Correlation, Acceptability and Options on Baskets

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Motivation

- Treat the top 50 stocks in the SPX as if they were the whole index.
- Build models of dependence on the 50 stocks and price options on this basket.
- Match market SPX options by pricing to acceptability at market implied stress levels.

Outline

- Pricing and Hedging to Acceptability

- Market Implied Surface of Stress Levels

- Time Changed Gaussian One Factor Copula Dependence

- Correlated Levy Dependence

Stress Surfaces

- Top 50 SPX Basket Stress Surface for Time Changed Gaussian Copula

- Stress Surface for Correlated Levy Dependence
  - VG and CGMY marginals
  - Physical Levy Marginals
  - Physical Scaled Marginals
  - Risk Neutral Marginals

Hedging Basket Options to Acceptability

- Static Hedging of Basket Options using single name options
  - Hedged and Unhedged Prices
  - Hedged and Unhedged Cash Flows

Pricing and Hedging to Acceptability

- The first principle to be understood is where risk neutral pricing is relevant and why for structured products risk neutral pricing is not relevant.

- The critical principle underlying risk neutral pricing is the idea of pricing all products under a single, so called risk neutral measure.

- The main motivation is linearity of the pricing operator backed by the recognition that in the absence of such a linearity there is a simple arbitrage, buy or sell the component cash flows A, B and sell or buy the package $(A + B)$.

- This argument requires trading in both directions at the same price.

- For structured products buying is at an ask price with sales at the bid and these are widely different.

The Relatively Liquid Hedging Assets

- We can view the structured product as a scenario or path contingent vector of total present value payouts $x = (x_s, s = 1, \cdots, M)$.

- Next we introduce the relatively liquid assets with bidirectional prices and by financing the trades we generate zero cost cash flows Y_{js} for asset j on scenario s.

Acceptable Risks

- If we charge the price a and adopt the hedge that takes the position α_j in liquid asset j then our residual cash flow is

$$a + \alpha' Y - x'$$

- If this position is zero or nonnegative, it is clearly acceptable.

- More Generally Acceptable Risks have been effectively defined as a convex cone containing the positive orthant.

- Intuitively, if a sufficient number of counterparties value the gains in excess of the losses, then the risk is acceptable.

- Let B be the matrix of such valuation measures used for testing acceptability. (See Carr, Geman, Madan JFE 2002 for greater details).

- For the risk to be acceptable we must have

$$a + (\alpha' Y - x')B \geq 0$$

The Ask Price Problem

- The Ask price problem is to find $a(x)$ such that

$$\begin{aligned} a(x) &= Min_{a,\alpha}\ a \\ & \quad S.T. \left(x' - \alpha' Y\right) B \leq a \end{aligned}$$

- The ask price is the smallest value needed to cover all the valuation shortfalls net of the hedge.

- By virtue of being a minimization problem defined with respect to a linear constraint set defined by x it is clear that $a(x)$ will be a **convex** functional of the cash flows x and linear or risk neutral pricing does not hold.

Law Invariant Cones of Acceptability

- Suppose we wish decide on the acceptablity of a random cash flow C based solely on its probability law or equivalently its distribution function $F(c)$.

- Cherny and Madan (2008) show how this is related to expectation under concave distortion.

- One introduces a collection of concave distribution functions $\Psi^{\alpha}(u)$ defined on the unit interval $0 \leq u \leq 1$ and indexed by the real number α such that we have acceptability at level α just if

$$\int_{-\infty}^{\infty} c d\Psi^{\alpha}(F(c)) \geq 0$$

- Equivalently we may write

$$\int_{-\infty}^{\infty} c\Psi^{\alpha\prime}(F(c))f(c)dc \geq 0$$

and we see that one is computing an expectation under the change of probability

$$\Psi^{\alpha\prime}(F(c))$$

that depends on the claim being priced via its distribution function $F(c)$.

The New Acceptability Cones: MINVAR

- The first family of concave distortions we considered was

$$\Psi^x(y) = 1 - (1 - y)^x$$

- It is simple to observe that X is acceptable under this distortion just if the expectation of the minimum of x independent draws from the distribution of X is still just positive.

- Hence we refer to this measure as $MINVAR$ as it is based on the expectation of minima.

- The measure change in this case is

$$\frac{dQ}{dP} = (x + 1)\,(1 - F_X(X))^x\,,\ x \in \mathbb{R}_+$$

- A potential drawback is that large losses have a maximum weight of $(x + 1)$.

- Asymptotically large gains receive a weight of zero.

MAXVAR

- The next concave distortion is based on the maxima of independent draws and is defined by

$$\Psi^x(y) = y^{\frac{1}{1+x}}$$

- Here we take expectations from a distribution G such that the law of the maxima of x independent draws from this distribution matches the distribution of X.

- The measure change now is

$$\frac{dQ}{dP} = \frac{1}{1+x}\left(F_X(X)\right)^{-\frac{x}{x+1}},\ x \in \mathbb{R}_+$$

- Large losses now receive unbounded large weights in the determining system, but large gains have a minimum weight of $(x+1)^{-1}$.

MAXMINVAR and MINMAXVAR

- We combine the two distortions in two ways to define MAXMINVAR by

$$\Psi^x(y) = \left(1 - (1-y)^{x+1}\right)^{\frac{1}{x+1}}$$

- and MINMAXVAR by

$$\Psi^x(y) = 1 - \left(1 - y^{\frac{1}{x+1}}\right)^{x+1}$$

- The densities in the determining system now have weights tending to infinity for large losses and zero for large gains.

- We shall use MINMAXVAR.

Acceptability Pricing and Distorted Expectations

- Consider now the pricing of a hedged or unhedged liability with cash flow C by distorted expectation up to some level α to charge the price a.

- We must then have that the cash flow

$$Y = a - C$$

with distribution function $F_Y(y)$ is just acceptable at distortion α.

- Hence

$$\int_{-\infty}^{\infty} y d\mathbf{\Psi}^{\alpha}(F_Y(y)) = 0$$

We now recognize that

$$F_Y(y) = F_{(-C)}(y - a)$$

and so we get that

$$\int_{-\infty}^{\infty} y d\Psi^{\alpha}\left(F_{(-C)}(y-a)\right) = 0$$

- Making the change of variable $c = y - a$ we get that

$$\int_{-\infty}^{\infty} (c+a) d\Psi^{\alpha}\left(F_{(-C)}(c)\right) = 0$$

or that

$$a = -\int_{-\infty}^{\infty} c d\Psi^{\alpha}\left(F_{(-C)}(c)\right)$$

Hence the price is the negative of the distorted expectation of the cash flow $-C$.

Market Implied Stress Levels

- We may choose a stress level and compute the negative of the α distorted expectation of $-C$ as the ask price.

- Alternatively, given the market price a we may solve for the market implied stress level, much like an implied volatility.

- This leads us to stress surfaces for options and we shall work with $MINMAXVAR$ stress surfaces.

Time Changed Gaussian One Factor Copula Dependence

- Qiwen Chen (2008), one of my students, proposed using the copula of the multivariate VG model in the original Madan and Seneta (1990) VG paper as a model of dependence. He reports positively on the performance of this model in terms of capturing the dependence in returns.

- The multivariate VG (MVG) time changes all coordinates of a multivariate Brownian motion by a single gamma time change.

