On the impact of correlation on option prices: a Malliavin Calculus approach

Stochastic volatility models allow us to describe the smiles and skews observed in real market data:

\[
dX_t = \left( r - \frac{1}{2} \sigma_t^2 \right) dt + \sigma_t \left( \rho dW_t^* + \sqrt{1 - \rho^2} dB_t^* \right)
\]

- Log-price
- Volatility (stochastic, adapted to the filtration generated by \( W \))

Implied volatility smile \( \rho = 0 \)

Implied volatility skew \( \rho \neq 0 \)
How to quantify the impact of correlation on option prices?

What about the term structure?

We will develop a formula of the form

\[
\text{Option price} = \text{option price in the uncorrelated case} + \text{correction due by correlation}
\]

(classical Hull and White formula)

This result will allow us to describe the impact of the correlation on the option prices. As an application, we can use it to construct option pricing approximation formulas, or to study the short-time behaviour of the implied volatility for stochastic volatility models with jumps.
Here our pourpose is to present the basic concepts on Malliavin calculus that have been used up to now in financial applications. Basically, we will see how to:

Calculate the Malliavin derivative of a diffusion process

Use the duality relationship between the Malliavin derivative and the Skorohod integral to develop adequate change-of-variable formulas for anticipating processes

MAIN IDEA: THE FUTURE INTEGRATED VOLATILITY IS AN ANTICIPATING PROCESS
Malliavin derivative: definition

\[ \{ W(h), h \in L^2([0,T]) \} \]

Gaussian process

\[ F = f(W(h_1), W(h_2), \ldots, W(h_n)) \]

random variable in \( L^2([0,T] \times \Omega) \)

\[ D_i F = \sum \frac{\partial f}{\partial x_i} (W(h_1), W(h_2), \ldots, W(h_n)) h_i(t) \]

Malliavin Derivative in \( L^2([0,T] \times \Omega) \)

(closable operator)
Malliavin derivative: examples

\[ D_t^W W_s = 1_{[0,s]}(t) \]

\[ S_t = \exp \left\{ \left( r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\} \text{(Black-Scholes)} \]

\[ D_r^W S_t = \sigma S_t 1_{[0,t]}(r) \]

\[ Y_t = m + (Y_0 - m)e^{-\alpha t} + c \int_0^t e^{-\alpha(t-r)} dW_r \]

\[ (Ornstein-Uhlenbeck) \]

\[ D_r^W Y_t = ce^{-\alpha(t-r)} 1_{[0,t]}(r) \]
Skorohod integral: definition

It is the adjoint of the Malliavin derivative operator:

\[ E \left( \delta^W(u) F \right) = E \int_0^T \left( D^W_s F \right) u_s ds , \text{ for all } F \in S \]

Example:

\[ h \in L^2([0,T]) \Rightarrow \delta^W(h) = W(h) \]
Skorohod integral: properties

The Skorohod integral of a process multiplied by a random variable

\[
\int_0^T F u_s dW_s = F \int_0^T u_s dW_s + \int_0^T \left( D^W_s F \right) u_s \, ds
\]

The Skorohod integral is an extension of the classical Itô integral
SOME PRELIMINARIES ON STOCHASTIC CALCULUS FOR
ANTICIPATING PROCESSES (VI)

THE ANTICIPATING ITÔ’S FORMULA

\[ X_t = X_0 + \int_0^t u_s dW_s + \int_0^t v_s ds \]

Non necessarily adapted

\[ F(X_t) = F(X_0) + \int_0^t F'(X_s)u_s dW_s \]
\[ + \int_0^t F'(X_s)v_s ds + \frac{1}{2} \int_0^t F''(X_s)(\nabla u)_s u_s ds, \]

where \((\nabla u)_s := u_s + 2\int_0^s D^W_s u_r dW_r + 2\int_0^s D^W_s v_r dr\)
Proof (sketch)

For the sake of simplicity we assume \( \nu \equiv 0 \).

