Optimal consumption-investment strategy under drawdown constraint

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Joint work with Nizar Touzi

Problem

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- Invest θ in a risky asset: $dS_t = \sigma S_t(dW_t + \lambda dt)$
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Is there any admissible strategy (C,θ) ?

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with $\int_0^T (c_t + {\pi_t}^2) dt < \infty$

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Literature on Drawdown

Wealth: $X_t^{x,C,\theta} = x - \int_0^t C_r dr + \int_0^t \sigma\theta_r (dW_r + \lambda dr)$

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- Long term growth rate [GZ93] [CK95]

$$u(x) = \sup_{\theta\in\mathcal{A}_D} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[\left(X_t^{x,C,\theta}\right)^p\right]$$

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- Intertemporal power utility [R06]

$$u(x) = \sup_{(C,\theta) \in \mathcal{A}_D} \mathbb{E}\left[\int_0^\infty e^{-\beta t} C_t{}^p dt\right]$$

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$$u(x) := \sup_{(C,\theta)\in\mathcal{A}_D} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t) dt\right]$$

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- Extra dependence on current maximum

Drawdown constraint: $X_t^{x,C,\theta} \geq \alpha Z_t^{x,z,C,\theta}$

with $Z_t^{x,z,C,\theta} := z \vee \left(X_t^{x,C,\theta}\right)^*$

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Assumptions: U is C^1, increasing, concave, satisf. Inada, $U(0) = 0$,

and $p := \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1 \wedge \frac{\gamma}{(1-\alpha)(1+\gamma)}$, with $\gamma := \frac{2\beta}{\lambda^2}$

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- $u(\alpha z, z) = 0$

Dynamic Programming

$$u(x,z) = \sup_{(C,\theta)_{[0,h]}} \mathbb{E}\left[\int_0^h e^{-\beta t} U(C_t)dt\right] + \mathbb{E}\left[e^{-\beta h} u\left(X_h^{x,C,\theta}, Z_h^{x,z,C,\theta}\right)\right]$$

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If u is regular,

$$u(x,z) + \mathbb{E}\left[\int_0^h \ldots dt\right] + \mathbb{E}\left[\int_0^h e^{-\beta t} u_z(X_t, Z_t) dZ_t\right]$$

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Term in "dZ_t" $\Rightarrow u_z(Z_t, Z_t)dZ_t = 0$

- $u_z = 0$ → $u_z(z,z) = 0$

or

- $dZ_t = 0$ → $\theta(z,z) = 0$

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- $\xrightarrow{u_z = 0}$ $u_z(z,z) = 0$
- or
- $\xrightarrow{dZ_t = 0}$ $\theta(z,z) = 0$

$\Rightarrow$ Boundary condition: $\boxed{[-u_z] \wedge [u_x/u_{xx}](z,z) = 0}$

Duality $v(y,z) := \sup_{x\geq 0}[u(x,z) - xy]$

Primal PDE

$$\alpha z < x < z$$

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$$\varphi(z) := u_x(z,z)$$

$$\varphi_\alpha(z) := u_x(\alpha z, z)$$

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$$\beta v - \beta y v_y - \frac{\lambda^2}{2} y^2 v_{yy} = V(y)$$

$$[-v_z] \wedge [y v_{yy}](\varphi(z), z) = 0$$

$$v_y(\varphi(z), z) = -z$$

$$v_y(\varphi_\alpha(z), z) = -\alpha z$$

Duality $v(y,z) := \sup_{x\geq 0}[u(x,z) - xy]$

Primal PDE

$$\alpha z < x < z$$

$$\beta u - V(u_x) + \frac{\lambda^2}{2}\frac{u_x^2}{u_{xx}} = 0$$

$$[-u_z] \wedge [u_x/u_{xx}](z,z) = 0$$

$$\varphi(z) := u_x(z,z)$$

$$\varphi_\alpha(z) := u_x(\alpha z, z)$$

Dual PDE

$$\varphi(z) < y < \varphi_\alpha(z)$$

$$\beta v - \beta y v_y - \frac{\lambda^2}{2} y^2 v_{yy} = V(y) \leftarrow$$

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- Generic form to v

Duality $v(y,z) := \sup_{x\geq 0}[u(x,z) - xy]$

Primal PDE	Dual PDE
$\alpha z < x < z$	$\varphi(z) < y < \varphi_\alpha(z)$
$\beta u - V(u_x) + \frac{\lambda^2}{2}\frac{u_x^2}{u_{xx}} = 0$	$\beta v - \beta y v_y - \frac{\lambda^2}{2} y^2 v_{yy} = V(y)$
$[-u_z] \wedge [u_x/u_{xx}](z,z) = 0$	$[-v_z] \wedge [y v_{yy}](\varphi(z), z) = 0$
$\varphi(z) := u_x(z,z)$	$v_y(\varphi(z), z) = -z \quad \longleftarrow$
$\varphi_\alpha(z) := u_x(\alpha z, z)$	$v_y(\varphi_\alpha(z), z) = -\alpha z \quad \longleftarrow$

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- $\varphi_\alpha = \infty$ gives v in terms of φ

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- Generic form to v
- $\varphi_\alpha = \infty$ gives v in terms of φ
- φ determined implicitly at the boundary

$\alpha < 1/(1+\gamma)$ → $u_z(z,z) = 0$

or

$\alpha > 1/(1+\gamma)$ → $\theta(z,z) = 0$

Duality $v(y,z) := \sup_{x\geq 0}[u(x,z) - xy]$

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- u determined implicitly

$\alpha < 1/(1+\gamma)$: $u_z(z,z) = 0$

or

$\alpha > 1/(1+\gamma)$: $\theta(z,z) = 0$

Verification

Assumptions: U is C^1, increasing, concave, satisf. Inada, $U(0) = 0$,

and $$p := \limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1 \wedge \frac{\gamma}{(1-\alpha)(1+\gamma)}, \quad \text{with } \gamma := \frac{2\beta}{\lambda^2}$$

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Verification theorem

- $u \in C^0\left(\overline{\mathbf{D}}_\alpha\right) \cap C^{2,1}\left(\mathbf{D}_\alpha\right)$ with $\mathbf{D}_\alpha := \{(x,z) : \alpha z < x < z\}$

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- $u(x, z) \leq K \left(1 + z^{\alpha p}(x - \alpha z)^{(1-\alpha)p}\right)$

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Azema-Yor martingale, non-linear constraint ? [EM06]

Explicit solution

Define $\delta := \frac{\gamma}{(1-\alpha)(1+\gamma)}$ and φ as the inverse on $\mathbb{R}^+$ of

$$g(\zeta) \;:=\; \frac{\delta}{\beta(1+\delta)} \left(\int_0^{\zeta} \frac{-V'(s)}{s} \left(\frac{s}{\zeta}\right)^{1+\delta} ds + \int_{\zeta}^{\infty} \frac{-V'(s)}{s}\, ds \right) .$$

