# Optimal portfolio liquidation

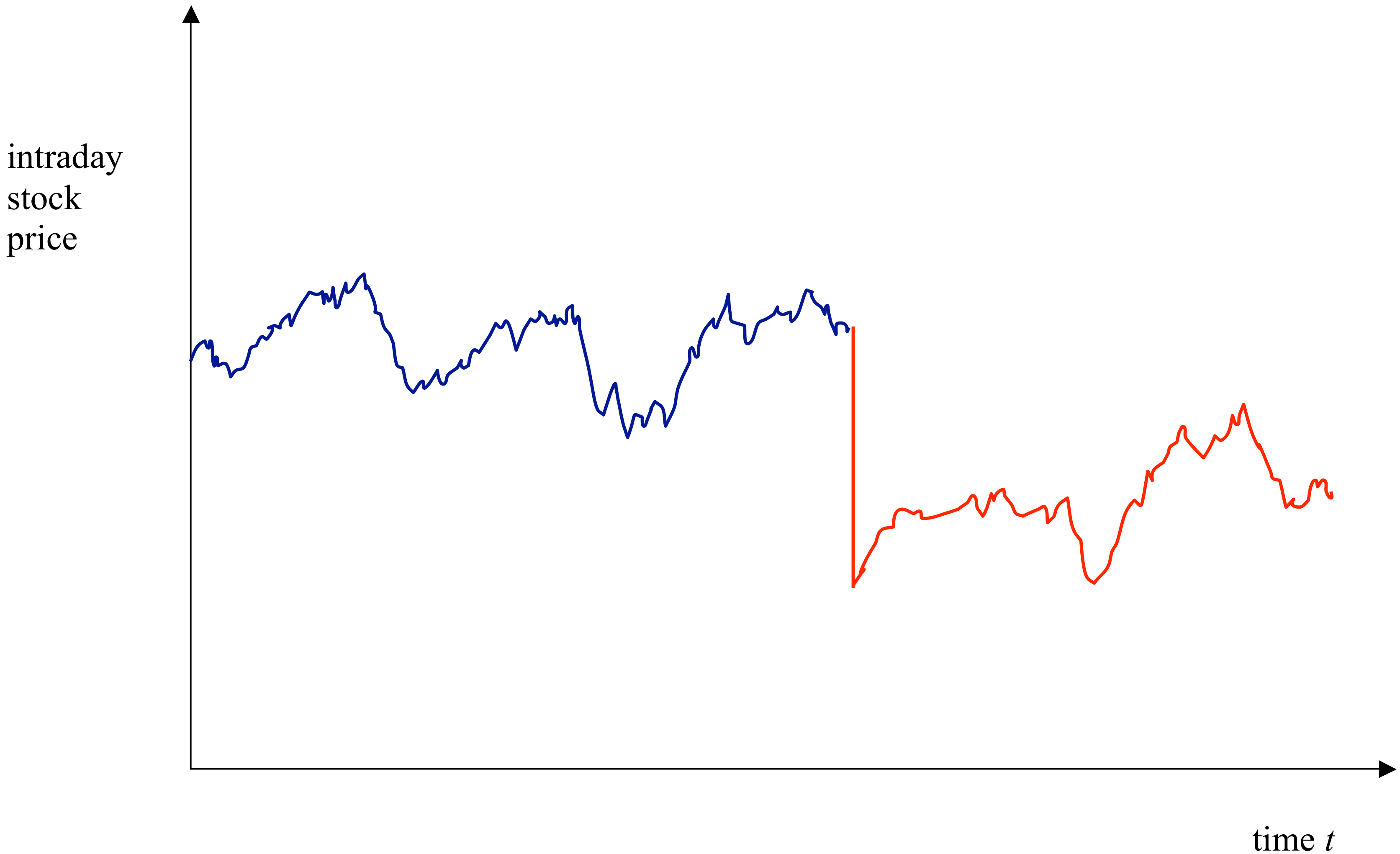
Alexander Schied
Operations Research and Information Engineering
Cornell University

**Advanced Modeling in Finance and Insurance**

RICAM Linz, September 26, 2008

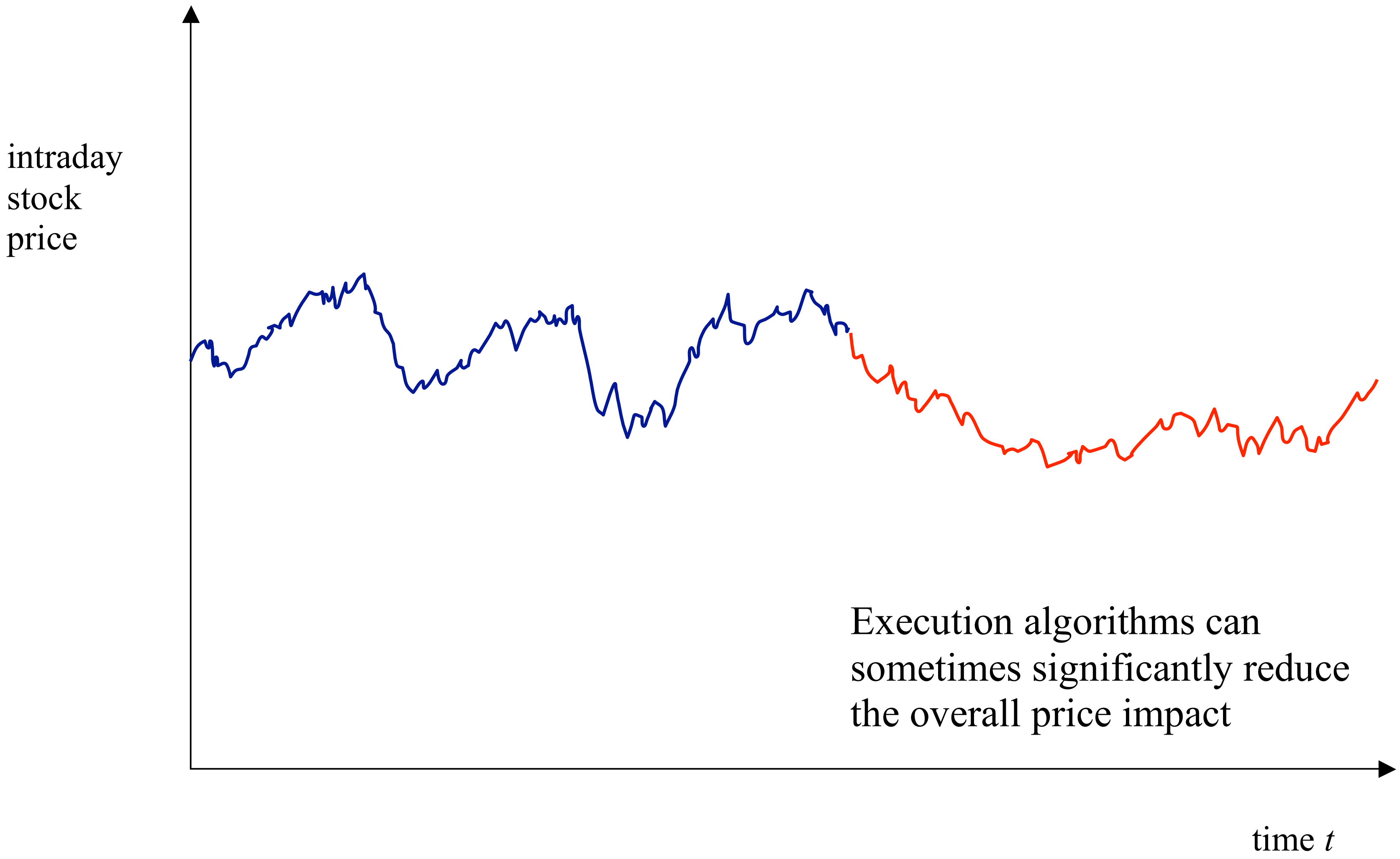
**Joint work with**
**Aurélien Alfonsi, Antje Fruth, and Torsten Schöneborn**

**Common problem:**
A trader needs to liquidate a large long (short) position by the end of the day



**Common problem:**

A trader needs to liquidate a large long (short) position by the end of the day



## How to optimally execute a large block order?

Need first:

- **How do stock prices react to large orders?**
- **What is a good model for this?**
- **How to formulate and solve the optimization problem?**

Interesting because:

- **Relevant in applications**
- **Liquidity risk in its purest form**
- **Interesting mathematics**

