

Stochastic Targets Problems with controlled Loss

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I. Problem formulation and examples

Problem Formulation

- **Dynamics:**

$$S^\pi = s + \int_t^\cdot \mu(S^\pi(u), \pi_u) du + \int_t^\cdot \sigma(S^\pi(u), \pi_u) dW_u$$

$$X^\pi = x + \int_t^\cdot \rho(S^\pi(u), X^\pi(u), \pi_u) du + \int_t^\cdot \beta(S^\pi(u), X^\pi(u), \pi_u) dW_u$$

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- **Loss function:** $\ell : (s, x) \in \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$, non-decreasing in x .

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- **Loss function:** $\ell : (s, x) \in \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$, non-decreasing in x .
- **Non standard control problem :** Given $p \in \text{Im}(\ell)$,

$$v(t, s; p) := \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[\ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right] \geq p \right\} .$$

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- **Aim:** Provide a PDE characterization of $v(\cdot; p)$.

Examples in Finance

Super Hedging

- Model specification

- S^π : stocks (possibly influenced by a large investor's strategy π).
- X^π : portfolio process of the (large) investor.
- $\ell(s, x) = \mathbf{1}\{x \geq g(s)\}$

$$v(t, s; 1) := \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} \left[X_{t,x,s}^\pi(T) \geq g(S_{t,s}^\pi(T)) \right] = 1 \right\}$$

Examples in Finance

Quantile Hedging

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Examples in Finance

Loss Functions

- Model specification

- S^π : stocks (possibly influenced by a large investor's strategy π).
- X^π : portfolio process of the (large) investor.
- $\ell(s, x) = -V([x - g(s)]^-)$ with V convex non decreasing

$$v(t, s; -p) := \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[V(\left[X_{t,x,s}^\pi(T) - g(S_{t,s}^\pi(T)) \right]^-) \right] \leq p \right\}$$

Examples in Finance

Indifference price

- Model specification

- S^π : stocks (possibly influenced by a large investor's strategy π).
- X^π : portfolio process of the (large) investor.
- $\ell(s, x) = U(x - g(s))$ with U concave non decreasing

$$v(t, s; p) := \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[U(X_{t, x_0 + x, s}^\pi(T) - g(S_{t, s}^\pi(T))) \right] \geq p \right\}$$

II. A toy financial model

Model

- Stock price:

$$\frac{dS(u)}{S(u)} = \mu(u, S(u), \pi_u) du + \sigma(u, S(u), \pi_u) dW_u$$

- Wealth process:

$$dX^\pi(u) = X^\pi(u) \pi_u \left(\mu(u, S(u), \pi_u) du + \sigma(u, S(u), \pi_u) dW_u^{\mathbb{Q}} \right)$$

$$v(t, s; p) := \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} \left[X_{t,x,s}^\pi(T) \geq g(S_{t,s}^\pi(T)) \right] \geq p \right\}$$

The case $p = 1$: super-hedging

1. Dual approach when S is independent of π :

$$\begin{aligned} v(t, s; 1) &:= \inf \left\{ x \geq 0 : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} \left[X_{t,s,x}^{\pi}(T) \geq g(S_{t,s}(T)) \right] = 1 \right\} \\ &= \sup_{\mathbb{Q} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}} \left[g(S_{t,s}(T)) \right] \end{aligned}$$

⇒ Standard dynamic programming and PDE derivation.

Problem: specific to “linear” problems, not robust to general specification (e.g. large investors,....).

The case $p = 1$: super-hedging

2. Direct approach of Soner and Touzi:

- (DP1): $x > v(t, s; 1) \Rightarrow \exists \pi \in \mathcal{A}$ s.t. for all stopping time $\tau \leq T$

$$X_{t,s,x}^{\pi}(\tau) \geq v(\tau, S_{t,s}^{\pi}(\tau); 1)$$

- (DP2): $x < v(t, s; 1) \Rightarrow$ for all stopping time $\tau \leq T$ and $\pi \in \mathcal{A}$

$$\mathbb{P}\left[X_{t,s,x}^{\pi}(\tau) > v(\tau, S_{t,s}^{\pi}(\tau); 1)\right] < 1$$

\Rightarrow Non-standard dynamic programming but sufficient for PDE derivation.

