# *Erlang(n) risk models with risky investments*

Corina Constantinescu

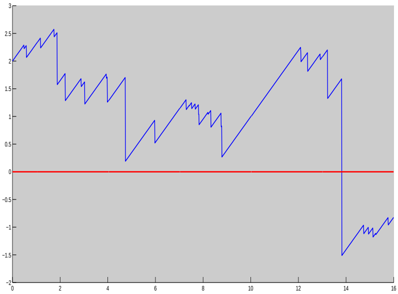
Johann Radon Institute for Computational and Applied Mathematics
Austrian Academy of Sciences

joint work with
H. Albrecher, University of Linz & RICAM
E. Thomann, Oregon State University

Special Semester on Stochastics with emphasis in Finance

RICAM, Linz, December 2nd, 2008

$$U_t = u + ct - \sum_{k=1}^{N(t)} X_k$$




- $u$: initial surplus
- $c$: premium rate
- $N(t)$ = number of claims up to time $t$ (Poisson/renewal)
- $X_k$: claim size (light/heavy)
- $T_k$: time of claim
- $\tau_n = T_n - T_{n-1}$: inter-arrival time ($\tau_0 = 0$)
- $X, \tau$ are independent
- the net profit condition

  $c - \lambda\mu > 0$ holds

- Time of ruin $T_u = \inf_{t\geq 0}\{t : U_t < 0 \mid U_0 = u\}$
- Probability of ruin in finite time $\Psi(u,t) = P(T_u < t)$
- Probability of ruin $\Psi(u) = P(T_u < \infty)$.

# Gerber-Shiu function

$$m_\delta(u) = E\left(e^{-\delta T_u} w(\underbrace{U(T_u-)}_{\text{surpl.im.bef.ruin}}, \underbrace{|U(T_u)|}_{\text{deficit at ruin}}) 1(T_u < \infty) \mid U(0) = u\right)$$

- $w = 1$ (LT of the time of ruin)

$$m_\delta(u) = E\left(e^{-\delta T_u} 1(T_u < \infty) \mid U(0) = u\right) = \phi_\delta(u)$$

$$\int_0^\infty e^{-\delta t} \Psi(u,t)dt = \frac{\phi_\delta(u)}{\delta}$$

- $w = 1$, $\delta = 0$ (Probability of ruin)

$$m_0(u) = E\left(1(T_u < \infty) \mid U(0) = u\right) = \Psi(u)$$

# Gerber-Shiu function on the Cramer-Lundberg model

By conditioning on the time and size of the first claim...

- **Integral equation.**

$$\begin{aligned} m_\delta(u) &= \int_0^\infty \lambda e^{-(\delta+\lambda)t} \int_0^{u+ct} m_\delta(u+ct-y)dF(y)dt \\ &+ \int_0^\infty \lambda e^{-(\delta+\lambda)t} \int_{u+ct}^\infty w(u+ct, y-u-ct)dF(y)dt \end{aligned}$$

with boundary condition, $\lim_{u\to\infty} m_\delta(u) = 0$.

# *Through integration by parts...*

**Integro-differential equation.**

- Gerber-Shiu function:

$$\begin{aligned}(-c\frac{d}{du} + \lambda + \delta)m_\delta(u) &= \lambda \int_0^u m_\delta(u-x) f_X(x)dx \\ &+ \underbrace{\lambda \int_u^\infty w(u, x-u) f_X(x)dx}_{=\omega(u)}\end{aligned}$$

with boundary conditions, $\begin{cases} \lim_{u\to\infty} m_\delta(u) = 0 \\ m_\delta(0) = \frac{\lambda}{c}\hat{\omega}(\rho) \end{cases}$

# *For special penalties...*

- **Gerber-Shiu function**

$$(-c\frac{d}{du} + \lambda + \delta)m_\delta(u) = \lambda \int_0^u m_\delta(u-x) f_X(x)dx + \lambda\omega(u)$$

