CONTRACTING FOR OPTIMAL INVESTMENT WITH RISK CONTROL

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Overview

- Contracting to align objectives
- Investing under constraints on the law of terminal wealth
- Investment under law-invariant coherent risk measure constraints
- Contracting for optimal investment under LI coherent risk measure constraints

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Insist $w_T \geq \underline{x}$, then make

$$U_P(x - \varphi(x)) = k\{\, U_A(\varphi(x)) - U_A(0)\,\} + U_P(\underline{x})$$

Proposition. *If U_P and U_A are strictly increasing, the function $\varphi : [\underline{x}, \infty) \to \mathbb{R}^+$ is well defined by*

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PROOF. Monotonicity obvious. If concavity fails, for some $x_1, x_2 \geq \underline{x}$, $p = 1 - q \in (0, 1)$, with $x = px_1 + qx_2$

$$\begin{aligned} u(x) &= U_A(\varphi(x)) \\ &< pu(x_1) + qu(x_2) \\ &= pU_A(\varphi(x_1)) + qU_A(\varphi(x_2)) \\ &\leq U_A(p\varphi(x_1) + q\varphi(x_2)), \end{aligned}$$

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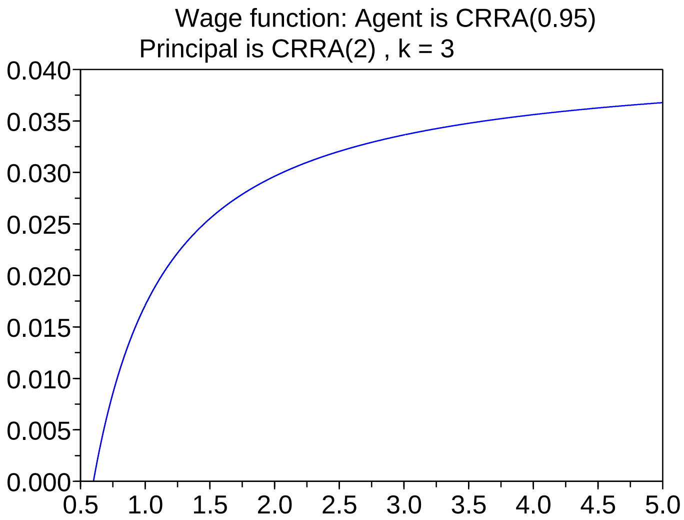
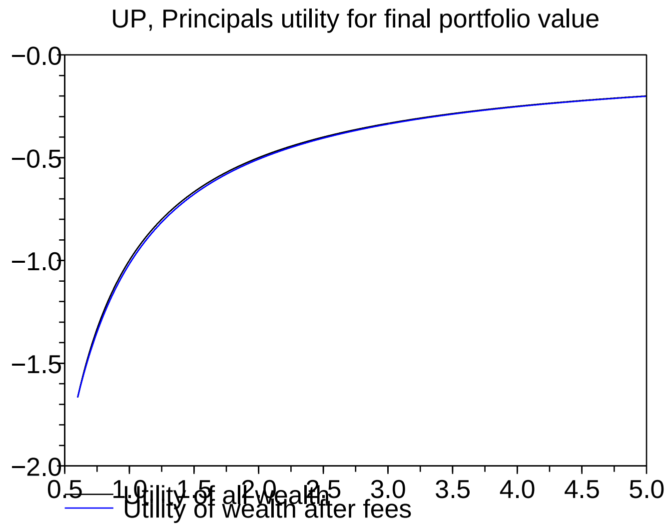
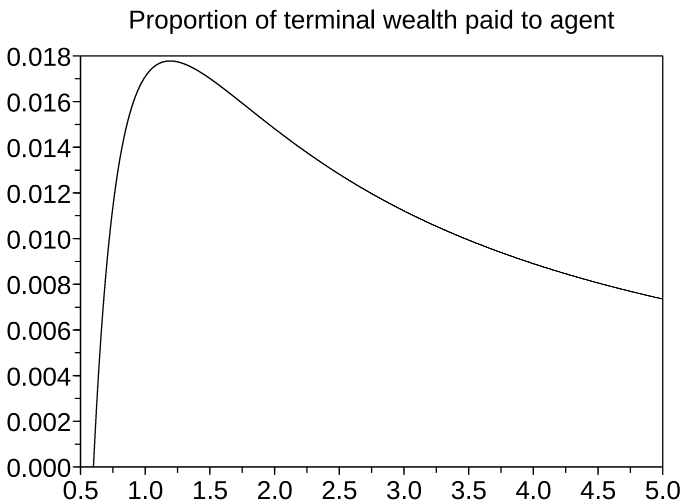
and so $\varphi(x) < p\varphi(x_1) + q\varphi(x_2)$. Hence

