CONTRACTING FOR OPTIMAL INVESTMENT WITH RISK CONTROL

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Overview

- Contracting to align objectives
- Investing under constraints on the law of terminal wealth
- Investment under law-invariant coherent risk measure constraints
- Contracting for optimal investment under LI coherent risk measure constraints
Aligning objectives.
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Insist $w_T \geq x$, then make

$$U_P(x - \varphi(x)) = k\{U_A(\varphi(x)) - U_A(0)\} + U_P(x)$$
Proposition. If $U_P$ and $U_A$ are strictly increasing, the function $\varphi : [x, \infty) \to \mathbb{R}^+$ is well defined by

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It is increasing, and $u \equiv U_A \circ \varphi$ is concave.
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Proof. Monotonicity obvious. If concavity fails, for some $x_1, x_2 \geq x$, $p = 1 - q \in (0, 1)$, with $x = px_1 + qx_2$

$$u(x) = U_A(\varphi(x))$$

$$< pu(x_1) + qu(x_2)$$

$$= p U_A(\varphi(x_1)) + q U_A(\varphi(x_2))$$

$$\leq U_A(p \varphi(x_1) + q \varphi(x_2)),$$

and so $\varphi(x) < p \varphi(x_1) + q \varphi(x_2)$. 


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and so $\varphi(x) < p\varphi(x_1) + q\varphi(x_2).$ Hence

$$u(x) = U_A(\varphi(x)) = U_P(x - \varphi(x))
> U_P(x - p\varphi(x_1) - q\varphi(x_2))
\geq pU_P(x_1 - \varphi(x_1)) + qU_P(x_2 - \varphi(x_2))
= pU_A(\varphi(x_1)) + qU_A(\varphi(x_2)).$$
How does it look?

Wage function: Agent is CRRA(0.95) Principal is CRRA(2), k = 3

UP, Principal's utility for final portfolio value

Proportion of terminal wealth paid to agent
Wage function: Agent is CRRA(0.95) Principal is $0.1 \times \text{CRRA}(2) + 0.9 \times \text{CRRA}(0.5)$, $k = 3$

Utility of all wealth

Utility of wealth after fees

Proportion of terminal wealth paid to agent

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Utility of wealth after fees
Investing under constraints on the law of $w_T$. 

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$$E[\zeta_T \psi(\zeta_T)] = w_0$$

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then an agent with initial wealth $w_0$ and maximizing $Eu(w_T)$ will choose $w_T = \psi(\zeta_T)$. If the risk-constrained principal offers the agent $\varphi$, where

$$kU_A(\varphi(x)) - a = u(x),$$

then the unconstrained agent implements the principal’s optimum.
Law-invariant coherent risk measures.
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These are

\[ \rho(X) = \sup \{ \rho^\mu(X) : \mu \in \mathcal{M} \}, \]

where \( \mathcal{M} \) is a collection of probability measures on \([0, 1]\).
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If \( X = \psi(\zeta) \equiv \psi(\zeta_T), \psi \text{ decreasing}, \) then \( F_X^{-1}(a) = \psi(F_\zeta^{-1}(1-a)). \)
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and

$$\rho^\mu(X) = -\int \psi(z) \left\{ \int_{1-F_\zeta(z)}^1 a^{-1} \mu(da) \right\} F_\zeta(dz)$$

$$= -E[\psi(\zeta) g_\mu(\zeta)]$$

for some non-negative increasing $g_\mu$. 
The optimization problem.

\[
\max_{\psi \downarrow, \psi \geq \zeta} EU(\psi(\zeta_T)), \quad w_0 = E[\zeta_T \psi(\zeta_T)], \quad E[\psi(\zeta_T)g_\mu(\zeta_T)] \geq b \quad \forall \mu \in \mathcal{M}
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\max_{\psi \downarrow, \psi \geq x} \mathbb{E}U(\psi(\zeta_T)), \quad w_0 = \mathbb{E}[\zeta_T \psi(\zeta_T)], \quad \mathbb{E}[\psi(\zeta_T)g_\mu(\zeta_T)] \geq b \quad \forall \mu \in \mathcal{M}
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L(\psi, z) \equiv E\left[ U(\psi(\zeta)) + \lambda(w_0 - \zeta \psi(\zeta)) + \sum_{i=1}^n \alpha_i \{\psi(\zeta)g_i(\zeta) - b_i - z_i\} \right]
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= E\left[ U(\psi(\zeta)) - \psi(\zeta)\{\lambda \zeta - \sum_{i=1}^n \alpha_i g_i(\zeta)\} - \alpha \cdot (z + b) \right] + \lambda w_0.
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Dual-feasibility: \( \alpha \geq 0 \).
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Complementary slackness: \(\alpha \cdot z = 0\).
\[
\sup L = \sup E \left[ U(\psi(\zeta)) - \psi(\zeta)h(\zeta) - \alpha \cdot b \right] + \lambda w_0
\]

where

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h(z) \equiv \lambda z - \sum_{i=1}^{n} \alpha_i g_i(z).
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Easy if \( h \) increasing. Else, set \( \tilde{h}(x) \equiv h(F^{-1}_\zeta(x)), \tilde{\psi}(x) \equiv \psi(F^{-1}_\zeta(x)) \), consider
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E \left[ U(\psi(\zeta)) - \psi(\zeta)h(\zeta) \right] = \int_0^1 \left\{ U(\tilde{\psi}(x)) - \tilde{\psi}(x) \tilde{h}(x) \right\} \, dx \equiv \Psi,
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say. Now set \( H(x) \equiv \int_0^x \tilde{h}(y) \, dy \), and let \( \overline{H} \) be the greatest convex minorant of \( H \), which we may express as
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\overline{H}(x) = H(x) + \eta(x)
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for some \( \eta \leq 0, \eta(0) = \eta(1) = 0 \). Now estimate

\[
\Psi = \int_0^1 \{ U(\tilde{\psi}(x)) - \tilde{\psi}(x)(\tilde{h}(x) + \eta'(x)) \} \, dx + \int_0^1 \tilde{\psi}(x) \eta'(x) \, dx \\
\leq \int_0^1 \tilde{U}(\tilde{h}(x) + \eta'(x)) \, dx + [\tilde{\psi}(x) \eta(x)]_0^1 - \int_0^1 \eta(x) \, d\tilde{\psi}(x).
\]
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- The numerical approach is to minimize the dual value over the Lagrange multipliers.
How does it look?

\[ \mu = \delta a \quad \text{for} \quad a = 0.05, \quad b = 0.9 \]
\( g_i(x) = a_i^{-1} I_{\{x > 1 - a_i\}} \), \( i = 1, 2 \), \( a_1 = 0.65 \), \( a_2 = 0.05 \), \( g_3(x) = \beta^{-1} \log \left( \frac{\beta}{\min(1-x, \beta)} \right) \); \( b = (1, 1.05, 1) \).
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Conclusions

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- Risk-measure constrained principal cares only about the law of terminal wealth, so can find his optimum as a decreasing function of $\zeta_T$
- Principal reverse-engineers a utility $u$ from his optimal wealth
- Principal offers a wage schedule to make the agent’s utility into $u$