Multivariate Lévy driven Stochastic Volatility Models

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Outline of this talk

- Motivation from finance and the univariate model
- Matrix subordinators
- Positive semi-definite Ornstein-Uhlenbeck type processes (based on Barndorff-Nielsen & St., 2007; Pigorsch & St., 2008a)
- Multivariate Ornstein-Uhlenbeck type stochastic volatility model (based on Pigorsch & St., 2008b)
- Multivariate COGARCH(1,1) (based on St., 2008)

Stylized Facts of Financial Return Data

- non-constant, stochastic volatility
- volatility exhibits jumps
- asymmetric and heavily tailed marginal distributions
- clusters of extremes
- log returns exhibit marked dependence, but have vanishing autocorrelations (squared returns, for instance, have non-zero autocorrelation)

Stochastic Volatility Models are used to cover these stylized facts.

Univariate BNS Model I

- Logarithmic stock price process $(Y_t)_{t \in \mathbb{R}^+}$:

$$dY_t = (\mu + \beta\sigma_{t-})\, dt + \sigma_{t-}^{1/2} dW_t$$

with parameters $\mu, \beta \in \mathbb{R}$ and $(W_t)_{t \in \mathbb{R}^+}$ being standard Brownian motion.

- Ornstein-Uhlenbeck-type volatility process $(\sigma_t)_{t \in \mathbb{R}^+}$:

$$d\sigma_t = -\lambda\sigma_{t-} dt + dL_t, \;\; \sigma_0 > 0$$

with parameter $\lambda > 0$ and $(L_t)_{t \in \mathbb{R}^+}$ being a Lévy subordinator.

Univariate BNS Model II

- Usually $E(\max(\log|L_1|, 0)) < \infty$ and σ is chosen as the unique stationary solution to $d\sigma_t = -\lambda\sigma_{t-}dt + dL_t$ given by

$$\sigma_t = \int_{-\infty}^{t} e^{-\lambda(t-s)} dL_s.$$

- Closed form expression for the integrated volatility

$$\int_0^t \sigma_s ds = \frac{1}{\lambda}(L_t - \sigma_t + \sigma_0).$$

Derivative Pricing via Laplace transforms possible.

The Need for Multivariate Models

Multivariate models are needed

- to study comovements and spill over effects between several assets.
- for optimal portfolio selection and risk management at a portfolio level.
- to price derivatives on multiple assets.

Desire:

Multivariate models that are flexible, realistic and analytically tractable.

Some Matrix Notation

- $M_d(\mathbb{R})$: the real $d \times d$ matrices.
- $\mathbb{S}_d$: the real symmetric $d \times d$ matrices.
- $\mathbb{S}_d^+$: the positive-semidefinite $d \times d$ matrices (covariance matrices) (a closed cone).
- $\mathbb{S}_d^{++}$: the positive-definite $d \times d$ matrices (an open cone).
- $A^{1/2}$: for $A \in \mathbb{S}_d^+$ the unique positive-semidefinite square root (functional calculus).

Matrix Subordinators

- Definition:

  An $\mathbb{S}_d$-valued Lévy process L is said to be a *matrix subordinator*, if $L_t - L_s \in \mathbb{S}_d^+$ for all $s, t \in \mathbb{R}^+$ with $t > s$.

  (Barndorff-Nielsen and Pérez-Abreu (2008)).

- The paths are $\mathbb{S}_d^+$-increasing and of finite variation.

- The characteristic function μ_{L_t} of L_t for $t \in \mathbb{R}^+$ is given by

$$\mu_{L_t}(Z) = \exp\left(t \left(i\mathrm{tr}(\gamma_L Z) + \int_{\mathbb{S}_d^+ \setminus \{0\}} \left(e^{i\mathrm{tr}(XZ)} - 1 \right) \nu_L(dX) \right) \right), \quad Z \in \mathbb{S}_d,$$

  where γ_L is the drift and ν_L the Lévy measure.

Examples of Matrix Subordinators

- Analogues of univariate subordinators can be defined via the characteristic functions: e.g. (tempered) stable, Gamma or IG matrix subordinators

- Diagonal matrix subordinators, i.e. off-diagonal elements zero, diagonal elements univariate subordinators

- Discontinuous part of the Quadratic (Co-)Variation process of any d-dimensional Lévy process $\tilde{L}$:

$$[\tilde{L}, \tilde{L}]_t^{\mathfrak{d}} = \sum_{s \leq t} \Delta \tilde{L}_s (\Delta \tilde{L}_s)^T$$

Linear Operators Preserving Positive-Semidefiniteness

Proposition Let $\mathbf{A} : \mathbb{S}_d \to \mathbb{S}_d$ be a linear operator. Then $e^{\mathbf{A}t}(\mathbb{S}_d^+) = \mathbb{S}_d^+$ for all $t \in \mathbb{R}$, if and only if $\mathbf{A}$ is representable as $X \mapsto AX + XA^T$ for some $A \in M_d(\mathbb{R})$. □

One has $e^{\mathbf{A}t}X = e^{At}Xe^{A^Tt}$ for all $X \in \mathbb{S}_d$.

In the above setting $\sigma(\mathbf{A}) = \sigma(A) + \sigma(A)$. Hence, $\mathbf{A}$ has only eigenvalues of strictly negative real part, if and only if this is the case for A.

Positive-semidefinite OU-type Processes

Theorem Let $(L_t)_{t\in\mathbb{R}}$ be a matrix subordinator with $E(\max(\log\|L_1\|,0))<\infty$ and $A\in M_d(\mathbb{R})$ such that $\sigma(A)\subset(-\infty,0)+i\mathbb{R}$.

Then the stochastic differential equation of Ornstein-Uhlenbeck-type

$$d\Sigma_t = (A\Sigma_{t-} + \Sigma_{t-}A^T)dt + dL_t$$

has a unique stationary solution

$$\Sigma_t = \int_{-\infty}^{t} e^{A(t-s)}dL_s e^{A^T(t-s)}$$

or, in vector representation, $\mathrm{vec}(\Sigma_t) = \int_{-\infty}^{t} e^{(I_d\otimes A + A\otimes I_d)(t-s)} d\mathrm{vec}(L_s)$.

Moreover, $\Sigma_t \in \mathbb{S}_d^+$ for all $t\in\mathbb{R}$. □

Stationary Distribution

Theorem Let γ_L be the drift of the driving matrix subordinator L and ν_L its Lévy measure.

The stationary distribution of the Ornstein-Uhlenbeck process Σ is infinitely divisible (even operator self-decomposable) with characteristic function

$$\hat{\mu}_\Sigma(Z) = \exp\left(i\mathrm{tr}(\gamma_\Sigma Z) + \int_{\mathbb{S}_d^+ \setminus \{0\}} (e^{i\mathrm{tr}(YZ)} - 1)\nu_\Sigma(dY) \right), \quad Z \in \mathbb{S}_d,$$

where

$$\gamma_\Sigma = -\mathbf{A}^{-1}\gamma_L \quad \text{and} \quad \nu_\Sigma(E) = \int_0^\infty \int_{\mathbb{S}_d^+ \setminus \{0\}} I_E(e^{As} x e^{A^T s}) \nu_L(dx) ds$$

for all Borel sets E in $\mathbb{S}_d^+ \setminus \{0\}$.
$\mathbf{A}^{-1}$ is the inverse of the linear operator $\mathbf{A} : \mathbb{S}_d(\mathbb{R}) \to \mathbb{S}_d(\mathbb{R})$, $X \mapsto AX + XA^T$ which can be represented as $\mathrm{vec}^{-1} \circ ((I_d \otimes A) + (A \otimes I_d))^{-1} \circ \mathrm{vec}$. □

Strict Positive-definiteness

Proposition If $\gamma_L \in \mathbb{S}_d^{++}$ or $\nu_L(\mathbb{S}_d^{++}) > 0$, then the stationary distribution P_Σ of Σ is concentrated on $\mathbb{S}_d^{++}$, i.e. $P_\Sigma(\mathbb{S}_d^{++}) = 1$. □

Theorem Let $\tilde{L}$ be a Lévy process in $\mathbb{R}^d$ with Lévy measure $\nu_{\tilde{L}} \neq 0$ and assume that $\nu_{\tilde{L}}$ is absolutely continuous (with respect to the Lebesgue measure on $\mathbb{R}^d$).