Define $f(.,z)$ as the inverse on $(\varphi(z), \infty)$ of

$$h(y,z) := \alpha z + \frac{\gamma}{\beta(1+\gamma)} \left(\frac{\varphi(z)}{y}\right)^{1+\gamma} \int_0^{\varphi(z)} \frac{-V'(s)}{s} \left(\frac{s}{\varphi(z)}\right)^{1+\delta} ds$$
$$+ \frac{\gamma}{\beta(1+\gamma)} \left\{ \int_{\varphi(z)}^{y} \frac{-V'(s)}{s} \left(\frac{s}{y}\right)^{1+\gamma} ds + \int_y^{\infty} \frac{-V'(s)}{s} ds \right\}$$

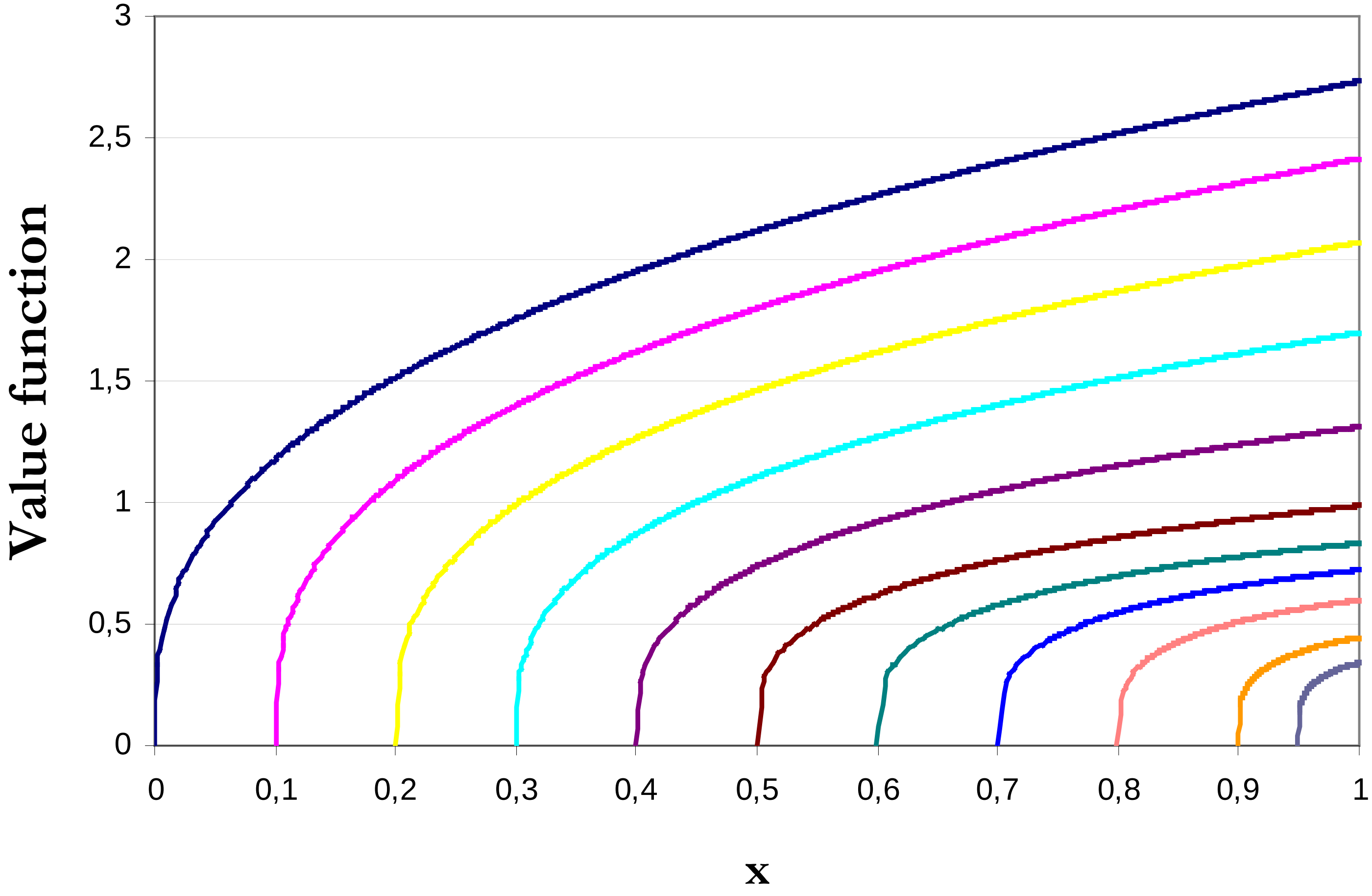
Optimal strategy: $C(x,z) = -[V' \circ f](x,z)$ and

$$\theta(x,z) = \frac{\lambda}{\sigma} \left((\gamma+1)(x-\alpha z) - \frac{\gamma}{\beta} \int_{f(x,z)}^{\infty} \frac{-V'(s)}{s} ds \right)$$

Value function: $u(x,z) \;=\; f(x,z) \left(\frac{\gamma+1}{\gamma} (x - \alpha z) \;+\; \frac{1}{\beta} \int_{f(x,z)}^{\infty} \frac{V(s)}{s^2} ds \right)$

