The Lévy-driven Continuous-Time Garch Model

Claudia Klüppelberg

Technische Universität München

email: cklu@ma.tum.de

http://www.ma.tum.de/stat/

Joint work with

Alexander Lindner, Ross Maller, Vicky Fasen, Stefan Haug, Gernot Müller

Question: How to model the volatility $(\sigma_t)_{t \geq 0}$.

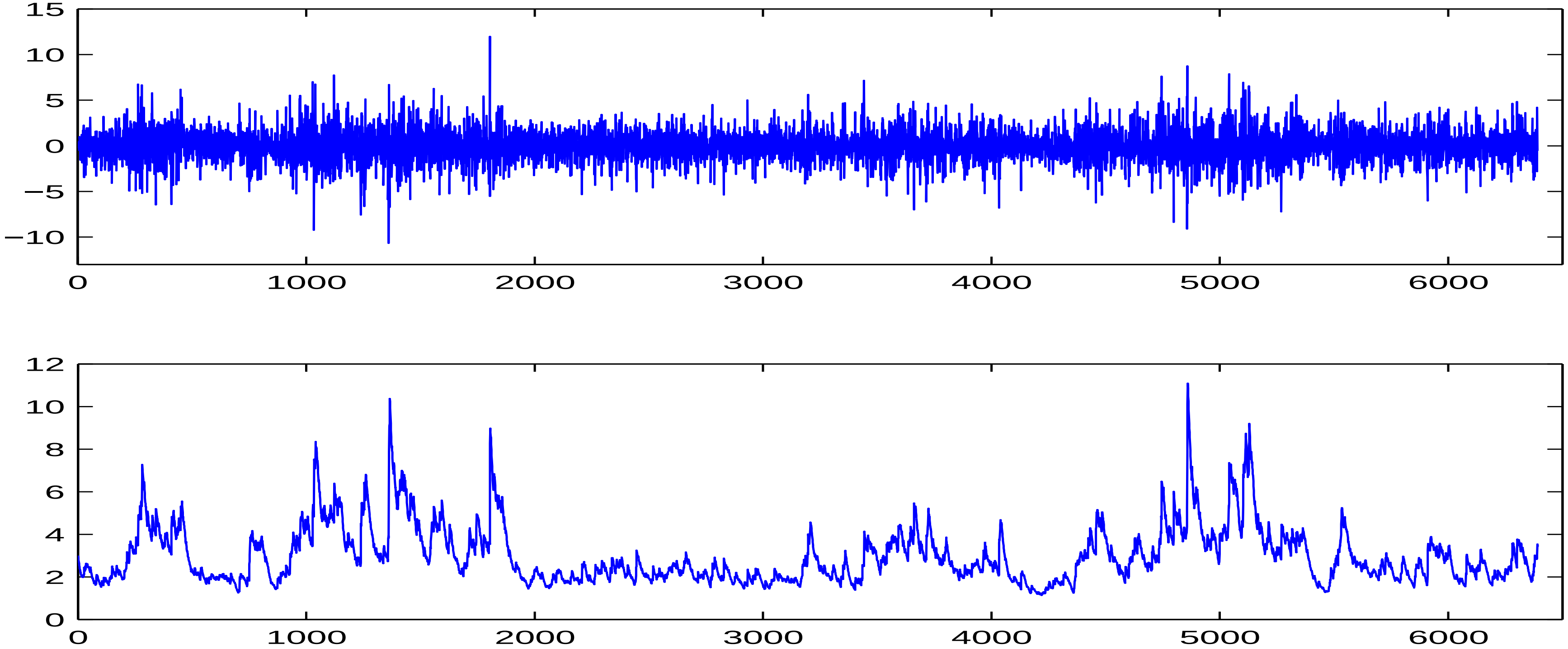


Figure 1: Deseasonalised 5 minutes log-returns of Intel (February 1 - May 31, 2002) and estimated volatility.

Stylized facts of volatility:

(1) volatility is random;

(2) volatility has heavy-tailed marginals (higher moments do not exist);

(3) volatility has skewed marginals (leverage effect);

(4) volatility is a stochastic process with long-range dependence effect;

(5) volatility is a stochastic process with clusters in the extremes.