Then the stationary distribution of the Ornstein-Uhlenbeck type process Σ_t driven by the discontinuous part of the quadratic variation $[\tilde{L}, \tilde{L}]_t^{\mathfrak{d}}$ is absolutely continuous with respect to the Lebesgue measure. Moreover, the stationary distribution P_Σ of Σ_t is concentrated on $\mathbb{S}_d^{++}$, i.e. $P_\Sigma(\mathbb{S}_d^{++}) = 1$. □

Marginal Dynamics

Assume that A is real diagonalisable and let $U \in GL_d(\mathbb{R})$ be such that $UAU^{-1} =: D$ is diagonal.

- $M_t := UL_tU^T$ is again a matrix subordinator.
- $(U\Sigma_tU^T)_{ij} = \left(\int_{-\infty}^t e^{D(t-s)}d(UL_sU^T)e^{D(t-s)}\right)_{ij} = \int_{-\infty}^t e^{(\lambda_i+\lambda_j)(t-s)}dM_{ij,s}$.
- Hence, the individual components of $U\Sigma_tU^T$ are stationary one-dimensional Ornstein-Uhlenbeck type processes with associated SDE $d(U\Sigma_tU^T)_{ij} = (\lambda_i + \lambda_j)(U\Sigma_tU^T)_{ij}dt + dM_{ij,t}$.

  M_{ii} for $1 \le i \le d$ are necessarily subordinators and $(U\Sigma_tU^T)_{ii}$ have to be positive OU type processes.
- The **individual components $\Sigma_{ij,t}$** of Σ_t **are superpositions of** (at most d^2) **univariate OU type processes.** The individual OU processes superimposed are in general not independent.

Second Order Structure

Theorem Assume that the driving Lévy process is square-integrable. Then the second order moment structure is given by

$$\begin{aligned} E(\Sigma_t) &= \gamma_\Sigma - \mathbf{A}^{-1} \int_{\mathbb{S}_d^+ \setminus \{0\}} y\nu(dy) = -\mathbf{A}^{-1} E(L_1) \\ \operatorname{var}(\operatorname{vec}(\Sigma_t)) &= -\mathcal{A}^{-1} \operatorname{var}(\operatorname{vec}(L_1)) \\ \operatorname{cov}(\operatorname{vec}(\Sigma_{t+h}), \operatorname{vec}(\Sigma_t)) &= e^{(A \otimes I_d + I_d \otimes A)h} \operatorname{var}(\operatorname{vec}(\Sigma_t)), \end{aligned}$$

where $t \in \mathbb{R}$ and $h \in \mathbb{R}^+$, $\mathbf{A} : M_d(\mathbb{R}) \to M_d(\mathbb{R}),\ X \mapsto AX + XA^T$ and $\mathcal{A} : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}),\ X \mapsto (A \otimes I_d + I_d \otimes A)X + X(A^T \otimes I_d + I_d \otimes A^T)$. □

The individual components of the autocovariance matrix do not have to decay exponentially, but may exhibit exponentially damped sinusoidal behaviour.

The Integrated Volatility

Theorem The integrated Ornstein-Uhlenbeck process Σ_t^+ is given by

$$\Sigma_t^+ := \int_0^t \Sigma_t dt = \mathbf{A}^{-1} (\Sigma_t - \Sigma_0 - L_t)$$

for $t \in \mathbb{R}^+$. □

Multivariate OU type Stochastic Volatility Model

d-dimensional logarithmic stock price process $(Y_t)_{t \in \mathbb{R}}$:

$$dY_t = (\mu + \Sigma_{t-}\beta)\, dt + \Sigma_{t-}^{1/2} dW_t$$

with

- $(W_t)_{t \in \mathbb{R}^+}$ being d-dimensional standard Brownian motion,
- $\mu, \beta \in \mathbb{R}^d$ and
- $(\Sigma_t)_{t \in \mathbb{R}^+}$ being a stationary $\mathbb{S}_d^+$-valued Ornstein-Uhlenbeck type process.

$\Longrightarrow$ Natural analogue of the univariate model

The Conditional Fourier Transform

Assume the driving matrix subordinator L has characteristic exponent ψ_L, i.e. $E(e^{i\mathrm{tr}(L_t z)}) = e^{t\psi_L(z)}$ for all $z \in M_d(\mathbb{R}) + i\mathbb{S}_d^+$. Let $(Y_0, \Sigma_0) \in \mathbb{R}^d \times \mathbb{S}_d^+$ be the initial values. Then we have for every $t \in \mathbb{R}^+$ and $(y, z) \in \mathbb{R}^d \times M_d(\mathbb{R})$

$$E\left(e^{i\left(Y_t^T y + \mathrm{tr}(\Sigma_t z)\right)}\middle|\Sigma_0, Y_0\right) = \exp\left\{i(Y_0 + \mu t)^T y + i\mathrm{tr}\left(\Sigma_0 e^{A^T t} z e^{At}\right)\right.$$
$$+ i\mathrm{tr}\left(\Sigma_0 e^{A^T t}\left[\mathbf{A}^{-*}\left(y\beta^T + \frac{i}{2}yy^T\right)\right]e^{At} - \Sigma_0\left[\mathbf{A}^{-*}\left(y\beta^T + \frac{i}{2}yy^T\right)\right]\right)$$
$$\left. + \int_0^t \psi_L\left(e^{A^T s} z e^{As} + e^{A^T s}\left[\mathbf{A}^{-*}\left(y\beta^T + \frac{i}{2}yy^T\right)\right]e^{As} - \mathbf{A}^{-*}\left(y\beta^T + \frac{i}{2}yy^T\right)\right) ds\right\}$$

with $\mathbf{A}^{-*}$ denoting the inverse of the adjoint of $\mathbf{A}$, i.e. $\mathbf{A}^{-*}$ is the inverse of the linear operator $\mathbf{A}^*$ given by $X \mapsto A^T X + XA$.

The Logarithmic Returns

Let $\Delta > 0$ (grid size). Define for $n \in \mathbb{N}$:

- log-returns over periods $[(n-1)\Delta, n\Delta]$ of length Δ:

$$\mathbf{Y}_n = Y_{n\Delta} - Y_{(n-1)\Delta} = \int_{(n-1)\Delta}^{n\Delta} (\mu + \Sigma_t \beta) dt + \int_{(n-1)\Delta}^{n\Delta} \Sigma_t^{1/2} dW_t.$$

- Integrated volatility over $[(n-1)\Delta, n\Delta]$:

$$\mathbf{\Sigma}_n := \int_{(n-1)\Delta}^{n\Delta} \Sigma_t dt.$$

It holds that

$$\mathbf{Y}_n \,|\, \mathbf{\Sigma}_n \sim N_d \left(\mu\Delta + \mathbf{\Sigma}_n \beta, \mathbf{\Sigma}_n\right)$$

with N_d denoting the d-dimensional normal distribution.

Second Order Structure of Σ_n

Assume henceforth $E(\|L_1\|^2) < \infty$.

$$\begin{aligned}
E(\mathbf{\Sigma}_n) &= \Delta E(\Sigma_0) = -\Delta \mathbf{A}^{-1} E(L_1) \\
\operatorname{var}(\operatorname{vec}(\mathbf{\Sigma}_n)) &= r^{++}(\Delta) + \left(r^{++}(\Delta)\right)^T \\
r^{++}(t) &= \left(\mathscr{A}^{-2}\left(e^{\mathscr{A}t} - I_{d^2}\right) - \mathscr{A}^{-1}t\right) \operatorname{var}(\operatorname{vec}(\Sigma_0)) \\
&= -\left(\mathscr{A}^{-2}\left(e^{\mathscr{A}t} - I_{d^2}\right) - \mathscr{A}^{-1}t\right)\mathcal{A}^{-1}\operatorname{var}(\operatorname{vec}(L_1)) \\
\operatorname{acov}_{\mathbf{\Sigma}}(h) &= e^{\mathscr{A}\Delta(h-1)}\mathscr{A}^{-2}\left(I_{d^2} - e^{\mathscr{A}\Delta}\right)^2 \operatorname{var}(\operatorname{vec}(\Sigma_0)) \\
&= -e^{\mathscr{A}\Delta(h-1)}\mathscr{A}^{-2}\left(I_{d^2} - e^{\mathscr{A}\Delta}\right)^2 \mathcal{A}^{-1}\operatorname{var}(\operatorname{vec}(L_1)), \ h \in \mathbb{N}.
\end{aligned}$$

where $\mathscr{A} = A \otimes I_d + I_d \otimes A$ and $\mathcal{A} : M_{d^2}(\mathbb{R}) \to M_{d^2}(\mathbb{R}), X \mapsto \mathscr{A}X + X\mathscr{A}^T$.