- Here we just use this procedure to generate correlated uniforms after transforming MVG outcomes to uniforms using their marginal VG distribution functions.

- We then generate actual coordinate outcomes using inverse uniform and prespecified marginal distributions.

- Following this suggestion, we consider here the restriction of the multivariate Brownian to that of a one factor Gaussian copula model.

- The model for the correlated uniforms is then obtained as

$$
\begin{aligned}
u_i &= F_{VG}(X_i) \\
X_i &= \sqrt{g}\left(\rho_i Z + \sqrt{1-\rho_i^2} Z_i\right) \\
&\quad Z, Z_i's \text{ independent Gaussians} \\
&\quad g \text{ is gamma distributed with} \\
&\quad \text{mean unity and variance } \nu
\end{aligned}
$$

- The actual centered data are then obtained as

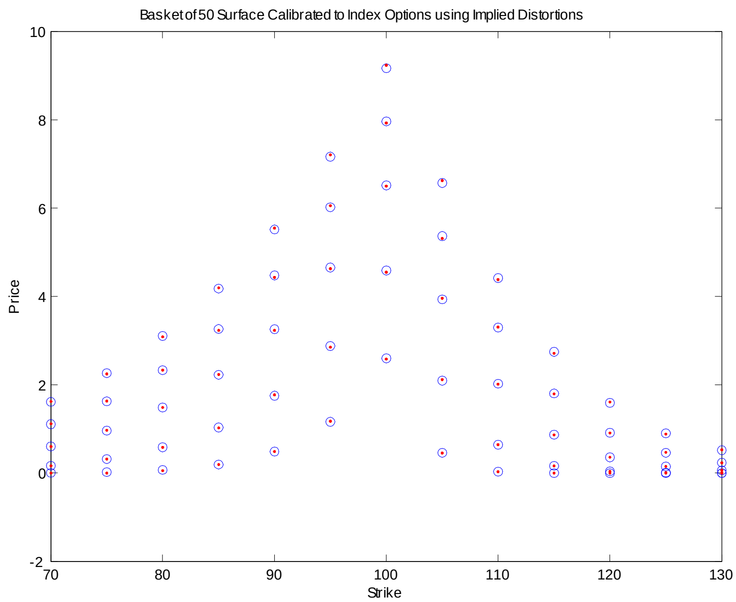
$$
Y_i = F_{VG_i}^{-1}(u_i).
$$

Results on time changed one factor MVG copula

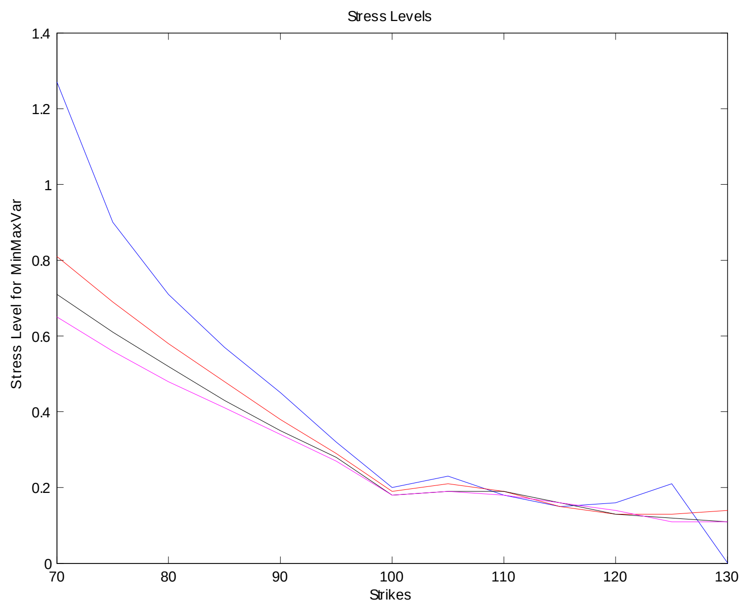
- We then generate 50 dependent uniforms and the inverse of the marginal distribution function to generate outcomes for the individual names with which we form the basket outcome and use it to price a basket option by computing discounted distorted expectations using one of the four distortions.

- It is unlikely that all strikes and maturities will be priced at the same stress levels

- We first extracted the market implied stress levels.



- We then graphed the stress levels as a function of strike and maturity



- A regression of log stress and log strike and maturity suggested a linear relationship at the log level or the functional form for the stress level

$$\alpha = A\left(\frac{K}{100}\right)^{-\beta} t^{-\gamma}$$

Calibration Results

- For the data of 20080220 we then adopted this stress level model with three parameters along with our dependence model with 6 parameters given by ν and five correlations with the latent systematic component to calibrate options on baskets of the top 50 stocks to the prices of index options. The estimated parameters were as follows.

Parameter	Value
α	0.0964
β	4.224
γ	0.2850
ν	0.0061
ρ_1	0.9964
ρ_2	0.3505
ρ_3	0.3214
ρ_4	0.3801
ρ_5	0.5319

We used 52 options with 13 strikes across four maturities and the fit statistics were

RMSE	0.0953
AAE	0.0813
APE	0.0325

Hedging Basket Options with Single Name Options

- Finally we consider a hedged option price where we seek positions in single name options against the basket liability to construct the residual cash flow as