Value function $u(x,1)$

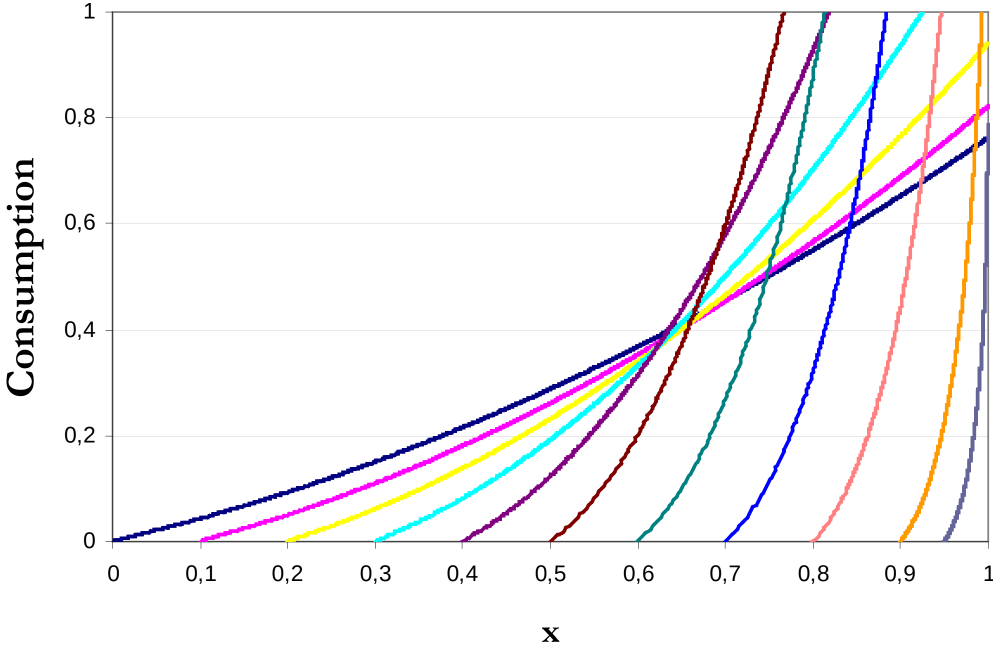
$$U(x) = x^p + x^q$$

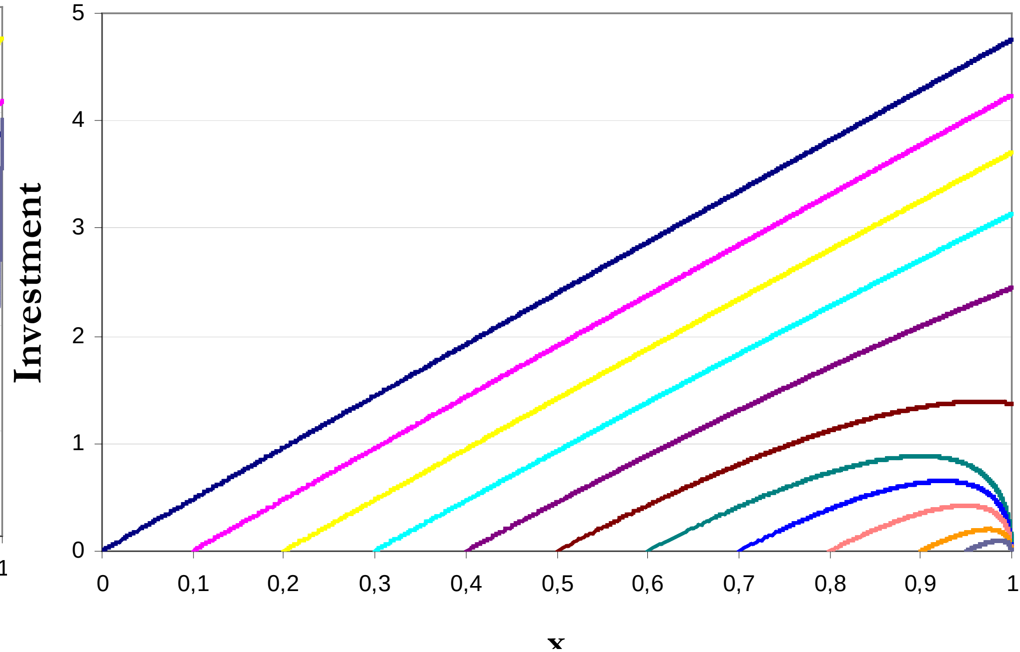


Value function $u(x,1)$ Vs x for different values of α.

Optimal Strategy

$$U(x) = x^p + x^q$$





Optimal strategy **vs** x for different values of α.

Finite horizon

Wealth: $$X_t^{x,C,\theta} = x - \int_0^t C_r dr + \int_0^t \sigma\theta_r \left(dW_r + \lambda dr\right)$$

Drawdown constraint: $$X_t^{x,C,\theta} \geq \alpha \left(X^{x,C,\theta}\right)_t^*$$

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Maximization with fixed horizon T

$$u(x) := \sup_{(C,\theta)\in\mathcal{A}_D} \mathbb{E}\left[\int_0^T e^{-\beta_s} U(C_s) ds\right]$$

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Temporal dependence of the value function

$$u(t,x,z) := \sup_{(C,\theta)\in\mathcal{A}_D} \mathbb{E}\left[\int_t^T e^{-\beta_s} U(C_s) ds\right]$$

Domain and properties

Domain of definition $\overline{\mathcal{O}}_\alpha$ of u is decomposed in

$$
\begin{aligned}
\mathcal{O}_\alpha &:= [0,T) \times \{(x,z) \ : \ 0 < \alpha z < x < z\}\,, \\
\mathcal{B}_0 &:= [0,T] \times \{(0,0)\}\,, \\
\mathcal{B}_\alpha &:= [0,T] \times \{(\alpha z, z) \ : z > 0\}\,, \\
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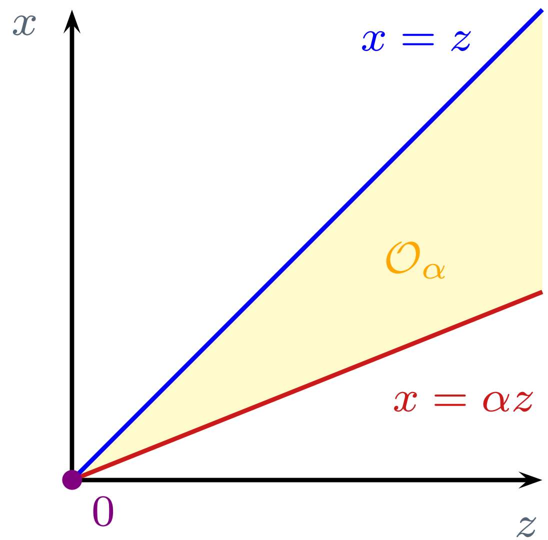

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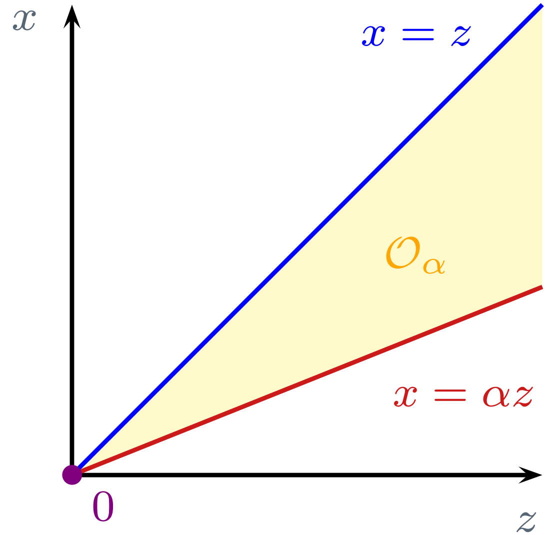

Properties of the value function

$$u(t \searrow, x \nearrow, z \searrow)$$

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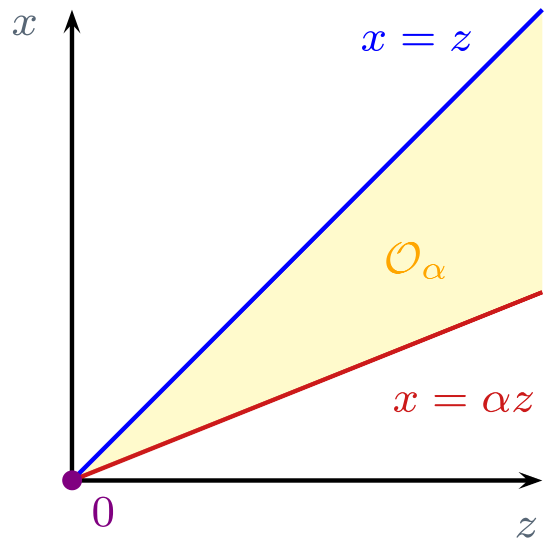

