Information Percolation with Equilibrium Search Dynamics

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Social Learning

- Learning from prices (Hayek (1945)):
  - Rational expectations equilibrium: Grossman (1981).
  - Strategic Foundations: Wilson (1977), Milgrom (1981), Pesendorfer and Swinkels (1997), and Reny and Perry (2006).
- Learning from Local Interactions:
  - Decentralized markets: Wolinsky (1990), Blouin and Serrano (2002).
  - Word-of-mouth learning: Banerjee and Fudenberg (2004).

Example: Federal Funds Market

Duffie and Ashcraft (2007)

- pricing of overnight loans of federal funds
- decentralized inter-bank market in which these loans are traded
- During a direct bilateral contact, counterparties exchange information and decide whether to forego a trade or to continue "shopping around".

Information Percolation

Duffie and Manso (2007):

- The cross-sectional distribution of information is a solution to a Boltzmann-type evolution equation!
- Explicit solution to the evolution equation = explicit cross-sectional distribution of information.

The Power of Decentralized Learning

Duffie, Giroux, and Manso (2008)

- convergence of beliefs is exponential
- extreme decentralization: the rate of convergence does not depend on the number of agents in each meeting.

This Paper: Endogenous Search Intensity

Two issues arise that may slow down, or even stop, learning:

- Externality Problems
- Coordination Problems

Other Papers with Failures of Social Learning

- Prices:
  - Grossman and Stiglitz (1976)
  - Vives (1993)
- Local interactions:
  - Bikhchandani, Hirshleifer, and Welch (1992)
  - Banerjee (1992)

Outline of the Talk

1. **Model**
2. **Stationary Measure**
3. **Optimality**
4. **Equilibrium**
5. **Policy Interventions**
   - Search Subsidy
   - Educating Agents at Birth

Outline of the Talk

Model Primitives

- continuum of agents
- random variable of interest to all agents: Y
- agents endowed with pairwise independent signals
- signals are jointly Gaussian with Y
- agent i is initially endowed with N_{i0} signals.

Model Primitives

- each agent stays in the market for an exponentially distributed time with parameter η'.
- at exit, agents choose an action A, with cost $(Y - A)^2$.
- optimal choice $A = E(Y \mid \mathcal{F}_{it})$, and expected exit cost equals $\mathcal{F}_{it}$-conditional variance

$$\sigma_{it}^2 = \frac{1 - \rho^2}{1 + \rho^2(N_{it} - 1)}$$

of Y.

Information Transmission

- Upon matching, agents exchange their information.
- Gaussian setting: enough to tell their mean $E(Y \mid \mathcal{F}_{it})$ and precision N_{it}.
- Post-meeting precision is just the sum of pre-meeting precisions. Agents i and j meet, their precisions become $N_{it} + N_{jt}$.

Search Technology

- Random matching
- Given current effort c, mean arrival rate is cbq_b, where b is a level of effort and q_b is the proportion of agents exerting effort level b.
- Exerting effort c costs $K(c)$ to the agent, where $c \in [c_L, c_H]$.

Externality and Coordination Problems

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Externality and Coordination Problems

Agent's Utility

Given a discount rate r, the agent's lifetime utility (measuring time from the point of that agent's market entrance) is

$$U(\phi) = E\left(-e^{-r\tau}\sigma_{i\tau}^2 - \int_0^\tau e^{-rt}K(\phi_t)\,dt\right),$$

where τ is the exit time and $K(c)$ is the cost rate for search effort level c, which is chosen at each time from some interval $[c_L, c_H] \subset \mathbb{R}_+$.

Entry and Exit Rates

- Agents enter the market at a rate proportional to the current mass q_t of agents in the market, for some proportional "birth rate" $\eta > 0$.
- Agents entering the market have precision distribution π.
- Agents exit the market pairwise independently at intensity η',
- The law of large numbers implies that the total quantity q_t of agents in the market at time t is $q_t = q_0 e^{(\eta - \eta')t}$ almost surely.

Cross-Sectional Distribution of Information Precision

The cross-sectional distribution μ_t of information precision at time t is defined, at any set B of positive integers, as the fraction

$$\mu_t(B) = \alpha(\{i : N_{it} \in B\})/q_t$$

of agents whose precisions are currently in the set B.

Dynamics of Information Transmission

Assuming that a search effort policy $C : \mathbb{N} \to [c_L, c_H]$ is used by all agents, the cross-sectional precision distribution satisfies (almost surely) the differential equation

$$\frac{d}{dt}\mu_t = \eta(\pi - \mu_t) + \mu_t^C * \mu_t^C - \mu_t^C \, \mu_t^C(\mathbb{N}),$$

where $\mu_t^C(n) = C_n \mu_t(n)$ is the effort-weighted measure and

$$\mu_t^C(\mathbb{N}) \; = \; \sum_{n=1}^{\infty} C_n \, \mu_t(n)$$

is the cross-sectional average search effort.

