{Error}_2| \le \left(\frac{M_0}{\sigma_{inf}}\right) M_1 M_0^2 (\sqrt{T})^3, \qquad M_1 = \max_{1\le i\le 4}(|\sigma^{(i)}|_\infty).$$

- Coefficient $C_{1,T}$ does not depend on the payoff but on the maturities.
- Recursive computation of C_1 for time-piecewise constant volatility.
- 2nd order approximation formula.
- The convergence order depends on the payoff smoothness (vanilla $\neq$ digital).
- Approximative prices depend on the level and slope of local volatility ATM.

Financial Modelling Corollary

Corollary 1. Two models with same jumps and two different local volatilities σ_1 and σ_2 give the same price for any Call option (up to an approximation error) if

$$\sigma_1(t, x_0) = \sigma_2(t, x_0) \text{ and } \partial_x \sigma_1(t, x_0) = \partial_x \sigma_2(t, x_0) \text{ for any } t.$$

Corollary 2 (CEV). The local volatilities of CEV (**Constant Elasticty of Variance**) type with time dependent parameters

$$\sigma(t, x) = \nu_t e^{(\beta_t - 1)x} \tag{1}$$

generate all possible model prices within this class of models.

Corollary 3 (Parameter Averaging Principle). A time dependent CEV model is equivalent to a time independent CEV model provided the parameters choice

$$\bar{\nu} = \sqrt{\frac{\int_0^T \nu_t^2 dt}{T}}, \qquad \bar{\beta} = \int_0^T \beta_t \rho_t dt, \quad \text{with} \quad \rho_t = \frac{\nu_t^2 \int_0^t \nu_s^2 ds}{\int_0^T \nu_t^2 \int_0^t \nu_s^2 ds}.$$

Theorem (3nd order formula)

Theorem (3nd order formula) with log-normal proxy (Black-Scholes). For vanilla payoff h,

$$\mathbb{E}[\mathbf{h}(\mathbf{X_T})] = \mathbb{E}[\mathbf{h}(\mathbf{X_T^{BS}})] + \sum_{\mathbf{i=1}}^{\mathbf{6}} \alpha_{\mathbf{i},\mathbf{T}} \mathrm{Greek}_{\mathbf{i}}^{\mathbf{h}}(\mathbf{X_T^{BS}}) + \mathrm{Error}_{\mathbf{3}}$$

with $|\mathrm{Error}_3| \leq (\frac{\mathrm{M}_0}{\sigma_{\inf}})\mathrm{M}_1\mathrm{M}_0^3(\sqrt{\mathrm{T}})^4$.

The coefficients $(\alpha_{i,T})_i$ are explicit and depend on $\sigma_t(x_0)$, $\partial_x\sigma_t(x_0)$ and $\partial^2_{x,x}\sigma_t(x_0)$.

⚠ Suitable for β_t close to 1.

Theorem (3nd order formula) with normal proxy (Bachelier). For vanilla payoff h,

$$\mathbb{E}[\mathbf{h}(\mathbf{X_T})] = \mathbb{E}[\mathbf{h}(\mathbf{X_T^{N}})] + \sum_{\mathbf{i=1}}^{\mathbf{6}} \alpha'_{\mathbf{i},\mathbf{T}} \mathrm{Greek}_{\mathbf{i}}^{\mathbf{h}}(\mathbf{X_T^{N}}) + \mathrm{Error}_{\mathbf{3}}.$$

⚠ Suitable for β_t close to 0.

Numerical tests with CEV model $\beta = 0.8$

Table: Errors on implied Black-Scholes volatilities (in bp) between the **second order** approximation formula (using **BS proxy**) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.8$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-1.63	-0.22	-0.08	-0.17	-0.86
1Y	-1.11	-0.26	-0.15	-0.22	-0.63
1.5Y	-0.98	-0.32	-0.21	-0.28	-0.60
2Y	-0.95	-0.38	-0.28	-0.34	-0.62
3Y	-0.98	-0.51	-0.41	-0.46	-0.69
5Y	-1.16	-0.77	-0.67	-0.70	-0.89
10Y	-1.70	-1.37	-1.26	-1.27	-1.40

Excellent results because model close to BS proxy.