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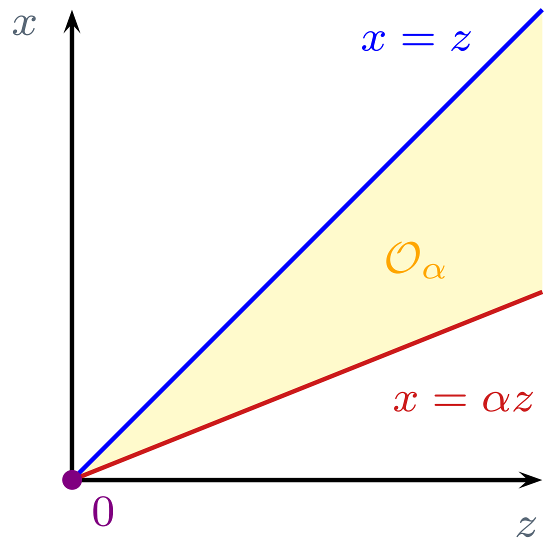

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$u(t \searrow, x \nearrow, z \searrow)$ $\qquad 0 \le u(t,x,z) \le K(1 + x^p)$

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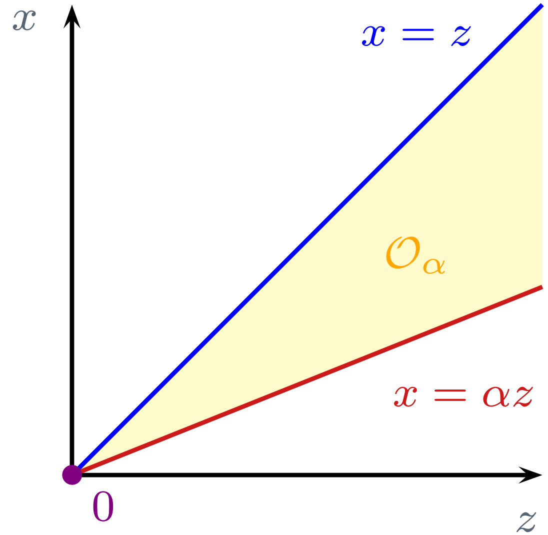

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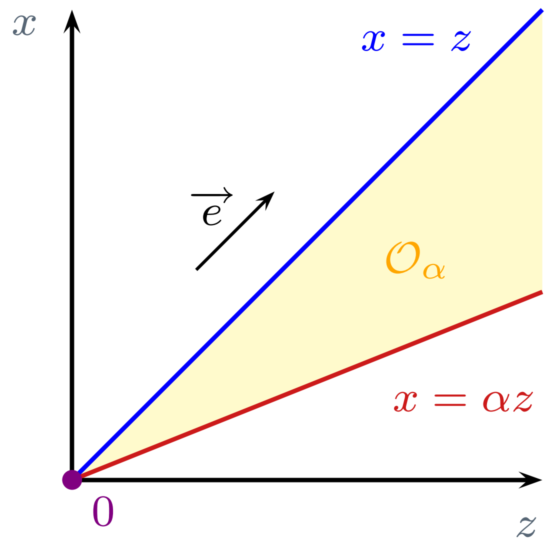

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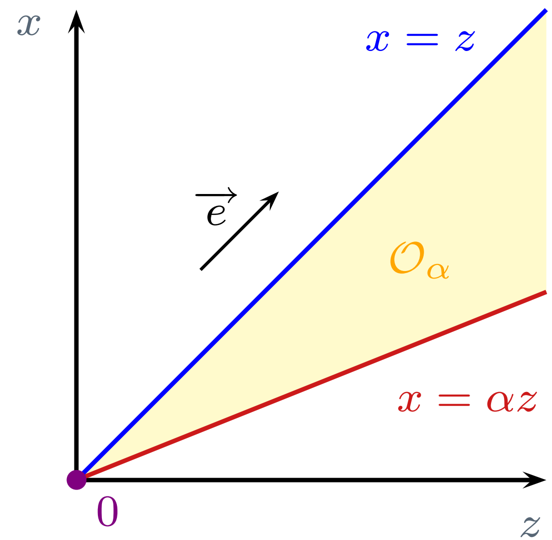

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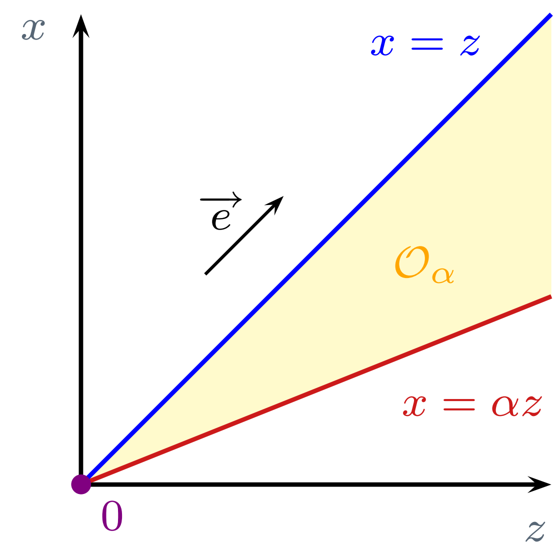

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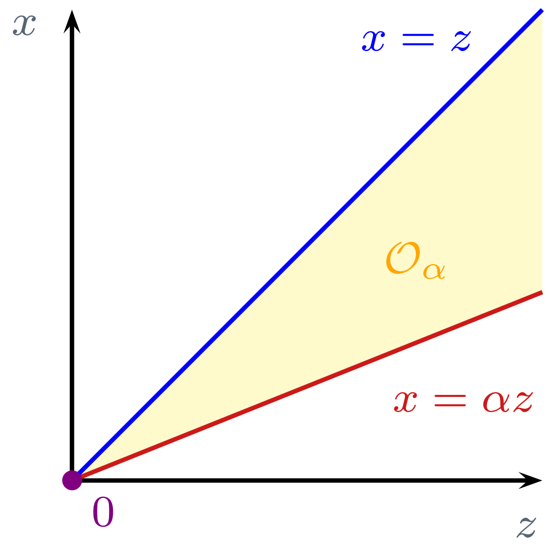
$0 \leq u(t,x,z) \leq u_\infty(x,z)$ $\quad \Rightarrow$ true on $\overline{\mathcal{O}}_\alpha$

Viscosity solution

$$\mathcal{L}\varphi := \varphi_t - \beta\varphi + V(\varphi_x) - \frac{\lambda^2}{2}\frac{\varphi_x^2}{\varphi_{xx}}$$

$$(E)\begin{cases} -\mathcal{L}\varphi = 0 & \text{on } \mathcal{O}_\alpha \\ -\varphi_z = 0 & \text{on } \mathcal{B}_1 \\ \varphi = 0 & \text{on } \mathcal{B}_T \cup \mathcal{B}_0 \end{cases}$$