# Overview

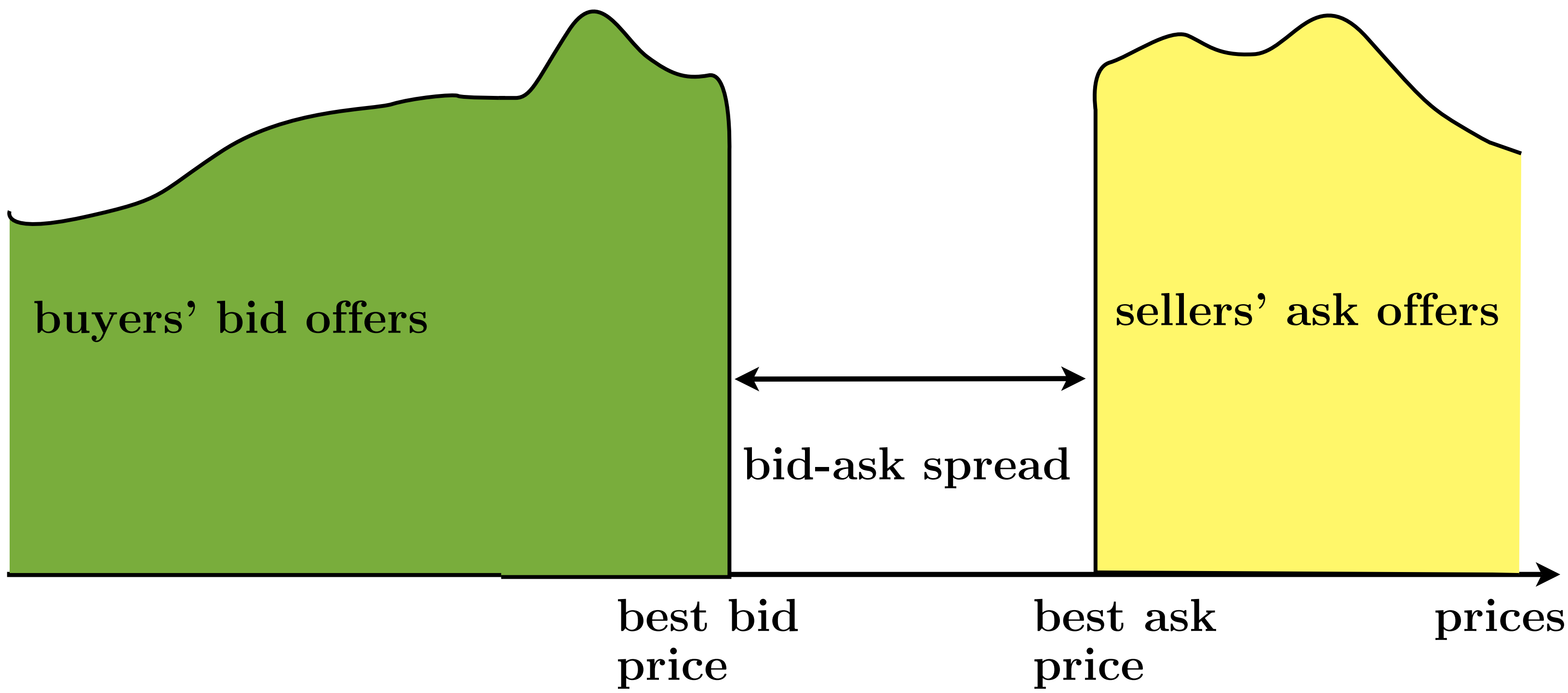
**0. Intuition for market impact**

**1. Optimal execution in a limit order book**

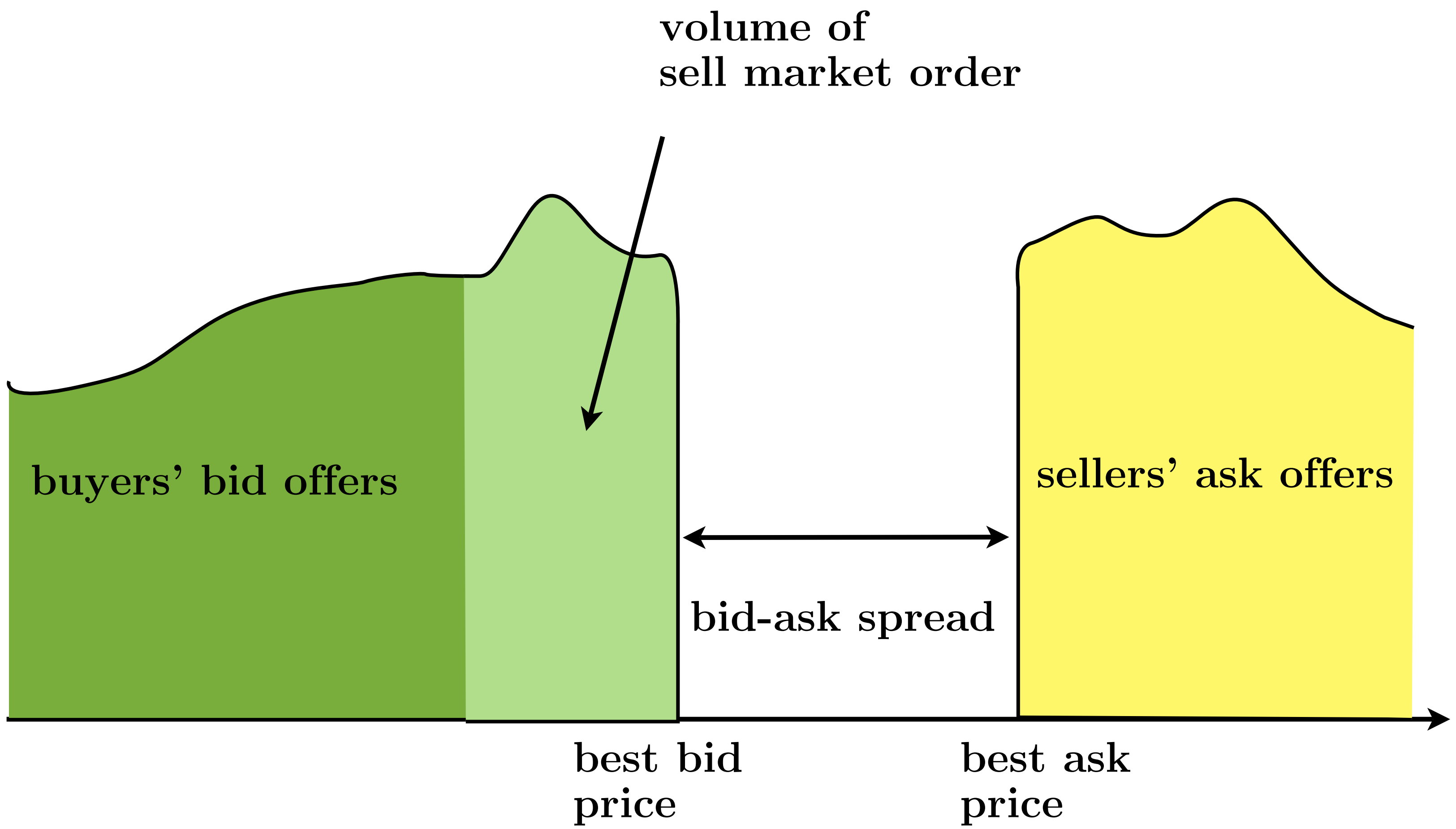
**2. Optimal execution and risk aversion**