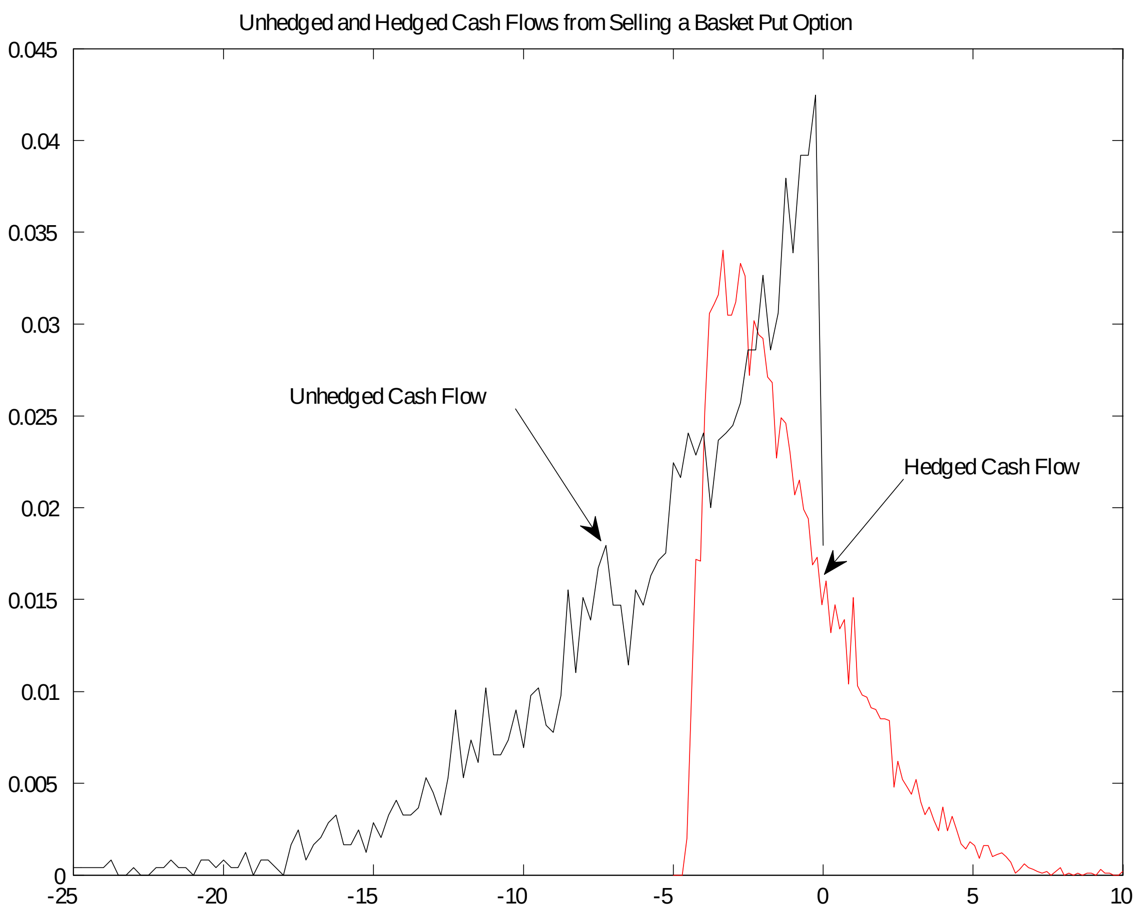
$$RCF = \alpha * HCF - TCF$$

- We find α to minimize the ask price for the residual cash flow defined as the negative of distorted expectation of this cash flow.

- For minmaxvar at stress level $5, .5$ the unhedged and hedged prices are

	5	.5
unhedged	24.4875	4.0047
hedged	4.5285	2.6904

- We present a graph of the unhedged and hedged cash flows and a graph of the hedge positions on a basket put struck 10% down.



Unhedged and Hedged Cash Flows from Selling a Basket Put Option
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
-25
-20
-15
-10
-5
0
5
10
Unhedged Cash Flow
Hedged Cash Flow

Correlated Levy Dependence

- We take the marginal processes to be zero mean univariate Lévy processes $(X_i(t), t \geq 0)$.

- These processes accomodate the possibility of being skewed by having a representation as Brownian with drift time changed by subordinators $(G_i(t), t \geq 0)$ with unit expectation that are independent across i and independent of the Brownian motions.

- We write

$$X_i(t) = \theta_i \left(G_i(t) - t\right) + \sigma_i W_i \left(G_i(t)\right)$$

for Brownian motions $(W_i(t), t \geq 0)$.

- The variance gamma model arises when $G_i(t)$ is a gamma process with unit mean rate, variance rate ν_i and density at unit time given by

$$f(x) = \frac{1}{\nu_i^{\frac{1}{\nu_i}} \Gamma\left(\frac{1}{\nu_i}\right)} x^{\frac{1}{\nu_i}-1} e^{-\frac{x}{\nu_i}}$$

- Many other subordinators are potential candidates including the inverse Gaussian for NIG, the generalized inverse Gaussian for GH, and the suitably shaved stable $Y/2, 1/2$ for the $CGMY$ and $Meixner$ processes.

- We shall work with $CGMY$ in addition to the VG.

- At unit time with $G_i = G_i(1)$ we may also express $X_i = X_i(1)$ as

$$X_i = \theta_i \left(G_i - 1 \right) + \sigma_i \sqrt{G_i} Z_i$$

where the $Z_i's$ are standard normal variates.

- In our correlated Lévy model we suppose that

$$E\left[Z_i Z_j \right] = \rho_{ij}.$$

- There is now dependence between unit returns as

$$E\left[X_i X_j \right] = \sigma_i \sigma_j E\left[\sqrt{G_i} \right] E\left[\sqrt{G_j} \right] \rho_{ij}$$

Return Correlation and Brownian Correlation

- We observe that

$$E\left[X_i^2\right] \geq \sigma_i^2 E[G_i] = \sigma_i^2$$

- It follows that observed return correlations

$$\frac{E\left[X_i X_j\right]}{\sqrt{E\left[X_i^2\right] E\left[X_j^2\right]}} \leq E\left[\sqrt{G_i}\right] E\left[\sqrt{G_j}\right] \rho_{ij} \leq \rho_{ij}$$

- Furthermore we estimate Brownian correlations as

$$\rho_{ij} = \frac{E\left[X_i X_j\right]}{\sigma_i \sigma_j E\left[\sqrt{G_i}\right] E\left[\sqrt{G_j}\right]}$$

- This estimate is readily available once marginal laws have been estimated as we then have the moments G_i and σ_i.

- When the estimates are greater than one and we have just a symmetric matrix that is not a correlation we follow Qi and Sun to construct the closest correlation matrix.

- We present a sample of VG marginals on the technology sector.

TABLE 1

VG parameter estimates
for the period 1/4/2002 to 6/18/2008
using daily log price relative returns

TICKER	σ	ν	θ in basis points
AAPL	0.0257	0.5737	13.6183
AMZN	0.0287	1.1043	30.6629
BAC	0.0166	2.7696	-22.1156
C	0.0201	2.4699	0.0004
CSCO	0.0218	0.7300	-9.5081
DELL	0.0188	0.7543	0.2387
F	0.0237	0.6088	25.1879
GM	0.0238	0.9076	24.0957
GS	0.0179	0.5790	0.0352
IBM	0.0146	0.8653	0.0167
INTC	0.0224	0.6473	-1.5322
KO	0.0109	0.7669	-0.2736
LEH	0.0275	2.6239	-31.2588
MER	0.0200	0.8421	0.0410
MMM	0.0126	0.8760	-0.0734
MS	0.0213	0.9177	-0.0457
MSFT	0.0202	2.7847	-23.4115
ORCL	0.0235	1.0347	-0.0021
QCOM	0.0239	0.6561	29.4362

Sample and Implied Brownian Correlation

	$AAPL$	$AMZN$	$CSCO$	$DELL$	IBM	$INTC$
$AAPL$	1	.2535	.3293	.3472	.3245	.3529
$AMZN$	.4009	1	.3522	.3517	.3089	.3294
$CSCO$	.4885	.4956	1	.5514	.5347	.6228
$DELL$	.5065	.4864	.7156	1	.5072	.5768
IBM	.4894	.4418	.7173	.6691	1	.5674
$INTC$	.5196	.4599	.8158	.7428	.7554	1
$MSFT$	.4437	.3768	.6676	.6064	.6684	.6937
$ORCL$	.4133	.3865	.6709	.5820	.6604	.6745
$QCOM$	.4662	.4391	.6698	.6089	.5709	.6747
$YHOO$	.3911	.6102	.6151	.5416	.5238	.6158

Gaussian and Levy Chisquare Statistics

chisquares	*AAPL*	*AMZN*	*CSCO*	*DELL*	*IBM*	*I*
AAPL	0	384.61	329.10	302.14	398.79	32
AMZN	18.72	0	324.42	312.77	403.43	33
CSCO	5.67	20.69	0	300.28	448.09	51
DELL	24.26	21.99	20.17	0	329.65	38
IBM	20.27	24.29	24.62	11.56	0	46
INTC	19.00	30.52	32.87	19.38	19.94	
MSFT	129.64	164.99	197.02	184.16	183.23	22
ORCL	13.91	18.62	13.15	25.58	25.91	3
QCOM	17.37	38.08	25.23	17.67	26.52	2
YHOO	164.41	161.21	125.50	127.06	122.97	15

p-values on arbitrary portfolios

- We exclude $MSFT$ and $YHOO$ as the chisquare statistics though a lot better than Gaussian were not that good.