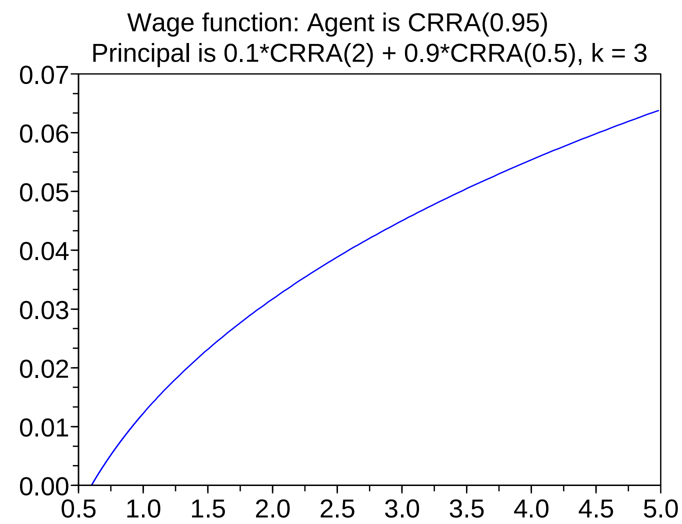
$$\begin{aligned} u(x) = U_A(\varphi(x)) &= U_P(x - \varphi(x)) \\ &> U_P(x - p\varphi(x_1) - q\varphi(x_2)) \\ &\geq pU_P(x_1 - \varphi(x_1)) + qU_P(x_2 - \varphi(x_2)) \\ &= pU_A(\varphi(x_1)) + qU_A(\varphi(x_2)). \end{aligned}$$

How does it look?

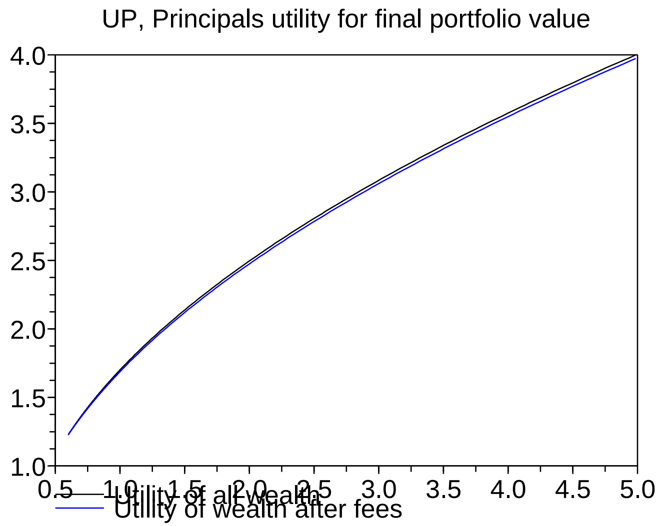




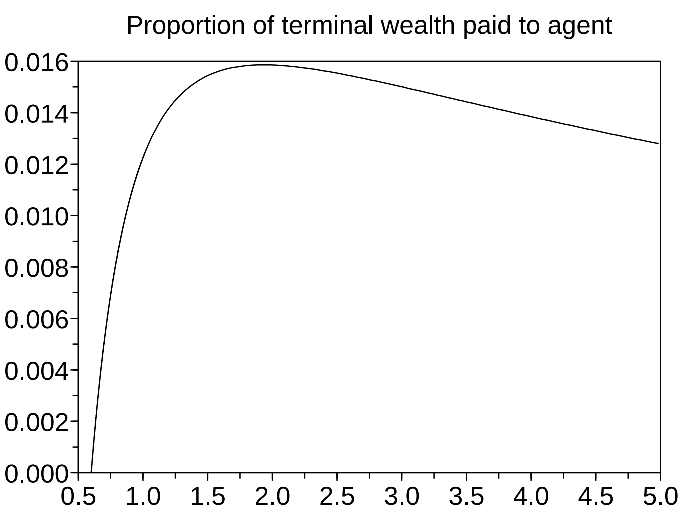




Wage function: Agent is CRRA(0.95)
Principal is 0.1*CRRA(2) + 0.9*CRRA(0.5), k = 3
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.00
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0