**3. Multi-agent equilibrium**

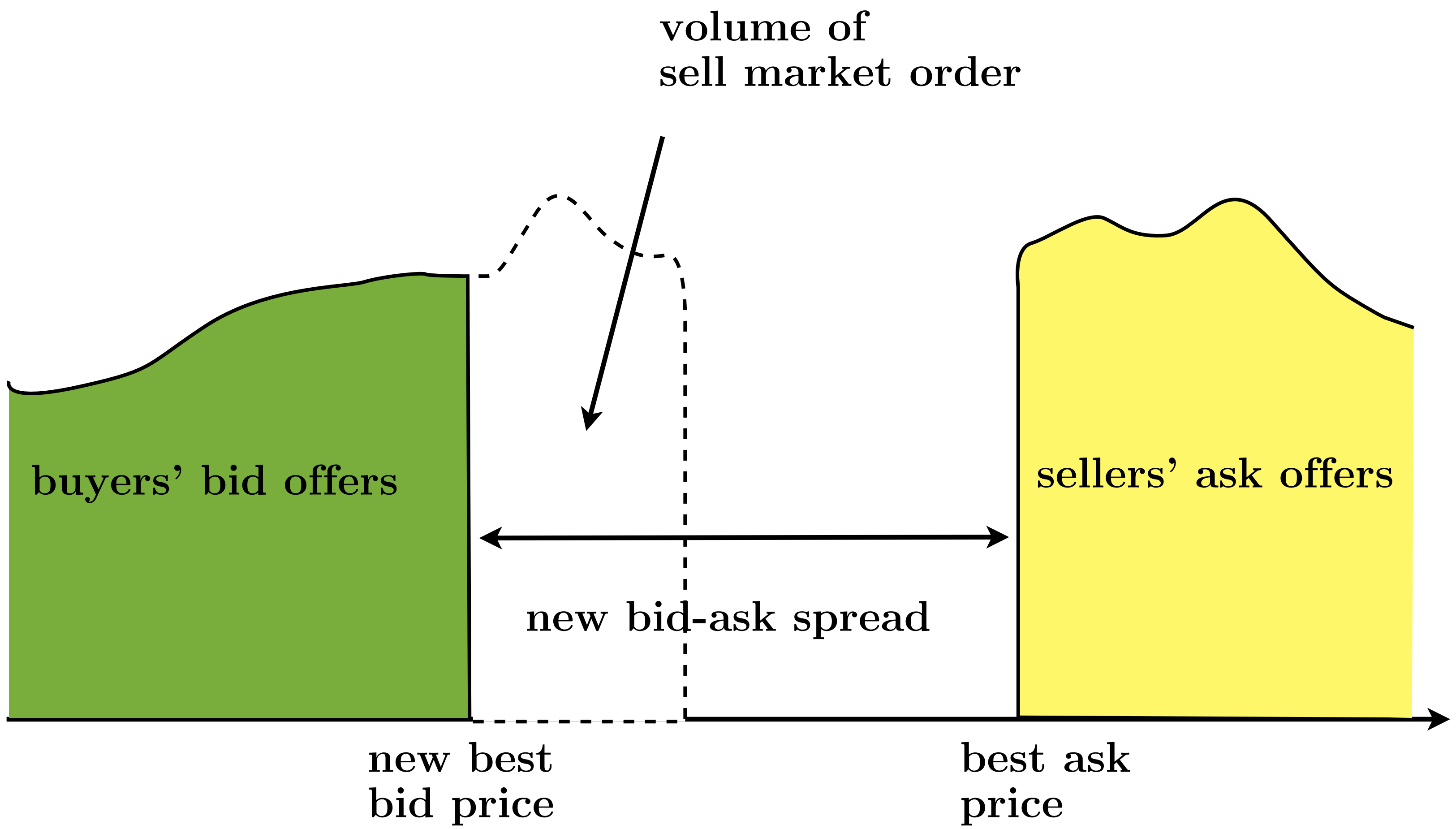
# Limit order book before market order



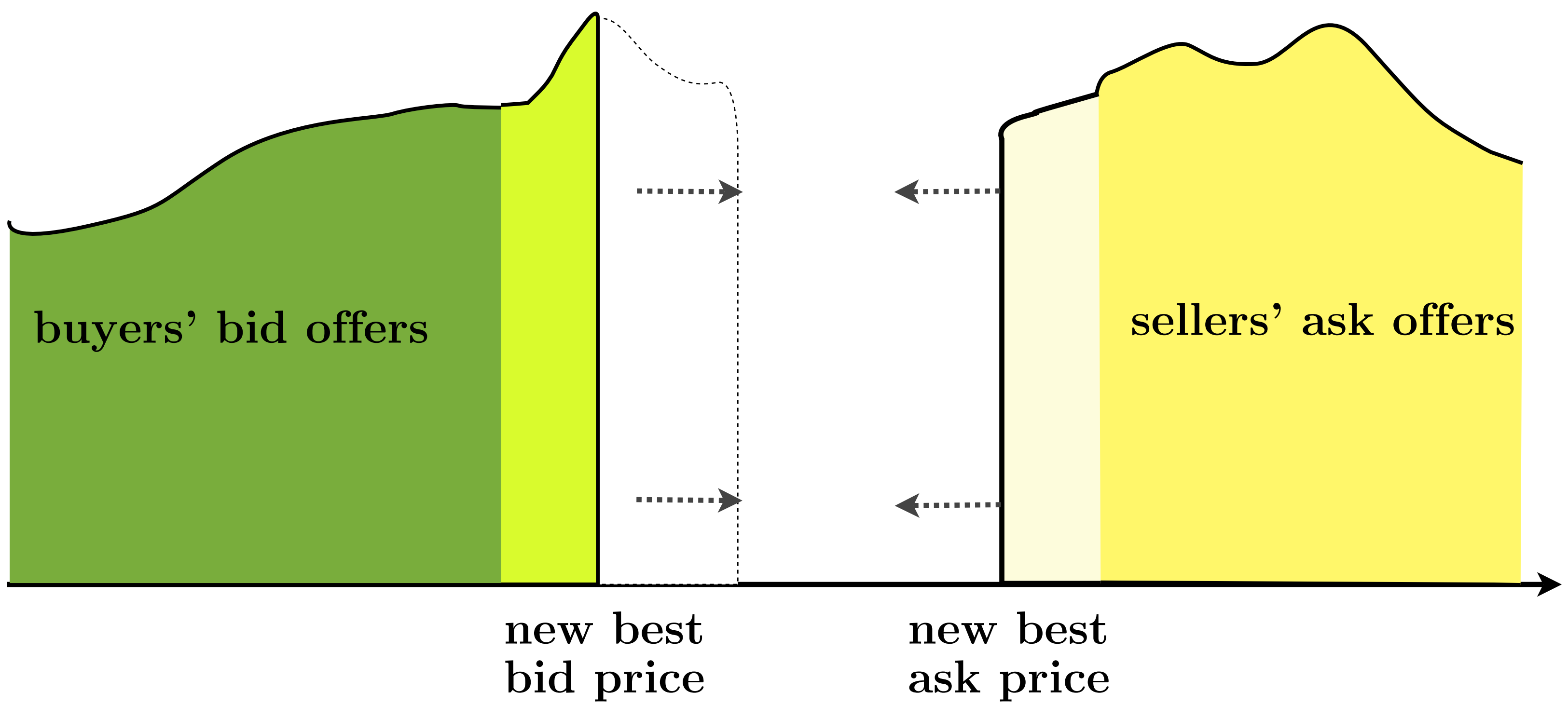
# Limit order book before market order



# Limit order book after market order



# Resilience of the limit order book after market order



# 1. Optimal execution in a limit order book model

**Block-shaped limit order book:**

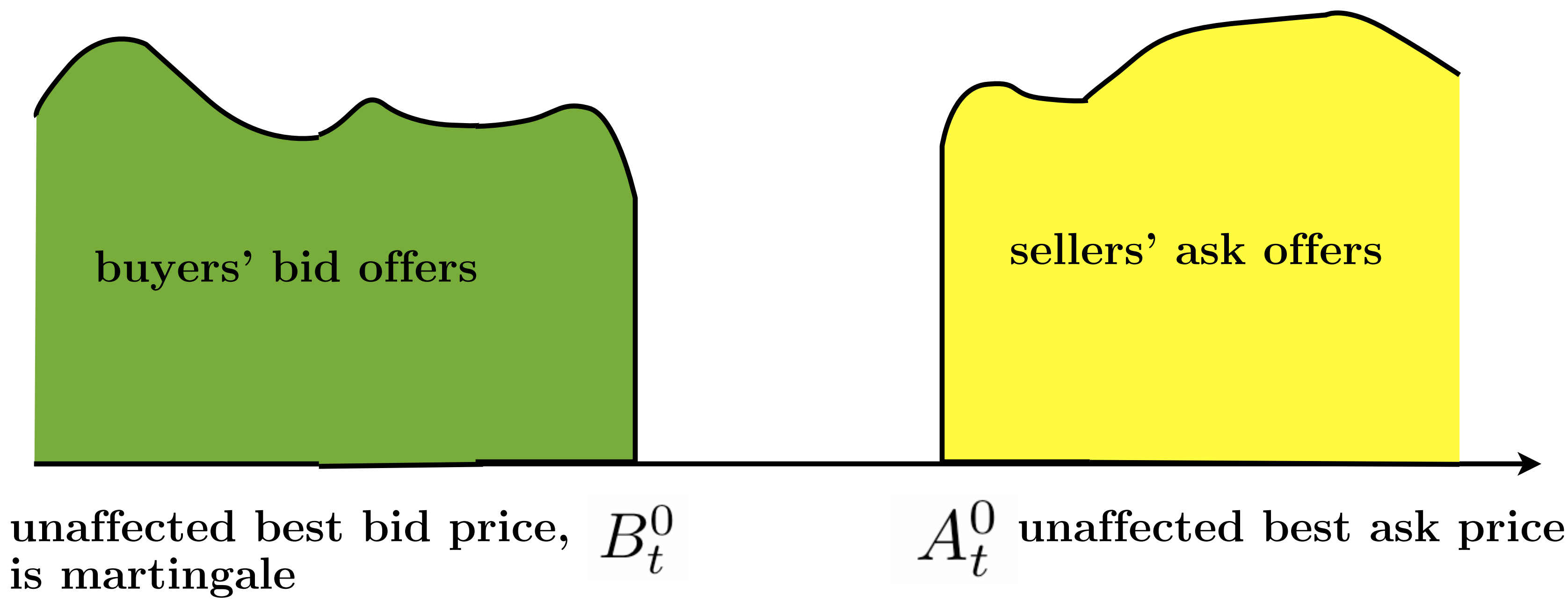
[1] A. Obizhaeva and J. Wang: *Optimal Trading Strategy and Supply/Demand Dynamics.* To appear in J. Financial Markets

**LOB with general shape function and linear constraints:**

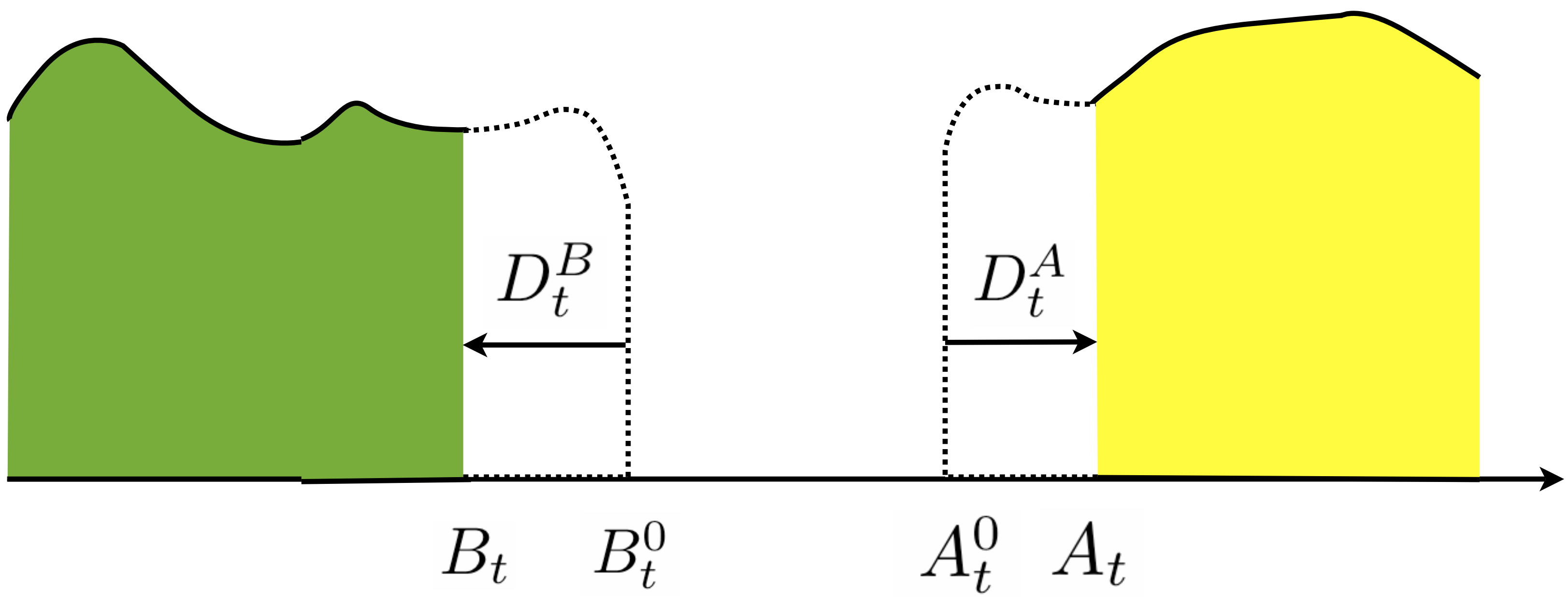
[2] (A. Alfonsi, A. Fruth, and A.S.): *Optimal execution strategies in limit order books with general shape functions.* Preprint, 2007

[3] (A. Alfonsi, A. Fruth, and A.S.): *Constrained portfolio liquidation in a limit order book model.* To appear in Banach Center Publications.

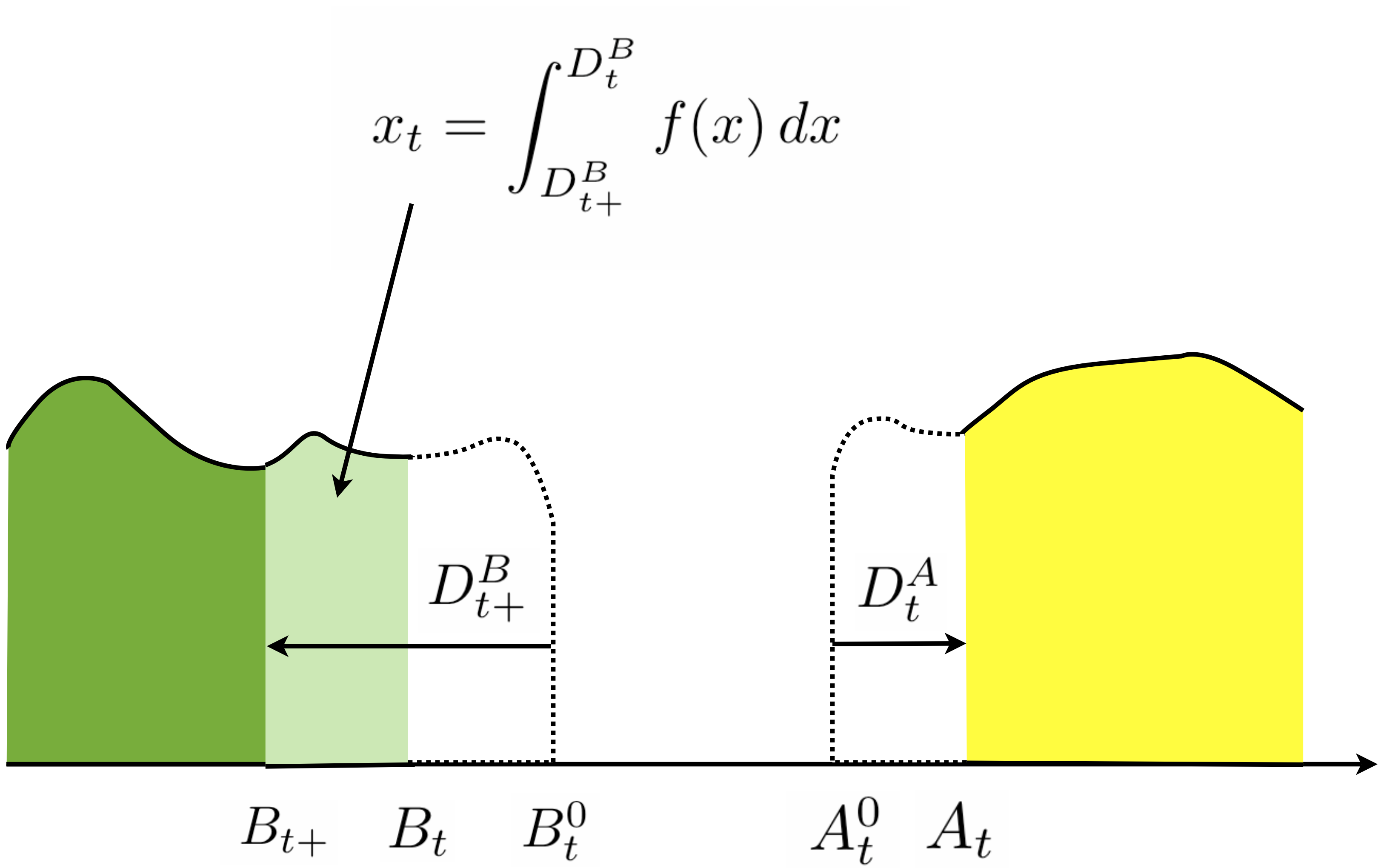
## Limit order book model without large trader