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- Take $x = v(t, s; 1)$. There is $\pi \in \mathcal{A}$ s.t. $X_{t,s,x}^\pi(\tau) \geq v(\tau, S_{t,s}^\pi(\tau); 1)$. Then,

$$\begin{aligned} dX_{t,s,x}^\pi(u) &= \pi_u [\mu(u, S^\pi(u), \pi_u) du + \sigma(u, S^\pi(u), \pi_u) dW_u] \\ &\geq dv(u, S_{t,s}^\pi(u); 1) \\ &= \mathcal{L}^\pi v(u, S_{t,s}^\pi(u); 1) du \\ &\quad + D_s v(u, S_{t,s}^\pi(u); 1) \sigma(u, S^\pi(u), \pi_u) dW_u \end{aligned}$$

This leads to

$$\max_{\pi \in \mathcal{G}(t,s)} \pi \mu(t, s, \pi) - \mathcal{L}^\pi v(t, s; 1) \geq 0$$

where $\mathcal{G}(t, s) := \{\pi : \pi \sigma(t, s, \pi) = D_s v(t, s; 1) \sigma(t, s, \pi)\}$

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• Take $x = v(t, s; 1) - \varepsilon$ and assume that there is

$$\pi(t, s) \in \mathcal{G}(t, s) := \{ \pi : \pi \sigma(t, s, \pi) = D_s v(t, s; 1) \sigma(t, s, \pi) \}$$

s.t.

$$\pi(t, s) \mu(t, s, \pi) - \mathcal{L}^{\pi(t, s)} v(t, s; 1) > \eta \quad (\text{"locally"}) .$$

Take $\pi_u := \pi(u, S_{t,s}^{\pi}(u))$. Then,

$$dX_{t,s,x}^{\pi}(u) - dv(u, S_{t,s}^{\pi}(u); 1) > \eta du .$$

This leads to $X_{t,s,x}^{\pi}(\tau) > v(\tau, S_{t,s}^{\pi}(\tau); 1)$ for τ well chosen.

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- Robust to the model specification !! See e.g. B. (2002) for application in Insurance.

The case $p < 1$: quantile hedging

$$v(t, s; p) := \inf \left\{ x \in \mathbb{R} : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} \left[X_{t,x,s}^{\pi}(T) \geq g(S_{t,s}^{\pi}(T)) \right] \geq p \right\}$$

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- Stock price under the (unique) Risk Neutral Measure \mathbb{Q} :

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- Problem Reformulation:

$$\max_{\pi \in \mathcal{A}} \mathbb{P} \left[X_{t,s,x}^{\pi}(T) \geq g(S_{t,s}(T)) \right]$$

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\Leftrightarrow (set formally $A = \{X_{t,s,x}^{\pi}(T) \geq g(S_{t,s}(T))\}$)

$$\max_{A \in \mathcal{F}} \mathbb{P}[A] \quad \text{under} \quad \mathbb{E}^{\mathbb{Q}} \left[g(S_{t,s}(T)) \mathbf{1}_A \right] \leq x$$

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• Foellmer and Leukert's solution:

$$\max \mathbb{P}[A] \quad \text{under} \quad \mathbb{P}^g[A] := \mathbb{E}^{\mathbb{Q}} \left[\frac{g(S_{t,s}(T))}{\mathbb{E}^{\mathbb{Q}}[g(S_{t,s}(T))]} \mathbf{1}_A \right] \leq \frac{x}{\mathbb{E}^{\mathbb{Q}}[g(S_{t,s}(T))]}.$$

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Solved by using **Neyman-Pearson's Lemma**: test \mathbb{P}^g against \mathbb{P}

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The case $p < 1$: quantile hedging

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- **Cons:**

- Explicit solution not known in general (need a way to compute it numerically...)
- Dual problem in incomplete markets is a control problem: how to solve it ?
- Relies heavily on the duality between super-hedgeable claims and risk neutral measures. How to extend this to large investor's problems, non financial problems, ... ?