Numerical tests with CEV model $\beta = 0.8$

Table: Errors in implied Black-Scholes volatilities (in bp) between the **third order** approximation formula (using **BS proxy**) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.8$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-0.08	-0.02	-0.01	0.00	0.00
1Y	-0.06	-0.03	-0.01	-0.01	0.00
1.5Y	-0.06	-0.03	-0.02	-0.01	0.00
2Y	-0.06	-0.04	-0.02	-0.01	0.00
3Y	-0.08	-0.05	-0.03	-0.01	0.00
5Y	-0.10	-0.06	-0.04	-0.01	0.01
10Y	-0.16	-0.10	-0.06	-0.02	0.01

Still excellent results because model close to BS proxy.

Numerical tests with CEV model $\beta = 0.2$

Table: Errors on implied Black-Scholes volatilities (in bp) between the **second order** approximation formula (using **BS proxy**) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.2$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-22.85	-3.33	-1.07	-2.61	-14.87
1Y	-16.60	-4.07	-2.14	-3.21	-10.20
1.5Y	-15.21	-5.11	-3.21	-4.03	-9.31
2Y	-15.13	-6.23	-4.27	-4.92	-9.29
3Y	-16.36	-8.53	-6.39	-6.74	-10.12
5Y	-20.47	-13.19	-10.60	-10.42	-12.74
10Y	-32.01	-24.45	-20.77	-19.45	-20.26

Less accurate results because model closer to normal proxy than log-normal.

Numerical tests with CEV model $\beta = 0.2$

Table: Errors in implied Black-Scholes volatilities (in bp) between the **third order** approximation formula (using **BS proxy**) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.2$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-1.23	-0.18	-0.01	0.12	0.53
1Y	-0.93	-0.34	-0.03	0.22	0.52
1.5Y	-1.19	-0.51	-0.06	0.31	0.68
2Y	-1.51	-0.68	-0.09	0.39	0.85
3Y	-2.22	-1.05	-0.19	0.52	1.17
5Y	-3.71	-1.87	-0.47	0.67	1.69
10Y	-7.32	-4.13	-1.56	0.55	2.38

Third order correction terms compensate very well the distance between the model and its log-normal proxy.

Numerical tests with CEV model $\beta = 0.8$

Table: Errors in implied Black-Scholes volatilities (in bp) between the **third order** approximation formula (using **Normal proxy**) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.8$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-1.61	-0.07	-0.01	0.03	0.77
1Y	-0.88	-0.08	-0.02	0.03	0.45
1.5Y	-0.61	-0.11	-0.02	0.04	0.31
2Y	-0.51	-0.15	-0.03	0.06	0.25
3Y	-0.49	-0.23	-0.05	0.10	0.23
5Y	-0.71	-0.44	-0.11	0.16	0.30
10Y	-1.70	-1.09	-0.37	0.22	0.56

Excellent results although model is not close to normal proxy.

Numerical tests with CEV model $\beta = 0.2$

Table: Errors in implied Black-Scholes volatilities (in bp) between the **third order** approximation formula (using **normal proxy**) and the closed formula for CEV model, expressed as a function of maturities in fractions of years and relative strikes. Parameters: $\beta = 0.2$, $\nu = 0.2$ and $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	0.22	0.06	-0.01	-0.06	-0.16
1Y	0.41	0.11	0.00	-0.10	-0.26
1.5Y	0.56	0.17	0.00	-0.13	-0.34
2Y	0.71	0.24	0.02	-0.16	-0.41
3Y	1.02	0.39	0.06	-0.20	-0.53
5Y	1.75	0.79	0.21	-0.23	-0.71
10Y	4.71	2.55	1.15	0.10	-0.84

Still excellent results (model is closed to normal proxy).

Numerical tests with time dependent coefficients

Piecewise constant functions ν and β are equal respectively on each interval of the form $[\frac{i}{20}, \frac{i+1}{20}[$ to $25\% - i \times 0.11\%$ and $100\% - i \times 0.75\%$.

