Modelling markets with transaction costs

Miklós Rásonyi

Computer and Automation Institute of
the Hungarian Academy of Sciences, Budapest

Based on joint work with Paolo Guasoni and Walter Schachermayer.

Topics in this talk

– No-arbitrage requirements restrict model choice.

– Discerning the relationship between arbitrage and the class of admissible trading strategies.

– From the point of view of arbitrage, which properties of stochastic processes matter ?

– Frictionless markets, markets with (proportional) transaction costs, liquidity constraints.

Semimartingales and free lunches I

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$

$(S_t)_{t\in[0,T]}$: adapted càdlàg process, locally bounded

Simple predictable integrands: τ_i increasing sequence of stopping times, $i = 1, \ldots, n+1$;

$$F = \sum_{i=1}^{n} f_i 1_{]\tau_i, \tau_{i+1}]}, \quad f_i \in \mathcal{F}_{\tau_i}, i = 1, \ldots, n.$$

Elementary stochastic integral:

$$(F \cdot S)_T := \sum_{i=1}^{n} f_i (S_{\tau_{i+1}\wedge T} - S_{\tau_i \wedge T}).$$

Frictionless model of trading

We assume 0 initial capital. Stock price: S , bond price $\equiv 1$.

F_t represents number of stock held in the portfolio at time t .

Interpretation: portfolio rebalanced at the stopping times τ_i in a predictable way.

Predictability: practical and technical justification.

Portfolio terminal value:

$V(F) := (F \cdot S)_T$

Semimartingales and free lunches II

Arbitrage: If there is F s.t. $V(F) \geq 0$ a.s., $P(V(F) > 0) > 0$.

Free lunch with vanishing risk for simple integrands: a simple predictable sequence F_n s.t. $V(F_n) \geq -1/n$ a.s. and $V(F_n) \to M \in [0, \infty]$ a.s., $P(M > 0) > 0$.

Theorem. (Delbaen and Schachermayer '94) No free lunch with vanishing risk for simple integrands implies that S is a semimartingale.

(Counter)example

Fractional Brownian motion with Hurst parameter $H \neq 1/2$: B^H .

Continuous centered Gaussian process satisfying

$$EB_s^H B_t^H = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$$

Not a semimartingale for $H \neq 1/2$: admits free lunches for simple integrands ! There is even arbitrage, Rogers '97, etc. . .

Restrictions on strategies I: time lag

Cheridito '03, Jarrow, Protter and Sayit '08: furthermore stipulate that $\tau_{i+1} \geq \tau_i + h$ for each i , for some $h > 0$.

Cheridito '03: (geometric) FBM has no arbitrage with respect to the restricted class.

"Discrete-time trading."

In practice S can be identified along a discrete sequence of time instants only. (Microstructure ?)

Model perturbation I

Theorem. (Jarrow, Protter and Sayit '08) If S is continuous, admits an equivalent local martingale measure, $\langle S\rangle$ satisfies a technical condition then $S+C$ has no arbitrage w.r.t. the restricted class for *any* adapted càdlàg bounded C .

Recurrent phenomenon: absence of arbitrage insensitive to certain perturbations of S .

Restrictions on strategies II: smoothness

Assume S continuous with a quadratic variation $d\langle S\rangle_t = \sigma^2(S_t)dt$ and satisfies a small ball condition.

$\sigma(\cdot)$ is C^1 with linear growth.

Forward integral: $F \cdot S$ is definable for e.g. $F_t = f(S_t)$ with $f \in C^1$ and Itô formula holds, Föllmer '81.

Theorem. (Bender, Sottinen and Valkeila '08) If F_t is a C^1 functional of t , S_t the average and the running maximum (minimum) of S at t then $V(F)$ cannot be an arbitrage.

Model perturbations II

Example. If $S = \exp\{B^H + W\}$ where W is BM and $H > 1/2$ then this model is arbitrage-free for the "smooth" strategies above.

$(\langle B^H + W\rangle = \langle W\rangle\ )$

If strategies are smooth, only quadratic variation of the process matters and finer probabilistic structure (i.e. long-range dependence) doesn't (from the arbitrage point of view).

Markets with friction

Bid- and ask prices: $\underline{S}_t \leq \overline{S}_t$, adapted and continuous (for simplicity)

Simple strategies:

$$F := \sum_{j=1}^{\infty} f_j 1_{]\tau_j, \tau_{j+1}]}, \quad f_j \in \mathcal{F}_{\tau_j}, \ j = 1, \ldots$$

where $\sup_j \tau_j > T$ a.s. and $F_0 = F_T = 0$.