UP, Principals utility for final portfolio value
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
Utility of all wealth
Utility of wealth after fees



Proportion of terminal wealth paid to agent
0.016
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0

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then an agent with initial wealth w_0 and maximizing $Eu(w_T)$ will choose $w_T = \psi(\zeta_T)$. If the risk-constrained principal offers the agent φ, where

$$kU_A(\varphi(x)) - a = u(x),$$

then *the unconstrained agent implements the principal's optimum.*

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for some non-negative increasing g_μ.

The optimization problem.

$$\max_{\psi\downarrow,\psi\geq\underline{x}} EU(\psi(\zeta_T)), \qquad w_0 = E[\zeta_T\psi(\zeta_T)], \quad E[\psi(\zeta_T)g_\mu(\zeta_T)] \geq b \quad \forall\mu\in\mathcal{M}$$

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Dual-feasibility: $\alpha \geq 0$, and

$$\lambda \geq \sup_{x>0} \frac{\sum_{i=1}^{n} \alpha_i\ g_i(x)}{x}.$$

Complementary slackness: $\alpha\cdot z = 0$.

$$\sup L = \sup E\left[U(\psi(\zeta)) - \psi(\zeta)h(\zeta) - \alpha \cdot b \right] + \lambda w_0$$

where

$$h(z) \equiv \lambda z - \sum_{i=1}^{n} \alpha_i \, g_i(z).$$

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Easy if h increasing. Else, set $\tilde h(x) \equiv h(F_\zeta^{-1}(x))$, $\tilde\psi(x) \equiv \psi(F_\zeta^{-1}(x))$, consider

$$E\big[\, U(\psi(\zeta)) - \psi(\zeta)h(\zeta) \,\big] = \int_0^1 \{U(\tilde\psi(x)) - \tilde\psi(x)\tilde h(x)\}\, dx \equiv \Psi,$$

say.

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say. Now set $H(x) \equiv \int_0^x \tilde{h}(y)\,dy$, and let $\underline{H}$ be the greatest convex minorant of H, which we may express as

$$\underline{H}(x) = H(x) + \eta(x)$$

for some $\eta \le 0$, $\eta(0) = \eta(1) = 0$.

$$\sup L = \sup E\Big[\, U(\psi(\zeta)) - \psi(\zeta)h(\zeta) - \alpha \cdot b \,\Big] + \lambda w_0$$

where

$$h(z) \equiv \lambda z - \sum_{i=1}^{n} \alpha_i \; g_i(z).$$

Easy if h increasing. Else, set $\tilde{h}(x) \equiv h(F_\zeta^{-1}(x))$, $\tilde{\psi}(x) \equiv \psi(F_\zeta^{-1}(x))$, consider

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$$\underline{H}(x) = H(x) + \eta(x)$$

for some $\eta \le 0$, $\eta(0) = \eta(1) = 0$. Now estimate

$$\begin{aligned} \Psi &= \int_0^1 \{U(\tilde{\psi}(x)) - \tilde{\psi}(x)(\tilde{h}(x) + \eta'(x))\} \; dx + \int_0^1 \tilde{\psi}(x)\eta'(x) \; dx \\ &\le \int_0^1 \tilde{U}(\tilde{h}(x) + \eta'(x)) \; dx + [\tilde{\psi}(x)\eta(x)]_0^1 - \int_0^1 \eta(x) \; d\tilde{\psi}(x). \end{aligned}$$