- The Blue line is for long short portfolios while the red is for long only portfolios.

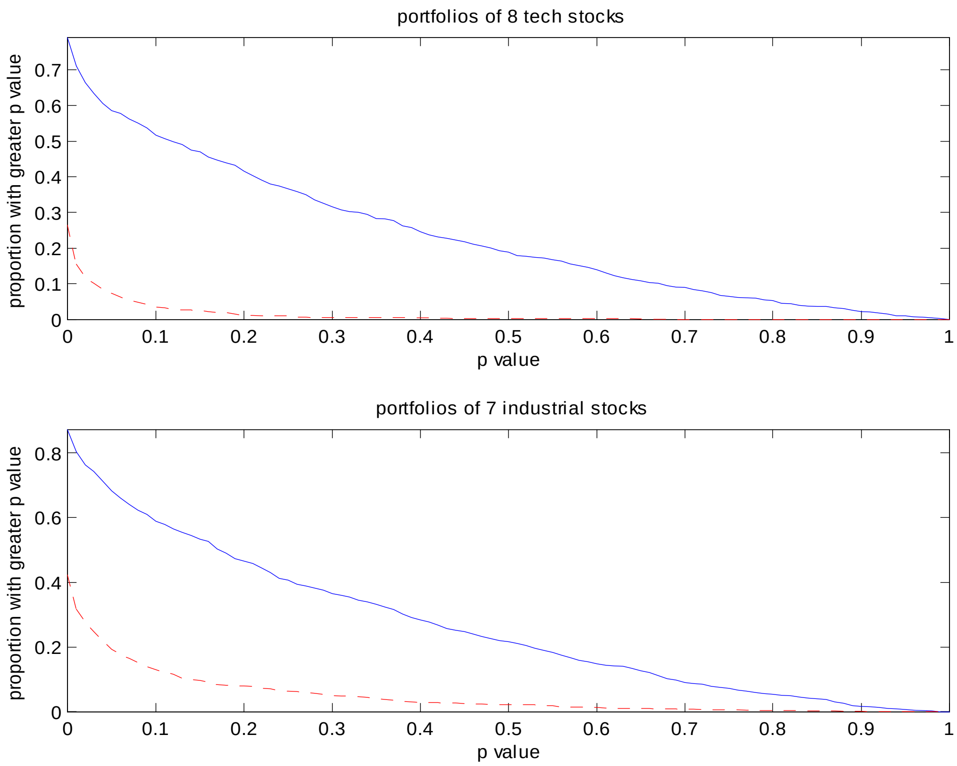
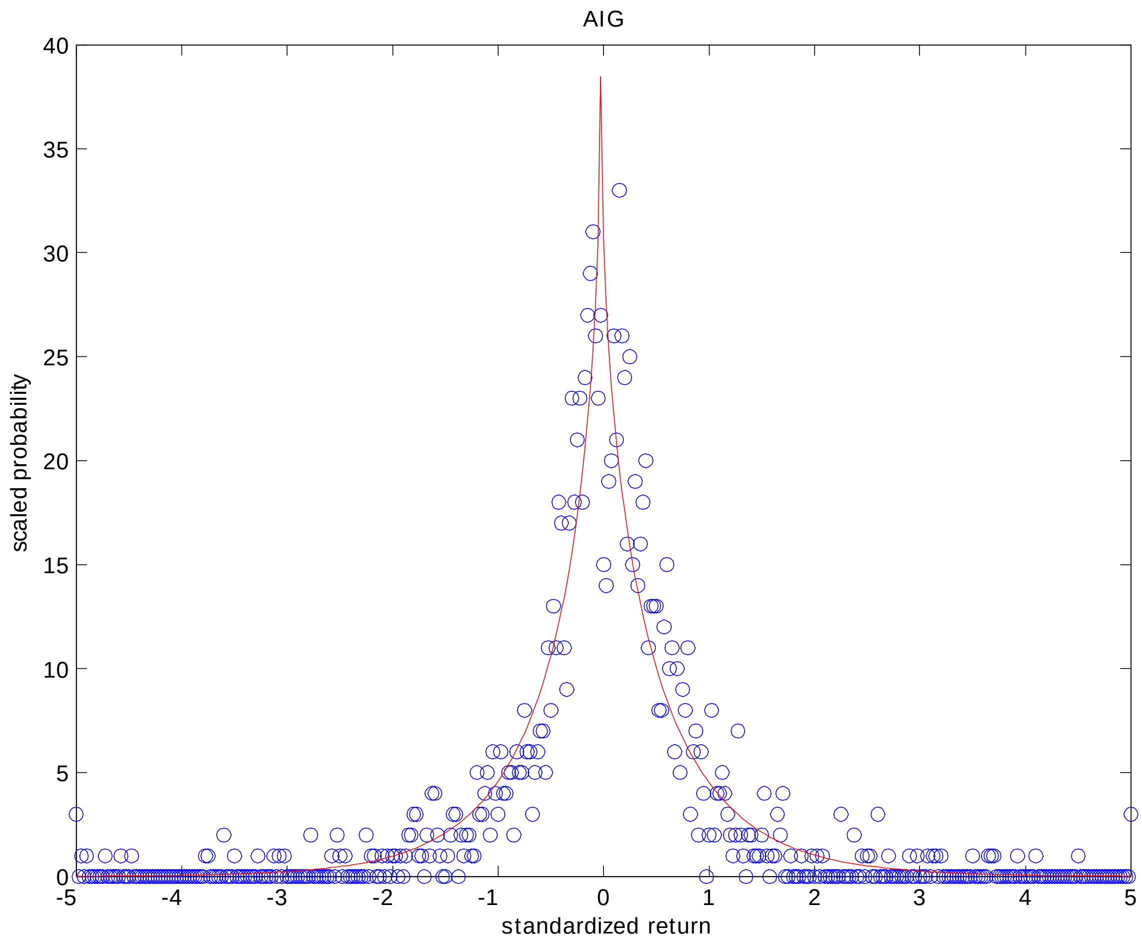