We proceed as in the proof of the classical Itô’s formula

\[
F(X_t) = F(X_0) + \sum_i F'(X_{t_i}) \left( \int_{t_i}^{t_{i+1}} u_s \, dW_s \right) + \frac{1}{2} \sum_i F''(X_{t_i}) \left( \int_{t_i}^{t_{i+1}} u_s \, dW_s \right)^2 + \frac{1}{2} \int_0^t u_s^2 \, ds
\]
\[
\sum F'(X_{t_i}) \left( \int_{t_i}^{t_{i+1}} u_s dW_s \right) \\
= \sum \int_{t_i}^{t_{i+1}} F'(X_{t_i}) u_s dW_s - \sum \int_{t_i}^{t_{i+1}} D_s^W F'(X_{t_i}) u_s ds \\
\frac{1}{2} \int_0^t F'(X_s) u_s dW_s \\
\int_0^t F''(X_s) \left( \int_0^s D_s^W u_r dW_r \right) u_s ds \\
\int_0^t D_s^W [F'(X_s)] u_s ds
\]
Basic idea

The Black-Scholes function final condition implies that

\[ V_T = BS \left( T, X_T; \nu_T \right) \]

Black-Scholes pricing formula

Log-price

\[ \nu_t^2 = \frac{1}{T-t} \int_t^T \sigma_s^2 ds \]

where

\[ V_t = E^* \left[ e^{-r(T-t)} H \bigg| F_t \right] \]

option price

payoff
Then

\[ V_t = e^{-r(T-t)} E^* \left( V_T | F_t \right) \]
\[ = E^* \left[ BS(T, X_T; \nu_T) | F_t \right] \]

The classical Hull and White term

\[ E^* \left( BS(t, X_t; \nu_t) | F_t \right) \]

is the option price in the uncorrelated case.
Then we want to evaluate the difference

\[ BS \left( T, X_T ; \nu_T \right) - BS \left( t, X_t ; \nu_t \right) \]

\( \nu_t \) is an anticipating process

We need to construct an adequate anticipating Itô’s formula
Anticipating Itô’s formula

\[ F(t, X_t, Y_t) = F(0, X_0, Y_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s, Y_s)ds \]
\[ + \int_0^t \frac{\partial F}{\partial x}(s, X_s, Y_s)dX_s + \int_0^t \frac{\partial F}{\partial y}(s, X_s, Y_s)dY_s \]
\[ + \int_0^t \frac{\partial^2 F}{\partial x \partial y}(s, X_s, Y_s)(D^-Y)_s u_s ds \]

additional term
\[ + \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s, Y_s)u_s^2 ds \]

\[ Y_t = \int_t^T \theta_s ds \]
\[ (D^-Y)_s := \int_s^T D^W_s \theta_r dr \]
Main result:

the extension of the Hull and White formula

We apply the above Itô’s anticipating formula to the process

\[ e^{-rt} BS\left(t, X_t; \nu_t\right) = e^{-rt} BS\left(t, X_t; \sqrt{\frac{1}{T-t}} Y_t\right) \]

and we obtain
AN EXTENSION OF THE HULL AND WHITE FORMULA (VI)

\[ e^{-rT} V_T = e^{-rT} BS(T, X_T, \nu_T) = e^{-rT} BS(t, X_t, \nu_t) \]

\[ + \int_t^T e^{-rs} \left[ L_{BS} (\nu_s) + \frac{1}{2} \left( \sigma_s^2 - \nu_s^2 \right) \left( \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial x} \right) \right] BS(s, X_s, \nu_s) ds \]

\[ + \int_t^T e^{-rs} \frac{\partial BS}{\partial x} (s, X_s, \nu_s) \sigma_s \left( \rho dW_s^* + \sqrt{1 - \rho^2} dZ_t^* \right) \]

\[ + \frac{\rho}{2} \int_0^t e^{-rs} \frac{\partial^2 BS}{\partial x \partial \sigma} (s, X_s, \nu_s) \frac{1}{\nu_s(T-s)} (D-Y)_s \sigma_s ds \]

\[ - \frac{1}{2} \int_t^T e^{-rs} \frac{\partial BS}{\partial \nu} (s, X_s, \nu_s) \left( \frac{\sigma_s^2 - \nu_s^2}{\nu_s(T-s)} \right) ds \]