$\Longrightarrow \operatorname{vec}(\mathbf{\Sigma}_n)$ is a causal ARMA(1,1) process with AR parameter $e^{\mathscr{A}\Delta}$.

Second Order Structure of $\mathbf{Y}_n$ and $\mathbf{Y}_n\mathbf{Y}_n^T$

$$
\begin{aligned}
E(\mathbf{Y}_n) &= (\mu + E(\Sigma_0)\beta)\Delta \\
\mathrm{var}(\mathbf{Y}_n) &= E(\Sigma_0)\Delta + (\beta^T \otimes I_d)\mathrm{var}(\mathrm{vec}(\boldsymbol{\Sigma}_n))(\beta \otimes I_d) \\
\mathrm{acov}_{\mathbf{Y}}(h) &= (\beta^T \otimes I_d)\mathrm{acov}_{\boldsymbol{\Sigma}}(h)(\beta \otimes I_d),\ h \in \mathbb{N}
\end{aligned}
$$

Assume $\mu = \beta = 0$. Then:

$$
\begin{aligned}
E(\mathbf{Y}_n\mathbf{Y}_n^T) &= E(\Sigma_0)\Delta \\
\mathrm{var}(\mathrm{vec}(\mathbf{Y}_n\mathbf{Y}_n^T)) &= (I_{d^2} + \mathbf{Q} + \mathbf{PQ})\,\mathrm{var}(\mathrm{vec}(\boldsymbol{\Sigma}_n)) \\
&\quad + (I_{d^2} + \mathbf{P})\,(E(\Sigma_0) \otimes E(\Sigma_0))\,\Delta^2 \\
\mathrm{acov}_{\mathbf{Y}\mathbf{Y}^T}(h) &= \mathrm{acov}_{\boldsymbol{\Sigma}}(h) \quad \text{for } h \in \mathbb{N}
\end{aligned}
$$

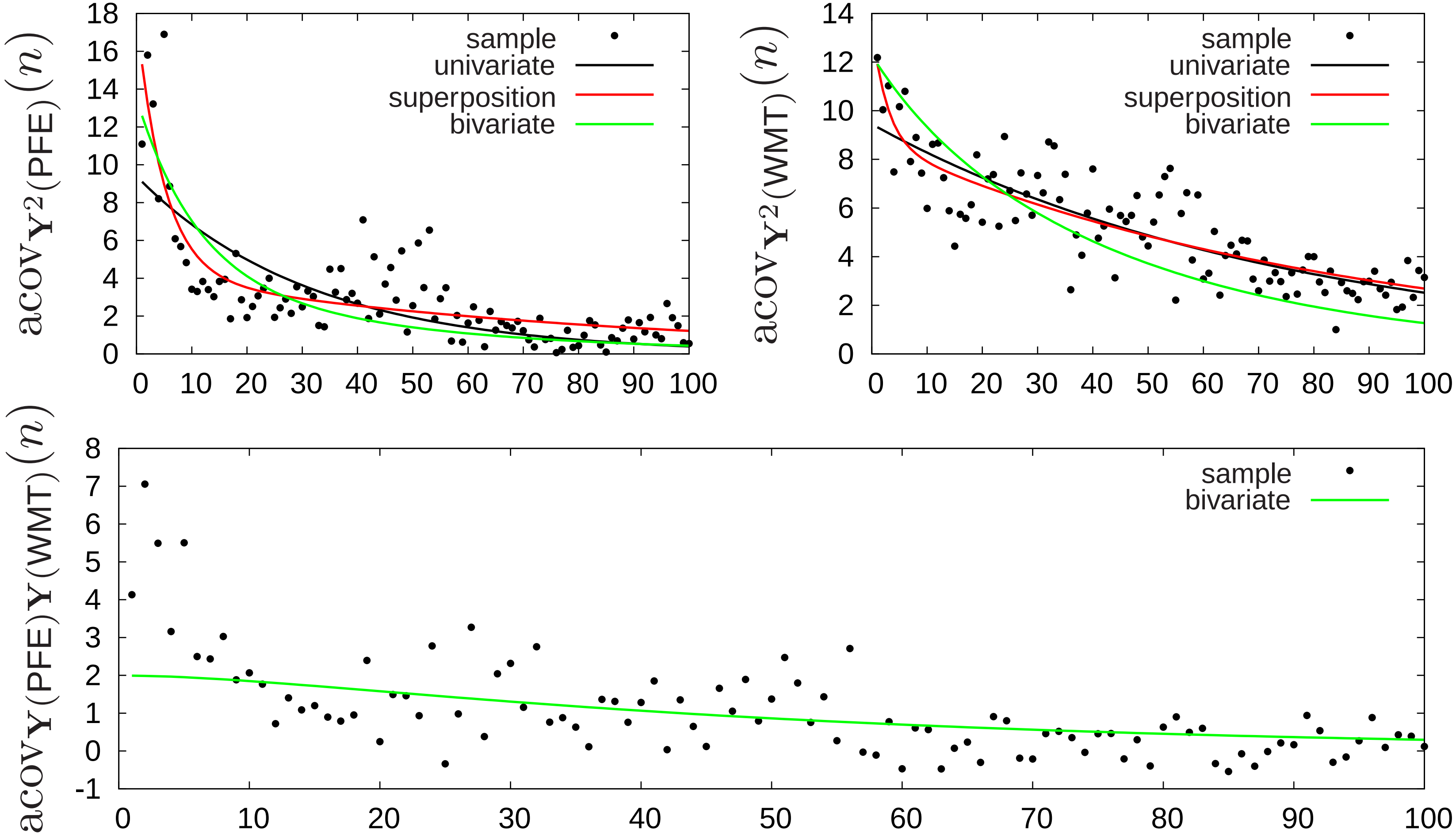
where $\mathbf{P}$ and $\mathbf{Q}$ are linear operators on $M_{d^2}(\mathbb{R})$ rearranging the entries.

$\Longrightarrow$ $\mathrm{vec}(\mathbf{Y}_n\mathbf{Y}_n^T)$ is a causal ARMA(1,1) process with AR parameter $e^{\mathscr{A}\Delta}$.