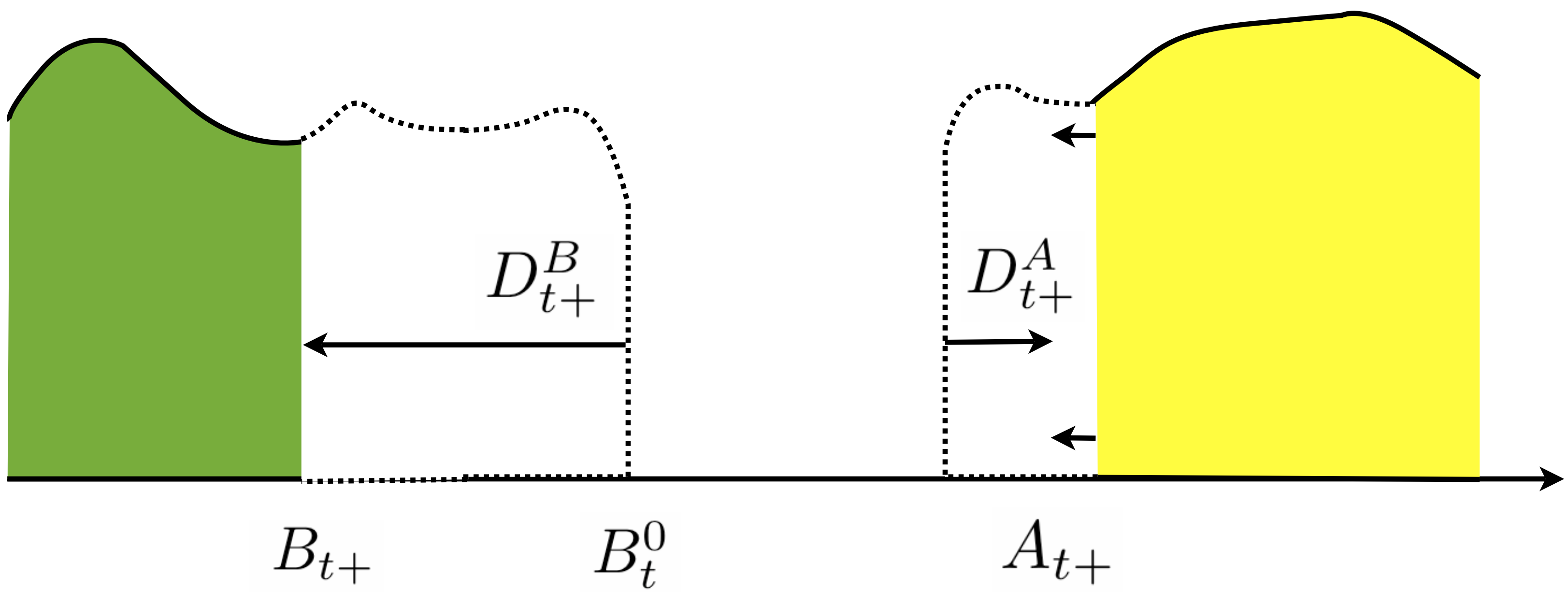
## Limit order book model after large trades



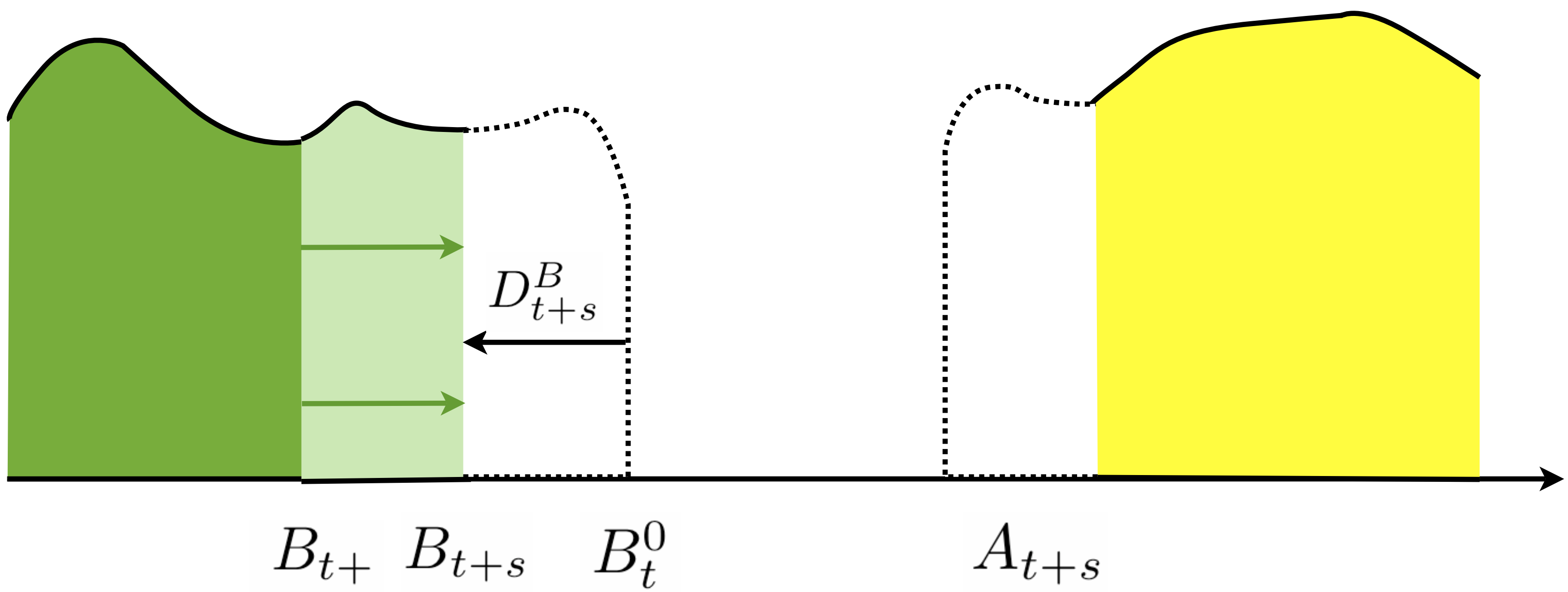
## Limit order book model at large trade



# Limit order book model immediately after large trade



# Limit order book model with resilience



**The LOB model:**

$f(x)$ = shape function = densities of bids for $x < 0$, asks for $x > 0$

$B_t^0$ = 'unaffected' bid price at time $t$, is martingale

$B_t$ = bid price after market orders before time $t$

$D_t^B = B_t - B_t^0$

If sell order of $\xi_t \geq 0$ shares is placed at time $t$:

$D_t^B$ changes to $D_{t+}^B$, where

$$\int_{D_{t+}^B}^{D_t^B} f(x)dx = \xi_t$$

and

$$B_{t+} := B_t + D_{t+}^B - D_t^B = B_t^0 + D_{t+}^B,$$

$\Longrightarrow$ nonlinear price impact

**The LOB model:**

$A_t^0$ = 'unaffected' ask price at time $t$, satisfies $B_t^0 \leq A_t^0$

$A_t$ = bid price after market orders before time $t$

$D_t^A = A_t - A_t^0$

If buy order of $\xi_t \leq 0$ shares is placed at time $t$:

$D_t^A$ changes to $D_{t+}^A$, where

$$\int_{D_t^A}^{D_{t+}^A} f(x)dx = -\xi_t$$

and

$$A_{t+} := A_t + D_{t+}^A - D_t^A = A_t^0 + D_{t+}^A,$$

If the large investor is inactive during the time interval $[t, t+s[$, there are *two* possibilities:

- **Exponential recovery of the extra spread**

$$D^B_{t+s} = e^{-\rho s} D^B_t$$

- **Exponential recovery of the order book volume**

$$E^B_{t+s} = e^{-\rho s} E^B_t$$

where

$$E^B_t = \int_{D^B_t}^{0} f(x)\, dx =: F(D^B_t).$$

**In both cases: analogous dynamics for $D^A$ or $E^A$**

**The optimization problem:**

- $N+1$ market orders: $\xi_n$ shares placed at time $t_n = n\tau$ s.th.
  1) $\xi_n$ is adapted and bounded from below (can be negative),
  2) we have $\sum_{n=0}^{N} \xi_n = X_0$

- When selling $\xi_n \geq 0$ shares, we sell $f(x)\,dx$ shares at price $B_t^0 + x$ with $x$ ranging from $D_{t+}^B$ to $D_t^B$, i.e., the revenues are positive:

$$\rho_{t_n}(\xi_n) := \int_{D_{t_n+}^B}^{D_{t_n}^B} (B_{t_n}^0 + x) f(x)\,dx = \xi_n B_{t_n}^0 + \int_{D_{t_n+}^B}^{D_{t_n}^B} x f(x)\,dx$$

- When buying shares ($\xi_n \leq 0$), the revenues are negative:

$$\rho_{t_n}(\xi_n) := \xi_n A^0_{t_n} - \int_{D^A_{t_n}}^{D^A_{t_n+}} x f(x)\, dx$$

- The expected revenues from the strategy $\xi = (\xi_n)$ are

$$\mathcal{R}(\xi) = \mathbb{E}\Big[ \sum_{n=0}^{N} \rho_{t_n}(\xi_n) \Big]$$

The goal is to maximize the expected revenues over all admissible strategies.

**Theorem 1** (Optimal strategy for exponential recovery of volume). *Suppose that the LOB has infinite depth, i.e., $|F(x)| \to \infty$ as $|x| \to \infty$, and that the function*

$$h(u) := e^{-\rho\tau} F^{-1}(e^{-\rho\tau} u) - F^{-1}(u)$$

*is one-to-one. Let $\xi_0^*$ be the unique solution of the equation*

$$F^{-1}\left(N\xi_0^*\left(1 - e^{-\rho\tau}\right) - X_0\right) = \frac{h(-\xi_0^*)}{1 - e^{-\rho\tau}},$$

*and*

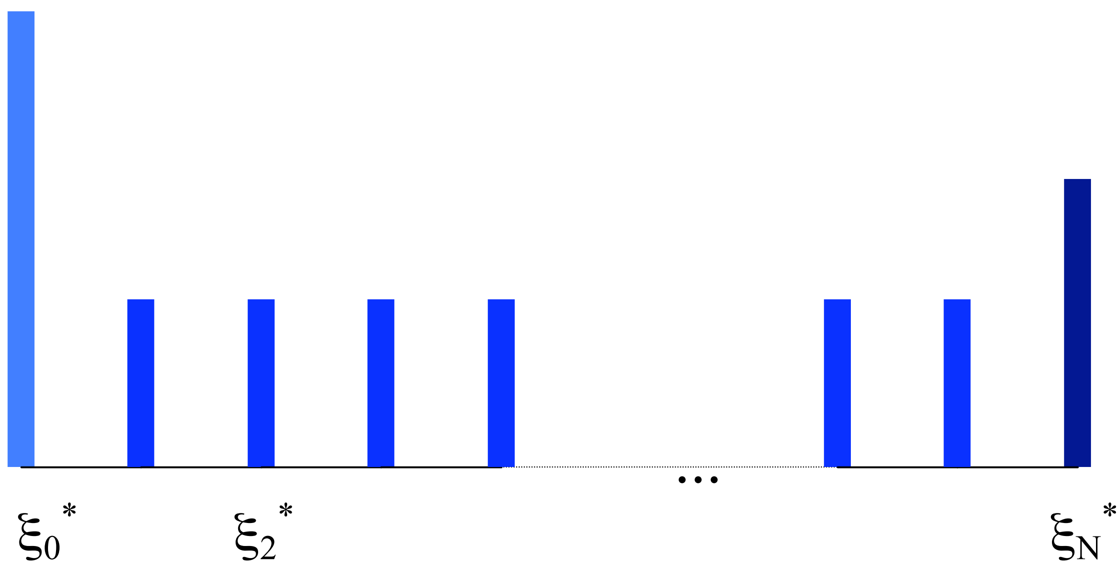
$$\xi_1^* = \cdots = \xi_{N-1}^* = \xi_0^*\left(1 - e^{-\rho\tau}\right),$$

*as well as*

$$\xi_N^* = X_0 - \xi_0^* - (N-1)\xi_0^*\left(1 - e^{-\rho\tau}\right).$$

*Then $\xi_n^* > 0$ for all $n$, and $\xi^*$ is the unique optimal strategy.*

The optimal strategy for the minimization
of the expected liquidation costs is deterministic:




Optimal sell market orders for *N+1* equidistant time steps

**Remark:** The impact of $\xi^*$ is such that

$$D^B_{t_{k+1}} = e^{-\rho\tau} D^B_{t_k+}$$

for all $k$.