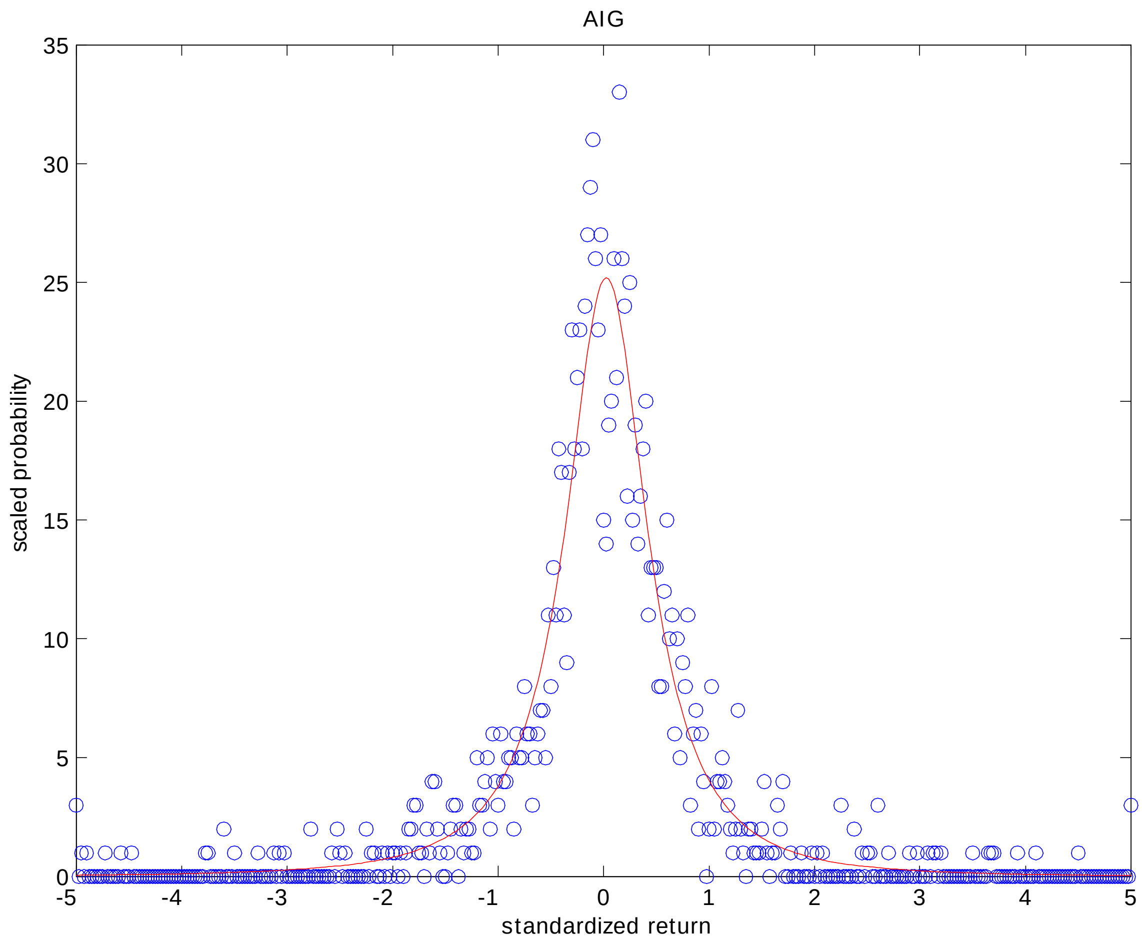
Figure 1: Long-Short portfolio complementary distribution function of p-values in blue. Long only portfolios are in red.