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- Formal DP:

$$x > v(t, s; \textcolor{blue}{p}) \not\Rightarrow \exists \pi \in \mathcal{A} \text{ s.t. } X_{t,s,x}^{\pi}(\tau) \geq v(\tau, S_{t,s}^{\pi}(\tau); \textcolor{blue}{p})$$

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- **Correction of the formal DP:** $x > v(t, s; p) \Rightarrow \exists \pi \in \mathcal{A}$ s.t.

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and $\mathbb{E}[P] = p$

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and $\mathbb{E}[\textcolor{blue}{P}] = p$ i.e.

$$\textcolor{blue}{P} = p + \int_t^\tau \alpha_u dW_u$$

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2. Direct approach as in Soner and Touzi

- **Dynamic Programming:** Set $P_{t,p}^\alpha = p + \int_t^\cdot \alpha_u dW_u$.

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$$X_{t,s,x}^\pi(\tau) \geq v(\tau, S_{t,s}^\pi(\tau); P_{t,p}^\alpha(\tau))$$

for all stopping time $\tau \leq T$.

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- **Dynamic Programming:** Set $P_{t,p}^\alpha = p + \int_t^\cdot \alpha_u dW_u$.
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$$X_{t,s,x}^\pi(\tau) \geq v(\tau, S_{t,s}^\pi(\tau); P_{t,p}^\alpha(\tau))$$
for all stopping time $\tau \leq T$.
 - (DP2): $x < v(t, s; p) \Rightarrow$ for all stopping time $\tau \leq T$, $\pi \in \mathcal{A}$ and $\alpha \in L^2(dt \times d\mathbb{P})$
$$\mathbb{P} \left[X_{t,s,x}^\pi(\tau) > v(\tau, S_{t,s}^\pi(\tau); P_{t,p}^\alpha(\tau)) \right] < 1$$

III. PDE derivation for stochastic target problems with controlled probability

PDE derivation (formally)

- Take $x = v(t, s; p)$. There is $\pi \in \mathcal{A}$ and $\alpha \in L^2(dt \times d\mathbb{P})$ s.t.

$$X_{t,s,x}^\pi(\tau) \geq v(\tau, S_{t,s}^\pi(\tau); P_{t,p}^\alpha(\tau))$$

PDE derivation (formally)

- Take $x = v(t, s; p)$. There is $\pi \in \mathcal{A}$ and $\alpha \in L^2(dt \times d\mathbb{P})$ s.t.

$$X_{t,s,x}^\pi(\tau) \geq v(\tau, S_{t,s}^\pi(\tau); P_{t,p}^\alpha(\tau))$$

Thus,

$$\begin{aligned} dX_{t,s,x}^\pi(u) &= \pi_u [\mu(u, S^\pi(u), \pi_u) du + \sigma(u, S^\pi(u), \pi_u) dW_u] \\ &\geq dv(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u)) \\ &= \mathcal{L}^{\pi,\alpha}v(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u))du \\ &\quad + D_s v(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u))\sigma(u, S^\pi(u), \pi_u) dW_u \\ &\quad + D_p v(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u))\alpha_u dW_u \end{aligned}$$

PDE derivation (formally)

- Take $x = v(t, s; p)$. There is $\pi \in \mathcal{A}$ and $\alpha \in L^2(dt \times d\mathbb{P})$ s.t.

$$X_{t,s,x}^\pi(\tau) \geq v(\tau, S_{t,s}^\pi(\tau); P_{t,p}^\alpha(\tau))$$

Thus,

$$\begin{aligned} dX_{t,s,x}^\pi(u) &= \pi_u [\mu(u, S^\pi(u), \pi_u) du + \sigma(u, S^\pi(u), \pi_u) dW_u] \\ &\geq dv(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u)) \\ &= \mathcal{L}^{\pi,\alpha}v(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u))du \\ &\quad + D_s v(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u))\sigma(u, S^\pi(u), \pi_u) dW_u \\ &\quad + D_p v(u, S_{t,s}^\pi(u); P_{t,p}^\alpha(u))\alpha_u dW_u \end{aligned}$$