Moment Estimators

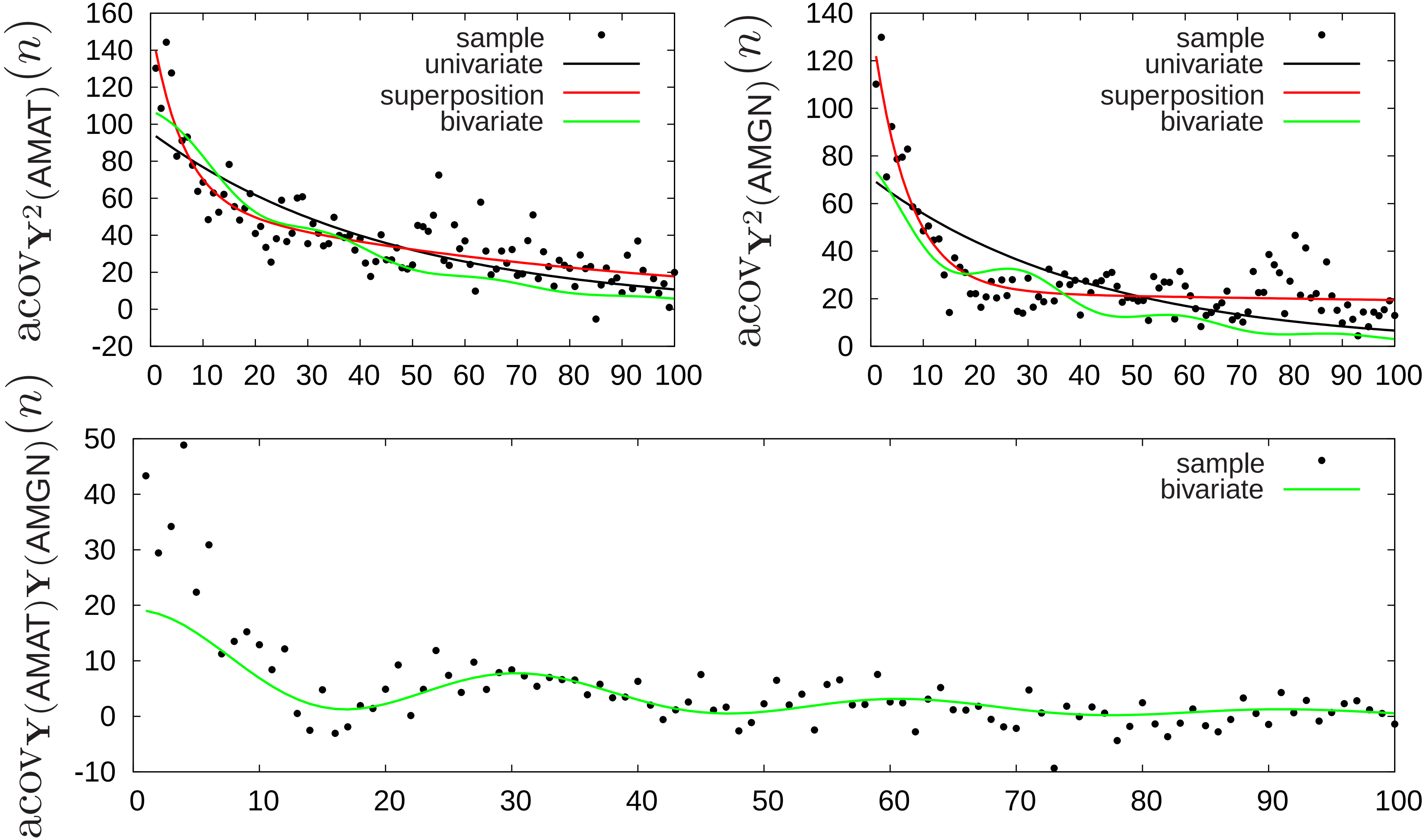
- Assume $\mu = \beta = 0$
- $E(L_1), \mathrm{var}(\mathrm{vec}(L_1))$ and A can be estimated from the empirically observed $E(\mathbf{Y}_n \mathbf{Y}_n^T), \mathrm{acov}_{\mathbf{Y}\mathbf{Y}^T}(1)$ and $\mathrm{acov}_{\mathbf{Y}\mathbf{Y}^T}(2)$.
- They are identified provided one assumes that $e^{\mathbf{A}_{\mathrm{vech}}\Delta}$ has a unique real logarithm and $\mathrm{var}(\mathrm{vech}(\Sigma_0))$ is invertible.
- In practice one uses more lags of the autocovariance function and GMM estimation.
- The log-returns $\mathbf{Y}$ are strongly mixing. Thus the estimators are under appropriate technical conditions consistent and asymptotically normal.

Empirical Illustration I



Empirical and estimated autocovariance functions: PFE and WMT

Empirical Illustration II



Empirical and estimated autocovariance functions: AMAT and AMGN

Extensions

Even more flexibility (and long memory) by considering superpositions of independent multivariate positive semi-definite OU type processes for Σ.

Possibilities:

- Superposition of finitely many OU type processes: Straightforward and (almost) all results easily extendible.
- Superposition of countably many OU type processes and convergence in L^2.
- Use of a $\mathbb{S}_d^+$-valued Lévy basis Λ on $\mathbb{R} \times M_d^-(\mathbb{R})$ with $M_d^-(\mathbb{R}) := \{X \in M_d(\mathbb{R}) : \sigma(X) \subset (-\infty, 0) + i\mathbb{R}\}$:

$$\Sigma_t = \int_{-\infty}^{t} \int_{M_d^-(\mathbb{R})} e^{A(t-s)} \Lambda(ds, dA) e^{A^T(t-s)}$$

Univariate BNS and COGARCH model

- The Ornstein-Uhlenbeck type stochastic volatility model (BNS model):

$$dY_t = \sqrt{\sigma_{t-}}dW_t$$

$$d\sigma_t = -\lambda\sigma_{t-}dt + dL_t$$

with $\lambda > 0$, W standard Brownian motion and L a subordinator.

- The COGARCH(1,1) model (Klüppelberg, Lindner, Maller (2004)):

$$dY_t = \sqrt{\sigma_{t-}}dL_t$$

$$\sigma_t = c + v_t, \quad dv_t = -\alpha v_{t-}dt + \beta\sigma_{t-}d[L,L]^{\mathfrak{d}}_t$$

with $\alpha, \beta, c > 0$, L a Lévy process and $[L,L]^{\mathfrak{d}}_t = \sum_{0<s\leq t}(\Delta L_s)^2$.

Multivariate COGARCH(1,1) – Definition

Definition Let L be a d-dimensional Lévy process and $A, B \in M_d(\mathbb{R})$, $C \in \mathbb{S}_d^+$ and set $[L, L]_t^{\mathfrak{d}} := \sum_{0<s\leq t} \Delta L_s (\Delta L_s)^T$. Then the process $Y = (Y_t)_{t\in\mathbb{R}^+}$ solving

$$dY_t = \Sigma_{t-}^{1/2} dL_t, \qquad \Sigma_t = C + V_t, \tag{1}$$

$$dV_t = (AV_{t-} + V_{t-}A^T)dt + B\Sigma_{t-}^{1/2} d[L, L]_t^{\mathfrak{d}} \Sigma_{t-}^{1/2} B^T \tag{2}$$

with initial values $Y_0 = 0$ in $\mathbb{R}^d$ and V_0 in $\mathbb{S}_d^+$ is called a *multivariate COGARCH(1,1) process*.

The process $V = (V_t)_{t\in\mathbb{R}^+}$ (or Σ) with paths in $\mathbb{S}_d^+$ is referred to as a *multivariate COGARCH(1,1) volatility process*. □

Agrees with the definition of the COGARCH(1,1) for $d = 1$ and inherits many of the properties of multivariate GARCH(1,1).

Multivariate COGARCH(1,1) – Equivalent Definitions

- One can directly define Σ via the SDE

$$d\Sigma_t = (A(\Sigma_{t-} - C) + (\Sigma_{t-} - C)A^T)dt + B\Sigma_{t-}^{1/2} d[L,L]_t^{\mathfrak{d}} \Sigma_{t-}^{1/2} B^T$$

which shows that Σ has a mean reverting structure (provided $\sigma(A) \subset (-\infty, 0) + i\mathbb{R}$) with “mean” C.

- The volatility process V (or Σ) is of finite variation and V satisfies for all $t \in \mathbb{R}^+$

$$V_t = e^{At} V_0 e^{A^T t} + \int_0^t e^{A(t-s)} B\Sigma_{s-}^{1/2} d[L,L]_s^{\mathfrak{d}} \Sigma_{s-}^{1/2} B^T e^{A^T(t-s)}.$$

Markovian Properties and Stationarity

Provided $C \in \mathbb{S}_d^{++}$, (Y,V) and V alone are temporally homogeneous strong Markov processes on $\mathbb{R}^d \times \mathbb{S}_d^+$ and $\mathbb{S}_d^+$, respectively. Moreover, both have the weak Feller property.