For each $\omega \in \Omega$ there are finitely many transactions.

Value and admissibility

$$V(F) = \sum_{j=1}^{\infty} \underline{S}_{\tau_j}(F_{\tau_{j+1}} - F_{\tau_j})^- \quad (1)$$

$$- \sum_{j=1}^{\infty} \overline{S}_{\tau_j}(F_{\tau_{j+1}} - F_{\tau_j})^+ \quad (2)$$

F is *simple* x *-admissible* if for each stopping time σ there is $\tau \geq \sigma$ such that $V(F 1_{[0,\sigma]} + F_\sigma 1_{(\sigma,\tau]}) \geq -x$ a.s.

F is simple admissible if it is simple x -admissible for some $x > 0$.

General trading strategies I

A process G is a (general) x *-admissible* strategy if $F_n(\omega, t) \to G(\omega, t)$ for each ω and t for some simple $x + 1/n$ -admissible F_n .

Robust no free lunch with vanishing risk (Schachermayer '04): there are $\underline{S}_t < \underline{S}'_t < \overline{S}'_t < \overline{S}_t$ such that the market $(\underline{S}', \overline{S}')$ has no free lunches with vanishing risk for simple admissible strategies. (RNFLVR)

Restrictions on strategies III: FV

Proposition. (Guasoni and Rásonyi '08) (RNFLVR) for simple strategies implies that each G is a finite variation process.

Proposition. If $\underline{S}, \overline{S}$ are bounded then a process G is an x -admissible strategy iff it is predictable with finite variation and for each $\delta > 0$ and each stopping time σ there is a stopping time $\tau \geq \sigma$ with

$$V(G1_{[0,\sigma]} + G_\sigma 1_{(\sigma,\tau]}) \geq -x - \delta$$

a.s.

General trading strategies II

Terminal value of trading with portfolio G :

$G = G^{+} - G^{-}$: minimal decomposition with G^{+}, G^{-} predictable increasing.

$V(G) := -\int_0^T \overline{S}_u dG_u^{+} + \int_0^T \underline{S}_u dG_u^{-}$

Stieltjes-integral.

Dual variables

A consistent price system is (Q, Z) s.t. $Q \sim P$, Z is a Q -martingale

$\underline{S}_t \leq Z_t \leq \overline{S}_t$ a.s. for all $t \in [0, T]$.

(Shadow price.)

Strictly consistent price system: strict inequalities.

Analogue of equivalent martingale measures (Jouini and Kallal '95, Kabanov and Stricker '00, Schachermayer '04).

Discrete-time: satisfactory multidimensional theory.

Basis of dual methods in utility maximisation (Kallsen and Muhle-Karbe '08).

Fundamental theorem I

Theorem. (Guasoni and Rásonyi '08) The following are equivalent.

– (RNFLVR) for simple strategies.

– No robust arbitrage for general strategies.

– Existence of strictly consistent price systems.

Fundamental theorem II

Special case: proportional transaction costs.

S_t positive, continuous and adapted. $\varepsilon > 0$ fixed.

$\underline{S}_t := (1-\varepsilon)S_t$, $\overline{S}_t = (1+\varepsilon)S_t$

Admissibility in the usual sense: $V(F1_{[0,t]}) \geq -x$ a.s. for all t .

Theorem. (Guasoni, Rásonyi and Schachermayer '08) There is absence of arbitrage for each $\varepsilon > 0$ iff there are strictly consistent price systems for each $\varepsilon > 0$.

Technical problems

Admissibility in the usual sense:

– Closedness of the set of attainable payoffs is problematic.

– Easy to check.

Our concept of admissibility:

– Economic interpretation, closedness.

– Difficult to check if a strategy is admissible.

Campi and Schachermayer ’06: one more concept.

Main ingredients

Lemma. If there is no arbitrage with simple admissible strategies and $V(F) \geq -x$ for a simple admissible strategy F then F is x -admissible.

Compare to: $V(F) \geq -x$ implies $V(F1_{[0,t]}) \geq -x$ for $t \in [0,T]$ in frictionless arbitrage-free markets.

(In discrete time: analogous condition implies existence of SCPS in a strong sense, Rásonyi '08, Kabanov and Stricker '02.)

Lemma. One can approximate G , $V(G)$ uniformly by some simple F (resp. $V(F)$).

Model classes with SCPS

Sufficient conditions (in the spirit of Levental-Skorohod '97, Guasoni '06 and Kabanov and Stricker '08).