Table: Errors on implied Black-Scholes volatilities (in bp) between the **second order** approximation formula and the PDE method, expressed as a function of maturities in fractions of years and relative strikes. Parameters: time dependent ν and β, $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	-0.67	-0.09	0.03	-0.07	-0.35
1Y	-0.44	0.10	0.06	-0.09	-0.26
1.5Y	-0.38	-0.13	0.09	0.11	-0.25
2Y	0.37	0.15	-0.11	-0.14	-0.26

Parameter averaging principle

Table: Errors on implied Black-Scholes volatilities (in bp) between the closed CEV formula applied to an equivalent CEV model and the PDE method, expressed as a function of maturities in fractions of years and relative strikes. Parameters: time dependent ν and β, $x_0 = 0$.

T/K	80%	90%	100%	110%	120%
6M	0.32	-0.08	-0.17	-0.09	0.09
1Y	0.09	-0.27	-0.20	-0.07	0.00
1.5Y	-0.28	-0.21	-0.38	-0.44	-0.25
2Y	-1.44	-0.74	-0.39	-0.43	-0.56

3) Applications to models with local volatility and jumps

Merton models with jumps and local volatilities (known also as Andersen-Andreasen model) on $X_t = \log(S_t)$:

$$dX_t = \sigma(t, X_{t-})dW_t + \mu(t, X_{t-})dt + dJ_t, X_0 = x_0,$$

where

- $(W_t)_{0\le t\le T}$ is a standard real Brownian motion;
- $J_t = \sum_{i=1}^{N_t} Y_i$;
- $(N_t)_t$ is a Poisson process with jump intensity λ;
- $(Y_i)_{i\in\mathbb{N}^*}$ are i.i.d. normal variables with mean η_J and volatility γ_J;
- $(J_t)_{0\le t\le T}$ and $(W_t)_{0\le t\le T}$ are independent;
- μ is adjusted to ensure martingale properties of the discounted asset under the risk-neutral measure: $\mu = \lambda(1 - e^{\eta_J + \frac{\gamma_J^2}{2}}) - \frac{\sigma^2}{2}$ (to simplify, no dividends).

☺ For time dependent σ, the Call prices within this model are given by Merton's formula.

Selection of a proxy

Dynamics of log-asset: $dX_t = \sigma(t, X_{t^-})dW_t + \mu(t, X_{t^-})dt + dJ_t, X_0 = x_0$.

One may replace $\sigma(t, X_{t^-}), \mu(t, X_{t^-})$ by $\sigma(t, x_0), \mu(t, x_0)$ because

1. σ does not vary much, ie $|\nabla_x \sigma|$ small;
2. coefficients are not large: $\sigma \approx 30\%$, small Gaussian jumps (size/frequency). Thus $X_t - x_0$ small.
3. small maturity T.

$\rightsquigarrow$ Merton's model as a proxy:

$$\boxed{dX_t^M = \sigma(t, x_0)dW_t + \mu(t, x_0)dt + dJ_t, X_0^M = x_0.}$$

Closed Merton's formula for the Call price.

2nd order approximation result

Theorem. The model price is expressed as a linear combination of Greeks (Delta, Gamma, Epsilon) in the proxy (Merton's model):

$$\mathbb{E}[\mathbf{h}(\mathbf{X_T})] = \mathbb{E}[\mathbf{h}(\mathbf{X_T^M})] + \sum_{\mathbf{i=1}}^{\mathbf{3}} \alpha_{\mathbf{i,T}} \mathrm{Greek}_{\mathbf{i}}^{\mathbf{h}}(\mathbf{X_T^M}) + \sum_{\mathbf{i=1}}^{\mathbf{3}} \beta_{\mathbf{i,T}} \mathrm{Greek}_{\mathbf{i}}^{\mathbf{h}}(\mathbf{X_T^M} + \mathbf{Y}') + \mathrm{Error}_{\mathbf{3}}$$

where $C_{1,T} = \int_0^T \sigma_t^2 \int_t^T \sigma \sigma_s^{(1)} ds$, $C_{2,T} = \int_0^T t\sigma_t \sigma_t^{(1)} dt$ and