Theorem 1. *Assume:*

- $C \in \mathbb{S}_d^{++}$, $A \in M_d(\mathbb{R})$ *is diagonalisable with* $S \in GL_d(\mathbb{C})$ *such that* $S^{-1}AS$ *is diagonal,*
- *the Lévy measure* ν_L *of* L *satisfies*

$$\int_{\mathbb{R}^d} \log\left(1 + \alpha_1 \|(S^{-1} \otimes S^{-1})\text{vec}(yy^T)\|_2\right) \nu_L(dy) \; < \; -2\max(\Re(\sigma(A))),$$

where $\alpha_1 := \|S\|_2^2 \|S^{-1}\|_2^2 K_{2,A} \|(S^{-1}BS) \otimes (S^{-1}BS)\|_2,$

$$K_{2,A} := \max_{X \in \mathbb{S}_d^+, \|X\|_2 = 1} \left(\frac{\|X\|_2}{\|(S^{-1} \otimes S^{-1})\text{vec}(X)\|_2} \right).$$

Then there exists a stationary distribution μ for the multivariate COGARCH(1,1) volatility process V having the following property:

If

$$\int_{\mathbb{R}^d} \left(\left(1 + \alpha_1 \|(S^{-1} \otimes S^{-1}) \mathrm{vec}(yy^T)\|_2\right)^k - 1 \right) \nu_L(dy) < -2k \max(\Re(\sigma(A)))$$

for some $k \in \mathbb{N}$, then

$$\int_{\mathbb{S}_d^+} \|x\|^k \mu(dx) < \infty,$$

i.e. the k-th moment of μ is finite.

The stationary distribution is not known to be unique or to be a limiting distribution.

Second Order Properties

Assume:

- The driving Lévy process L has finite fourth moments and ν_L satisfies

$$\int_{\mathbb{R}^d} xx^T \nu_L(dx) = \sigma_L I_d,$$

$$\int_{\mathbb{R}^d} \mathrm{vec}(xx^T)\mathrm{vec}(xx^T)^T \nu_L(dx) = \rho_L(I_{d^2} + K_d + \mathrm{vec}(I_d)\mathrm{vec}(I_d)^T)$$

  for some $\sigma_L, \rho_L \in \mathbb{R}^+$ and with K_d being the commutation matrix,

- $\sigma(A), \sigma(\mathscr{A}), \sigma(\mathscr{C}) \subset (-\infty, 0) + i\mathbb{R}$ with

$$\mathscr{A} = A \otimes I_d + I_d \otimes A + \sigma_L B \otimes B,$$

$$\mathscr{C} := \mathscr{A} \otimes I_{d^2} + I_{d^2} \otimes \mathscr{A} + \sigma_L \left((B \otimes B) \otimes I_{d^2} + I_{d^2} \otimes (B \otimes B)\right) + \mathscr{B}\mathcal{R},$$

$$\mathscr{B} = (B \otimes B) \otimes (B \otimes B),\ \mathcal{R} = \rho_L \left(\mathcal{Q} + \mathcal{K}_d \mathcal{Q} + I_{d^4}\right),$$

  where $\mathcal{K}_d$ and $\mathcal{Q}$ are certain permutation matrices.

- V_0 has finite second moments.

Then V is asymptotically second order stationary with

- mean

$$E(\mathrm{vec}(V_\infty)) = -\sigma_L \mathscr{A}^{-1}(B \otimes B)\mathrm{vec}(C),$$

- autocovariance function

$$\mathrm{acov}_{\mathrm{vec}(V_\infty)}(h) = e^{\mathscr{A}h}\mathrm{var}(\mathrm{vec}(V_\infty))$$

for $h \in \mathbb{R}^+$

- and variance

$$\begin{aligned}\mathrm{vec}(\mathrm{var}(\mathrm{vec}(V_\infty))) = -\,\mathscr{C}^{-1}\,&\left[\left(\sigma_L^2\mathscr{C}(\mathscr{A}^{-1}\otimes\mathscr{A}^{-1})\mathscr{B}+\mathscr{B}\mathcal{R}\right)(\mathrm{vec}(C)\otimes\mathrm{vec}(C))\right.\\ &+(\sigma_L(B\otimes B)\otimes I_{d^2}+\mathscr{B}\mathcal{R})\,\mathrm{vec}(C)\otimes E(\mathrm{vec}(V_\infty))\\ &\left.+(\sigma_L I_{d^2}\otimes(B\otimes B)+\mathscr{B}\mathcal{R})\,E(\mathrm{vec}(V_\infty))\otimes\mathrm{vec}(C)\right].\end{aligned}$$

The Increments of Y

For $\Delta > 0$ the sequence of increments $\mathbf{Y} = (\mathbf{Y}_n)_{n\in\mathbb{N}}$ defined by

$$\mathbf{Y}_n = \int_{(n-1)\Delta}^{n\Delta} \Sigma_{s-}^{1/2} dL_s$$

gives the log-returns over consecutive time periods of length Δ in a financial context.

Stationarity:

If Σ (or V) is stationary, then $\mathbf{Y}$ is stationary.

Stationary Second Order Structure of the ("Squared") Increments

Assume that the previous assumptions regarding the second order behaviour are satisfied and $E(L_1) = 0$, $\mathrm{var}(L_1) = (\sigma_L + \sigma_W)I_d$ for some $\sigma_W \in \mathbb{R}^+$, then:

- $(\mathbf{Y}_n)_{n\in\mathbb{N}}$ has finite fourth moments, mean zero and is uncorrelated. The increments $\mathbf{Y}$ are white noise with variance:

$$\mathrm{vec}(\mathrm{var}(\mathbf{Y}_1)) = (\sigma_L + \sigma_W)\Delta\mathscr{A}^{-1}(A \otimes I_d + I_d \otimes A)\mathrm{vec}(C)$$

- **but** the sequence of "squared" increments $(\mathbf{Y}_n\mathbf{Y}_n^T)_{n\in\mathbb{N}}$ has non-zero autocorrelations which decrease exponentially (from lag one onwards):

$$\mathrm{acov}_{\mathbf{YY}}(h) = e^{\mathscr{A}\Delta h}\mathscr{A}^{-1}\left(I_{d^2} - e^{-\mathscr{A}\Delta}\right)(\sigma_L + \sigma_W)\mathrm{cov}(\mathrm{vec}(V_\Delta), \mathrm{vec}(\mathbf{Y}_1\mathbf{Y}_1^*))$$

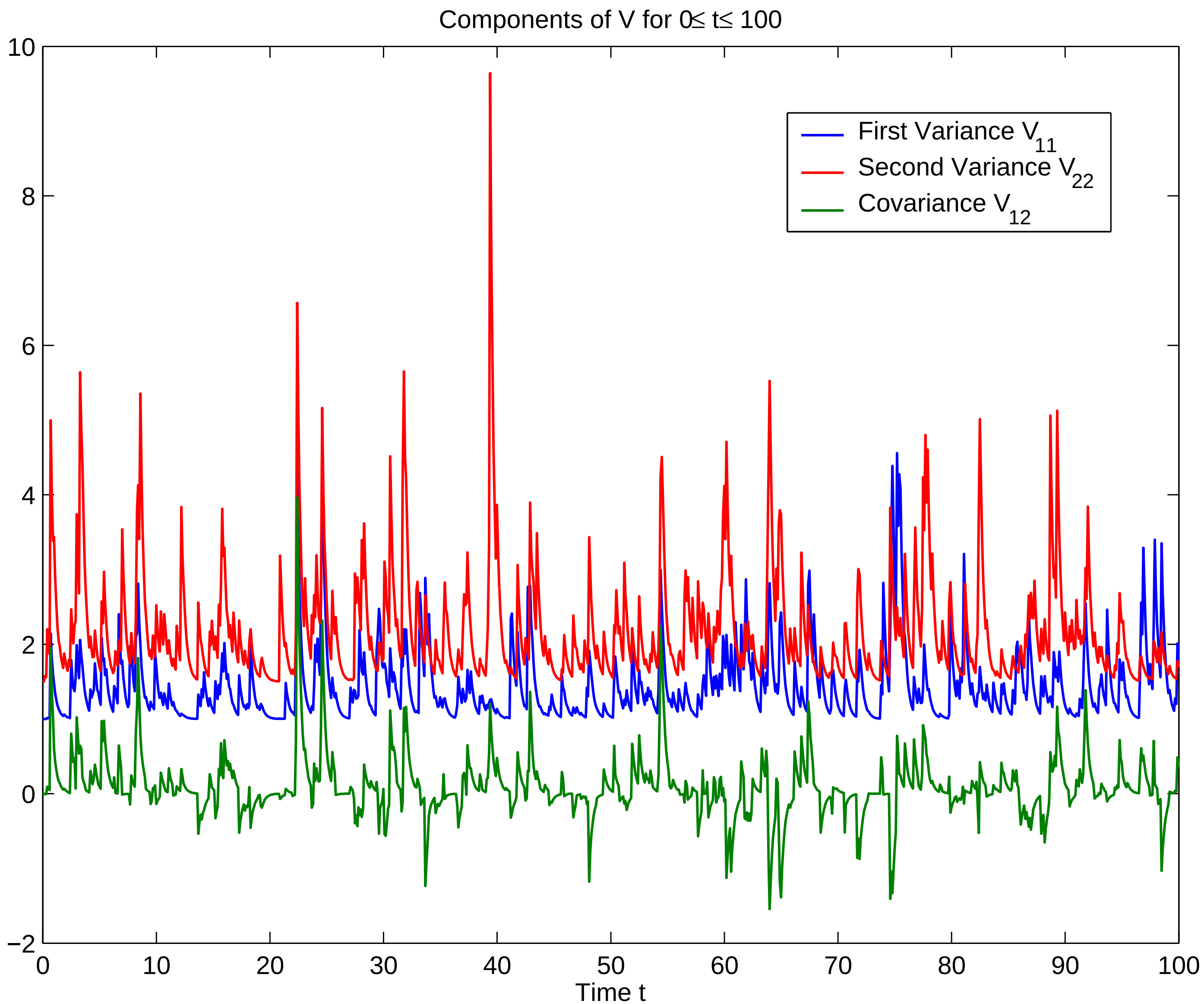
This is the autocovariance structure of an ARMA(1,1) process.