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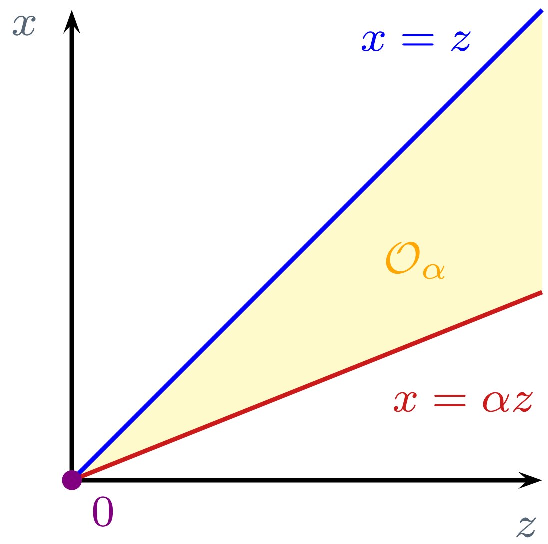

u is a constrained viscosity solution of (E)

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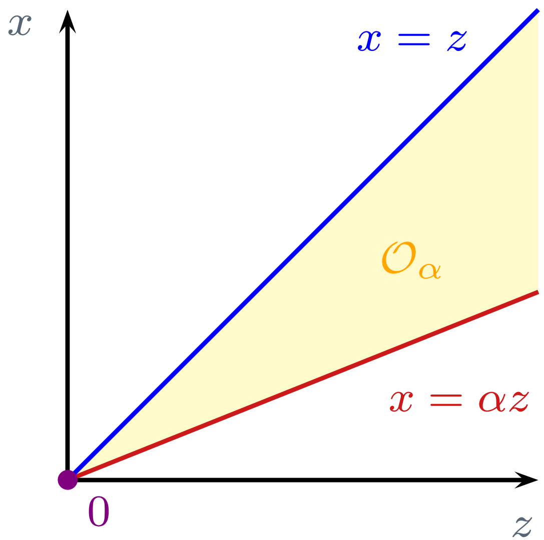


u is a constrained viscosity solution of (E), i.e.

super-solution

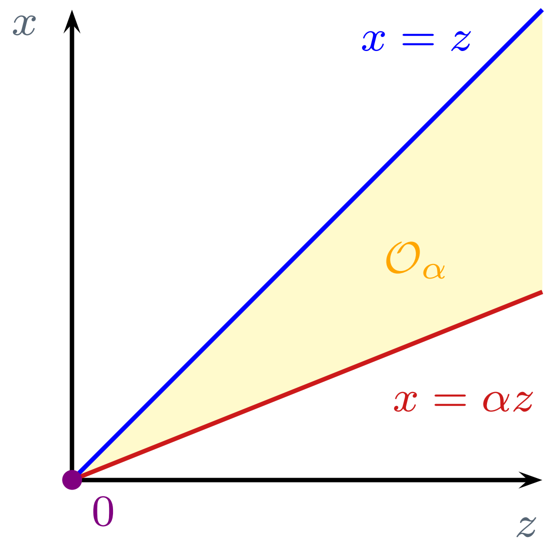
- $u_* \geq 0$ on $\mathcal{B}_T \cup \mathcal{B}_0$,
- $\forall\, y_0 \in \overline{\mathcal{O}}_\alpha$, $\varphi \in C^{1,2,1}(\overline{\mathcal{O}}_\alpha)$ satisf. $0 = (u_* - \varphi)(y_0) = \inf_{\overline{\mathcal{O}}_\alpha}(u_* - \varphi)$:

  $-\mathcal{L}\varphi(y_0) \geq 0$ if $y_0 \in \mathcal{O}_\alpha$ and $-\varphi_z(y_0) \geq 0$ if $y_0 \in \mathcal{B}_1$.

Viscosity solution

$$\mathcal{L}\varphi := \varphi_t - \beta\varphi + V(\varphi_x) - \frac{\lambda^2}{2}\frac{\varphi_x^2}{\varphi_{xx}}$$

$$(E)\begin{cases} -\mathcal{L}\varphi = 0 & \text{on } \mathcal{O}_\alpha \\ -\varphi_z = 0 & \text{on } \mathcal{B}_1 \\ \varphi = 0 & \text{on } \mathcal{B}_T \cup \mathcal{B}_0 \end{cases}$$



u is a constrained viscosity solution of (E), i.e.

super-solution

- $u_* \geq 0$ on $\mathcal{B}_T \cup \mathcal{B}_0$,
- $\forall\, y_0 \in \overline{\mathcal{O}}_\alpha$, $\varphi \in C^{1,2,1}(\overline{\mathcal{O}}_\alpha)$ satisf. $0 = (u_* - \varphi)(y_0) = \inf_{\overline{\mathcal{O}}_\alpha}(u_* - \varphi)$:

  $-\mathcal{L}\varphi(y_0) \geq 0$ if $y_0 \in \mathcal{O}_\alpha$ and $-\varphi_z(y_0) \geq 0$ if $y_0 \in \mathcal{B}_1$.

sub-solution

- $u^* \leq 0$ on $\mathcal{B}_T \cup \mathcal{B}_0$,
- $\forall\, y_0 \in \overline{\mathcal{O}}_\alpha$, $\varphi \in C^{1,2,1}(\overline{\mathcal{O}}_\alpha)$ satisf. $0 = (u^* - \varphi)(y_0) = \sup_{\overline{\mathcal{O}}_\alpha}(u^* - \varphi)$:

  $-\mathcal{L}\varphi(y_0) \leq 0$ if $y_0 \in \mathcal{O}_\alpha \cup \mathcal{B}_\alpha$ and $\min\{-\mathcal{L}\varphi, -\varphi_z\}(y_0) \leq 0$ if $y_0 \in \mathcal{B}_1$

Unicity of the solution

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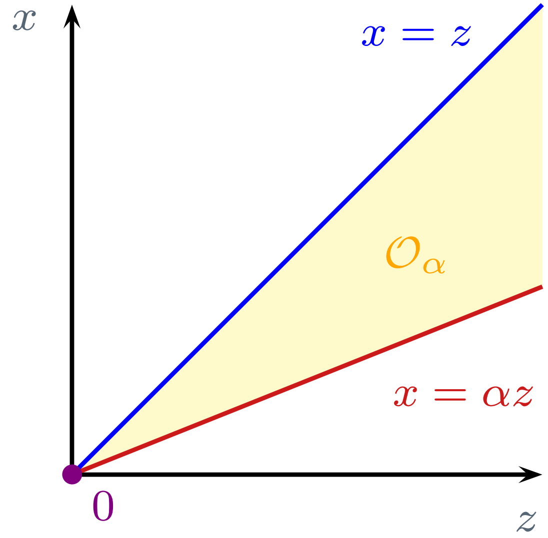

Comparison theorem

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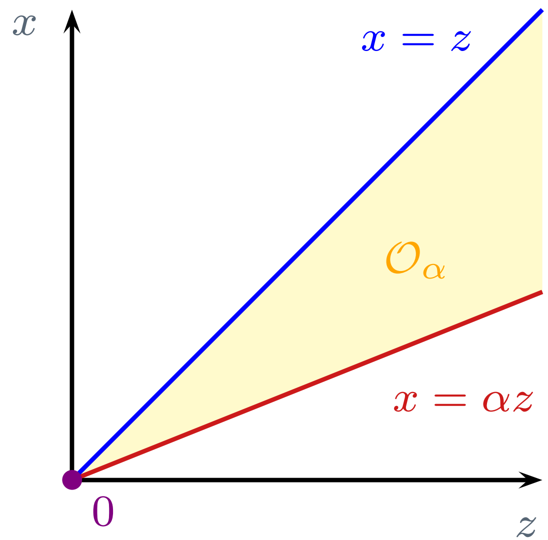

Comparison theorem [Z94]

Let w u.s.c. sub-solution and v l.s.c. super-solution of (E) s.t.