This leads to

$$\max_{(\pi,\alpha) \in \mathcal{G}(t,s,p)} \pi \mu(t, s, \pi) - \mathcal{L}^{\pi,\alpha}v(t, s; p) = 0$$

where $\mathcal{G}(t, s, p) := \{(\pi, \alpha) : \pi \sigma(t, s, \pi) = D_s v(t, s; p) \sigma(t, s, \pi) + D_p v(t, s; p) \alpha\}$

Verification in the quantile hedging problem

- **The Problem:**

$$v(t, s; p) := \inf \left\{ x \in \mathbb{R}_+ : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} \left[X_{t,x,s}^\pi(T) \geq g(S_{t,s}(T)) \right] \geq p \right\} .$$

where

$$dS_{t,s}(r) = S_{t,s}(r) (\mu dt + \sigma dW_r) \quad \text{and} \quad dX_{t,x,s}^\pi(r) = \pi_r dS_{t,s}(r)$$

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- **Associated PDE:**

$$0 = \sup_{\pi \sigma s = \sigma s v_s + \alpha v_p} \left(\pi \mu s - \mu s v_s - \frac{1}{2} \sigma^2 s^2 v_{ss} - \alpha \sigma s v_{sp} - \alpha^2 v_{pp} \right)$$

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where

$$dS_{t,s}(r) = S_{t,s}(r) (\mu dt + \sigma dW_r) \quad \text{and} \quad dX_{t,x,s}^\pi(r) = \pi_r dS_{t,s}(r)$$

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$$0 = \sup_{\substack{\pi \sigma s = \sigma s v_s + \alpha v_p}} \left(\pi \mu s - \mu s v_s - \frac{1}{2} \sigma^2 s^2 v_{ss} - \alpha \sigma s v_{sp} - \alpha^2 v_{pp} \right)$$

- This implies that $v_{pp} \geq 0$ and

$$\begin{aligned} 0 &= \sup_\alpha \left(\frac{\mu}{\sigma} \alpha v_p - v_t - \frac{1}{2} \sigma^2 s^2 v_{ss} - \alpha \sigma s v_{sp} - \alpha^2 v_{pp} \right) \\ &= -v_t - \frac{1}{2} \sigma^2 s^2 v_{ss} + \frac{1}{2} \frac{\left(\frac{\mu}{\sigma} v_p - \sigma s v_{sp} \right)^2}{v_{pp}} \end{aligned}$$

Verification in the quantile hedging problem

- **Associated PDE (bis):** $0 = -v_t - \frac{1}{2}\sigma^2 s^2 v_{ss} + \frac{1}{2} \frac{(\frac{\mu}{\sigma}v_p - \sigma s v_{sp})^2}{v_{pp}}$
- **Boundary conditions:** $v(T-, s, 1) = g(s)$, $v(T-, s, 0) = 0$ and v concave in $p \Rightarrow v(T-, s, p) = pg(s)$

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- **Legendre-Fenchel transform of v with respect to the p -variable:**
$$u(t, s, q) := \sup_{p \in [0, 1]} \{pq - v(t, s, p)\} .$$

Verification in the quantile hedging problem

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- **Legendre-Fenchel transform of v with respect to the p -variable:**
 $u(t, s, q) := \sup_{p \in [0, 1]} \{pq - v(t, s, p)\}$.
a- Boundary conditions: $u(T-, s, q) = (q - g(s))^+$

Verification in the quantile hedging problem

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- **Legendre-Fenchel transform of v with respect to the p -variable:**
 $u(t, s, q) := \sup_{p \in [0, 1]} \{pq - v(t, s, p)\}$.

a- Boundary conditions: $u(T-, s, q) = (q - g(s))^+$

b- Associated PDE:

$$-u_t - \frac{1}{2}\sigma^2 u_{ss} - (\mu/\sigma)q\sigma s u_{sq} - \frac{1}{2}(\mu/\sigma)^2 q^2 u_{qq} = 0$$