Illustrative Simulations

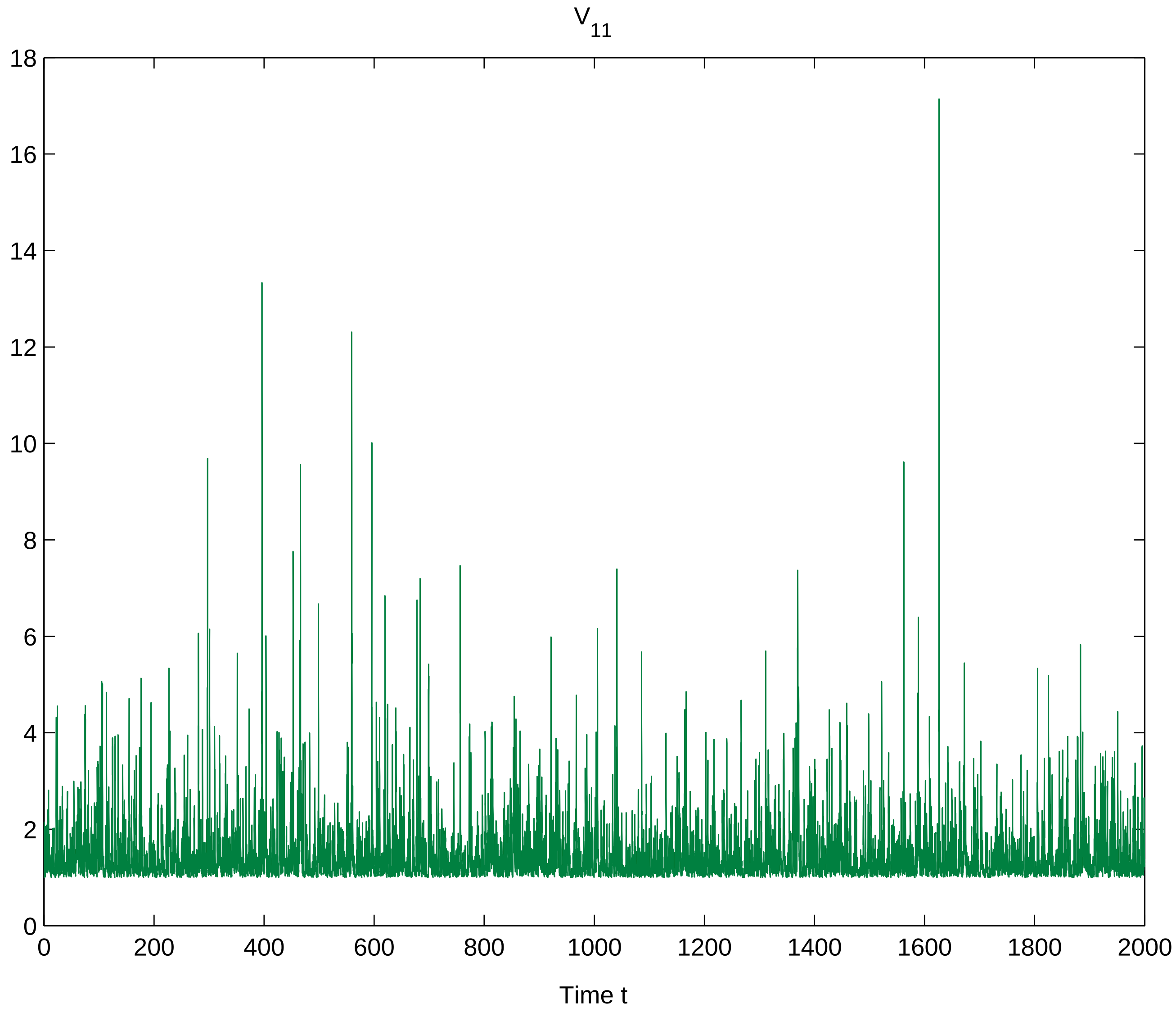
In the following a simulation of a two-dimensional COGARCH(1,1) process is shown where:

- the driving Lévy process is the sum of a standard Brownian motion and a compound Poisson process in $\mathbb{R}^2$ with rate 4 and $N(0, I_2/4)$-distributed jumps.
- Hence, $[L, L]^{\mathfrak{d}}$ is a compound Poisson process with Wishart-distributed jumps.
- $A = -1.6 I_2$, $B = I_2$ and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}$ (corresponds to a "mean" correlation of zero).