- $([w]^+ + [v]^-)(t,x,z) \le K(1 + x^p)$ on $\overline{\mathcal{O}}_\alpha$

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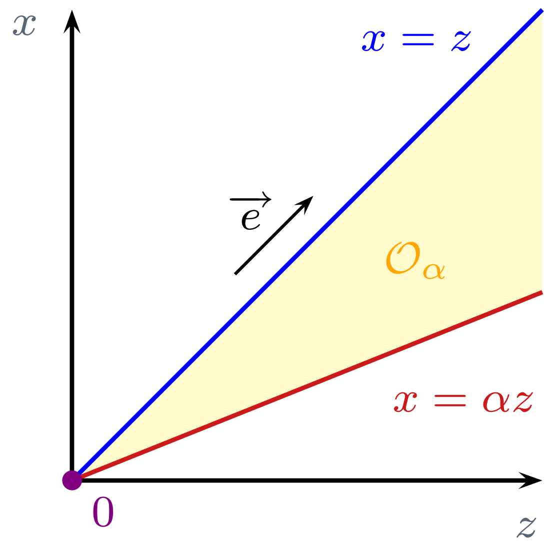

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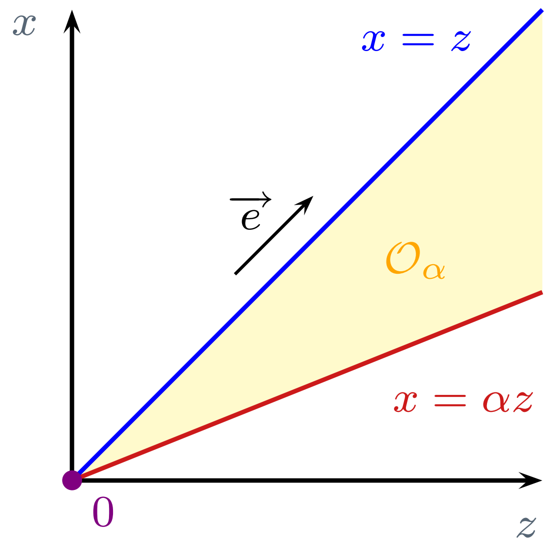

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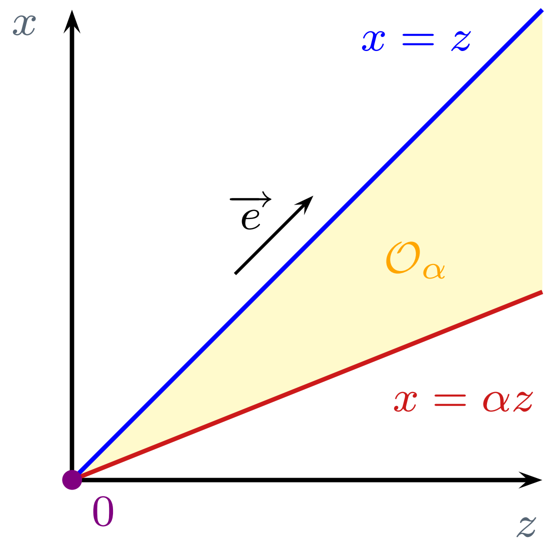

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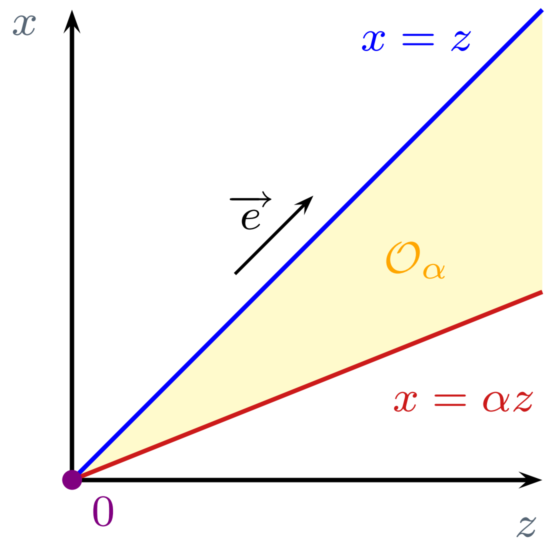

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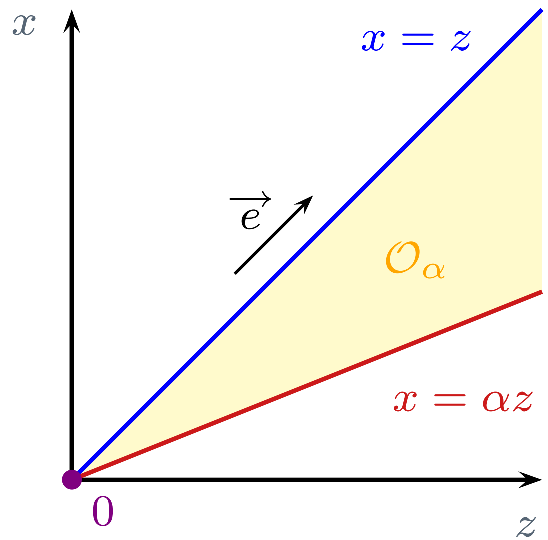

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$\Rightarrow$ Numerical approximation [BS91] [BDR94]

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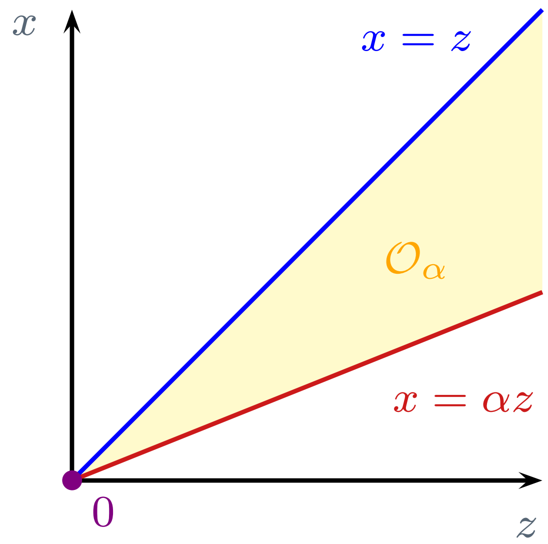