Verification in the quantile hedging problem

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 - a- Boundary conditions: $u(T-, s, q) = (q - g(s))^+$
 - b- Associated PDE:
$$-u_t - \frac{1}{2}\sigma^2 u_{ss} - (\mu/\sigma)q\sigma s u_{sq} - \frac{1}{2}(\mu/\sigma)^2 q^2 u_{qq} = 0$$
 - c- Feynman-Kac:

$$u(t, s, q) = \mathbb{E}^{\mathbb{Q}} \left[(Q_{t,q}(T) - g(S_{t,s}(T)))^+ \right] \quad \text{where} \quad \frac{dQ(r)}{Q(r)} = (\mu/\sigma)dW_r^{\mathbb{Q}}$$

Verification in the quantile hedging problem

- Optimal controls: solution to

$$0 = \sup_{\pi \sigma s = \sigma s v_s + \alpha v_p} \left(\pi \mu s - \mu s v_s - \frac{1}{2} \sigma^2 s^2 v_{ss} - \alpha \sigma s v_{sp} - \alpha^2 v_{pp} \right)$$

is given by

$$\hat{\pi} := v_s + \frac{\hat{\alpha}}{s\sigma} v_p , \quad \hat{\alpha} := \frac{\frac{\mu}{\sigma} v_p - \sigma s v_{sp}}{v_{pp}} .$$

Verification in the quantile hedging problem

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$$0 = \sup_{\pi \sigma s = \sigma s v_s + \alpha v_p} \left(\pi \mu s - \mu s v_s - \frac{1}{2} \sigma^2 s^2 v_{ss} - \alpha \sigma s v_{sp} - \alpha^2 v_{pp} \right)$$

is given by

$$\hat{\pi} := v_s + \frac{\hat{\alpha}}{s\sigma} v_p , \quad \hat{\alpha} := \frac{\frac{\mu}{\sigma} v_p - \sigma s v_{sp}}{v_{pp}} .$$

⇒ Retrieve also the dynamics of the probability of hedging $P^{\hat{\alpha}}$!

IV. General results

General framework

- **Dynamics:**

$$S^\pi = s + \int_t^\cdot \mu(S^\pi(u), \pi_u) du + \int_t^\cdot \sigma(S^\pi(u), \pi_u) dW_u$$

$$X^\pi = x + \int_t^\cdot \rho(S^\pi(u), X^\pi(u), \pi_u) du + \int_t^\cdot \beta(S^\pi(u), X^\pi(u), \pi_u) dW_u$$

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- **Problems:** Given ℓ from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} and $p \in \text{Im}(\ell)$ (assumed to be convex),

$$v(t, s; p) := \inf \left\{ x \in \mathbb{R}_+ : \exists \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[\ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right] \geq p \right\}.$$

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- **DP based on the reformulation:**

$$v(t, s; p) = \inf \left\{ x \in \mathbb{R}_+ : \exists (\pi, \alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \geq P_{t,p}^\alpha(T) \right\}$$

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$$\begin{aligned} v(t, s; p) &= \inf \left\{ x \in \mathbb{R}_+ : \exists (\pi, \alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \ell(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T)) \geq P_{t,p}^\alpha(T) \right. \\ &\quad \left. =: w(t, x; p) \right\} \end{aligned}$$

$v(t, s; p) \geq w(t, x; p)$:

For $x > v(t, s; p)$ $\exists \pi$ s.t. $\mathbb{E} [\ell(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T))] \geq p$. Take α so that $P_{t,p}^\alpha(T) = \ell(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T)) + p - \mathbb{E} [\ell(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T))]$.

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$v(t, s; p) \leq w(t, x; p)$:

For $x > w(t, s; p)$ $\exists (\pi, \alpha)$ s.t. $\ell(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T)) \geq P_{t,p}^\alpha(T)$ where $\mathbb{E}[P_{t,p}^\alpha(T)] = p$.

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⇒ Back to Stochastic Target Problems !!!

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⇒ Back to Stochastic Target Problems !!!

⇒ Main difficulty: unbounded controls (new technics...)

Results

- Viscosity characterization for stochastic target problems with unbounded controls (with minimal assumptions)
- Derivation of the boundary conditions for stochastic target problems with controlled probability of loss.