The following two conditions imply the existence of SCPS for all $\varepsilon > 0$:

– 0 is a.s. in the (relative) interior of the convex hull of the support of the conditional distribution of $S_\tau - S_\sigma$ w.r.t. $\mathcal{F}_\sigma$, for all stopping times $\sigma \leq \tau$.

– For all stopping times τ and for all $\delta > 0$,

$$P(\sup_{u \in [\tau, T]} |S_u - S_\tau| < \delta | \mathcal{F}_\tau) > 0$$

a.s. on $\{\tau < T\}$.

How to check these conditions ?

Conditional full support

$C_x^+[u,v]$: continuous positive functions on $[u,v]$ starting from $x > 0$

We say that S has conditional full support if for all $u < T$,

$$\text{supp} P(S|_{[u,T]} \in \cdot | \mathcal{F}_u) = C_{S_u}^+[u,T]$$

almost surely.

Example. Any Markov process S with full support on $C_{S_0}^+[0,T]$ satisfies this. (Stroock and Varadhan '72 support theorem.)

Theorem. (Guasoni, Rásonyi and Schachermayer '08) C. f. s. implies the existence of SCPS for all $\varepsilon > 0$.

FBM & co.

$S_t = \exp\{B_t^H\}$ has conditional full support (Guasoni et al. '08).

Gaussian moving averages (Cherny '08).

Mixture models (products of independent processes with c.f.s.).

A digression back to frictionless models

Take S with conditional full support (satisfying previous assumptions) and F simple predictable

$$F = \sum_{i=1}^{n} f_i \mathbf{1}_{]\tau_i, \tau_{i+1}]}, \quad f_i \in \mathcal{F}_{\tau_i}, i = 1, \ldots, n.$$

where τ_i are hitting times of continuous boundaries by S ($\max S$).

Then $V(F)$ cannot be an arbitrage. (Bender, Sottinen and Valkeila '08)

Smooth trajectories

Lemma. If $X_0 = 0$ and X has c.f.s. in the sense

$$\text{supp}P(X|_{[u,T]} \in \cdot|\mathcal{F}_u) = C_{X_u}[u,T] \text{ a.s. for each } u < T,$$

then $Y_t := \int_0^t X_s ds$ also has c.f.s. in the above sense.

Corollary: $\exp\{Y\}$ has SCPS *and* smooth trajectories.

Under proportional transaction costs, trajectorial properties do not matter from the arbitrage point of view (while probabilistic properties do).

Hedging

Theorem. If g is lower semicontinuous and bounded from below, the asymptotic ($\varepsilon \to 0$) superreplication price of $g(S_T)$ is $\widehat{g}(S_0)$ where $\widehat{g}$ is the concave envelope of g .

It follows that the superreplication price of $(S_T - K)^+$ is S_0 . (Soner, Shreve and Cvitanic '95; Levental and Skorohod '97)

This shows how investors' hands are tied by transaction costs.

To price options utility-based approach needed. Duality theory. (Kabanov, Last, Stricker, Campi, Schachermayer)

Illiquid markets - an example

Price process replaced by supply curve.

Hypothetical price: $dS_t = S_t\mu(S_t)dt + S_t\sigma(S_t)dW_t$.

Buying ν units of stock at time t costs

e.g. $\mathcal{S}(t,\nu) := S_t e^{\alpha\nu}$

with some parameter $\alpha > 0$.

Illiquid markets -trading

Discrete-time heuristics leads to terminal wealth

$$V(F) = (F \cdot S)_T - \int_0^T S_u^2 \sigma^2(S_u) \gamma_u^2 \frac{\partial}{\partial \nu} \mathcal{S}(u,0) du$$

where $(\partial/\partial\nu)\mathcal{S}(u,0) = \alpha S_u$ and strategies are of the form

$$F_t = \int_0^t \beta_u du + (\gamma \cdot S)_t$$

with β, γ progressively measurable.

Thus it seems that trading strategies should have finite *quadratic* variation in this context.

(Liquidation function is smooth at the origin while its derivative jumps at 0 in the case of proportional transaction costs.)

Moral I

In frictionless market models discretized trading strategies allow for (bold) perturbations of the probability as well as the trajectorial structure.

Smooth trading and pricing of “smooth” options is indifferent to (certain) probabilistic perturbations as long as quadratic variation remains unchanged.

Moral II

Under (proportional) transaction costs trajectorial properties seem to be irrelevant (jump case: on-going research). Probabilistic properties are important.

Illiquid case: strategies with finite *quadratic* variation appear (transition from frictionless to transaction cost world).