Recall discrete time GARCH(1,1) model

$$Y_n = \sigma_n Z_n \quad \text{i.i.d. } \textbf{innovations } (Z_n)_{n\in\mathbb{N}_0},$$

Volatility process: Define for σ^2 the random recurrence equation

$$\sigma_n^2 = \beta + \lambda Y_{n-1}^2 + \delta\sigma_{n-1}^2, \quad n \in \mathbb{N}.$$

Reorganise and iterate the recurrence:

$$\begin{aligned}\sigma_n^2 &= \beta + \lambda Y_{n-1}^2 + \delta\sigma_{n-1}^2 = \beta + (\delta + \lambda Z_{n-1}^2)\sigma_{n-1}^2 \\ &= \beta \sum_{i=0}^{n-1} \prod_{j=i+1}^{n-1} (\delta + \lambda Z_j^2) + \sigma_0^2 \prod_{j=0}^{n-1} (\delta + \lambda Z_j^2) \qquad (1)\end{aligned}$$

Under appropriate conditions: $\sigma_n^2 \quad \xrightarrow{d} \quad \sigma_\infty^2 \stackrel{d}{=} \beta \sum_{i=0}^{\infty} \prod_{j=1}^{i} (\delta + \lambda Z_j^2).$

Continuous time GARCH(1,1)

Idea: start with **(1)** and **replace** the sum by an integral

$$\Leftrightarrow \quad \sigma_n^2 = \left(\beta \int_0^n \exp\left(-\sum_{j=0}^{[s]} \log(\delta + \lambda Z_j^2)\, ds\right) + \sigma_0^2\right) \exp\left(\sum_{j=0}^{n-1} \log(\delta + \lambda Z_j^2)\right)$$

Replace $\boldsymbol{Z_j}$ by jumps of a Lévy process L and take $\beta, \eta = -\log\delta, \varphi = \lambda/\delta$.
Then for a finite r.v. σ_0^2 define the **volatility process**

$$\sigma_t^2 = \left(\beta \int_0^t e^{X_s} ds + \sigma_0^2\right) e^{-X_t} \quad t \geq 0.$$

with **auxiliary process**

$$X_t = t\eta - \sum_{0<s\leq t} \log(1 + \varphi(\Delta L_s)^2) \quad t \geq 0.$$

Recall: $(L_t)_{t\geq 0}$ is Lévy process if $\quad Ee^{isL_t} = e^{t\psi_L(s)}$, $s \in \mathbb{R}$, with