The Terms in the Equation

- The term $\eta(\pi - \mu_t)$ represents the replacement of agents with newly entering agents;
- the convolution term

$$(\mu_t^C * \mu_t^C)(n) = \sum_{k=1}^{n-1} \mu_t(k)C(k)C(n-k)\mu_t(n-k)$$

  is the rate at which new agents of a given precision are created through matching and information sharing;
- the term $\mu_t^C(n)\,\mu_t^C(\mathbb{N})$ is the rate of replacement of agents with prior precision n with those of some new posterior precision.

Separability Between Posterior Precision and Mean

Proposition For any search-effort policy function C, the cross-sectional distribution f_t of precisions and posterior means of the agents is almost surely given by

$$f_t(n, x, \omega) = \mu_t(n)\, p_n(x \,|\, Y(\omega)), \tag{1}$$

where μ_t is the unique solution of the differential equation for the evolution of the cross-sectional distribution of information precision and $p_n(\,\cdot\,|\,Y)$ is the Y-conditional Gaussian density of $E(Y \,|\, X_1, \ldots, X_n)$, for any n signals $X_1, \ldots, X_n$.

Outline of the Talk

Stationary Measure

In a stationary setting, this precision distribution μ solves

$$0 = \eta(\pi - \mu) + \mu^C * \mu^C - \mu^C \, \mu^C(\mathbb{N}),$$

which can be viewed as a form of algebraic Ricatti equation.

Stationary Measure

Lemma Given a policy C, there is a unique measure μ satisfying the stationary-measure equation. This measure μ is characterized as follows. For any $\bar{C} \in [c_L, c_H]$, construct a measure $\bar{\mu}(\bar{C})$ by the algorithm:

$$\bar{\mu}_1(\bar{C}) \;=\; \frac{\eta\,\pi_1}{\eta \;+\; C_1\,\bar{C}}$$

and then, inductively,

$$\bar{\mu}_k(\bar{C}) \;=\; \frac{\eta\,\pi_k \;+\; \sum_{l=1}^{k-1} C_l\,C_{k-l}\,\bar{\mu}_l(\bar{C})\,\bar{\mu}_{k-l}(\bar{C})}{\eta \;+\; C_k\,\bar{C}}.$$

There is a unique solution $\bar{C}$ to the equation $\bar{C} = \sum_{n=1}^{\infty} \bar{\mu}_n(\bar{C})\bar{C}$. Given such a $\bar{C}$, we have $\mu = \bar{\mu}(\bar{C})$.

Stability

Proposition Suppose that there is some integer N such that $C_n = C_N$ for $n \geq N$ and that $\eta \geq c_H C_N$. Then the unique solution μ_t of the evolution equation converges pointwise to the unique stationary measure μ.

Outline of stability proof

- Denote

$$c_H - C_i = f_i \geq 0;$$

- Rewrite the equation as

$$\mu_k' = \eta\,\pi_k - (\eta + c_H^2)\,\mu_k + c_H\, f_k\, \mu_k + C_k\, \mu_k \sum_{i=1}^{\infty} f_i\, \mu_i + \sum_{l=1}^{k-1} C_l \mu_l\, C_{k-l}\mu_{k-l}; \quad (2)$$

- Taylor expand in "powers" of $f = (f_i)$:

$$\mu_k = \sum_{j=0}^{\infty} \mu_{k\,j}(t),$$

with

$$\mu_{k\,j} = \frac{1}{j!}\frac{\partial^j \mu_k}{\partial f^j}\Big|_{f=0}(f, \cdots, f).$$

Convergence

- **Key idea:** $\mu_{k\,j}$ are nonnegative and solve simpler ODEs. Comparison theorem for ODEs implies that

$$\sum_{j=0}^{\infty} \mu_{k\,j}(t) \;\leq\; \mu_k \tag{3}$$

and hence the expansion converges;

- $\mu_{k\,j}$ solve simple, linear ODEs and $\lim_{t\to\infty} \mu_{k\,j}(t)$ exists;
- use comparison theorem for ODEs to get uniform tail estimates for (3) and get

$$\lim_{t\to\infty} \sum_{j=0}^{\infty} \mu_{k\,j}(t) \;=\; \sum_{j=0}^{\infty} \lim_{t\to\infty} \mu_{k\,j}(t).$$

Trigger Policies

A trigger policy C^N, for some integer $N \geq 1$, is defined by

$$\begin{aligned} C_n^N &= c_H, \quad n < N, \\ &= c_L, \quad n \geq N. \end{aligned}$$

Condition for convergence in the previous proposition becomes $\eta \geq c_H c_L$.

Information Sharing Opportunities

Proposition Let μ^M and ν^N be the unique stationary measures corresponding to trigger policies C^M and C^N respectively. Let $\mu_n^{C,N} = \mu_n^N C_n^N$ denote the associated search-effort-weighted measure. If $N > M$, then $\mu^{C,N}$ has the first order dominance (FOSD) property over $\mu^{C,M}$.

This is only true for trigger policies! Just the opposite can occur for general policies. More intensive search at given levels of information can in some cases lead to a poorer information sharing.