- $\alpha_{1,T} = \frac{1}{2} C_{1,T} + \lambda(e^{\eta_J + \frac{\gamma_J^2}{2}} - 1) C_{2,T}$,
- $\alpha_{3,T} = C_{1,T}$,
- $\alpha_{2,T} = -(\alpha_{1,T} + \alpha_{3,T})$,
- $\beta_{1,T} = -\lambda \eta_J C_{2,T}$,
- $\beta_{3,T} = \lambda \gamma_J C_{2,T}$,
- $\beta_{2,T} = -(\beta_{1,T} + \beta_{3,T})$.

Computation of the *price* $\mathbb{E}[h(X_T)]$: instantaneous !! (in fact 4 milliseconds on a 2, 6 GHz Pentium PC)

Upper bounds of the error

Theorem (cont'd). Error estimates for vanilla options (call/put):

$$|\text{Error}| \leq_c \frac{M_0}{\sigma_{inf}}(M_1\sqrt{T})\big((M_0\sqrt{T})^2 + M_J^2\sqrt{\lambda T}\big).$$

$\Longrightarrow$ Second order approximation formula.

For binary options, first order approximation formula.

No error for time-dependent coefficients ($M_1 = 0$).

Here again, proof relies on (heavy) Malliavin calculus techniques.

Numerical results: approximation accuracy

- The jump parameters have been set to: $\lambda = 30\%, \eta_J = -8\%, \gamma_J = 35\%$.
- The piecewise constant functions ν and β are equal respectively at each interval of the form $[\frac{i}{20}, \frac{i+1}{20}]$ to $25\% - i \times 0.11\%, 1 - i \times 0.75\%$.
- The spot, the risk free rate and the dividend yield are set respectively to $100, 4\%, 0\%$.

Error in implied Black Scholes volatilities between the approximation formula and the PIDE method.

T/K	70%	85%	100%	120%	150%
3M	0.02	-0.03	-0.92	-0.07	-0.12
1Y	0.04	0.06	0.15	-0.11	0.01
3Y	0.22	-0.23	0.11	0.41	0.31
5Y	1.39	1.06	-0.01	1.85	1.76

$\Longrightarrow$ Errors do not exceed **2 bp** for various strikes and maturities!!

Results with bumped parameters

Initial parameters:

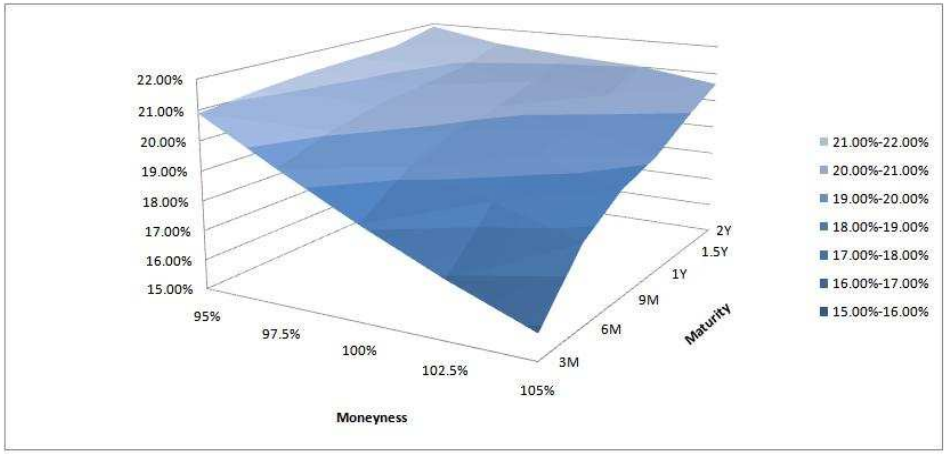
- $S_0 = 1$,
- $r = q = 0$,
- $T = 1Y$,
- $\sigma = 0.25$,
- $\beta = 0.95$,
- $\lambda = 0.3$,
- $\mu_J = -0.08$,
- $\sigma_J = 0.35$.