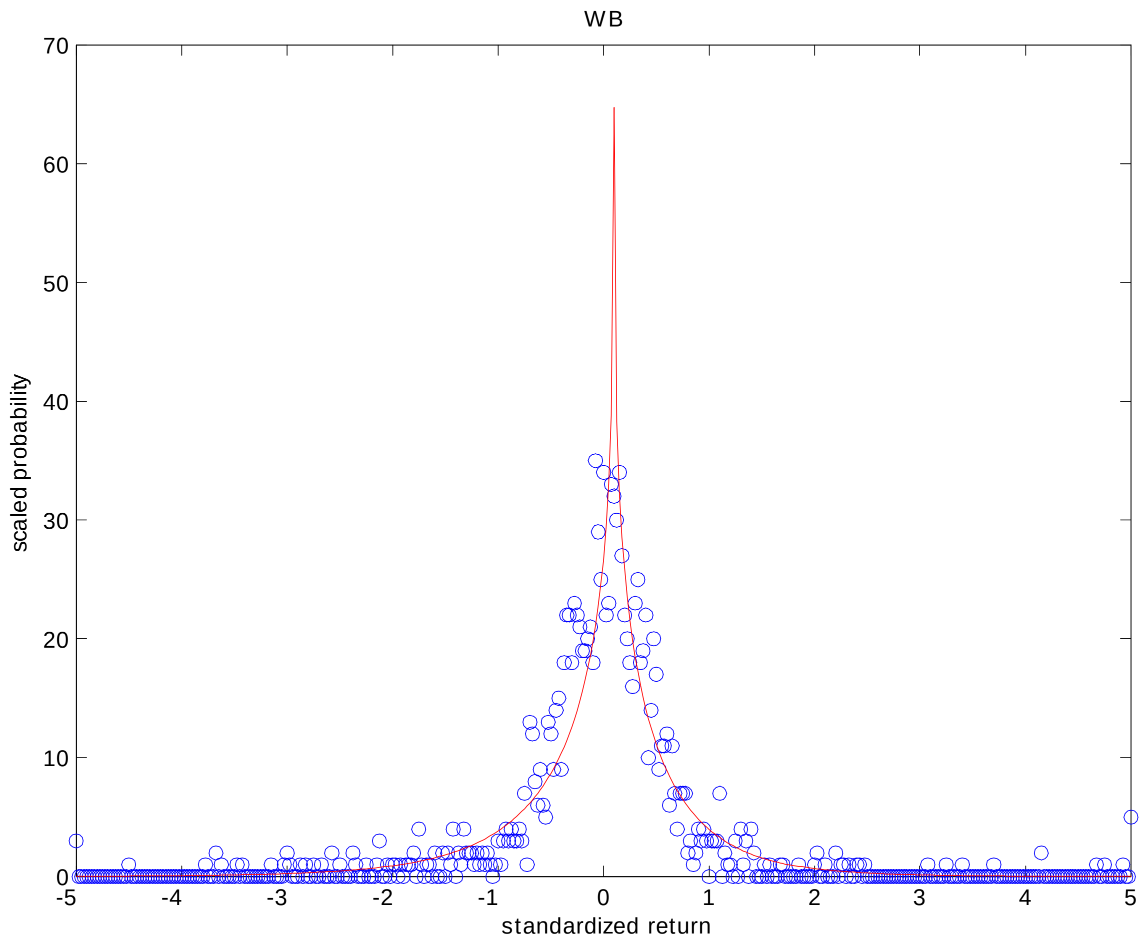
Top 50 Levy Correlation

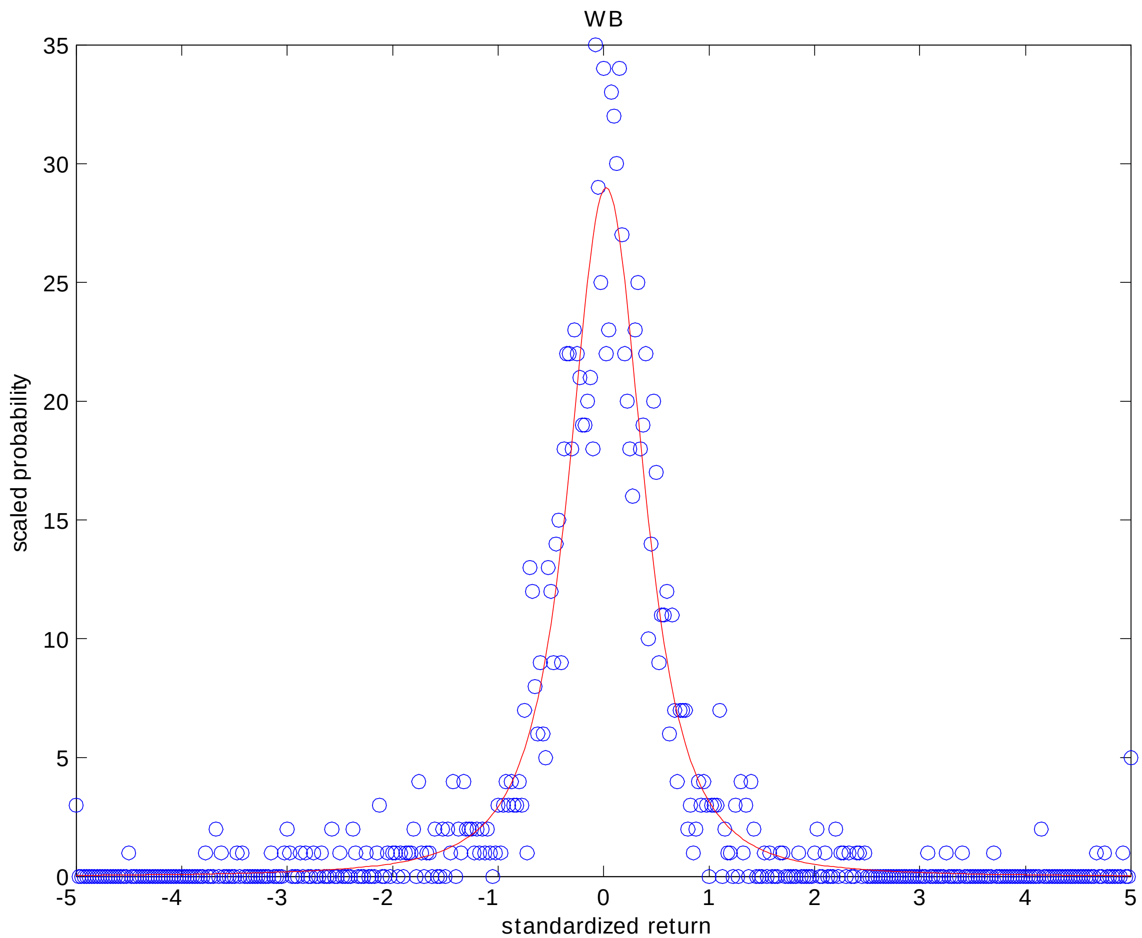
- We constructed the Brownian correlation matrix of the top 50 stocks using VG and CGMY marginals.

- We present a sample of the VG and CGMY density fits.


AIG
scaled probability
standardized return
40
35
30
25
20
15
10
5
0
-5
-4
-3
-2
-1
0
1
2
3
4
5


AIG
scaled probability
standardized return
35
30
25
20
15
10
5
0
-5
-4
-3
-2
-1
0
1
2
3
4
5


WB
scaled probability
standardized return
70
60
50
40
30
20
10
0
-5
-4
-3
-2
-1
0
1
2
3
4
5


WB
scaled probability
standardized return
35
30
25
20
15
10
5
0
-5
-4
-3
-2
-1
0
1
2
3
4
5

- We then constructed marginals at option maturities by running the Lévy process or by scaling.

- We also extracted risk neutral marginals and using our Brownian correlations we constructed top 50 basket option cash flows.

- Finally we worked out implied stress levels for VG and CGMY, run, scaled and risk neutral as function of strike for three maturities, six months, 9 months and one year.

- We now present the implied stress functions for SPX as at 20080220.