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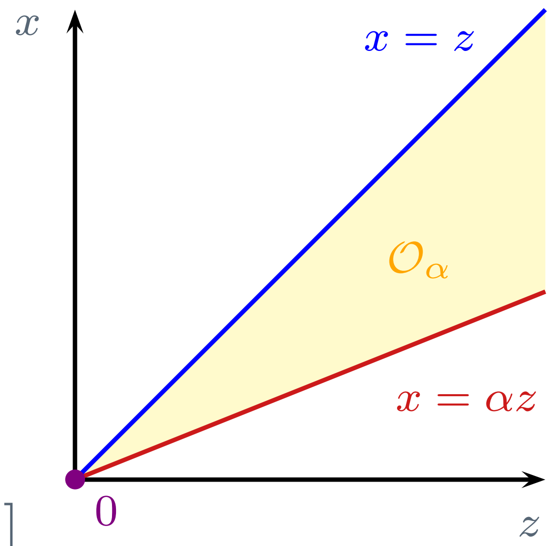

Numerical Approximation [BS91] [BDR94]

Pt of interest $z_0 \Rightarrow$ regular grid $(z_i)_{i \leq N_z}$ of $[0, 2z_0]$

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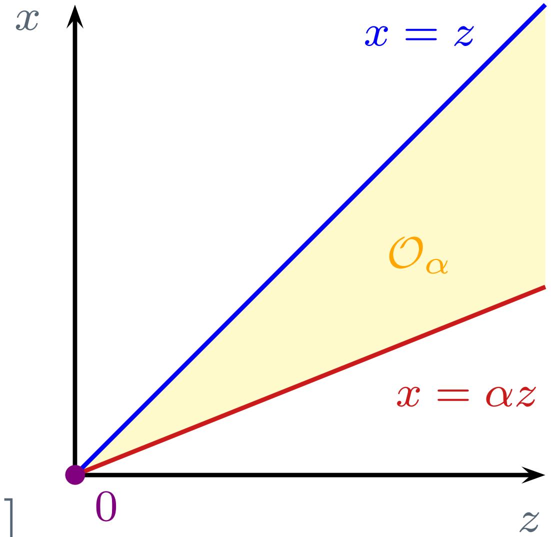

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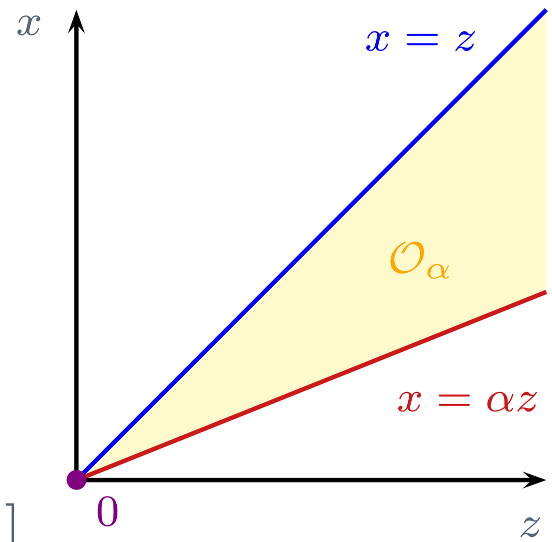

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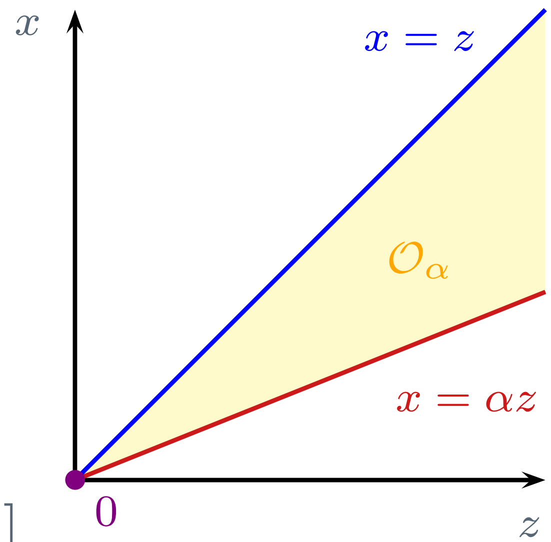

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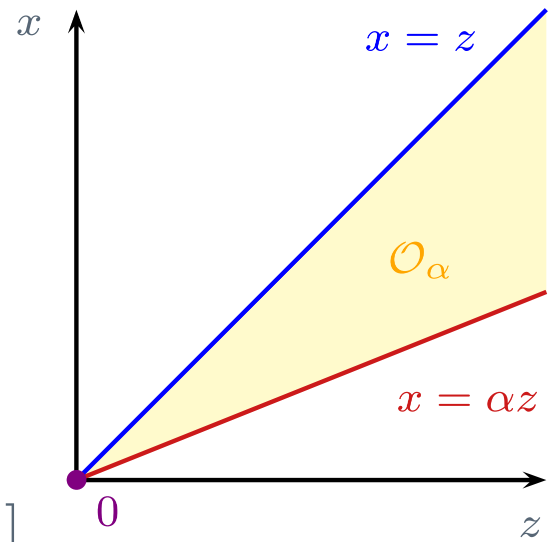

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$$-\frac{\lambda^2}{2}\frac{[u(t_{n+1},x_i^{j+1},z_i)-u(t_{n+1},x_i^j,z_i)]^2}{u(t_{n+1},x_i^{j+1},z_i)+2u(t_n,x_i^j,z_i)-u(t_{n+1},x_i^{j-1},z_i)}\Delta t\,.$$

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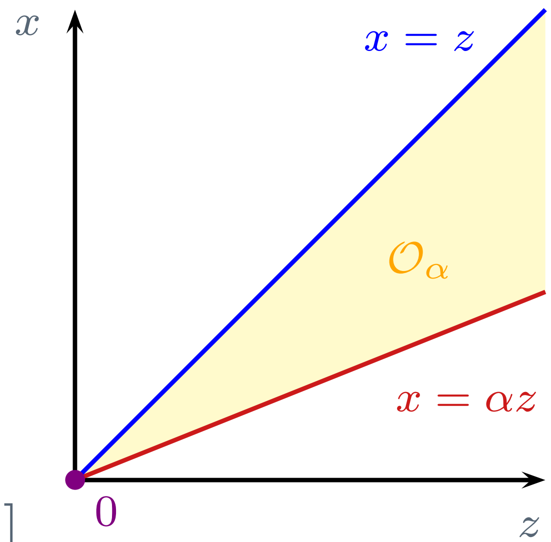