Black-Scholes differential operator

Cancel

Zero expectation
AN EXTENSION OF THE HULL AND WHITE FORMULA (VII)

\[
e^{-rT} E^* \left( V_T \mid F_t \right) = e^{-rt} E^* \left( BS(t, X_t, \nu_t) \mid F_t \right) + \frac{\rho}{2} \int_0^t e^{-rs} \frac{\partial^2 BS}{\partial \chi \partial \sigma} (s, X_s, \nu_s) \left( \frac{1}{\nu_s (T - s)} \right) (D^{-Y})_s \sigma_s ds
\]

\[
\left( \frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2} \right) (s, X_s, \nu_s) = H(s, X_s, \nu_s) \quad \Lambda_s = \left( \int_s^T D_s^w \sigma_r^2 dr \right) \sigma_s
\]

\[
V_t = e^{-r(T-t)} E^* \left( V_T \mid F_t \right)
= E^* \left( BS(t, X_t, \nu_t) \mid F_t \right) + \frac{\rho}{2} E^* \left( \int_0^t e^{-r(s-t)} H(s, X_s, \nu_s) \Lambda_s ds \mid F_t \right)
\]
The above arguments do not require the volatility to be Markovian.

The main contribution of this formula is to describe the effect of the correlation as the term

$$\frac{\rho}{2} E^* \left( \int_0^t e^{-r(s-t)} H(s, X_s, \nu_s) \Lambda_s \, ds \bigg| F_t \right)$$
Consider the approximation

\[ V_{approx} = BS\left(t, X_t, \nu_t^*\right) + \frac{\rho}{2} H\left(t, X_t, \nu_t^*\right) E^*\left(\int_t^T \Lambda_s dS \bigg| F_t \right), \]

\[ \nu_t^* = \sqrt{\frac{1}{T-t} \int_t^T E^*\left(\sigma_s^2 \bigg| F_t \right) dS} \]
Using Malliavin calculus again we can see that, in the case

\[ \sigma_s = f(Y_s), \text{ with} \]

\[ dY_t = \alpha(m - Y_t)dt + \lambda\sqrt{\alpha}dW_t \]

and \( f \) regular enough

\[ |V_t - V_{\text{aprox}}| \leq C \frac{\lambda^2}{\alpha} (1 + |\ln \alpha|) \]

A similar result was proven by Alòs and Ewald (2008) for the Heston volatility model.
APPLICATIONS TO OPTION PRICING APPROXIMATION (III)

Application:

the Stein and Stein model with correlation

\[
\sigma_t = Y_t, \quad c = \lambda \sqrt{\alpha},
\]

\[
\left( \nu_t^* \right)^2 = \frac{1}{T-t} \int_t^T \left( M_t^2(s) + c^2 F(s-t) \right) ds
\]

\[
M_t(s) = m + (\sigma_t - m) e^{-\alpha(s-t)}, \quad F(u) = \int_0^u e^{-2\alpha \theta} d\theta
\]

\[
E^* \left( \int_t^T \left( \int_s^T Y_r e^{-\alpha(r-s)} dr \right) Y_s ds \bigg| F_s \right)
\]

\[
= c \int_t^T M_t(s) \left( \int_s^T e^{-\alpha(r-s)} M_t(r) dr \right) ds + c^2 \int_t^T F(T-s) F(s) ds
\]
Numerical results


t = 0.5, X_t = \ln100, \alpha = 4, m = 0.2, \lambda = 0.05, r = 0.0953, \sigma_t = 0.2

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Example: long-memory volatilities

Assume that (see for example Comte, Coutin and Renault (2003)),

$$\sigma_t^2 = \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} \tilde{\sigma}_s^2 ds,$$

where $\tilde{\sigma}_s = f(Y_s)$

$$\int_s^T \sigma_r^2 dr = \frac{1}{\Gamma(\beta + 1)} \int_s^T (T - r)^\beta \tilde{\sigma}_r^2 du,$$

$$+ \int_s^T \left( \int_0^r \frac{(r-u)^{\beta-1}}{\Gamma(\beta)} \tilde{\sigma}_u^2 du \right) dr$$
\[
\int_s^T D_s^{W^*} \sigma_r^2 dr = \frac{1}{\Gamma(\beta + 1)} \int_s^T (T - r)^\beta D_s^{W^*} \tilde{\sigma}_r^2 dr
\]