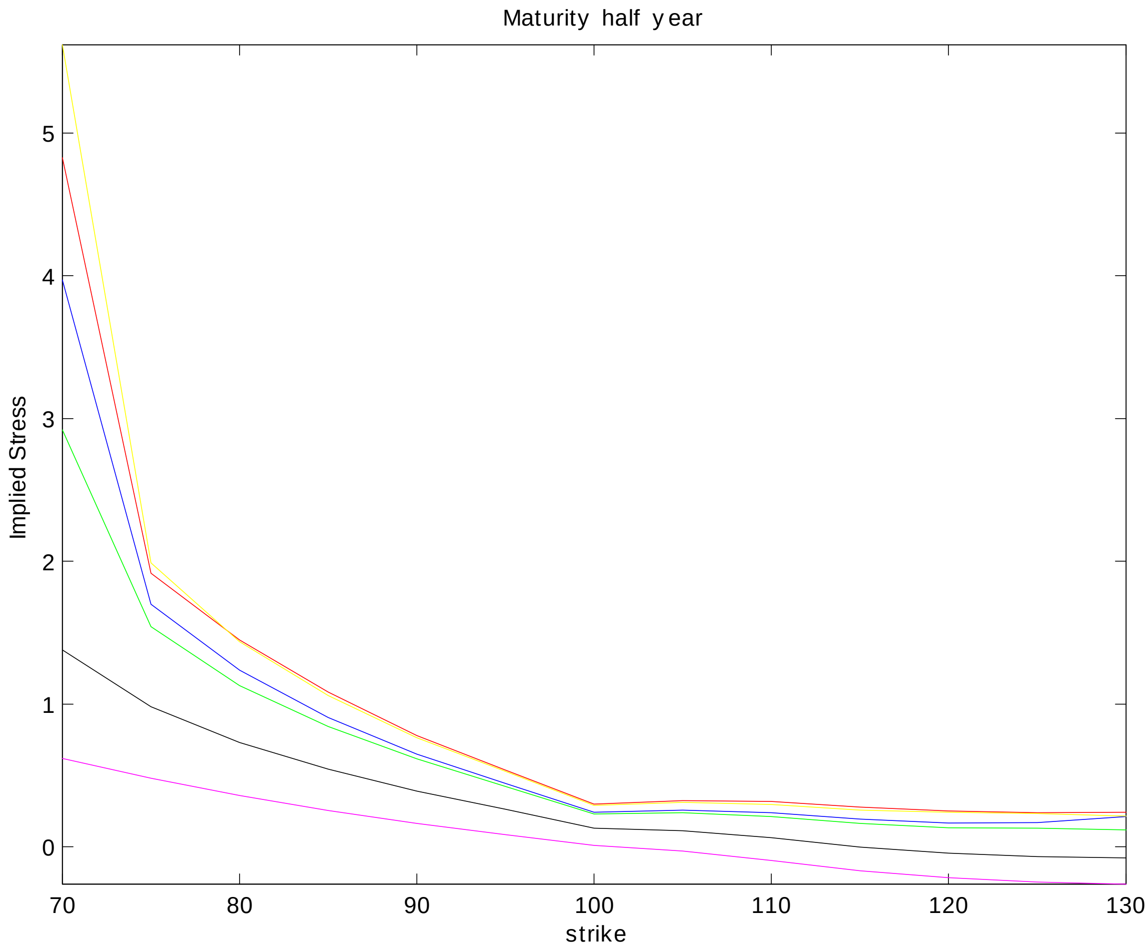
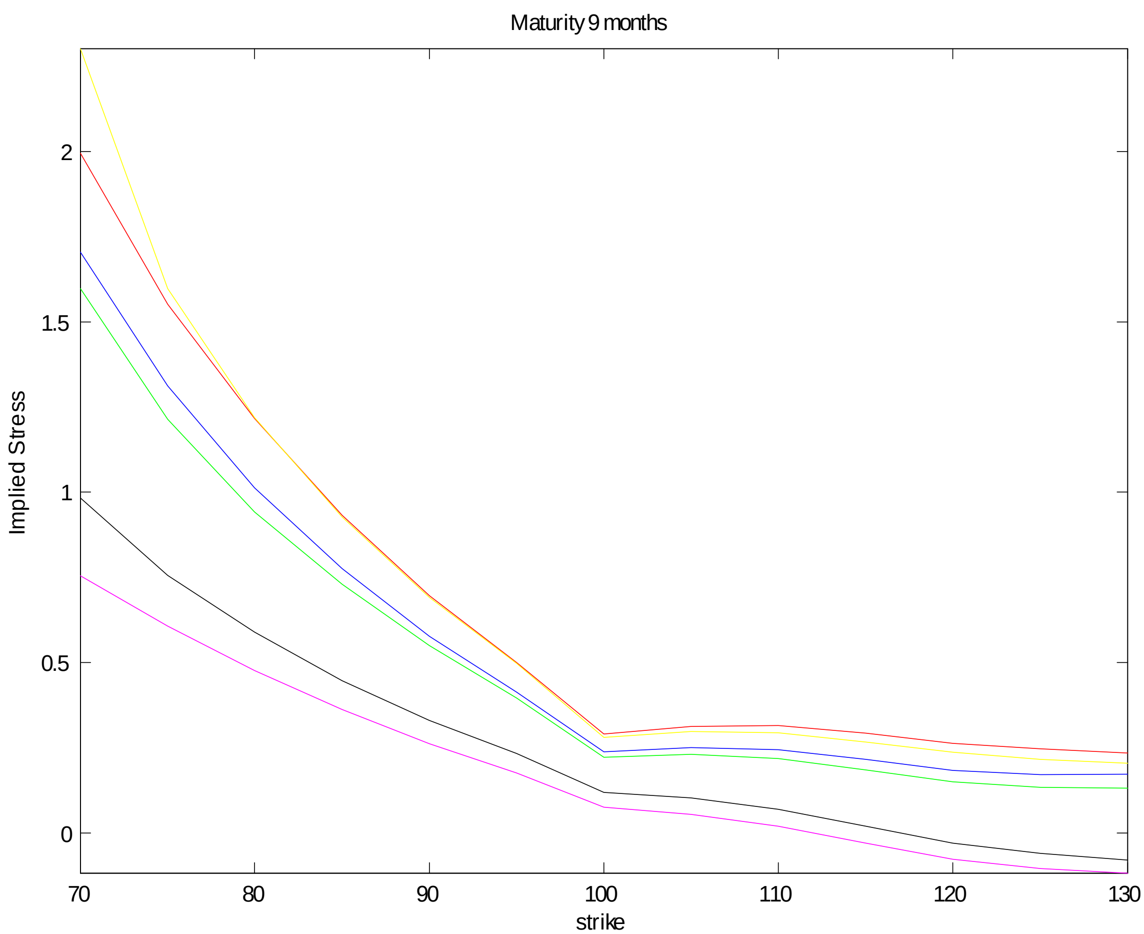
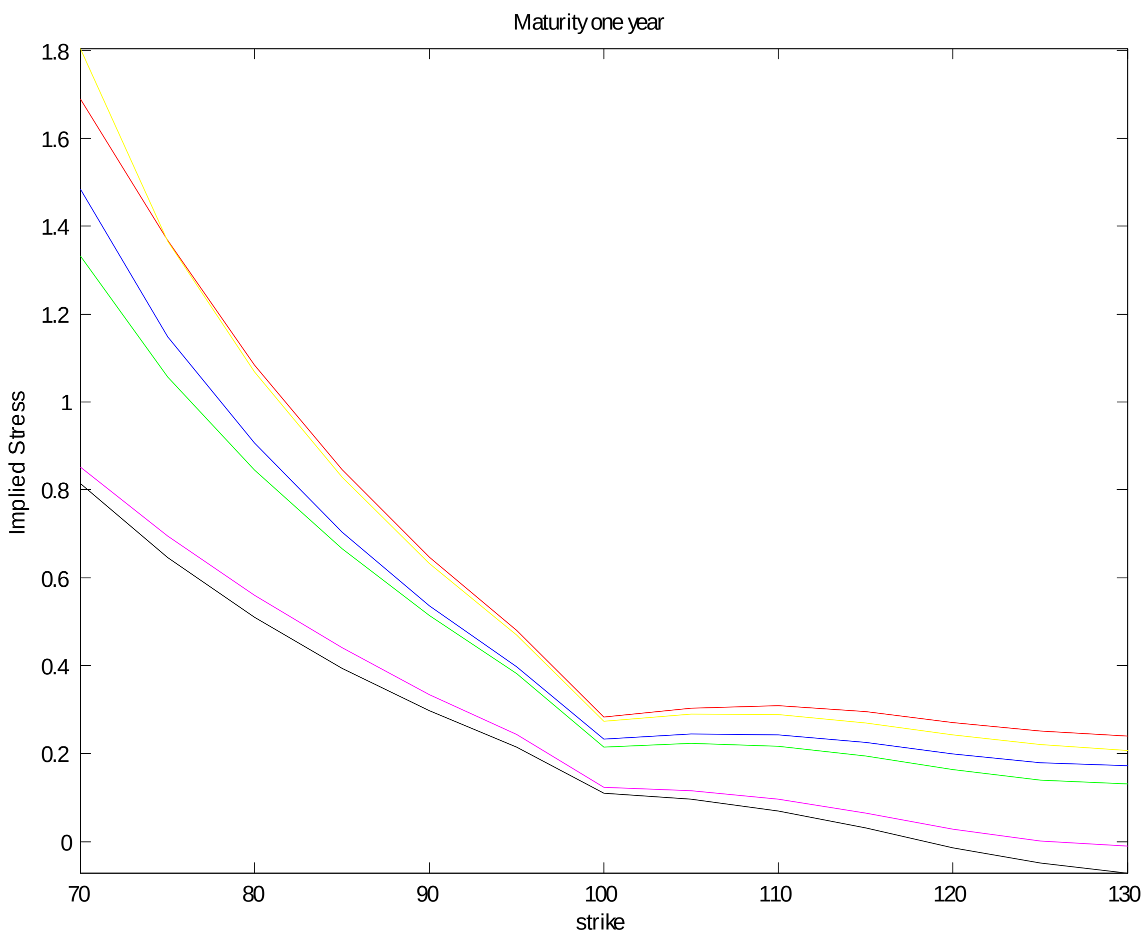


Figure 2: Blue, Red Black are VG Levy Scaled and Risk Neutral Magenta, Green and Yellow are CGMY