App. form.	80%	90%	100%	110%	120%
initial parameters	30.98%	30.36%	30.09%	30.05%	30.21%
bumped parameters: $\beta = 0.5$	32.10%	30.86%	29.97%	29.36%	29.00%
β=0.1	33.08%	31.31%	29.86%	28.75%	27.91%
σ=0.5	53.54%	53.33%	53.16%	53.02%	52.91%
$\lambda = 1$	41.71%	40.88%	40.46%	40.38%	40.56%
$\lambda = 3$	63.77%	63.24%	62.91%	62.76%	62.75%
$\beta = 0.1$, $\lambda = 0.0$	27.42%	26.18%	25.00%	23.92%	22.90%
PDE Method	80%	90%	100%	110%	120%
initial parameters	30.96%	30.36%	30.09%	30.07%	30.24%
bumped parameters: β=0.5	32.04%	30.91%	30.12%	29.63%	29.42%
β=0.1	33.11%	31.45%	30.18%	29.28%	28.74%
$\sigma = 0.5$	53.54%	53.34%	53.19%	53.07%	52.98%
$\lambda = 1$	41.69%	40.89%	40.50%	40.45%	40.65%
$\lambda = 3$	63.81%	63.32%	63.03%	62.91%	62.93%
$\beta = 0.1$, $\lambda = 0.0$	27.66%	26.26%	25.05%	23.99%	23.04%
Errors in bp	80%	90%	100%	110%	120%
initial parameters	1.97	0.54	-0.49	-1.75	-3.19
bumped parameters: β=0.5	5.88	-4.36	-14.94	-26.77	-41.94
β=0.1	-3.37	-14.25	-31.32	-52.86	-83.81
σ=0.5	0.25	-1.64	-3.24	-5.34	-6.74
λ=1	2.02	-1.17	-3.57	-6.68	-8.99
λ=3	-4.23	-8.47	-11.44	-15.08	-18.29
$\beta = 0.1$, $\lambda = 0.0$	-24.00	-8.25	-4.98	-6.52	-14.45

Calibration results on market data

DATA: 30 Call prices on EURO STOXX Index.

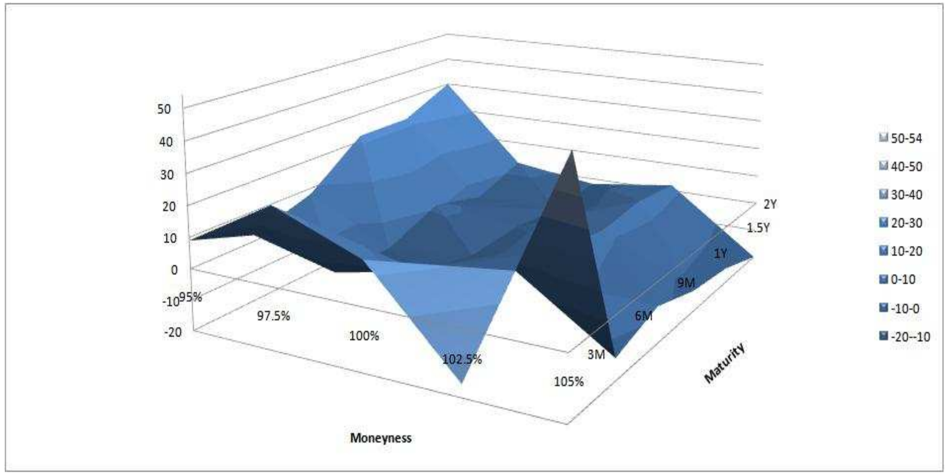
T/K	95%	97.5%	100%	102.5%	105%
3M	20.88%	19.47%	18.13%	16.91%	15.85%
6M	21.12%	20.07%	19.26%	18.55%	17.70%
9M	21.30%	20.47%	19.86%	19.33%	18.65%
1Y	21.39%	20.67%	20.16%	19.71%	19.11%
1.5Y	21.46%	20.90%	20.61%	20.40%	19.92%
2Y	21.89%	21.41%	21.18%	21.02%	20.61%



Implied Black-Scholes volatilities

Errors between implied B-S volatilities for the EURO STOXX 50 Index and those calculated within the calibrated model (in bp).