\[
E^* \left( \int_t^T \Lambda_s ds \bigg| F_t \right)
\]

\[
= E^* \left( \frac{\lambda \sqrt{\alpha \rho}}{\Gamma(\beta + 1)} \int_t^T \left( \int_s^T (T - r)^\beta D_s^{W^*} \tilde{\sigma}_r^2 dr \right) \tilde{\sigma}_s ds \bigg| F_t \right)
\]

\[
= \frac{2\lambda \sqrt{\alpha \rho}}{\Gamma(\beta + 1)}
\]

\[
\times E^* \left( \int_t^T \left( \int_s^T (T - r)^\beta e^{-\alpha(r-s)} f(Y_r)f'(Y_r)dr \right) f(Y_s)ds \bigg| F_t \right)
\]
In Alòs, León and Vives (2007) we considered the following model for the log-price of a stock under a risk-neutral probability $Q$:

$$X_t = x + (r - \lambda k)t - \frac{1}{2} \int_0^t \sigma_s^2 ds + \int_0^t \sigma_s \left( \rho dW_s + \sqrt{1 - \rho^2} dB_s \right) + Z_t, \; t \in [0, T]$$

Adapted to the filtration generated by $W$

Independent

Compound Poisson with intensity $\lambda$

and Lévy measure $\nu = \lambda f(y)$,

with $k = \frac{1}{\lambda} \int (e^y - 1) \nu(dy) < \infty$
APPLICATIONS TO THE STUDY OF THE SHORT-TIME BEHAVIOUR OF THE IMPLIED VOLATILITY (II)

Similar arguments as in the previous paper give us the following extension of the Hull and White formula:

\[
V_t = E\left( BS\left( t, X_t, \nu_t \right) \bigg| F_t \right) \quad \text{Hull and White}
\]

\[
+ \frac{\rho}{2} E\left( \int_0^t e^{-r(s-t)} \partial_x G(s, X_s, \nu_s) \Lambda_s \, ds \bigg| F_t \right) \quad \text{Correlation}
\]

\[
+ E\left( \int_t^T \int_R e^{-r(s-t)} \left( BS\left( s, X_s + y, \nu_s \right) - BS\left( s, X_s, \nu_s \right) \right) \nu(dy)ds \bigg| F_t \right)
\]

\[
- \lambda k E\left( \int_t^T e^{-r(s-t)} \partial_x BS\left( s, X_s, \nu_s \right) ds \bigg| F_t \right) \quad \text{Jumps}
\]

\[
\nu(t) = \sqrt{\frac{Y_t}{T-t}} ; \quad G(t, x, \sigma) = \left( \partial_{xx} - \partial_x \right) BS\left( t, x, \sigma \right)
\]
APPLICATIONS TO THE STUDY OF THE SHORT-TIME BEHAVIOUR OF THE IMPLIED VOLATILITY (III)

After some algebra, we can prove from this expression that:

If \( E \left[ \left( D_s \sigma_r \right)^2 \bigg| F_t \right] \leq C(r - s)^{2\delta}, \delta \geq 0 \)

and

\[
\lim_{r \downarrow t} \left[ D_t \sigma_r \right] = D_t^+ \sigma_t
\]

If \( E \left[ \left( D_s \sigma_r \right)^2 \bigg| F_t \right] \leq C(r - s)^{2\delta}, \delta < 0 \)

and

\[
\frac{\partial I_t}{\partial X_t} \left( x_t^* \right) \rightarrow -\frac{\rho}{\sigma_t} D_t^+ \sigma_t - \frac{\lambda k}{\sigma_t}
\]

\[
\lim_{(T-t)^{-\delta}} \frac{\partial I_t}{\partial X_t} \left( x_t^* \right) = -\frac{\rho}{\sigma_t} L_t^{\delta,+} \sigma_t
\]
Example 1: classical jump-diffusion models