- **Laplace transform of the time of ruin**

$$(-c\frac{d}{du} + \lambda + \delta)\phi_\delta(u) = \lambda \int_0^u \phi_\delta(u-x) f_X(x)dx + \lambda\overline{F}_X(u)$$

- **Probability of ruin**

$$(-c\frac{d}{du} + \lambda)\Psi(u) = \lambda \int_0^u \Psi(u-x) f_X(x)dx + \lambda\overline{F}_X(u)$$

## Classical results-Probability of ruin

$$(-c\frac{d}{du} + \lambda)\Psi(u) = \lambda \int_0^u \Psi(u-x) f_X(x)dx + \lambda \overline{F}_X(u)$$
$$\lim_{u\to\infty} \Psi(u) = 0$$

- If claim sizes are exponentially bounded (light claims) then

$$\Psi(u) \sim \frac{c-\lambda\mu}{-\lambda \hat{f}_X'(-R) - c} e^{-Ru}, u \to \infty$$

Cramer(1930)

- If claims sizes are heavy-tailed (heavy claims) then

$$\Psi(u) \sim k\overline{F}_I(u), u \to \infty$$

Embrechts et al(1997)

# Classical results-LT of finite-time ruin probability

$$\left(-c\frac{d}{du}+\lambda+\delta\right)\phi_\delta(u)=\lambda\int_0^\infty \phi_\delta(u-x)f_X(x)dx+\lambda\overline{F}_X(u)$$

$$\lim_{u\to\infty}\phi_\delta(u)=0$$

- If light claims then
  - Laplace transform of time of ruin

$$\phi_\delta(u)\sim\frac{\delta}{-\lambda\hat{f}_X'(-R)-c}\left(\frac{1}{R}+\frac{1}{\rho}\right)e^{-Ru},\quad u\to\infty,$$

$$\lim_{\delta\to 0}\phi_\delta(u)=\Psi(u)$$

  - (single) Laplace transform of the finite-time ruin probability

$$\int_0^\infty e^{-\delta t}\Psi(u,t)dt\sim\frac{1}{-\lambda\hat{f}_X'(-R)-c}\left(\frac{1}{R}+\frac{1}{\rho}\right)e^{-Ru},\quad u\to\infty.$$

(Gerber & Shiu, 1998)

# Classical results-Gerber-Shiu functions

$$(-c\frac{d}{du} + \lambda + \delta)m_\delta(u) = \lambda \int_0^u m_\delta(u-x)f_X(x)dx + \lambda\omega(u)$$

$$\lim_{u\to\infty} m_\delta(u) = 0$$

- If light claims then

$$m_\delta(u) \sim \frac{\lambda \int_0^\infty \int_0^\infty w(x,y)(e^{Rx} - e^{-\rho x})f_X(x+y)dxdy}{-\lambda \hat{f}'_X(-R) - c} e^{-Ru}$$

Gerber&Shiu(1998)

# *Sparre Andersen model with investments*

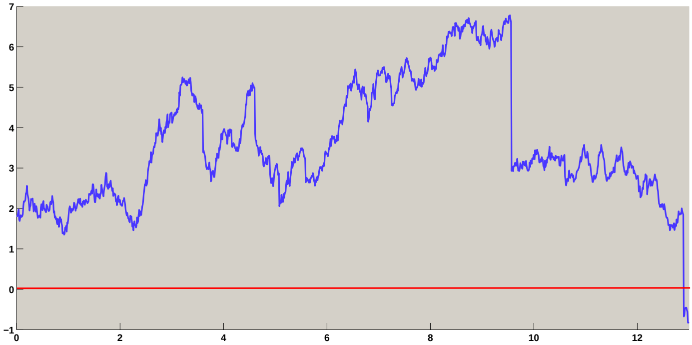
- The claim number process $N(t)$ is a renewal process
- We allow an additional non-traditional feature: investments in a risky asset with returns modeled by a stochastic process $Z_t$, described by an SDE
- Denote $U_k := U(T_k)$. The model

$$U_k = Z_{\tau_k}^{U_{k-1}} - X_k$$

is a discrete Markov process.