Maturity 9 months
Implied Stress
strike
0
0.5
1
1.5
2
70
80
90
100
110
120
130



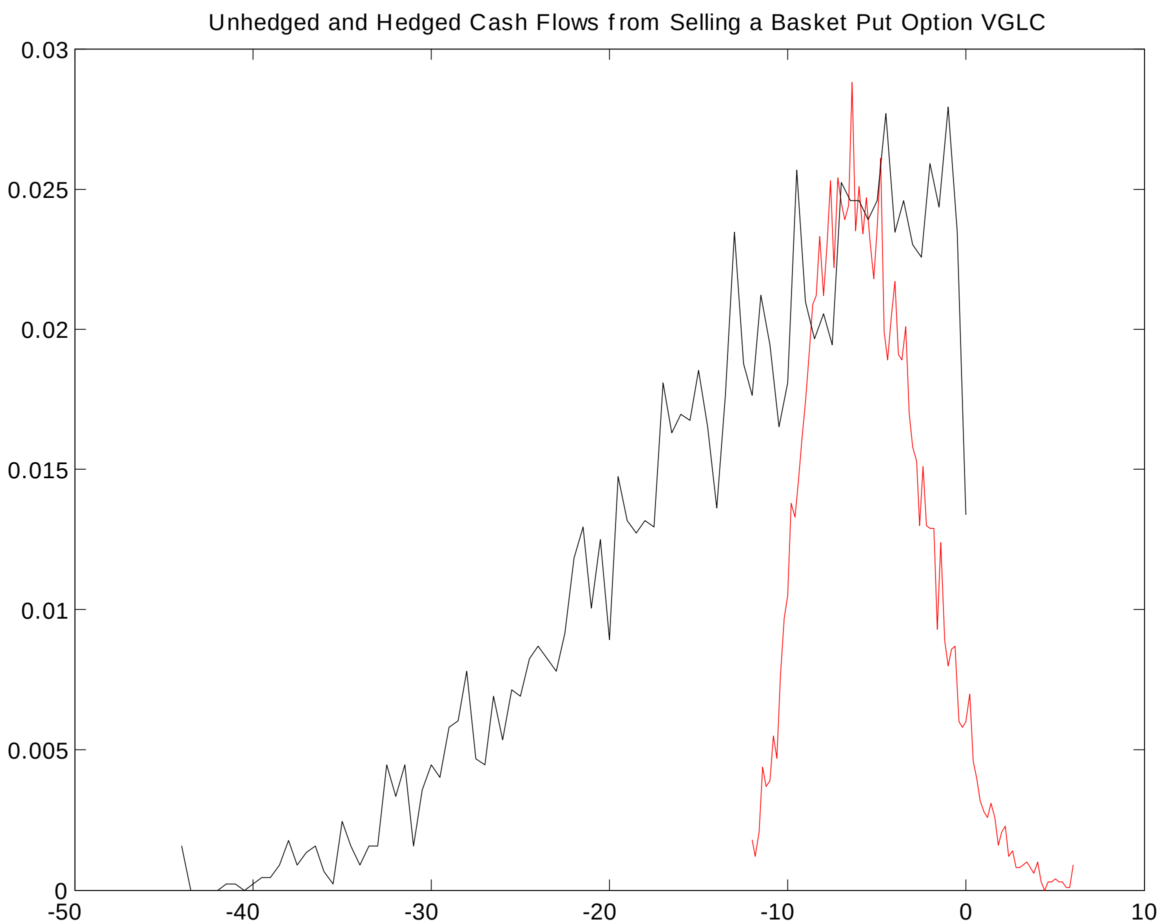
Maturity one year
Implied Stress
strike
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
70
80
90
100
110
120
130

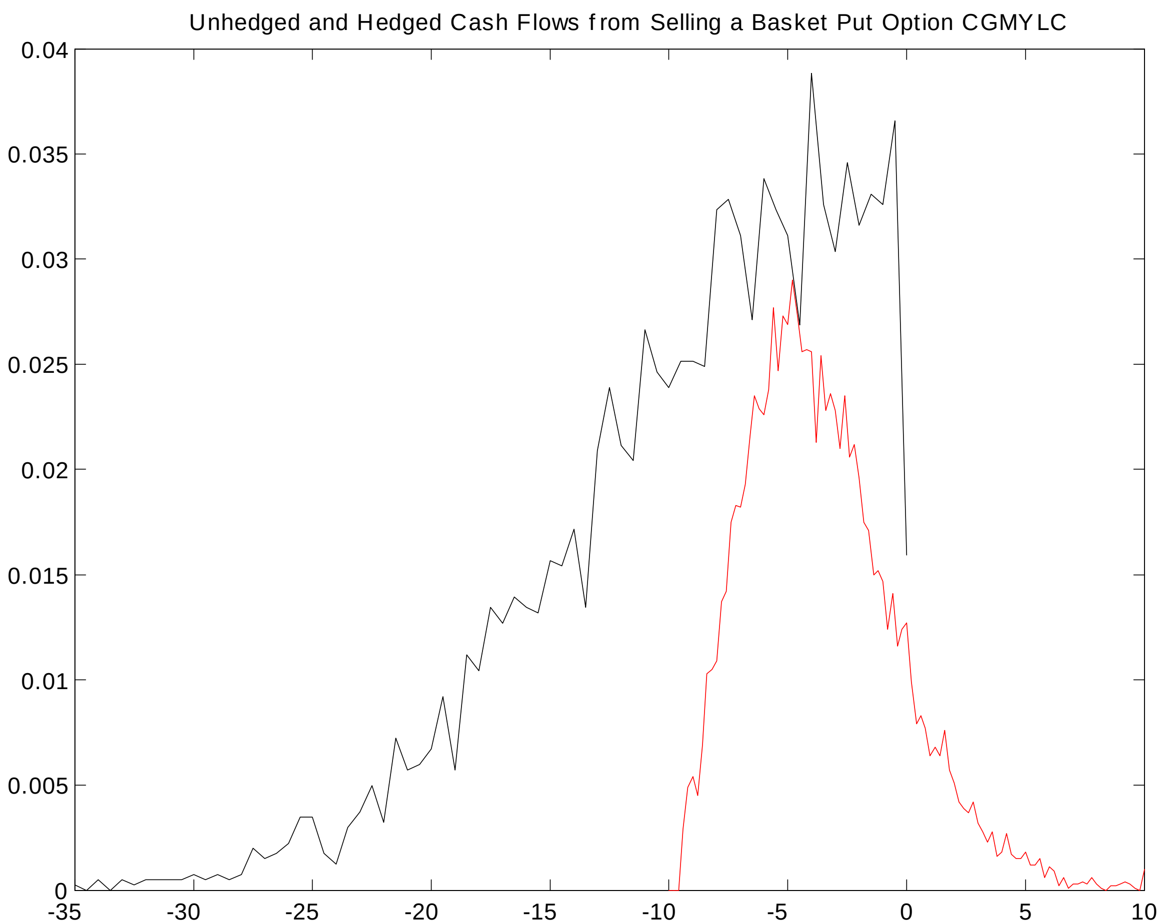
Hedging with Single Name Options

- We considered a high stress level of 5 for MINMAX-VAR and obtained the following hedged and unhedged prices for a 95 put using risk neutral marginals.

	Time Change Copula	VG Levy Correlation	C
UnHedged	24.4875	43.5140	3
Hedged	4.5285	12.5934	9

- Additionally we graph the unhedged and hedged cash flows.


Unhedged and Hedged Cash Flows from Selling a Basket Put Option VGLC
0.03
0.025
0.02
0.015
0.01
0.005
0
-50
-40
-30
-20
-10
0
10


Unhedged and Hedged Cash Flows from Selling a Basket Put Option CGMYLC
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
-35
-30
-25
-20
-15
-10
-5
0
5
10

Conclusion

- We have introduced two new form of dependence modeling, the multivariate VG copula and correlated Lévy processes via time change and Brownian correlation

- We have evaluated these models in the context of Basket option pricing using implied stress functions as a metric.

- Considerable stress has to be used with physical Lévy or scaled marginals.

- The required stress is reduced with risk neutral marginals but it is still present for down side puts.

- Pricing to acceptability was shown to be an engine for hedging with hedged prices substantially reduced for an attainment of the same level of acceptability.

- The hedged cash flows are also a lot less exposed to negatives.

- These results are conditioned by the path space used to construct the hedge.