T/K	95%	97.5%	100%	102.5%	105%
3M	9	25	14	-16	54
6M	2	-5	1	7	-13
9M	8	-6	-3	5	-8
1Y	22	1	-2	2	-13
1.5Y	22	-4	-4	4	-15
2Y	30	2	-2	2	-20



Calibrated values of the piecewise constant functions ν and β.

T	ν	β
3M	21.48%	94.36%
6M	18.73%	95.63%
9M	21.46%	93.81%
1Y	21.41%	93.39%
1.5Y	18.06%	96.60%
2Y	18.15%	98.38%

Calibrated jump parameters:

- $\lambda = 28.52\%$,
- $\eta_J = -31.32\%$,
- $\gamma_J = 5.11\%$.

Calibration time: 400 ms!!

Models under consideration (Cont'd)

2) **Time dependent Heston model**:

$$\frac{dS_t}{S_t} = r_t dt + \sqrt{V_t} dW_t, \quad dV_t = \alpha(\beta_t - V_t)dt + \sigma_t \sqrt{V_t} dB_t.$$

For these 2 models, usually model prices are computed using PDE methods of Fourier methods $\rightsquigarrow$ the calibration takes at least one minute.

Our objective: real time calibration (<1s)

- For this, we avoid the use of PDE .... and we directly use weak approximation techniques.
- Advantages: the approximations are explicit even if the coefficients are time dependent.

4) Applications to time dependent Heston model

The log asset price X solves:

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t dt + \sqrt{V_t} dW_t, \\ dV_t &= \kappa(\theta_t - V_t)dt + \xi_t \sqrt{V_t} dB_t, \\ d\langle W, B\rangle_t &= \rho_t dt. \end{aligned}$$

See Lewis'00 for approximative formulas for time independent $(\theta_t)_t$ and $(\xi_t)_t$.

Our model proxy related to $\xi_t = 0$.

Assumptions: 1) $\sup_{t\le T} |\rho_t| < 1$; 2) $\xi_t > 0$; 3) $\inf_{t\le T} \frac{2\kappa\theta_t}{\xi_t^2} \ge 1$.

Theorem. For an explicit Black-Scholes equivalent volatility $\hat\sigma$ and explicit coefficients $(\alpha_i)_i$, one has

$$\mathrm{Call}^{Heston}(T,K) = \mathrm{Call}^{BS}(T,K,\hat\sigma) + \sum_{i=1}^{4} \alpha_i \mathrm{Greek}_i^{BS}(T,K,\hat\sigma) + O(|\xi|_\infty \sqrt{T})^3.$$

Numerical results

Parameters: $V_0 = (20\%)^2$, $\theta = (24.5\%)^2$, $\kappa = 3, \xi = 30\%$, $S_0 = 100$, $r = q = 0\%$.

Error (in bp) on implied Black Scholes volatilities.

$\rho = 0$:

T/K	80%	90%	100%	110%	120%
3M	-4.86	0.53	0.38	0.63	-2.68
6M	-1.89	1.02	0.93	1.05	-0.27
1Y	0.26	1.07	1.12	1.09	0.72
2Y	0.49	0.65	0.68	0.66	0.57
5Y	0.16	0.17	0.17	0.17	0.16

$\rho = -20\%$:

T/K	80%	90%	100%	110%	120%
3M	-7.65	-1.25	0.38	2.35	3.68
6M	-3.27	1.60	-0.23	-0.87	1.25
1Y	-2.39	0.29	0.05	0.37	2.22
2Y	-0.92	0.29	0.63	0.98	1.66
5Y	-0.01	0.19	0.30	0.39	0.52

$\rho = -50\%$:

T/K	80%	90%	100%	110%	120%
3M	-13.87	-2.55	0.19	4.38	23.20
6M	-12.34	-0.95	0.81	3.39	15.29
1Y	-5.01	-0.01	1.53	3.25	7.48
2Y	-1.18	0.53	1.43	2.30	3.59
5Y	0.13	0.42	0.62	0.82	1.03

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