**Corollary 2** (Closed-form solution for block-shaped LOB).
*In a block-shaped LOB, the unique optimal strategy $\xi^*$ is*

$$\xi_0^* = \xi_N^* = \frac{X_0}{(N-1)(1-e^{-\rho\tau})+2} \quad \textit{and} \quad \xi_1^* = \cdots = \xi_{N-1}^* = \frac{X_0 - 2\xi_0^*}{N-1}.$$

Improves the entangled forward-backward recursive scheme by Obizhaeva and Wang ($X_t$ = number of shares at time $t$):

$$
\text{(1)}\quad
\begin{aligned}
\xi_n^* &= \frac{1}{2}\delta_{n+1}\left[\epsilon_{n+1}X_{t_n} - \phi_{n+1}D_{t_n}\right], \qquad n = 0, \ldots, N-1, \\
\xi_N^* &= X_T,
\end{aligned}
$$

$$
\text{(2)}\quad
\begin{aligned}
\delta_n &:= \left(\frac{1}{2q} + \alpha_n - \beta_n\kappa e^{-\rho\tau} + \gamma_n\kappa^2 e^{-2\rho\tau}\right)^{-1} \\
\epsilon_n &:= \lambda + 2\alpha_n - \beta_n\kappa e^{-\rho\tau} \\
\phi_n &:= 1 - \beta_n e^{-\rho\tau} + 2\gamma_n\kappa e^{-2\rho\tau}.
\end{aligned}
$$

$$
\text{(3)}\quad
\begin{aligned}
\alpha_N = \frac{1}{2q} - \lambda \quad \text{and} \quad \alpha_n &= \alpha_{n+1} - \frac{1}{4}\delta_{n+1}\epsilon_{n+1}^2, \\
\beta_N = 1 \quad \text{and} \quad \beta_n &= \beta_{n+1}e^{-\rho\tau} + \frac{1}{2}\delta_{n+1}\epsilon_{n+1}\phi_{n+1}, \\
\gamma_N = 0 \quad \text{and} \quad \gamma_n &= \gamma_{n+1}e^{-2\rho\tau} - \frac{1}{4}\delta_{n+1}\phi_{n+1}^2.
\end{aligned}
$$

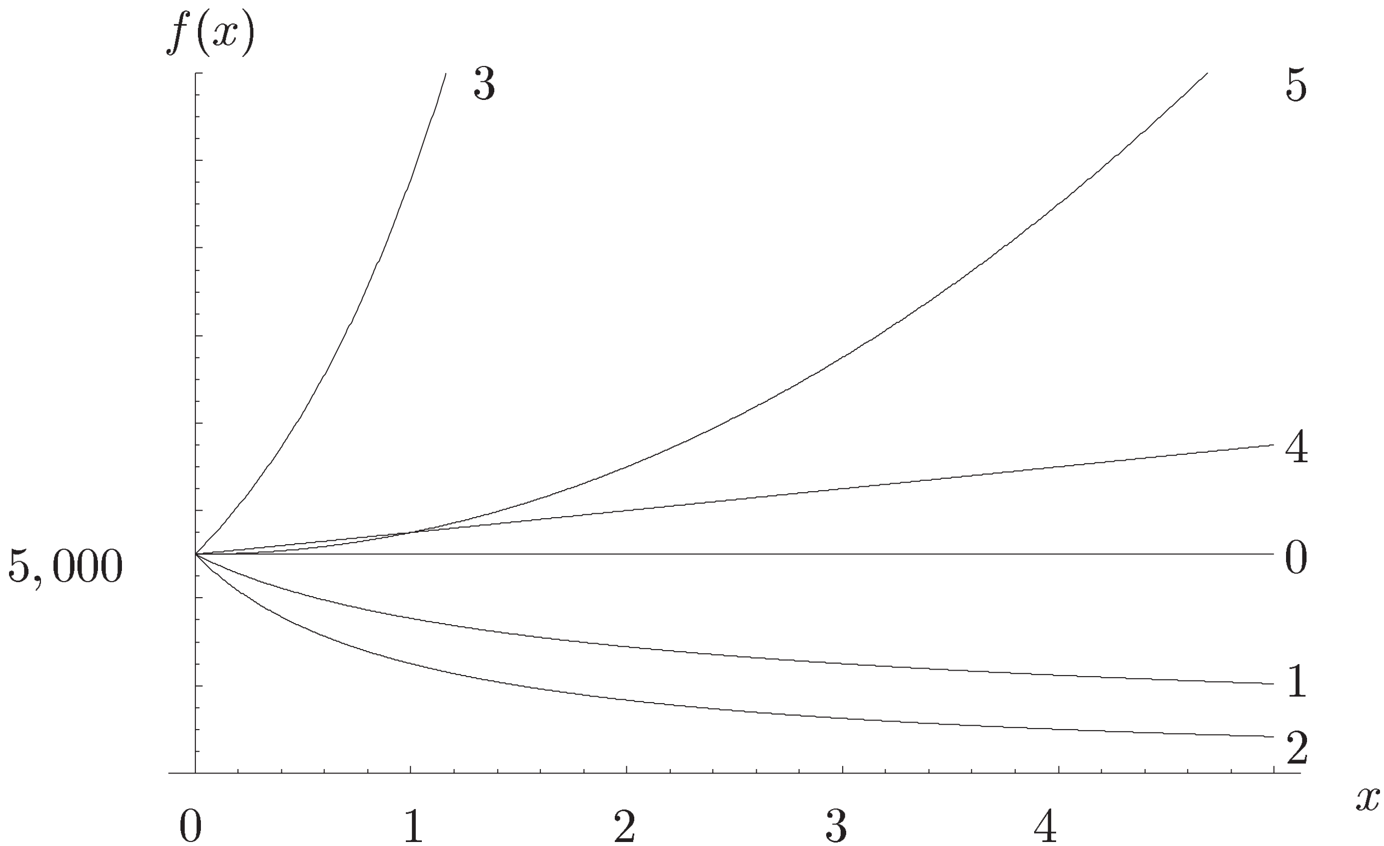
Further **Theorems:**

- Analogous (but different) result for exponential recovery of $E^B$.

- Explicit solution for inhomogeneous recovery (block-shaped LOB)

- Explicit solutions under linear constraints (block-shaped LOB).

**Example shape functions:**



| Example number | $f(x)$ | Recovery of $E$ | | | Recovery of $D$ | | |
|---|---|---|---|---|---|---|---|
| | | $\xi_0^*$ | $\xi_1^*$ | $\xi_N^*$ | $\xi_0^*$ | $\xi_1^*$ | $\xi_N^*$ |
| 0 | $q$ | 10,223 | 8,839 | 10,223 | 10,223 | 8,839 | 10,223 |
| 1 | $\frac{q}{\sqrt{\lvert x\rvert+1}}$ | 10,257 | 8,869 | 9,925 | 10,756 | 8,724 | 10,726 |
| 2 | $\frac{q}{\lvert x\rvert+1}$ | 10,303 | 8,909 | 9,520 | 13,305 | 8,154 | 13,305 |
| 3 | $qe^{\lvert x\rvert}$ | 10,139 | 8,767 | 10,962 | 9,735 | 8,947 | 9,741 |
| 4 | $\frac{q}{10}\lvert x\rvert+q$ | 10,211 | 8,829 | 10,326 | 10,130 | 8,860 | 10,131 |
| 5 | $\frac{q}{10}x^2+q$ | 10,192 | 8,812 | 10,498 | 10,101 | 8,868 | 10,091 |

Table 1: The table shows optimal strategies for various choices of the shape function $f$. We set $X_0 = 100,000$ and $q = 5,000$ shares, $\rho = 20$, $T = 1$ and $N = 10$.

## Sketch of proof:

### 1. Step: reduction to deterministic optimization problem

Introduce simplified dynamics with collapsed bid-ask spread

- $E_0 = D_0 = 0$ and

$$E_t = F(D_t) \qquad \text{and} \qquad D_t = F^{-1}(E_t).$$

- Regardless of the sign of $\xi_n$,

$$E_{t_n+} = E_{t_n} + \xi_n \qquad \text{and} \qquad D_{t_n+} = F^{-1}\left(\xi_n + F\left(D_{t_n}\right)\right).$$

- For $k = 0, \ldots, N-1$,

$$D_{t_{k+1}} = e^{-\rho\tau} D_{t_k+}$$

Then

$$E_t^B \leq E_t \leq E_t^A \qquad \text{and} \qquad D_t^B \leq D_t \leq D_t^A.$$

Introduce simplified revenues

$$\overline{\rho}_{t_n}(\xi_n) := B^0_{t_n}\xi_n + \int_{D_{t_n+}}^{D_{t_n}} x f(x)\, dx,$$

They satisfy

$$\overline{\rho}_{t_n}(\xi_n) \geq \rho_{t_n}(\xi_n) \qquad \text{with equality if } \xi_k \geq 0 \text{ for all } k.$$

The simplified revenues functional is

$$\overline{\mathcal{R}}(\xi) = \mathbb{E}\Big[\sum_{n=0}^{N} \overline{\rho}_{t_n}(\xi_n)\Big].$$

With $X_t = X_0 - \sum_{t_k<t} \xi_k$ for $t \leq T$ and $X_{t_{N+1}} := 0$,

$$\sum_{n=0}^{N} B^0_{t_n}\xi_n = -\sum_{n=0}^{N} B^0_{t_n}(X_{t_{n+1}} - X_{t_n}) = X_0 B_0 + \sum_{n=1}^{N} X_{t_n}(B^0_{t_n} - B^0_{t_{n-1}}).$$

Hence,

$$\overline{\mathcal{R}}(\xi) = B_0 X_0 + \mathbb{E}\big[\, R(\xi_0, \ldots, \xi_N)\,\big],$$

where for any deterministic strategy $(x_0, \ldots, x_N)$

$$R(x_0, \ldots, x_N) := \sum_{n=0}^{N} \int_{D_{t_n+}}^{D_{t_n}} x f(x)\, dx.$$

## 2. Step: deterministic optimization problem

Let

$$\Xi := \Big\{\, (x_0, \ldots, x_N) \in \mathbb{R}^{N+1} \,\Big|\, \sum_{n=0}^{N} x_n = X_0 \Big\}.$$

We will show that $R$ has a unique minimum on $\Xi$ that coincides with the optimal strategy.

With

$$\widetilde{F}(x) = \int_0^x y f(y)\, dy, \qquad G(x) = \widetilde{F}(F^{-1}(x)), \qquad \text{and } a := e^{-\rho\tau}$$

we have

$$\begin{aligned}
R(x_0, \ldots, x_N) &= \sum_{n=0}^{N} \left( \widetilde{F}\left(F^{-1}\left(E_{t_n}\right)\right) - \widetilde{F}\left(F^{-1}\left(E_{t_n+}\right)\right) \right) \\
&= \sum_{n=0}^{N} \left( G\left(E_{t_n}\right) - G\left(E_{t_n} + x_n\right) \right) \\
&= \quad G(0) - G(x_0) \\
&\quad + G(a x_0) - G(a x_0 + x_1) \\
&\quad + G\left(a^2 x_0 + a x_1\right) - G\left(a^2 x_0 + a x_1 + x_2\right) \\
&\quad + \ldots \\
&\quad + G\left(a^N x_0 + \cdots + a x_{N-1}\right) - G\left(a^N x_0 + \cdots + x_N\right).
\end{aligned}$$

**Lemma 3** *We have* $R(x_0, \ldots, x_N) \longrightarrow -\infty$ *for* $|\xi| \to \infty$, *and therefore there exists a local maximum of* $R$ *in* $\Xi$.