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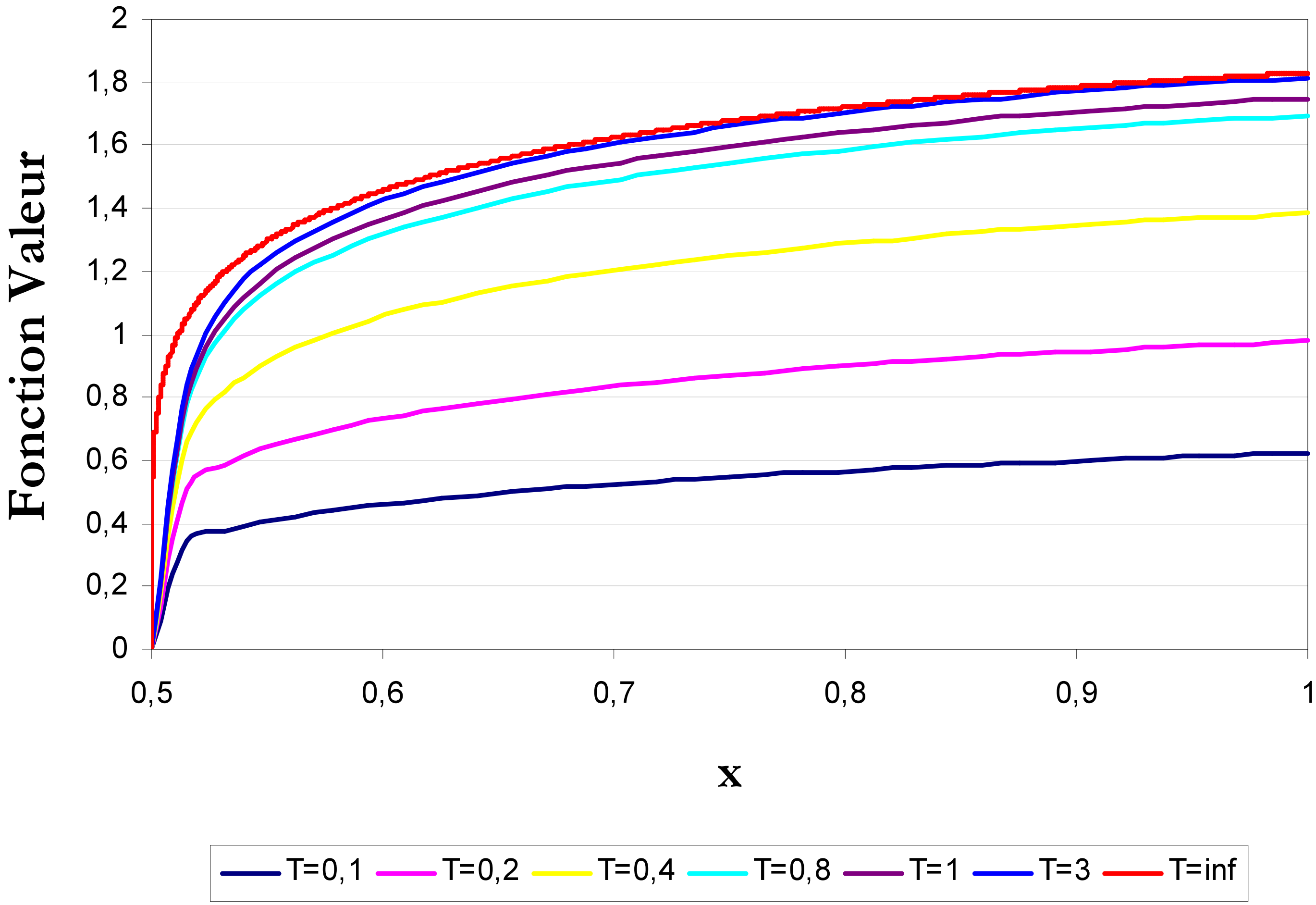
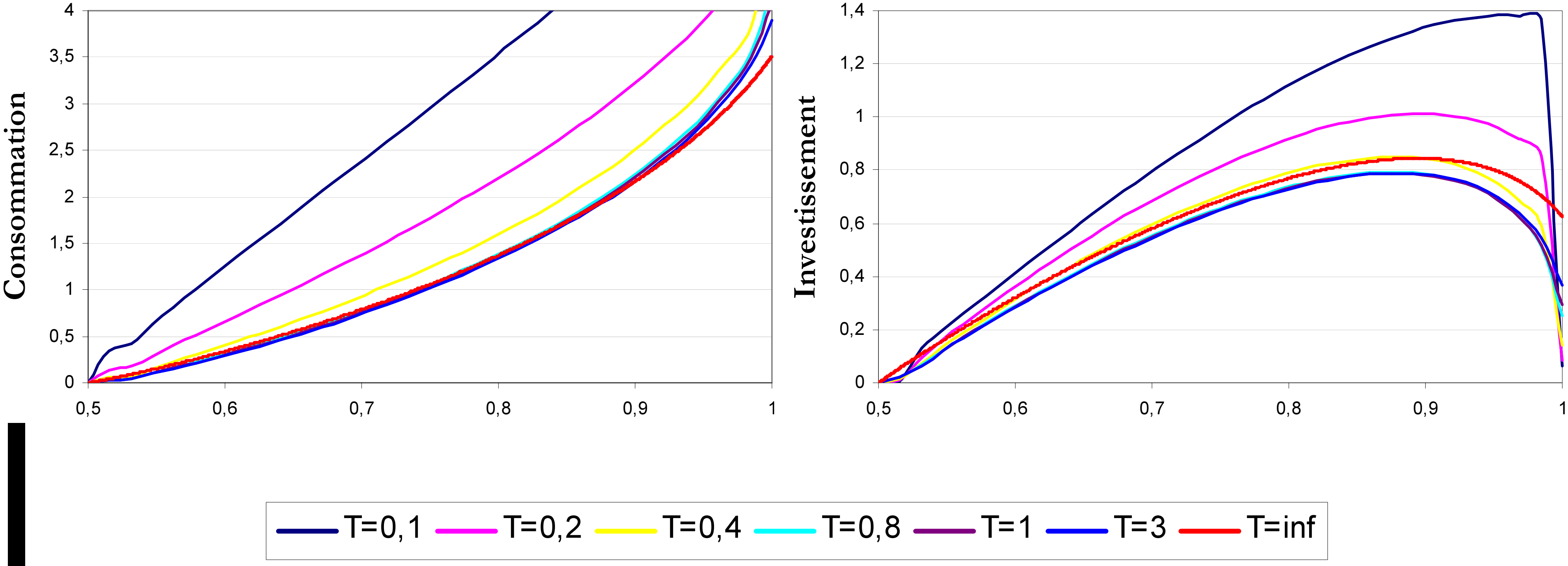
$$\hat{u}(t_n, x_i^j, z_i) := \tilde{u}(t_n, x_i^j, z_i)\mathbf{1}_{x_i^j \le z_i} + \tilde{u}(t_n, x_i^j, x_i^j)\mathbf{1}_{x_i^j > z_i}$$

Value Function $u(T, x, 1)$



Value function $u(T, x, 1)$ **Vs** x for different horizons T

Optimal strategy



Optimal strategy vs x for different horizons T.

Conclusion

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Open questions

- Direct resolution of the primal non linear PDE ?
- Probabilistic resolution of the stochastic control problem ?
- Non linear constraints ?
- Utility of final wealth criterion ?
- Interest rate influence ?

Thanks !

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