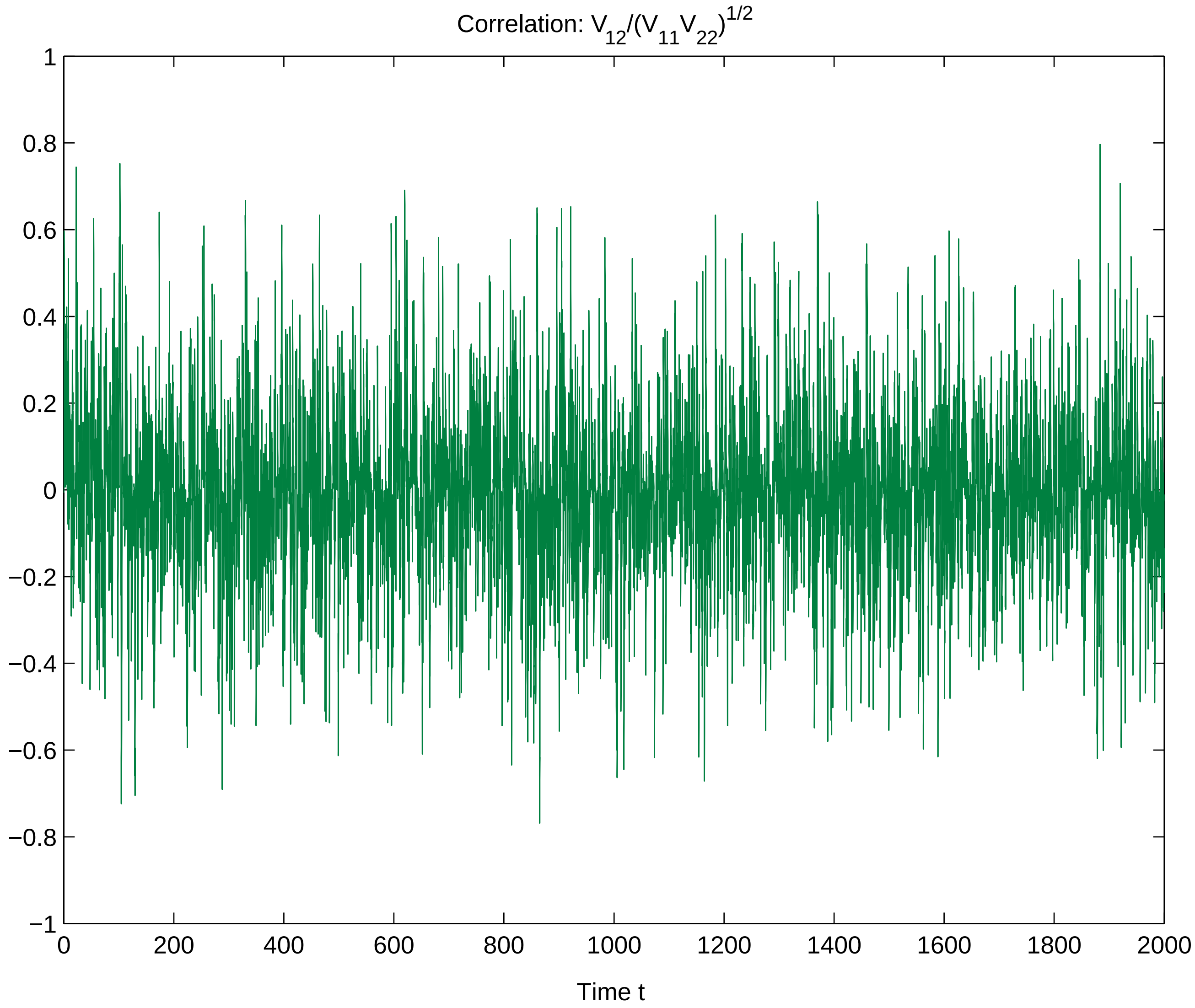
Stochastic Volatility Process Σ



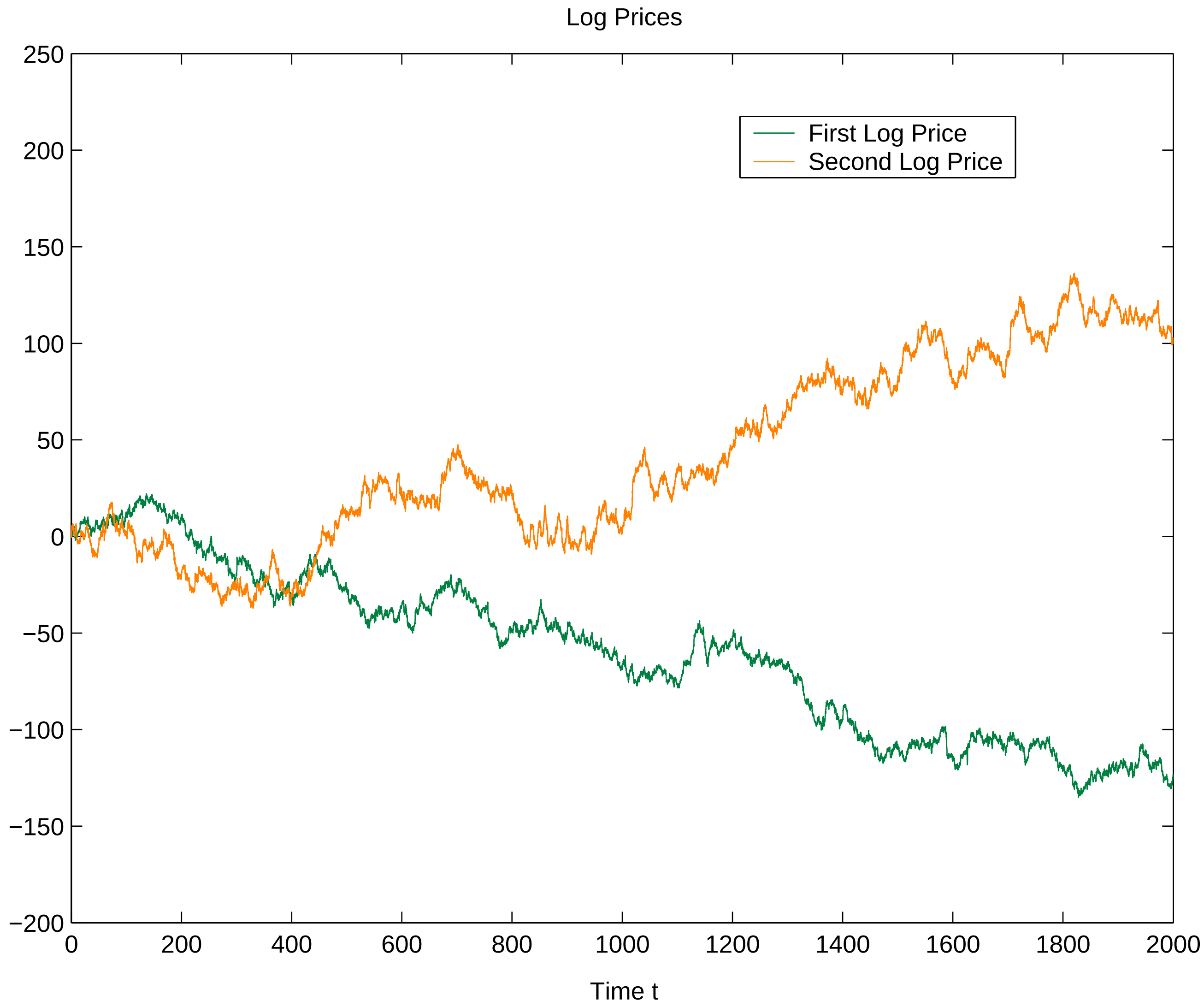
First Stochastic Variance Process Σ_{11}



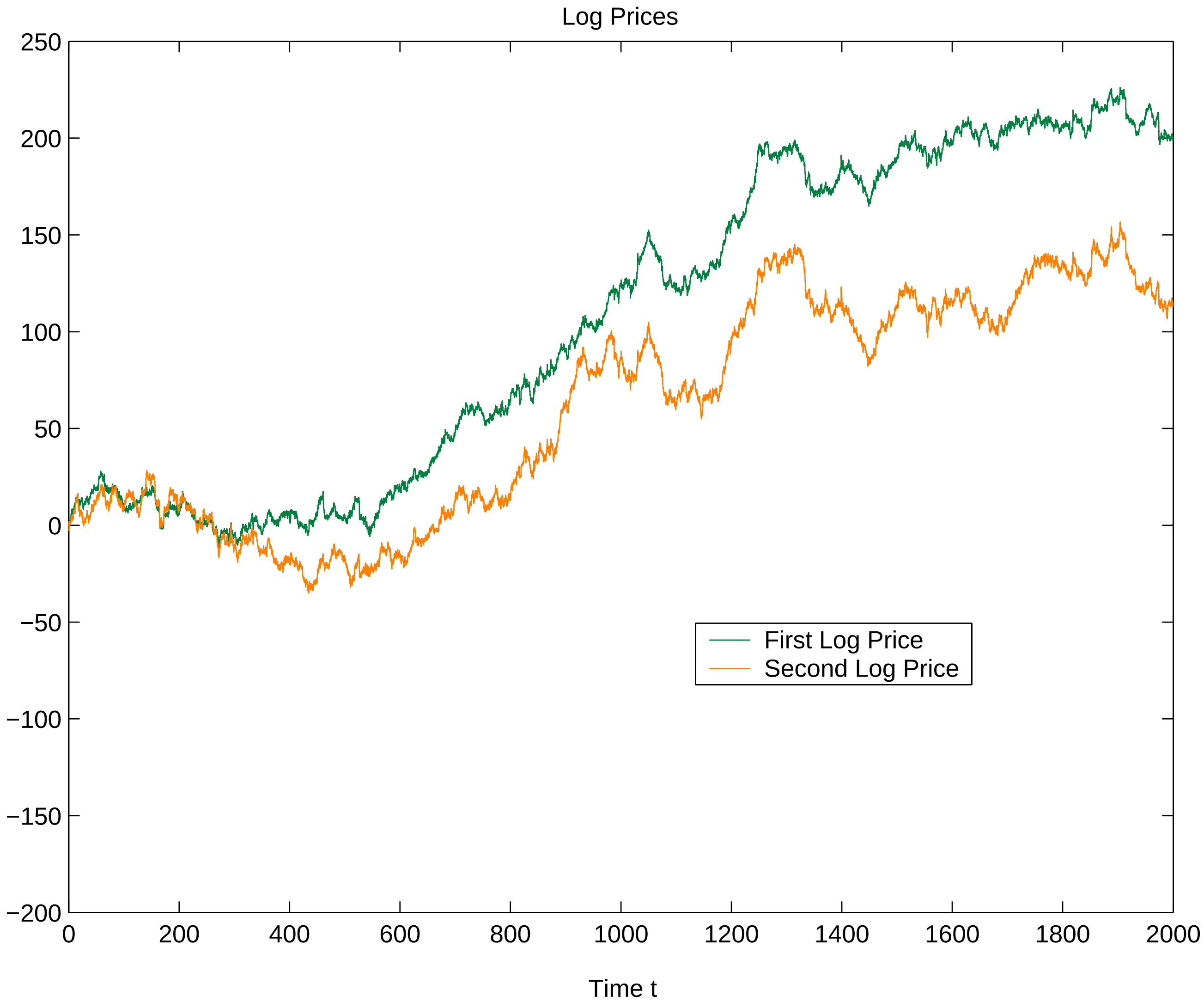
Stochastic Correlation Process $\Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$