Thanks to Lemma 3, there is at least one optimal strategy $\xi^* = (x_0^*, \ldots, x_N^*) \in \Xi$, and so there exists a Lagrange multiplier $\nu \in \mathbb{R}$ such that

$$\frac{\partial}{\partial x_j} R(x_0^*, \ldots, x_N^*) = \nu \qquad \text{for } j = 0, \ldots, N.$$

Note

$$G'(x) = F^{-1}(x).$$

$$
\begin{aligned}
\frac{\partial}{\partial x_j} R(x_0, \ldots, x_N) \quad &= \quad a \left[ \frac{\partial}{\partial x_{j+1}} R(x_0, \ldots, x_N) + G' \left( a(a^j x_0 + \cdots + x_j) \right) \right] \\
& \qquad - G' \left( a^j x_0 + \cdots + x_j \right)
\end{aligned}
$$

Hence

$$
\nu = a\nu + h(a^j x_0^* + \cdots + x_j^*).
$$

and

$$
\begin{aligned}
x_0^* \quad &= \quad h^{-1} \left( \nu \left( 1 - a \right) \right) \\
x_j^* \quad &= \quad x_0^* \left( 1 - a \right) \qquad \text{for } j = 1, \ldots, N-1 \\
x_N^* \quad &= \quad X_0 - x_0^* - (N-1) x_0^* \left( 1 - a \right).
\end{aligned}
$$

More computations now yield equation for $x_0^*$...................

## 2. Optimal execution and risk aversion

Risk-neutral evaluation does not capture volatility risk. Must introduce risk aversion.

The limit order book model is too complicated.

Results are based on simpler model.

- Almgren & Chriss, Almgren & Lorenz: mean-variance optimization.

- with Torsten Schöneborn: expected utility maximization
  - absolute risk aversion determines qualitative behavior of optimal strategies.
  - CARA case is special.

[1] Almgren, R., Chriss, N. *Value under liquidation.* Risk, Dec. 1999.

[2] Almgren, R., Chriss, N. *Optimal execution of portfolio transactions.* J. Risk 3, 5-39 (2000).

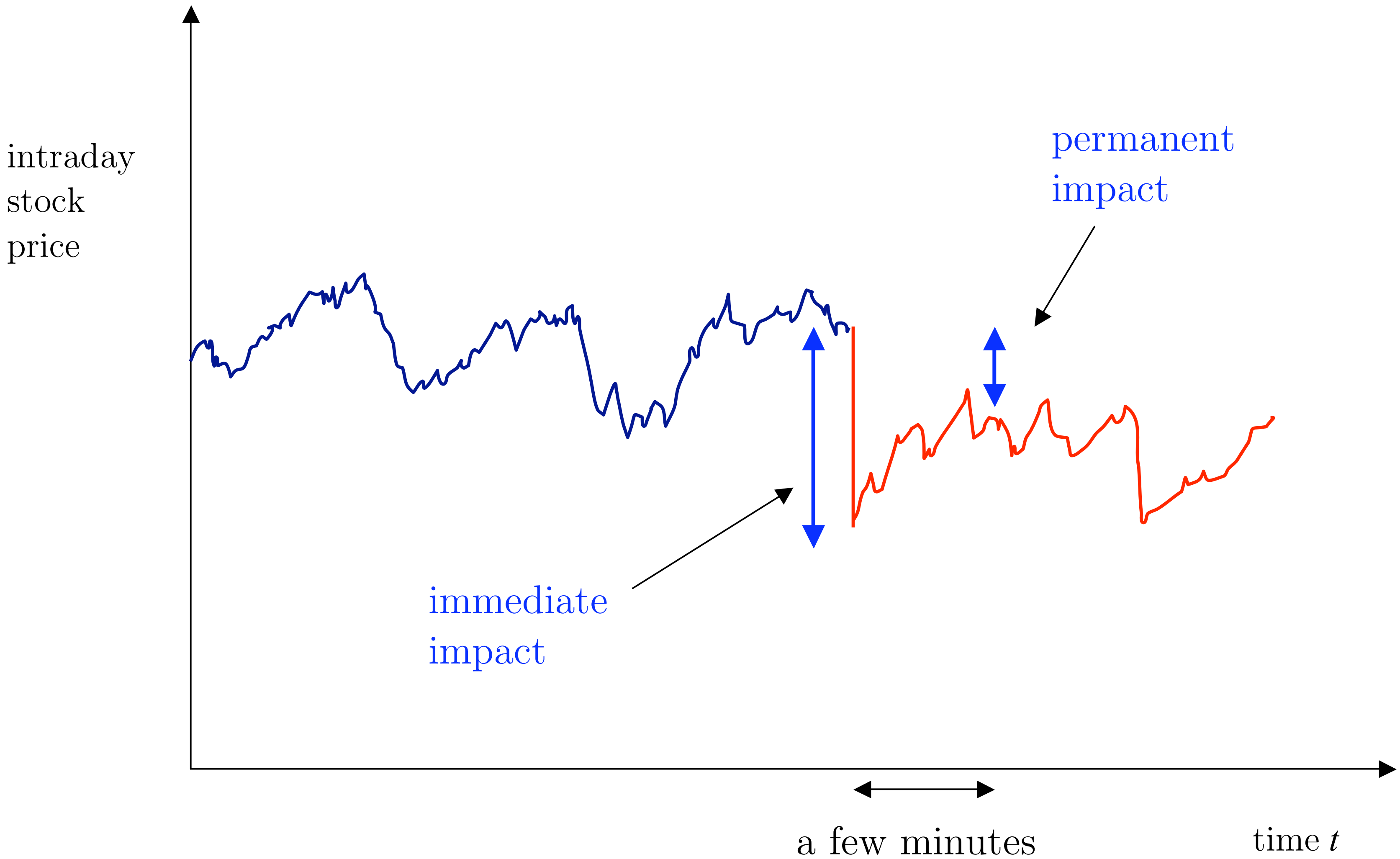
[3] Almgren, R. *Optimal execution with nonlinear impact functions and trading-enhanced risk.* Applied Mathematical Finance 10, 1-18 (2003).

[4] Almgren, R., Lorenz, J. *Adaptive arrival price.* In: Algorithmic Trading III: Precision, Control, Execution, Brian R. Bruce, editor, Institutional Investor Journals (2007).

[5] (A.S. and T. Schöneborn) *Optimal basket liquidation with finite time horizon for CARA investors.* Preprint (2007).

[6] (A.S. and T. Schöneborn) *Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets.* Preprint (2008).

# Temporary and permanent price impacts



**Liquidation time:** $T \in [0, \infty]$.

**Strategy:** $X$ adapted with $X_0 > 0$ fixed and $X_T = 0$.
Admissible: $X_t$ bounded, absolutely continuous in $t$.

**Market impact model:** Following Almgren (2003),

$$\begin{array}{ccccccccc} S_t^X & = & S_0 & + & \sigma B_t & + & \gamma\,(X_t - X_0) & + & h(-\dot{X}_t) \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & \text{initial} & & \text{Brownian} & & \text{permanent} & & \text{temporary} \\ & & \text{price} & & \text{motion} & & \text{impact} & & \text{impact} \\ & & & & & & \text{(linear)} & & \text{(nonlinear)} \end{array}$$

Most common model in practice, *drift, multiple assets* possible.

**Assumption:**

$$f(x) := xh(x)$$

is convex, $C^1$, and satisfies $f(x)/x \to \infty$ for $|x| \to \infty$.
E.g., $h(x) = \alpha \operatorname{sign}(x)\sqrt{|x|} + \beta x$.

**Sales revenues:**

$$\begin{aligned}\mathcal{R}_T(X) &= \int_0^T (-\dot{X}_t) S_t^X \, dt = \dots \\ &= S_0 X_0 - \frac{\gamma}{2} X_0^2 + \sigma \int_0^T X_t \, dB_t - \int_0^T f(-\dot{X}_t) \, dt.\end{aligned}$$

**Goal:** maximize expected utility

$$\mathbb{E}[\, u(\mathcal{R}_T(X)) \,]$$

over admissible strategies for $u$ increasing, concave.

## Setup as control problem

- Control process $\xi_t$, corresponding to $-\dot{X}_t$.
- $X_t^\xi := X_0 - \int_0^t \xi_s \, ds$
- controlled diffusion:

$$R_t^\xi = R_0 + \sigma \int_0^t X_s^\xi \, dB_s - \int_0^t f(\xi_s) \, ds$$

- value function

$$v(T, X_0, R_0) = \sup_{\xi \in \mathcal{X}_T(X_0)} \mathbb{E}\big[\, u(R_T^\xi) \,\big],$$

where

$$\mathcal{X}_T(X_0) = \Big\{ \xi \,|\, X^\xi \text{ is bounded and } \int_0^T \xi_t \, dt = X_0 \Big\}$$

## Heuristic derivation of HJB equation

$$dv(T-t, X_t^\xi, R_t^\xi) = \sigma v_R X_t^\xi \, dB_t$$
$$+ \Big( - v_t - \xi_t v_X + v_R f(\xi_t) + \frac{\sigma^2}{2} (X_t^\xi)^2 v_{RR} \Big) \, dt$$

Hence

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_\xi \big( \xi v_X + v_R f(\xi) \big)$$

## Heuristic derivation of HJB equation

$$dv(T-t, X_t^\xi, R_t^\xi) = \sigma v_R X_t^\xi \, dB_t$$
$$+ \Big( - v_t - \xi_t v_X + v_R f(\xi_t) + \frac{\sigma^2}{2} (X_t^\xi)^2 v_{RR} \Big) \, dt$$

Hence

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_\xi \big( \xi v_X + v_R f(\xi) \big)$$

What about the constraint $\int_0^T \xi_t \, dt = X_0$?

## Heuristic derivation of HJB equation

$$dv(T-t, X_t^\xi, R_t^\xi) = \sigma v_R X_t^\xi \, dB_t$$
$$+ \Big( -v_t - \xi_t v_X + v_R f(\xi_t) + \frac{\sigma^2}{2}(X_t^\xi)^2 v_{RR} \Big) \, dt$$

Hence

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_\xi \big( \xi v_X + v_R f(\xi) \big)$$

What about the constraint $\int_0^T \xi_t \, dt = X_0$? It is in the initial condition:

$$\lim_{T \downarrow 0} v(T, X, R) = \begin{cases} u(R) & \text{if } X = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

## Heuristic derivation of HJB equation

$$dv(T-t, X_t^\xi, R_t^\xi) = \sigma v_R X_t^\xi \, dB_t$$
$$+ \Big( -v_t - \xi_t v_X + v_R f(\xi_t) + \frac{\sigma^2}{2}(X_t^\xi)^2 v_{RR} \Big) \, dt$$

Hence

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_\xi \big( \xi v_X + v_R f(\xi) \big)$$

What about the constraint $\int_0^T \xi_t \, dt = X_0$? It is in the initial condition:

$$\lim_{T \downarrow 0} v(T, X, R) = \begin{cases} u(R) & \text{if } X = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

**Theorem:** If $u(x) = -e^{-\alpha x}$ for some $\alpha > 0$, then the unique optimal strategy $\xi^*$ is a deterministic function of $t$. Moreover, $v$ is a classical solution of the singular HJB equation.