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The Source of Problems with the FOSD Property

- **average search intensity** $\bar{C} = \bar{C}((C_i))$ is **increasing in** (C_i);
- Components

$$\mu_k = \mu_k((C_i), \bar{C})$$

of the stationary measure are increasing in (C_i) but **decreasing in** $\bar{C}$;

- two competing mechanisms, determining the change of the upper tail

$$\sum_{k \geq n} C_k \, \mu_k((C_i), \bar{C}),$$

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Hamilton-Jacobi-Bellman Equation

The value function V_n for precision n satisfies the Hamilton-Jacobi-Bellman equation for optimal search effort given by

$$0 = -(r+\eta')V_n + \eta' u_n + \sup_{c\in[c_L,c_H]} \{-K(c) + c\sum_{m=1}^{\infty}(V_{n+m}-V_n)\mu_m^C\}.$$

Monotonicity of the Policy Function

Proposition: Suppose that K is increasing, convex, and differentiable. Then, given (μ, C), the optimal search effort Γ_n is monotone decreasing in the current precision n.

Corollary: Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. Then, given (μ, C), there is a trigger policy C^N that is optimal for all agents.

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Equilibrium Definition

An equilibrium is a search-effort policy function C satisfying:

1. there is a unique stationary cross-sectional precision measure μ induced by C;
2. taking as given the market properties (μ, C), the policy function C is indeed optimal for each agent.

Existence of Equilibrium

Theorem Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. Then there exists a trigger policy that is an equilibrium.

Sketch of the Proof

1. We let $\mathcal{N}(N) \subset \mathbb{N}$ be the set of trigger levels that are optimal given the conjectured market properties (μ^N, C^N) associated with a trigger level N.
2. We can look for an equilibrium in the form of a fixed point of the optimal trigger-level correspondence $\mathcal{N}(\,\cdot\,)$, that is, some N such that $N \in \mathcal{N}(N)$.
3. **Lemma:** The correspondence $\mathcal{N}(N)$ is increasing in N.
4. **Lemma:** There exists a uniform upper bound on $\mathcal{N}(N)$, independent of N, given by

$$\overline{N} = \max\{j : \, c_H \eta'(r + \eta')(\overline{u} - u(j)) \geq \kappa\}.$$

Algorithm to Compute Equilibria

Start with $N = \overline{N}$.

1. Compute $\mathcal{N}(N)$. If $N \in \mathcal{N}(N)$, then output C^N (an equilibrium of the game). Go to the next step.
2. If $N > 0$, go back to Step 1 with $N = N - 1$. Otherwise, quit.

There is Never an Equilibrium with "Too Much" Search

Proposition If C^N is an equilibrium of the game then it Pareto dominates any outcome in which all agents employ a policy $C^{N'}$ for a trigger level $N' < N$.

Externality problem:

- An agent with a high search intensity produces an indirect benefit to other agents;
- agents do not take this externality into account $\Longrightarrow$ social learning may slow down or even collapse.

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Equilibrium with Minimal Search

Let V^0 be the values function corresponding to the minimal search case.
Theorem The minimal-search policy C, that with $C(n) = c_L$ for all n, is an equilibrium if and only if $\kappa \geq B$, where

$$B = c_L \sum_{m=1}^{\infty} (V^0_{1+m} - V^0_1)\, \mu^0_m. \tag{4}$$

In particular, if $c_L = 0$, then $B = 0$ and minimal search is always an equilibrium.

Coordination Problem:

For sufficiently small κ there always exist multiple equilibria, both with and without search.

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Format of the Search Subsidy

- a tax τ is charged of each agent entering the market
- the proceeds are used to subsidize search so that the search cost for each agent becomes $K(c) = (\kappa - \delta)c$.

Effects on Search

Proposition If C^N is an equilibrium with subsidy δ, then for any $\delta' \geq \delta$, there exists some $N' \geq N$ such that $C^{N'}$ is an equilibrium with subsidy δ'.

Example

1. For some integer $N > 1$, $\pi_0 = 1/2$, $\pi_N = 1/2$, and $c_L = 0$.
2. Choose parameters so that, given market conditions (μ^N, C^N) agents slightly prefer policy C^0 over C^N.
3. Each agent is now taxed at entry and given the search subsidy δ.
4. We can choose this subsidy so that, given market conditions (μ^N, C^N), agents strictly prefer C^N to C^0.
5. For sufficiently large N all agents have strictly higher indirect utility.

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Format of the Education Policy

- $M \geq 1$ additional public signals
- every agent observes those signals when they enter the market

Effects of Public Signals on Search

Proposition If C^N is an equilibrium with M public signals, then for any $M' \leq M$, there exists some $N' \geq N$ such that $C^{N'}$ is an equilibrium with M' public signals.

Example

1. Suppose, for some integer $N > 1$, that $\pi_0 = 1/2$, $\pi_N = 1/2$, and $c_L = 0$.
2. Choose parameters so that, given market conditions (μ^N, C^N) agents are indifferent between policies C^N and C^0.
3. Give each agent $M = 1$ public signal at entry.
4. All agents strictly prefer C^0 to C^N
5. For sufficiently large N all agents have strictly lower indirect utility.

Conclusion

- Model of social learning with endogenous search intensity.
- Social learning may slow down or even collapse:
  - coordination problems.
  - externality problems.
- Two policy interventions:
  - search subsidy
  - education at entry

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