We refer to this process as **renewal jump-diffusion process**.

# *Renewal jump-diffusion process*




We assume that the company invests all its money continuously in a risky asset with the price modeled by a geometric Brownian motion.

**Question:**

When we invest everything in a risky asset, do the ruin probabilities have a faster decay than when there is no investment?

**Answer:**

When investments in an asset whose price follows a GBM, the ruin probabilities have a power decay

$$\Psi(u) \sim Cu^{-k}, \quad \text{as} \quad u \to \infty,$$

where $k$ depends on the parameters of the investments or on those of the claim sizes.

# *Objective*

- Analyze the asymptotic behavior of the ruin probability $\Psi(u)$ and the Laplace transform of the time of ruin $\phi_\delta(u)$ (implicitly the Laplace transform of the finite-time ruin probability) as the initial capital (surplus) $u \to \infty$.
- Determine a general integro-differential equation for $m_\delta(u)$.

Main tools

- integration by parts
- regular variation theory

## *Assumptions (equation)*

- **Inter-arrival times** $\{\tau_k\}_{k\geq 0}$ have densities $f_\tau$ satisfying an ODE with constant coefficients

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = 0$$

  with homogeneous or non-homogeneous boundary conditions. (example: $f_\tau(t) = \lambda e^{-\lambda t} \implies (\frac{d}{dt} + \lambda)f_\tau(t) = 0$)

- The price of the **investments** $Z_t^u$ up to time $t$ starting with an initial capital $u$ is modeled by a non-negative stochastic process with an infinitesimal generator $A$

# *Assumptions (asymptotic behavior)*

We identify two cases:

- **Light claims.** Claim sizes $\{X_k\}_{k\geq 0}$ have well-behaved distributions $F_X$ with exponentially bounded tails

$$1 - F_X(x) \leq ce^{-\alpha x}, \ \alpha, c \in \mathcal{R}, \ \forall x \geq 0$$

- **Heavy-tailed claims.** Claim sizes $\{X_k\}_{k\geq 0}$ have regularly varying distribution

$$1 - F_X(x) \sim Cx^{-\alpha} l(x), \quad \text{as} \quad x \to \infty$$

(Notation: $1 - F_X(x) \in \mathcal{R}(-\alpha)$)

where $C$ is a positive constant and $l(x)$ is a slowly varying function.

$$U_k = Z_{\tau_k}^{U_{k-1}} - X_k$$

**Theorem.** Let $h$ be a sufficiently smooth function of the risk process such that $E(h(U_1) \mid U_0 = u) = h(u)$. If $f_\tau$ satisfies the ODE of order $n$, with constant coefficients

$$\mathcal{L}\left(\frac{d}{dt}\right) f_\tau(t) = \sum_{k=0}^{n} \alpha_k \left(\frac{d}{dt}\right)^k f_\tau(t) = 0$$

and homogeneous boundary conditions, then

$$\boxed{\mathcal{L}^*(A)h(u) = \alpha_0 \left(\int_0^u h(u-x) f_X(x)dx + \omega(u)\right)}$$

The **proof** uses semigroup theory, Kolmogorov backward equation and integration by parts.

# *Probability of ruin*

- Since the probability of non-ruin $\Phi(u)$ satisfies the hypothesis and then the IDE.
- As a consequence the probability of ruin also satisfies this IDE

$$\mathcal{L}^*(A)\Psi(u) = \alpha_0 \left( \int_0^u \Psi(u-x) f_X(x)dx + \overline{F}_X(u) \right)$$

$$\begin{cases} \Psi(u) = 1 \quad \text{if} \quad u < 0 \\ \lim_{u\to\infty} \Psi(u) = 0 \end{cases}$$

Recall:

- $A$: infinitesimal generator of the investment process,
- $F_X$ claim sizes distribution
- $\mathcal{L}^*(\frac{d}{dt})$ is adjoint to $\mathcal{L}(\frac{d}{dt}) = \sum_{k=0}^n \alpha_k \left(\frac{d}{dt}\right)^k$

# More IDEs

**Laplace transform of the time of ruin**

$$\mathcal{L}^*(A-\delta)\phi_\delta(u)=\alpha_0\left(\int_0^u \phi_\delta(u-x)f_X(x)dx+\overline{F}_X(u)\right)$$

**Gerber-Shiu function**

$$\mathcal{L}^*(A-\delta)m_\delta(u)=\alpha_0\left(\int_0^u m_\delta(u-x)f_X(x)dx+\omega(u)\right)$$

Recall:

- $A$ infinitesimal generator of the investment process,
- $F_X$ claim sizes distribution
- $\mathcal{L}^*(\frac{d}{dt})$ is adjoint to $\mathcal{L}(\frac{d}{dt})=\sum_{k=0}^n \alpha_k\left(\frac{d}{dt}\right)^k$

# *Classical Cramer-Lundberg model*

- The surplus model:

$$U(t) = u + ct - \sum_{k=0}^{N(t)} X_k.$$

- The **ODE** satisfied by the exponential inter-arrival times

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt} + \lambda)f_\tau(t) = 0 \implies \mathcal{L}^*(\frac{d}{dt}) = (-\frac{d}{dt} + \lambda)$$

- The **SDE** satisfied by the investment process

$$dZ_t^u = cdt; A = c\frac{d}{du}$$

- Then the **IDE** for Gerber-Shiu function

$$\underbrace{(-c\frac{d}{du} + \delta + \lambda)}_{\mathcal{L}^*(A-\delta)} m_\delta(u) = \lambda \int_0^u m_\delta(u-x) f_X(x)dx + \lambda\omega(u)$$

# *Cramer-Lundberg model with investments*

- The surplus model:

$$U(t) = u + ct + \mathbf{a}\int_0^{\mathbf{t}} \mathbf{U(s)ds} + \sigma \int_0^{\mathbf{t}} \mathbf{U(s)dW_s} - \sum_{k=0}^{N(t)} X_k.$$

- The **ODE** satisfied by the exponential inter-arrival times

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt} + \lambda)f_\tau(t) = 0 \implies \mathcal{L}^*(\frac{d}{dt}) = (-\frac{d}{dt} + \lambda)$$

- The **SDE** satisfied by the investment process

$$dZ_t^u = (c + aZ_t^u)dt + \sigma Z_t^u dW_t; A = \frac{\sigma^2 u^2}{2}\frac{d^2}{du^2} + (c + au)\frac{d}{du}$$

- Then the **IDE** satisfied by the probability of ruin

$$(-A + \lambda)\Psi(u) = \lambda \int_0^u \Psi(u-y)dF_X(y)dy + \lambda \overline{F}_X(u)$$

## *Asymptotic behavior of the probability of ruin*

$$\left(-\frac{\sigma^2 u^2}{2}\frac{d^2}{du^2} - (c + au)\frac{d}{du} + \lambda\right)\Psi(u) = \lambda \int_0^u \Psi(u-y)dF_X(y)dy + \lambda \overline{F}_X(u)$$

- For **small volatility** ($\frac{2a}{\sigma^2} > 1$): $\Psi(u) \sim Cu^{-k}$, $u \to \infty$
  - If claim sizes are exponentially bounded (light claims)
    (Norberg&Kalashnikov(2002), Frolova et.al(2002), C.&Thomann(2005))

$$k = \frac{2a}{\sigma^2} - 1$$

  - If claims sizes are regularly varying (heavy-tailed claims)
    (Paulsen(2002))

$$k = \max\left(\alpha, \frac{2a}{\sigma^2} - 1\right)$$

- For **large volatility** ($\frac{2a}{\sigma^2} < 1$): $\Psi(u) = 1$, $\forall u > 0$.
  (Norberg&Kalashnikov(2002), Frolova et.al(2002))