**Theorem:** If $u(x) = -e^{-\alpha x}$ for some $\alpha > 0$, then the unique optimal strategy $\xi^*$ is a deterministic function of $t$. Moreover, $v$ is a classical solution of the singular HJB equation.

The fact that optimal strategies for CARA investors are deterministic is very robust. Is also true

- if Brownian motion is replaced by a Lévy process;
- for the limit order book model in part 1.

(A.S., T. Schöneborn, and M. Tehranci, in preparation).

**Example:** For linear temporary impact, $f(x) = \lambda x^2$, the optimal strategy is

$$\xi_t^* = X_0 \sqrt{\frac{\alpha\sigma^2}{2\lambda}} \cdot \frac{\cosh\left((T-t)\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}{\sinh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}$$

$$X_t^{\xi^*} = X_0 \cdot \frac{\cosh\left(t\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)\sinh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right) - \cosh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)\sinh\left(t\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}{\sinh\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)}$$

The value function is

$$v(T, R_0, X_0) = -\exp\left[-\alpha(R_0 + \widetilde{S}_0 X_0 - \frac{\gamma}{2} X_0^2) + X_0^2 \sqrt{\frac{\lambda\alpha^3\sigma^2}{2}} \coth\left(T\sqrt{\frac{\alpha\sigma^2}{2\lambda}}\right)\right]$$

Results for other utility functions not available at this time, because of difficult PDE.

$\Longrightarrow$ Consider infinite-time horizon instead

- Assume also linear temporary impact:

$$f(x) = \lambda x^2$$

- Utility function $u \in C^6(\mathbb{R})$ such that the absolute risk aversion,

$$A(R) := -\frac{u''(R)}{u'(R)} \qquad \text{(= constant for exponential utility),}$$

satisfies

$$0 < A_{min} \leq A(R) \leq A_{max} < \infty.$$

Recall

$$R_t^{\xi} = R_0 + \sigma \int_0^t X_s^{\xi}\, dB_s - \lambda \int_0^t \xi_s^2\, ds.$$

- Optimal liquidation:

$$\text{maximize} \quad \mathbb{E}[\, u(R_\infty^{\xi})\,]$$

- Maximization of asymptotic portfolio value:

$$\text{maximize} \quad \lim_{t\uparrow\infty} \mathbb{E}[\, u(R_t^{\xi})\,]$$

**Note:** Liquidation enforced by the fact that a risk-averse investor does not want to hold a stock whose price process is a martingale.

Recall

$$R_t^\xi = R_0 + \sigma \int_0^t X_s^\xi \, dB_s - \lambda \int_0^t \xi_s^2 \, ds.$$

- Optimal liquidation:

$$\text{maximize} \quad \mathbb{E}[\, u(R_\infty^\xi) \,]$$

- Maximization of asymptotic portfolio value:

$$\text{maximize} \quad \lim_{t \uparrow \infty} \mathbb{E}[\, u(R_t^\xi) \,]$$

**Note:** Liquidation enforced by the fact that a risk-averse investor does not want to hold a stock whose price process is a martingale.

**Theorem:** Both problems are equivalent.

HJB equation for finite time horizon:

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_{\xi} \left( \xi v_X + v_R \xi^2 \right)$$

Guess for infinite time horizon:

$$0 = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_{\xi} \left( \xi v_X + v_R \xi^2 \right)$$

Initial condition:

$$v(0, R) = u(R).$$

HJB equation for finite time horizon:

$$v_t = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_{\xi} \left( \xi v_X + v_R \xi^2 \right)$$

Guess for infinite time horizon:

$$0 = \frac{\sigma^2}{2} X^2 v_{RR} - \inf_{\xi} \left( \xi v_X + v_R \xi^2 \right)$$

Initial condition:

$$v(0, R) = u(R).$$

Corresponding reduced-form equation:

$$v_X^2 = -2\lambda\sigma^2 X^2 v_R \cdot v_{RR}$$

Not very nice either......

**Way out:** consider optimal Markov control in HJB equation

$$\widehat{\xi}(X,R) = -\frac{v_X(X,R)}{2\lambda v_R(X,R)}$$

and let

$$\widetilde{c}(Y,R) = \frac{\widehat{\xi}(\sqrt{Y},R)}{\sqrt{Y}}.$$

If $v$ solves the HJB equation, then $\widetilde{c}$ solves

$$(*) \qquad \begin{cases} \widetilde{c}_Y = \dfrac{\sigma^2}{4\widetilde{c}}\widetilde{c}_{RR} - \dfrac{3}{2}\lambda\widetilde{c}\widetilde{c}_R \\ \\ \widetilde{c}(0,R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \end{cases}$$

**Way out:** consider optimal Markov control in HJB equation

$$\widehat{\xi}(X,R) = -\frac{v_X(X,R)}{2\lambda v_R(X,R)}$$

and let

$$\widetilde{c}(Y,R) = \frac{\widehat{\xi}(\sqrt{Y},R)}{\sqrt{Y}}.$$

If $v$ solves the HJB equation, then $\widetilde{c}$ solves

$$(*) \qquad \begin{cases} \widetilde{c}_Y = \dfrac{\sigma^2}{4\widetilde{c}}\widetilde{c}_{RR} - \dfrac{3}{2}\lambda\widetilde{c}\widetilde{c}_R \\ \\ \widetilde{c}(0,R) = \sqrt{\frac{\sigma^2 A(R)}{2\lambda}} \end{cases}$$

**Theorem:** $(*)$ admits a unique classical solution $\widetilde{c} \in C^{2,4}$ satisfying

$$\sqrt{\frac{\sigma^2 A_{min}}{2\lambda}} \le \widetilde{c}(Y,R) \le \sqrt{\frac{\sigma^2 A_{max}}{2\lambda}}$$

Next, there exists a $C^{2,4}$-solution $\widetilde{w}$ of the transport equation

$$\begin{cases} \widetilde{w}_Y = -\lambda \widetilde{c} \widetilde{w}_R \\ \widetilde{w}(0, R) = u(R). \end{cases}$$

Next, there exists a $C^{2,4}$-solution $\widetilde{w}$ of the transport equation

$$\begin{cases} \widetilde{w}_Y = -\lambda \widetilde{c} \widetilde{w}_R \\ \widetilde{w}(0, R) = u(R). \end{cases}$$

**Theorem:** $w(X, R) := \widetilde{w}(X^2, R)$ is a smooth solution of the HJB equation

$$0 = \frac{\sigma^2}{2} X^2 w_{RR} - \inf_{\xi} \big( \xi w_X + w_R \xi^2 \big), \qquad w(0, R) = u(R).$$

The unique minimizer is

$$\widehat{\xi}(X, R) = -\frac{w_X(X, R)}{2\lambda w_R(X, R)} = \widetilde{c}(X^2, R) X$$

Estimates on $w$ and a verification argument yield

**Theorem:** $w$ is the value function:

$$w(X_0, R_0) = v(X_0, R_0) = \max_{\xi} \mathbb{E}[\, u(R^{\xi}_{\infty})\,].$$

The unique optimal strategy is given by

$$\xi^*_t = -\dot{X}^*_t = \widehat{\xi}\big(X^*_t, R^{X^*}_t\big).$$

Moreover, $R \mapsto v(X, R)$ is a utility function for each $X$ and

$$\widetilde{c}(X, R) = \sqrt{\frac{\sigma^2 A(X, R)}{2\lambda}},$$

where $A(X, R)$ is the absolute risk aversion of $v(X, \cdot)$ at $R$.

**Corollary:** If $u(R) = -e^{-AR}$, then

$$X_t^{\xi^*} = X_0 \exp\Big( -t\sqrt{\frac{\sigma^2 A}{2\lambda}}\Big).$$

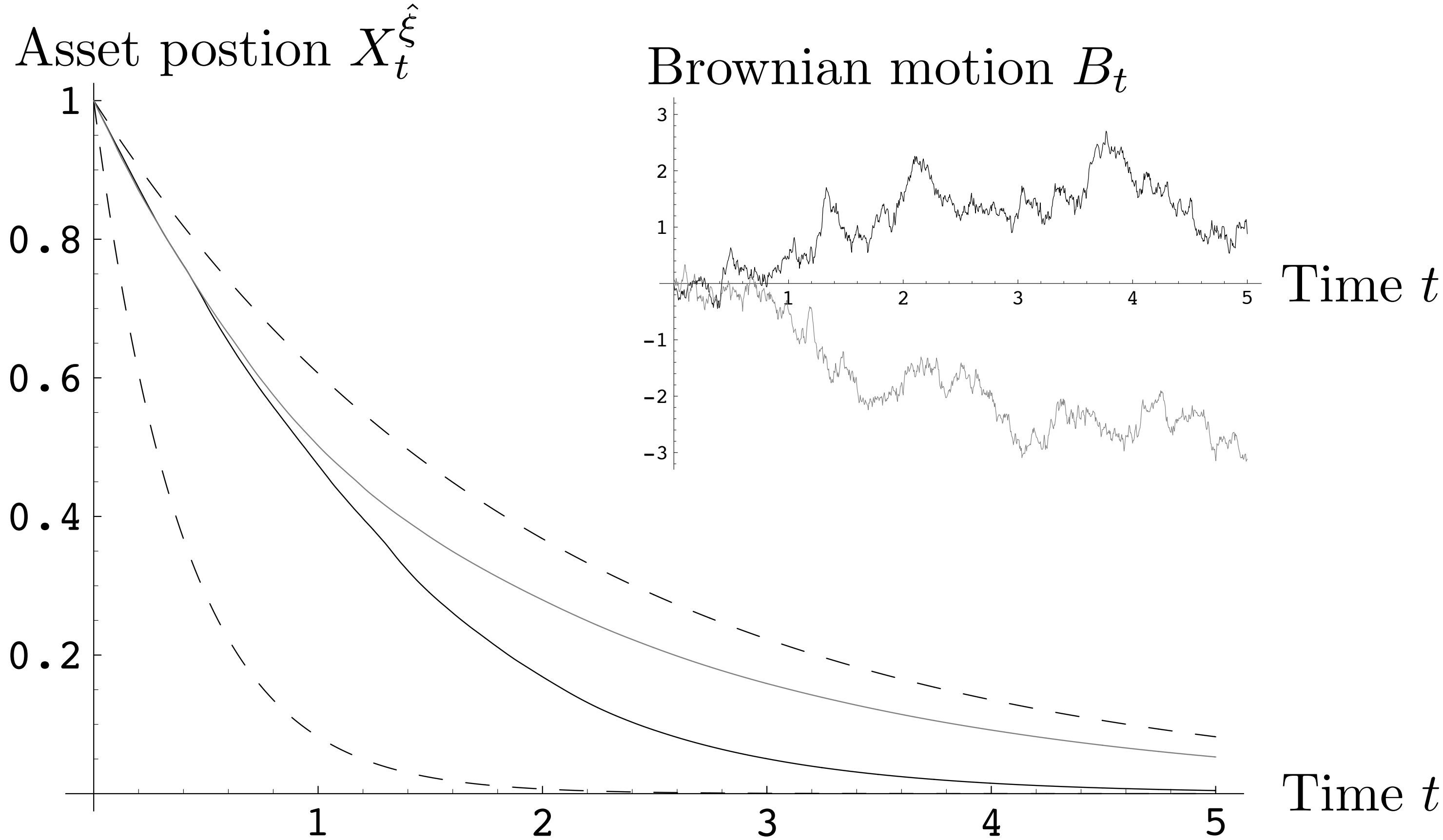
**Corollary:** If $u(R) = -e^{-AR}$, then

$$X_t^{\xi^*} = X_0 \exp\Big( -t\sqrt{\frac{\sigma^2 A}{2\lambda}}\Big).$$

General result:

**Theorem:** The optimal strategy $\hat{\xi}(X, R)$ is increasing (decreasing) in $R$ iff $A(R)$ is increasing (decreasing). I.e.,

| Utility function | | Optimal trading strategy |
|---|---|---|
| DARA | $\Longleftrightarrow$ | Passive in the money |
| CARA | $\Longleftrightarrow$ | Neutral in the money |
| IARA | $\Longleftrightarrow$ | Aggresive in the money |




Two optimal strategies for the IARA utility function with

$$A(R) = 2(1.5 + \tanh(R - 100))^2.$$

**Theorem:** If $u^1$ and $u^0$ are such that $A^1 \geq A^0$ then $\widehat{\xi^1} \geq \widehat{\xi^0}$.

**Theorem:** If $u^1$ and $u^0$ are such that $A^1 \geq A^0$ then $\widehat{\xi}^1 \geq \widehat{\xi}^0$.

**Idea of Proof:** $g := \widetilde{c}^1 - \widetilde{c}^0$ solves

$$g_Y = \frac{1}{2} a g_{RR} + b g_R + V g,$$

where

$$a = \frac{\sigma^2}{2\widetilde{c}^0}, \qquad b = -\frac{3}{2}\lambda \widetilde{c}^1, \qquad \text{and} \qquad V = -\frac{\sigma^2 \widetilde{c}^1_{RR}}{4\widetilde{c}^0 \widetilde{c}^1} - \frac{3}{2}\lambda \widetilde{c}^0_R.$$

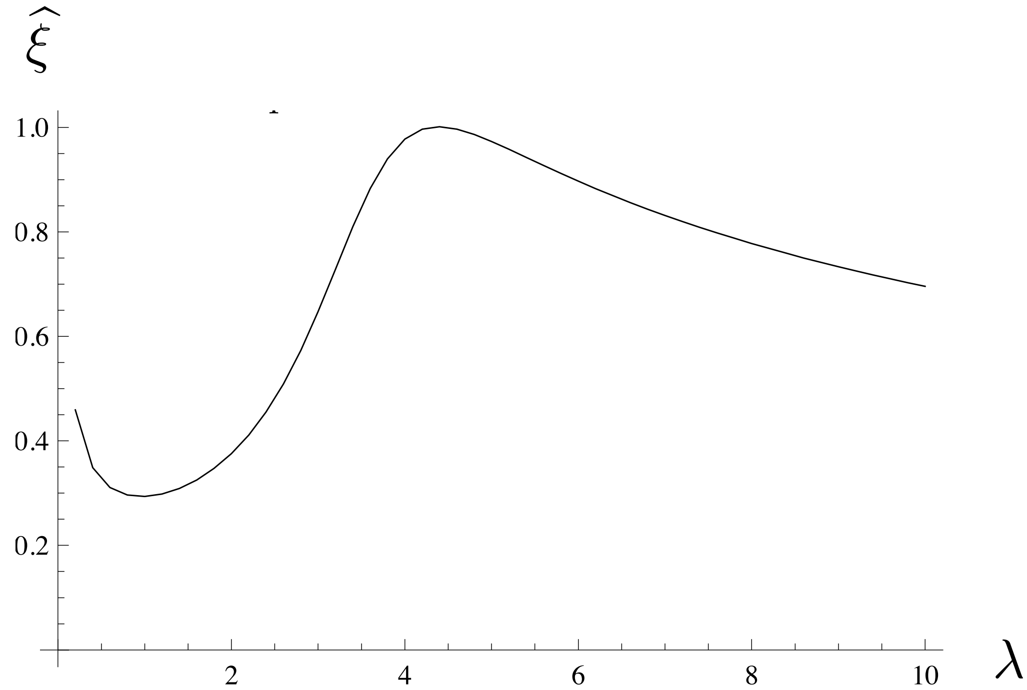
The boundary condition of $g$ is

$$g(0,R) = \sqrt{\frac{\sigma^2 A^1(R)}{2\lambda}} - \sqrt{\frac{\sigma^2 A^0(R)}{2\lambda}} \geq 0$$

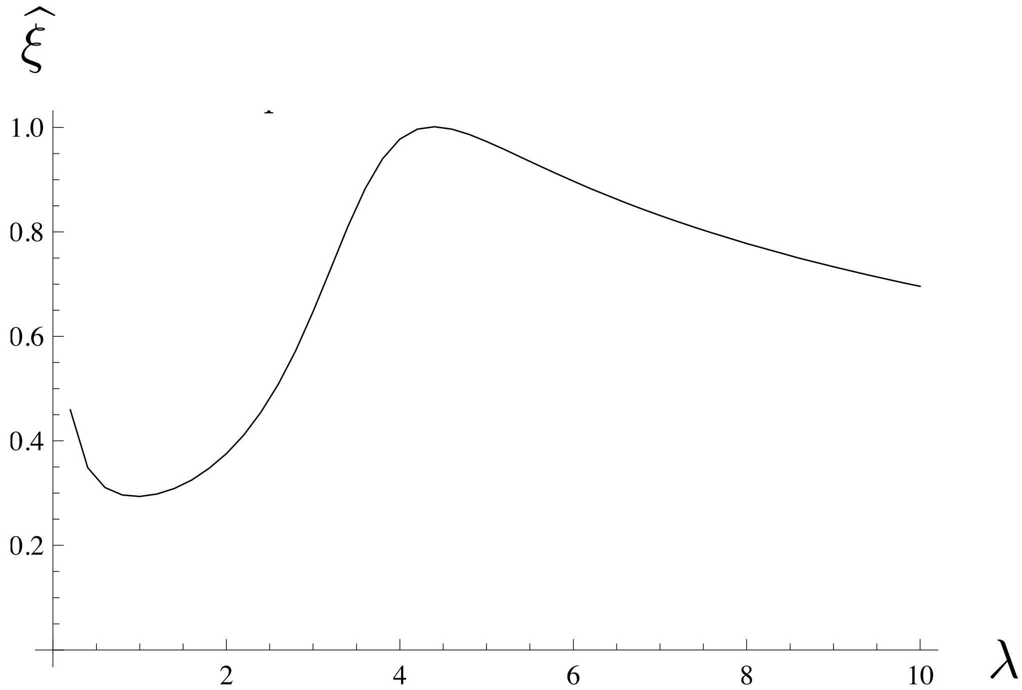
Now maximum principle or Feynman-Kac argument....
(plus localization)

**What about other monotonicity relations?**

• Monotonicity in $\lambda$: intuitively, an increase in liquidaton costs should lead to a decrease of liquidation speed.




Dependence of the optimal strategy $\widehat{\xi}$ on $\lambda$ for the DARA utility function with $A(R) = 2(1.2 - \tanh(15R))^2$.



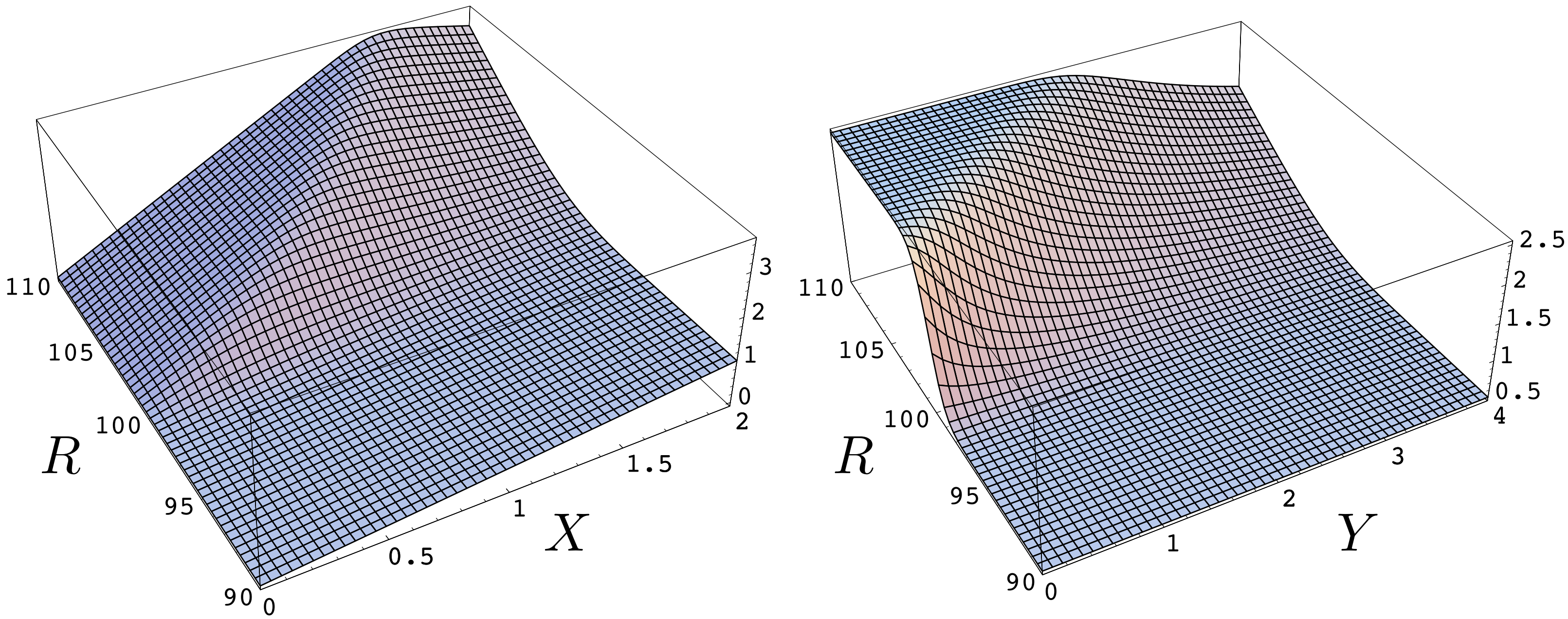

Dependence of the optimal strategy $\widehat{\xi}$ on $\lambda$ for the DARA utility function with $A(R) = 2(1.2 - \tanh(15R))^2$.

**Theorem:** IARA $\Longrightarrow \widehat{\xi}$ is decreasing in $\lambda$.

**What about other monotonicity relations?**

• Monotonicity in $\lambda$: intuitively, an increase in liquidaton costs should lead to a decrease of liquidation speed.

• Monotonicity in $X$: intuitively, larger asset position should lead to an *increased* liquidation speed.




$\hat{\xi}(X, R)$ and $\widetilde{c}(Y, R)$

IARA utility function with $A(R) = 2(1.5 + \tanh(R - 100))^2$ and parameter $\lambda = \sigma = 1$.

**What about other monotonicity relations?**

• Monotonicity in $\lambda$: intuitively, an increase in liquidaton costs should lead to a decrease of liquidation speed.

• Monotonicity in $X$: intuitively, larger asset position should lead to an *increased* liquidation speed.

• Monotonicity in $\sigma$: intuitively, an increase in volatility should lead to an increase in the liquidation speed.

**What about other monotonicity relations?**

• Monotonicity in $\lambda$: intuitively, an increase in liquidaton costs should lead to a decrease of liquidation speed.

• Monotonicity in $X$: intuitively, larger asset position should lead to an *increased* liquidation speed.

• Monotonicity in $\sigma$: intuitively, an increase in volatility should lead to an increase in the liquidation speed.

?

**Thank you for your attention**