Optimal Control with Stochastic Target Constraints

B. Bouchard, R. Elie and C. Imbert[‡]

2008

[‡]Crest and Ceremade, Paris-Dauphine

I. Problem Formulation and Examples

General framework

- **Dynamics:**

$$\begin{aligned} S^\pi &= s + \int_t^\cdot \mu(S^\pi(u), \pi_u) du + \int_t^\cdot \sigma(S^\pi(u), \pi_u) dW_u \\ X^\pi &= x + \int_t^\cdot \rho(S^\pi(u), X^\pi(u), \pi_u) du + \int_t^\cdot \beta(S^\pi(u), X^\pi(u), \pi_u) dW_u \end{aligned}$$

- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} :

$$V(t, s, x) := \sup_{\pi \in \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}}} \mathbb{E} \left[F \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right]$$

where

$$\bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} := \left\{ \pi \in \mathcal{A} \text{ s.t. } \bar{\ell} \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \geq 0 \right\} .$$

Example 1: Index tracking constraint

- $F(s, x) = U(x)$: utility function. $S^{\pi, 1}$: an index. X^{π} : wealth process.
- Portfolio optimization problem

$$V(t, s, x) := \sup_{\pi \in \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}}} \mathbb{E} \left[U \left(X_{t,x,s}^{\pi}(T) \right) \right]$$

where

$$\bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} := \left\{ \pi \in \mathcal{A} \text{ s.t. } X_{t,x,s}^{\pi}(T)/x_0 \geq 90\% \times S_{t,s}^{\pi, 1}(T)/s_0^1 \right\} .$$

Here, $\bar{\ell}(s, x) := x/x_0 - 90\% \times s^1/s_0^1$.

Example 2: Moment constraints

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} :

$$V(t, s, x; p) := \sup_{\pi \in \mathcal{A}_{t,s,x,p}^\ell} \mathbb{E} \left[F \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right]$$

where $\mathcal{A}_{t,s,x,p}^\ell := \left\{ \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[\ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right] \geq p \right\}$.

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where $\mathcal{A}_{t,s,x,p}^\ell := \left\{ \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[\ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right] \geq p \right\}$.

- **Reformulation:** We have

$$\mathcal{A}_{t,s,x,p}^\ell := \left\{ \pi \in \mathcal{A} \text{ s.t. } \exists \alpha \in L^2 \text{ s.t. } \ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \geq P_{t,p}^\alpha(T) \right\},$$

where $P_{t,p}^\alpha(r) := p + \int_t^r \alpha_u dW_u$.

Example 2: Moment constraints

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} :

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where $\mathcal{A}_{t,s,x,p}^\ell := \left\{ \pi \in \mathcal{A} \text{ s.t. } \mathbb{E} \left[\ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right] \geq p \right\}$.

- **Reformulation:** We have

$$\mathcal{A}_{t,s,x,p}^\ell := \left\{ \pi \in \mathcal{A} \text{ s.t. } \exists \alpha \in L^2 \text{ s.t. } \ell \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \geq P_{t,p}^\alpha(T) \right\}.$$

- Set $\bar{\ell}(s, x, p) := \ell(s, x) - p$ and

$$\bar{\mathcal{A}}_{t,s,x,p}^\ell := \left\{ (\pi, \alpha) \in \mathcal{A} \times L^2 \text{ s.t. } \bar{\ell} \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T), P_{t,p}^\alpha(T) \right) \geq 0 \right\}$$

$$\text{then } V(t, s, x; p) := \sup_{(\pi, \alpha) \in \bar{\mathcal{A}}_{t,s,x,p}^\ell} \mathbb{E} \left[F \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right].$$

Example 2: Moment constraints (special case)

- **Problems:** Given F, ℓ from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} :

$$V(t, s, x; p) := \sup_{\pi \in \mathcal{A}_{t,s,x,p}^\ell} \mathbb{E} \left[F \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right]$$

where $\mathcal{A}_{t,s,x,p}^\ell := \left\{ \pi \in \mathcal{A} \text{ s.t. } \mathbb{P} \left[X_{t,x,s}^\pi(T) \geq g((S_{t,s}^\pi(T))) \right] \geq p \right\}$,

for $\ell(s, x) := \mathbf{1}_{x \geq g(x)}$.