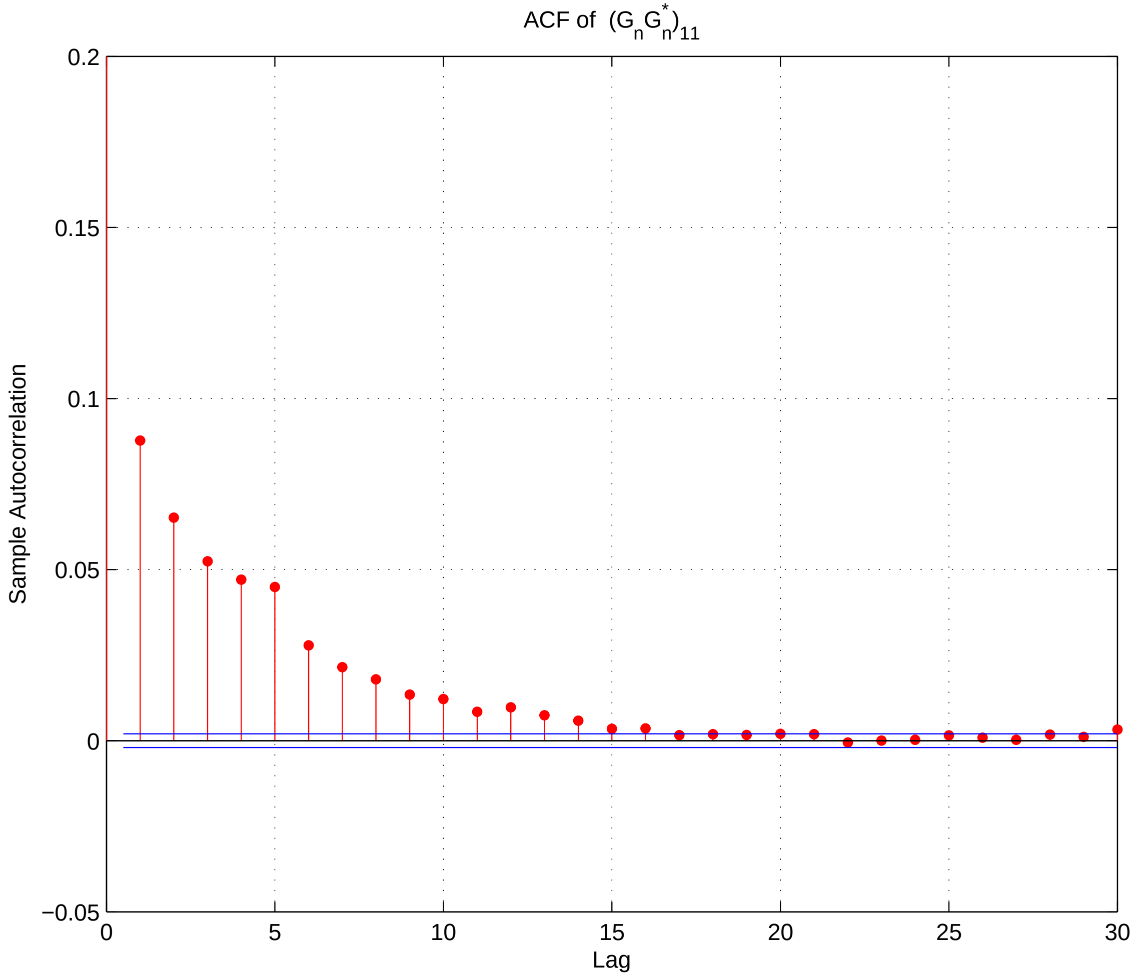
Log-Price Process Y



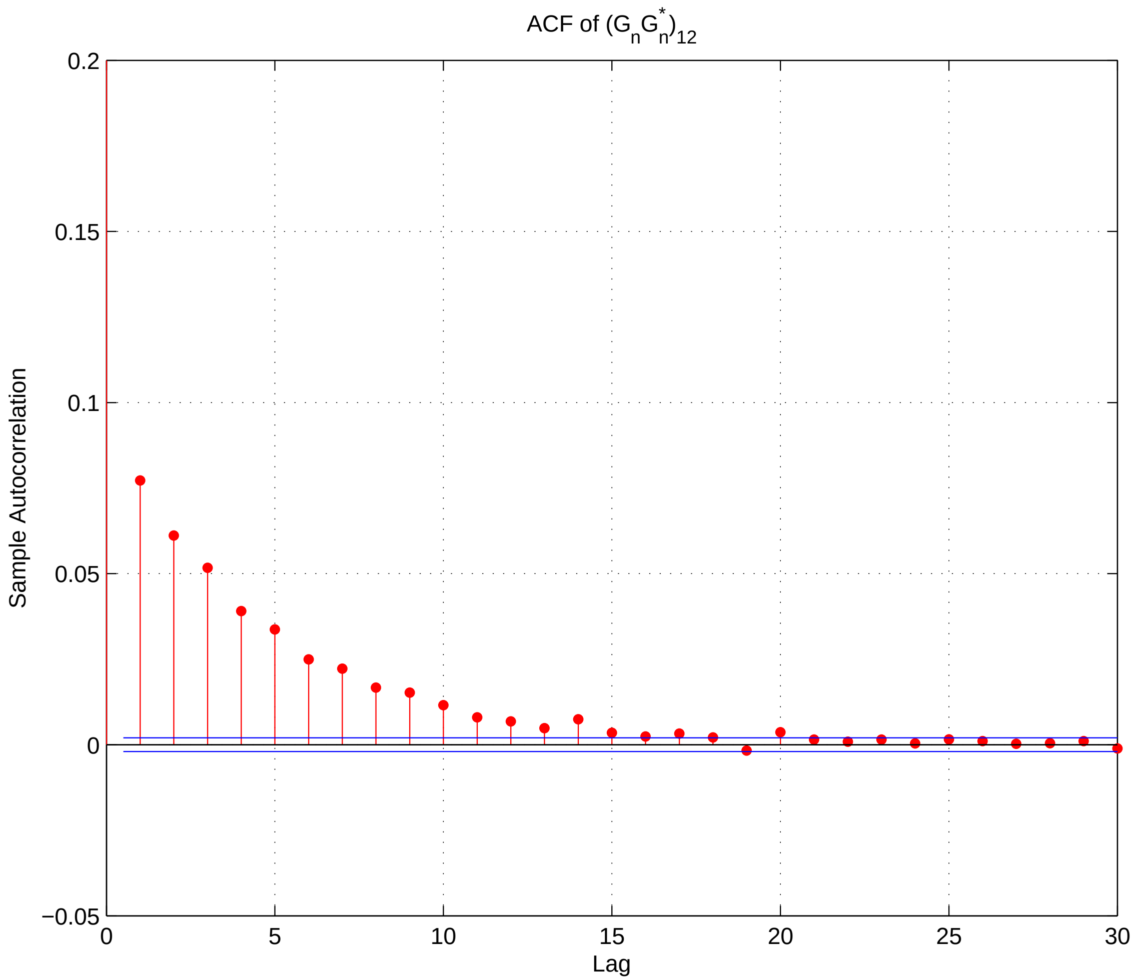
Alternative C: Log-Price Process Y



ACF "Squared Returns" $(\mathbf{YY}^T)_{11}$



ACF "Squared Returns" $(\mathbf{YY}^T)_{12}$



Thank you very much for your attention!