Assume that the volatility process can be written as

$$\sigma_r = f(Y_r),$$

$$dY_r = a(r, Y_r)dr + b(r, Y_r)dW_r$$

$$D_s Y_r = \int_s^r \frac{\partial a}{\partial x}(u, Y_u)D_s Y_u du + b(s, Y_s)$$

$$+ \int_s^r \frac{\partial b}{\partial x}(u, Y_u)D_s Y_u dW_u$$

$$D_t^+ \sigma_t = f'(Y_t)b(t, Y_t)$$
APPLICATIONS TO THE STUDY OF THE SHORT-TIME BEHAVIOUR OF THE IMPLIED VOLATILITY (V)

\[
\lim_{T \to t} \frac{\partial I_t}{\partial x}(x_t^*) = -\frac{1}{\sigma_t} \left( \lambda k + \rho f'(Y_t)b(t, Y_t) \right)
\]

If \( Y \) is an Ornstein-Uhlenbeck process of the form

\[
Y_r = m + (Y_t - m)e^{-\alpha(r-t)} + c \int_t^r \sqrt{2\alpha}e^{-\alpha(r-s)} dW_s
\]

\[
\lim_{T \to t} \frac{\partial I_t}{\partial x}(x_t^*) = -\frac{1}{\sigma_t} \left( \lambda k + \rho c \sqrt{2\alpha} f'(Y_t) \right)
\]

(this agrees with the results in Medvedev and Scaillet (2004))
Example 2: fractional stochastic volatility models with $H > 1/2$

Assume that the volatility process can be written as

$$
\sigma_r = f(Y_r); \quad Y_r = m + (Y_t - m)e^{-\alpha(r-t)} + c\sqrt{2\alpha} \int_t^r e^{-\alpha(r-s)}dW_r^H
$$

$$
\left( H - \frac{1}{2} \right) \int_t^r \left( \int_s^r e^{-\alpha(r-u)}(u-s)^{H-\frac{1}{3}}du \right) dW_s
$$

$$
D_t^+ \sigma_t = 0
$$

That is, the at-the-money short-dated skew slope is not affected by the correlation in this case.
Example 3: fractional stochastic volatility models with $H<1/2$

Assume that the volatility process can be written as

$$\sigma_r = f(Y_r); \quad Y_r = m + (Y_t - m)e^{-\alpha(r-t)} + c\sqrt{2\alpha}\int_t^r e^{-\alpha(r-s)}dW_r^H$$

$$+ \int_t^r e^{-\alpha(r-s)}(r-s)^{H-\frac{1}{2}}dW_s$$

$$+ \left(\frac{1}{2} - H\right)\int_t^r \left(\int_s^r (e^{-\alpha(r-u)} - e^{-\alpha(r-s)}) (u-s)^{H-\frac{1}{3}} du\right) dW_s$$
That is, the introduction of fractional components with Hurst index $H<1/2$ in the definition of the volatility process allows us to reproduce a skew slope of order

$$O(T - t)^{\delta}, \quad \delta > -\frac{1}{2}$$

More similar to the ones observed in empirical data (see Lee (2004))
Example 4: Time-varying coefficients

(Fouque, Papanicolaou, Sircar and Solna (2004))

Assume that the volatility process can be written as

\[ \sigma_r = f(Y_r) \]

\[ Y_r = m + (Y_t - m)e^{-\int_t^r \alpha(s)ds} + c \int_t^r \sqrt{2\alpha(s)}e^{-\alpha(r-s)} dW_r , \]

\[ \sqrt{\alpha(s)} = (T - s)^{-\frac{1}{2} + \varepsilon}, \varepsilon > 0 \]

Next maturity date
Then

\[
\frac{1}{(T-t)^{2+\left(1-\varepsilon\right)}} \int_t^T \int_s^T E\left(D_s \sigma_r | F_t\right) dr ds \\
+ \rho c \left( \frac{1}{-1/2 + \varepsilon} \right) \left( \frac{1}{1/2 + \varepsilon} \right) \frac{f'(Y_t)}{2}
\]

tends to zero as \( T \to t \)

In this case, the short-date skew slope of the implied volatility is of the order

\[O(T-t)^{-\frac{1}{2}+\varepsilon}\]