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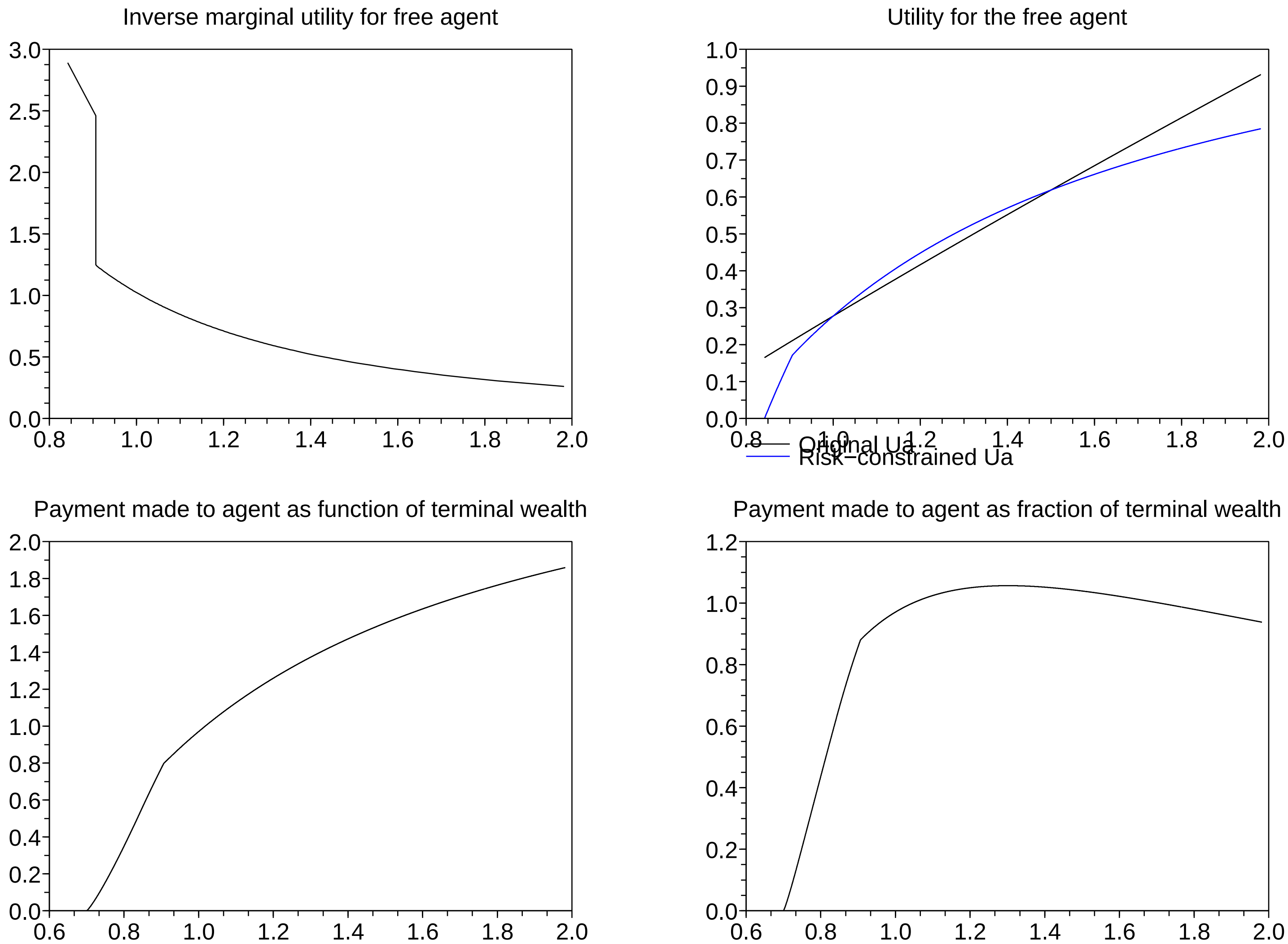
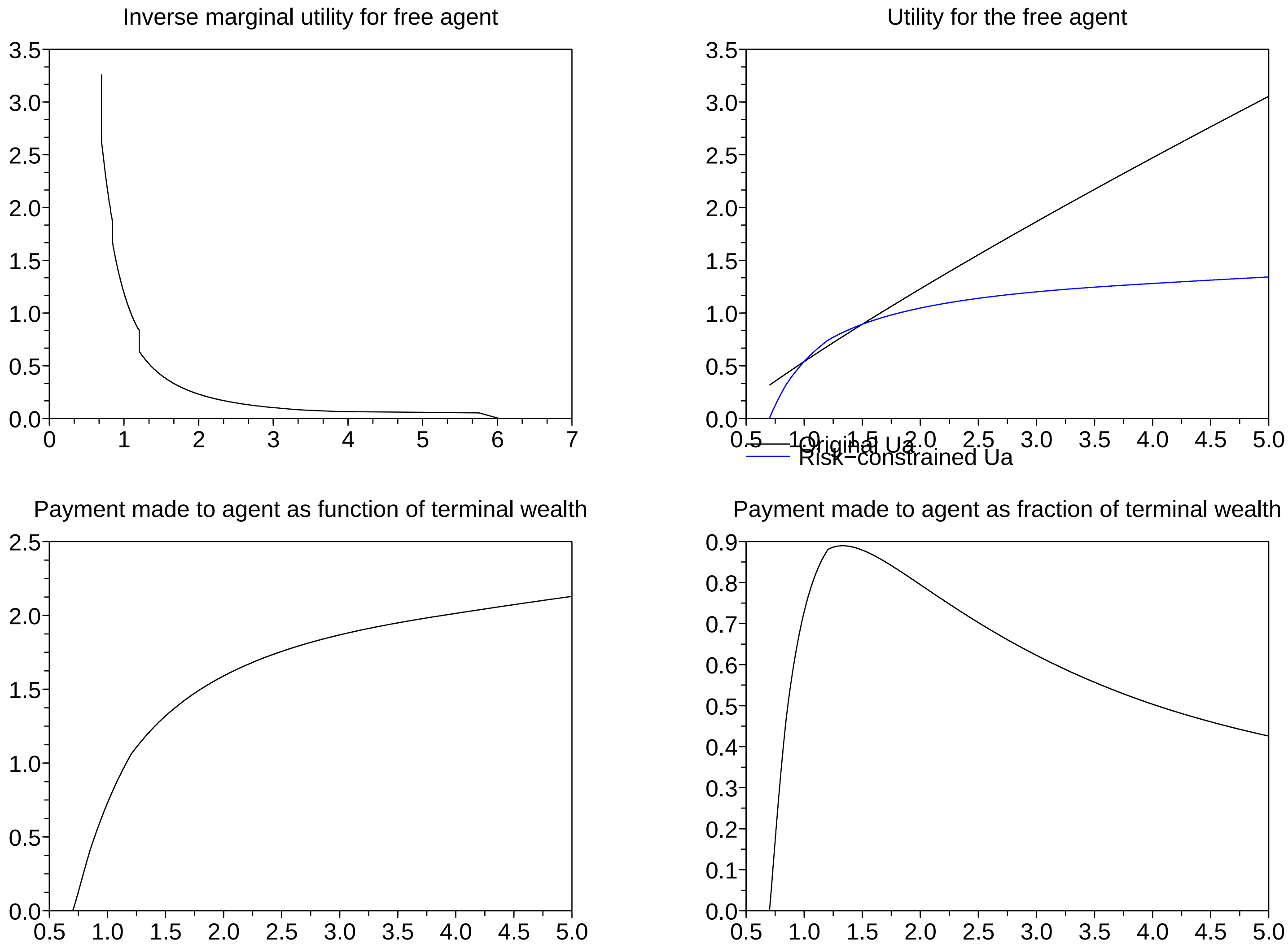
- This allows us to replace the prinicpal's constrained problem with an unconstrained problem for the agent. (Slight mismatch irrelevant in practice).
- The numerical approach is to minimize the dual value over the Lagrange multipliers.

How does it look?



$\mu = \delta_a$ for $a = 0.05$, $b = 0.9$



$g_i(x) = a_i^{-1} I_{\{x>1-a_i\}}$, $i = 1, 2$, $a_1 = 0.65$, $a_2 = 0.05$, $g_3(x) = \beta^{-1}\log(\frac{\beta}{\min(1-x,\beta)})$; $b = (1, 1.05, 1)$.

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- Risk-measure constrained principal cares only about the law of terminal wealth, so can find his optimum as a decreasing function of ζ_T
- Principal reverse-engineers a utility u from his optimal wealth
- Principal offers a wage schedule to make the agent's utility into u