# *Asymptotic behavior of the Laplace transform of ruin*

$$\left(-\frac{\sigma^2 u^2}{2}\frac{d^2}{du^2} - (c + au)\frac{d}{du} + \lambda + \delta\right)\phi_\delta(u) = \lambda\int_0^u \phi_\delta(u-y)dF_X(y)dy + \lambda\overline{F}_X(u)$$

- For **small volatility** ($\frac{2a}{\sigma^2} > 1$): $\phi_\delta(u) \sim Cu^{-k}$, as $u \to \infty$
  - Light claims

$$k = \left(\frac{a}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta}{\sigma^2}}$$

  - Heavy-tailed claims, $\mathcal{R}(-\alpha), \alpha < 0$

$$k = \max\left(\alpha, \left(\frac{a}{\sigma^2} - \frac{1}{2}\right) + \sqrt{\left(\frac{a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta}{\sigma^2}}\right)$$

# *Asymptotic behavior of the Gerber-Shiu function*

$$\left(-\frac{\sigma^2 u^2}{2}\frac{d^2}{du^2}-(c+au)\frac{d}{du}+\lambda+\delta\right)m_\delta(u)=\lambda\int_0^u m_\delta(u-y)dF_X(y)dy+\lambda\omega(u)$$

- For **small volatility** ($\frac{2a}{\sigma^2}>1$): Interplay between penalty and claim size distribution

# *Erlang(n) risk model with investments*

- The **ODE** satisfied by the Erlang(n) inter-arrival times

$$\mathcal{L}(\frac{d}{dt})f_\tau(t) = (\frac{d}{dt} + \lambda)^n f_\tau(t) = 0 \implies \mathcal{L}^*(\frac{d}{dt}) = (-\frac{d}{dt} + \lambda)^n$$

- The **SDE** satisfied by the investment process

$$dZ_t^u = (c + aZ_t^u)dt + \sigma Z_t^u dW_t; A = (c + au)\frac{d}{du} + \frac{\sigma^2 u^2}{2}$$

- Then the **IDE** satisfied by the Gerber-Shiu function

$$(-A+\lambda+\delta)^n m_\delta(u) = \lambda^n \int_0^u m_\delta(u)(u-y)dF_X(y)dy + \lambda^n \omega(u)$$

# *Asymptotic behavior of the probability of ruin*

$$(-A+\lambda)^n\Psi(u) = \lambda^n \int_0^u \Psi(u-y)dF_X(y)dy + \lambda^n \overline{F}_X(u)$$

- For small volatility, $\frac{2a}{\sigma^2} > 1$, $\Psi(u) \sim Cu^{-k}$, as $u \to \infty$
  - If exponentially bounded (**light**) claims

$$k = \frac{2a}{\sigma^2} - 1$$

  - If regularly varying (**heavy-tailed**) claims $(\mathcal{R}(-\alpha), \alpha > 0)$

$$k = \max\left(\alpha, \frac{2a}{\sigma^2} - 1\right)$$

# *Asymptotic behavior of the Laplace transform of ruin*

$$(-A + \lambda + \delta)^n \phi_\delta(u) = \lambda^n \int_0^u \phi_\delta(u-y) dF_X(y) dy + \lambda^n \overline{F}_X(u)$$

- For **"small volatility"** $\frac{2a}{\sigma^2} > 1$, $\phi_\delta(u) \sim Cu^{-k}$, as $u \to \infty$
  - If claim sizes are exponentially bounded (light claims)

$$k = \frac{a}{\sigma^2} - \frac{1}{2} + \sqrt{\left(\frac{a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta}{\sigma^2}}$$

  - If claims sizes are regularly varying (heavy-tailed claims)

$$k = \max\left(\alpha, \frac{a}{\sigma^2} - \frac{1}{2} + \sqrt{\left(\frac{a}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\delta}{\sigma^2}}\right)$$

# *Our method*

Steps in all examples to follow:

1. IDE $(\Psi(u), \phi_\delta(u), m_\delta(u))$
2. Take Laplace transform of the IDE
3. Exploit regularity at zero of the homogeneous part of the ODE satisfied by the Laplace transform $\hat{\Psi}(s), \hat{\phi}_\delta(s), \hat{m}_\delta(s)$
4. Obtain the particular solution of the non-homogeneous ODE
   - Laplace transform of the tail of the claims distribution $\overline{F}_X(u)$
   - Laplace transform of $\omega(u) = \int_u^\infty w(u, x-u) f_X(x) dx$
5. Use Karamata -Tauberian arguments to establish decay rate

# *Cramer-Lundberg with investments*

$$(-\frac{\sigma^2 u^2}{2}\frac{d^2}{du^2}-(c+au)\frac{d}{du}+\lambda)\Psi(u) = \lambda\int_0^u \Psi(u-y)dF_X(y)dy+\lambda\overline{F}_X(u)$$

- Laplace transform

$$(-\hat{A}+\lambda)\hat{\Psi}(s) - \lambda\hat{f}_X\hat{\Psi}(s) = c\Psi(0) + \lambda(\frac{1}{s} - \frac{\hat{f}_X(s)}{s})$$

- 2-nd order ODE: $s^2y^2 + p_1(s)sy + p_2(s) = p_3(s)$
- Homogeneous equation is regular at zero solutions of the form

$$y(s) = \mathbf{s}^{\rho}\sum_{k=0}^{\infty} c_k s^k$$

## Regularity at zero

Determine $\rho$ :

- The coefficient of the $s^\rho$ term should be zero, reduces to the equation

$$\left(\sigma^2(\rho+2)-a\right)(\rho+1)=0$$

- with solutions:
  1. $\rho_1=-1$
  2. $\rho_2=-2+\frac{2a}{\sigma^2}$

$$\Longrightarrow\ \hat{\Psi}(s)=C_1 s^{\rho_1} I_1(s)+C_2 s^{\rho_2} I_2(s)+C_3 P(s)$$

where $P(s)$ is the particular solutions of the non-homogeneous equation obtained through perturbation analysis

## Particular solution

- Light claims: analytic function (does not produce a candidate for the decay)

$$\Longrightarrow \hat{\Psi}(s) = C_1 s^{\rho_1} l_1(s) + C_2 s^{\rho_2} l_2(s) + C_3 l_3(s)$$

- Regularly varying: $\rho_3 = -1 + \alpha$

$$\Longrightarrow \hat{\Psi}(s) = C_1 s^{\rho_1} l_1(s) + C_2 s^{\rho_2} l_2(s) + C_3 s^{\rho_3} l_3(s)$$

where $l_1 l_2, l_3$ are slowly varying functions.

- Cases: $\rho_1 < \rho_2 < \rho_3$ or $\rho_1 < \rho_3 < \rho_2$

## Extensions

1. Same arguments work for Erlang(n) or mixture of exponentials inter-arrival times
2. Stochastic ordering for asymptotic analysis to Gamma of non-integers
3. Fractional investments

$$U(t) = u + ct + \gamma a \int_0^t U(s) ds + \gamma \sigma \int_0^t U(s) dW_S - \sum_{k=0}^{N(t)} X_k.$$

## Conclusions

1. For a Sparre Andersen model, perturbed by a continuous stochastic proces, if the inter-claim arrivals density satisfies a ODE with constant coefficients (Laplace transform is a rational function) a general integro-differential equation can be derived for functions of the risk process
2. For exponential bounded claim sizes (light claims), in an Erlang(n) risk model with investments in a stock modeled by a GBM with small volatility, $\Psi(u)$ has an algebraic decay rate, depending on the parameter of the investments only.
3. For regularly varying claim size distributions (heavy-tailed claims), in Erlang(n) risk models with investments in a GBM with small volatility, the decay rate depends on the parameters of the investment or of the claim size, whichever is larger.

THANK YOU FOR YOUR ATTENTION!