(see Boyle and Tian 07 for dual approach in complete market)

II. PDE Derivation

- **Dynamics:**

$$\begin{aligned} S^\pi &= s + \int_t^\cdot \mu(S^\pi(u), \pi_u) du + \int_t^\cdot \sigma(S^\pi(u), \pi_u) dW_u \\ X^\pi &= x + \int_t^\cdot \rho(S^\pi(u), X^\pi(u), \pi_u) du + \int_t^\cdot \beta(S^\pi(u), X^\pi(u), \pi_u) dW_u \end{aligned}$$

- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} :

$$V(t, s, x) := \sup_{\pi \in \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}}} \mathbb{E} \left[F \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \right]$$

$$\text{where } \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} := \left\{ \pi \in \mathcal{A} \text{ s.t. } \bar{\ell} \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \geq 0 \right\} .$$

- **Dynamics:**

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- **Problems:** Given $F, \bar{\ell}$ from $\mathbb{R}^d \times \mathbb{R}$ into \mathbb{R} :

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where $\bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} := \left\{ \pi \in \mathcal{A} \text{ s.t. } \bar{\ell} \left(S_{t,s}^\pi(T), X_{t,x,s}^\pi(T) \right) \geq 0 \right\}$.

- Set $v(t, s) := \inf \{x \in \mathbb{R} : \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} \neq \emptyset\}$ and $D := \{(t, s, x) : \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} \neq \emptyset\}$. It $\bar{\ell}$ is non-decreasing in x , then $\text{cl}(D) = \text{int}_p D \cup \partial_p D \cup \partial_T D$ with

$$\begin{aligned} \text{int}_p D &:= \{t < T, x > v^*(t, s)\}, \quad \partial_p D := \{t < T, x \in [v_*(t, s), v^*(t, s)]\}, \\ \partial_T D &:= \{t = T, x \geq w_*(T, s)\}. \end{aligned}$$

PDE in the domain $\text{int}_p D$

- Recall that

$$\text{int}_p D := \{t < T, x > v^*(t, s)\} \quad \text{with} \quad v(t, s) := \inf\{x \in \mathbb{R} : \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} \neq \emptyset\}$$

- $x > v^*(t, s) \Rightarrow X_{t,x,s}^\pi(\tau) > v^*(\tau, S_{t,s}^\pi(\tau))$ for $\tau > t$ well chosen and $\pi \in \mathcal{A}$ given.
- Locally can choose any control !
- Associated PDE

$$\inf_{\pi \in A} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0$$

PDE on the spacial boundary $\partial_p D$

- Recall that

$\partial_p D := \{t < T, x \in [v_*(t, s), v^*(t, s)]\}$ with $v(t, s) := \inf\{x \in \mathbb{R} : \bar{\mathcal{A}}_{t,s,x}^{\bar{\ell}} \neq \emptyset\}$

- Assume v is smooth.

If $x = v(t, s)$, we should have $dX_{t,x,s}^{\pi}(t) \geq dv(t, S_{t,s}^{\pi}(t))$.

This implies that

$$\begin{aligned}\pi(t) \in \mathcal{N}(t, s, x, v) := \{&\pi \in A : \beta(s, x, \pi) = \sigma(s, \pi)Dv(t, s), \\ &\rho(s, x, \pi) - \mathcal{L}_S^{\pi}v(t, s) \geq 0\}.\end{aligned}$$

- PDE on $\partial_p D$

$$\inf_{\pi \in \mathcal{N}(t, s, x, v)} \left(-\mathcal{L}_{(S,X)}^{\pi} V(t, s, x) \right) = 0.$$

On the boundary $\partial_T D$

- Recall that

$$\partial_T D := \{t = T, x \geq v_*(t, s)\} \quad \text{with} \quad v(t, s) := \inf\{x \in \mathbb{R} : \bar{\mathcal{A}}_{t,s,x}^{\ell} \neq \emptyset\}$$

- We have the natural boundary condition: $V(T-, s, x) = F(s, x)$.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T, x > v^*(t, s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

PDE formulation: sum up

- On $\text{int}_p D := \{t < T, x > v^*(t, s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

- On $\partial_p D := \{t < T, x \in [v_*(t, s), v^*(t, s)]\}$

$$\inf_{\pi \in \mathcal{N}(t, s, x, v)} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

with $\mathcal{N}(t, s, x, v) :=$

$$\{\pi \in A : \beta(s, x, \pi) = \sigma(s, \pi) Dv(t, s), \rho(s, x, \pi) - \mathcal{L}_S^\pi v(t, s) \geq 0\}.$$

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- On $\partial_T D := \{t = T, x \geq v_*(t, s)\}$: $V(T-, s, x) = F(s, x)$.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T, x > v^*(t, s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

- On $\partial_p D := \{t < T, x \in [v_*(t, s), v^*(t, s)]\}$

$$\inf_{\pi \in \mathcal{N}(t, s, x, v)} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

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$$\{\pi \in A : \beta(s, x, \pi) = \sigma(s, \pi) Dv(t, s), \rho(s, x, \pi) - \mathcal{L}_S^\pi v(t, s) \geq 0\}.$$

- On $\partial_T D := \{t = T, x \geq v_*(t, s)\}$: $V(T-, s, x) = F(s, x)$.

- Already proved:

On $\text{int}_p D$ after relaxing the operator (A may be unbounded).

PDE formulation: sum up

- On $\text{int}_p D := \{t < T, x > v^*(t, s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

- On $\partial_p D := \{t < T, x \in [v_*(t, s), v^*(t, s)]\}$

$$\inf_{\pi \in \mathcal{N}(t, s, x, v)} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

with $\mathcal{N}(t, s, x, v) :=$

$$\{\pi \in A : \beta(s, x, \pi) = \sigma(s, \pi) Dv(t, s), \rho(s, x, \pi) - \mathcal{L}_S^\pi v(t, s) \geq 0\}.$$

- On $\partial_T D := \{t = T, x \geq v_*(t, s)\}$: $V(T-, s, x) = F(s, x)$.

- Already proved:

On $\partial_p D$ when v is continuous (need to express the constraint \mathcal{N} in terms of test functions for v).

PDE formulation: sum up

- On $\text{int}_p D := \{t < T, x > v^*(t, s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

- On $\partial_p D := \{t < T, x \in [v_*(t, s), v^*(t, s)]\}$

$$\inf_{\pi \in \mathcal{N}(t, s, x, v)} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

with $\mathcal{N}(t, s, x, v) :=$

$$\{\pi \in A : \beta(s, x, \pi) = \sigma(s, \pi) Dv(t, s), \rho(s, x, \pi) - \mathcal{L}_S^\pi v(t, s) \geq 0\}.$$

- On $\partial_T D := \{t = T, x \geq v_*(t, s)\}$: $V(T-, s, x) = F(s, x)$.

- Already proved:

On $\partial_p D$ when v is not continuous: the constraints does not appear in the subsolution property.

PDE formulation: sum up

- On $\text{int}_p D := \{t < T, x > v^*(t, s)\}$:

$$\inf_{\pi \in A} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

- On $\partial_p D := \{t < T, x \in [v_*(t, s), v^*(t, s)]\}$

$$\inf_{\pi \in \mathcal{N}(t, s, x, v)} \left(-\mathcal{L}_{(S,X)}^\pi V(t, s, x) \right) = 0 .$$

with $\mathcal{N}(t, s, x, v) :=$

$$\{\pi \in A : \beta(s, x, \pi) = \sigma(s, \pi) Dv(t, s), \rho(s, x, \pi) - \mathcal{L}_S^\pi v(t, s) \geq 0\}.$$

- On $\partial_T D := \{t = T, x \geq v_*(t, s)\}$: $V(T-, s, x) = F(s, x)$.

- Already proved:

On $\partial_T D$ after relaxing the operator (A may be unbounded).

Remaining points to study

1. Comparison principle
2. Numerical schemes
3. Examples