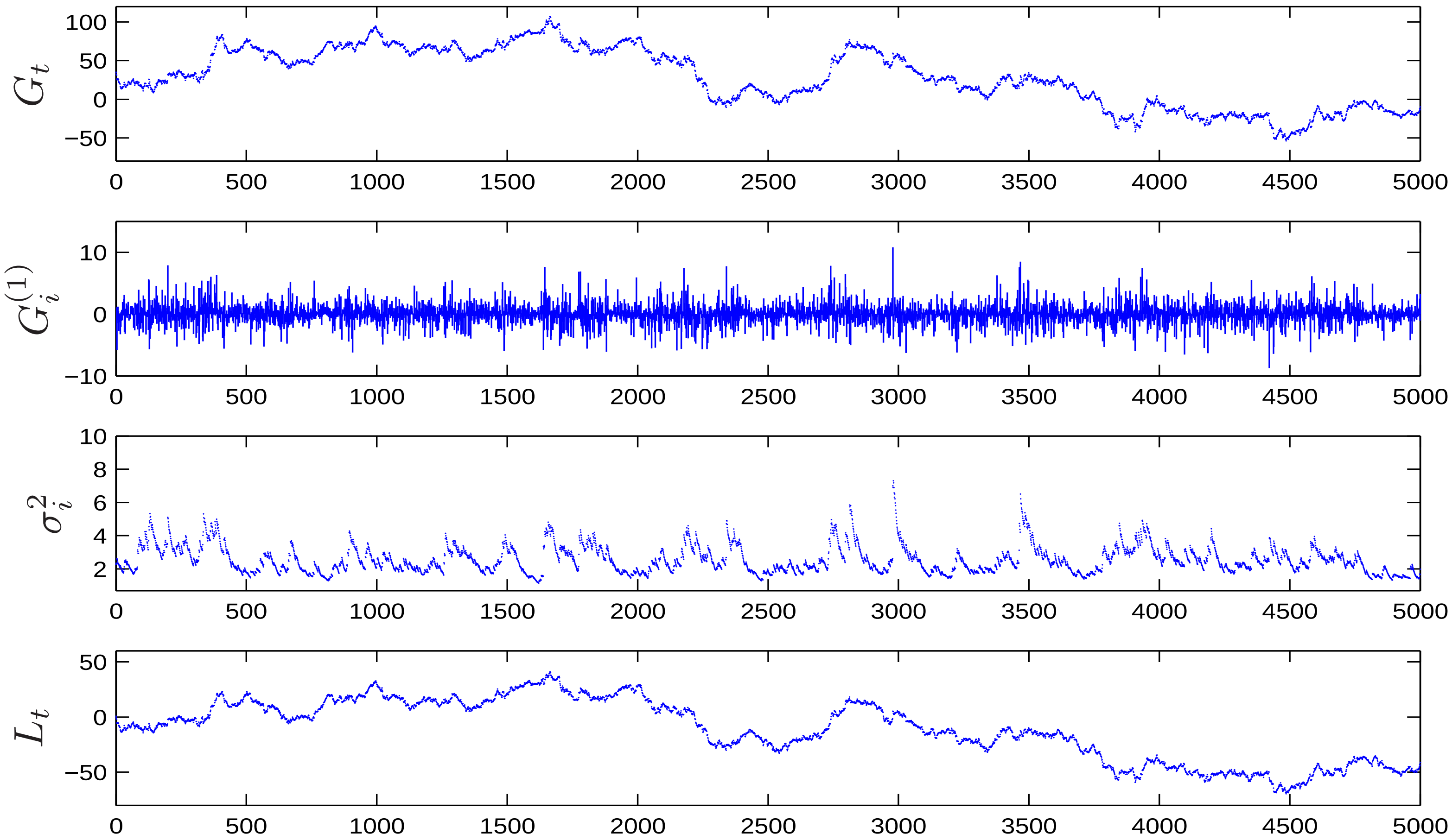
$$\psi_L(s) = i\gamma_L s - \tau_L^2 \frac{s^2}{2} + \int_{\mathbb{R}} (e^{isx} - 1 - isxI_{\{|x|<1\}})\Pi_L(dx), \quad s \in \mathbb{R}.$$

$(\gamma_L, \tau_L, \Pi_L)$ **characteristic tripel**, Π_L **Lévy measure**: $\int_{|x|<\varepsilon} x^2 \Pi_L(dx) < \infty$.

Define the **COGARCH(1,1)process** by

$$G_t = \int_{(0,t]} \sigma_{t-} \, dL_t \quad t \geq 0.$$

(Note: this defines the martingale part of the price process.)



First: Simulated VG driven COGARCH(1,1) process with $\beta = 0.04$, $\eta = 0.053$ and $\varphi = 0.038$;

second: differenced COGARCH process $(G_t^{(1)})$;

third: volatility process (σ_t);

last: VG process (L_t) with characteristic function $Ee^{iuL_1} = (1 + u^2/(2C))^{-C}$ and $C = 1$;

Properties

- G jumps at the same times as L with jump size $\Delta G_t = \sigma_t \Delta L_t$.
- $(X_t)_{t\geq 0}$ is spectrally negative, has drift η, no Gaussian part, Lévy measure

$$\Pi_X([0,\infty)) = 0 \quad \Pi_X((-\infty,-x]) = \Pi_L(\{|y| \geq \sqrt{(e^x-1)\varphi}\}) \text{ for } x > 0.$$

- $d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)\,dt + \varphi\,\sigma_{t-}^2 d[L,L]_t^{(d)}$

where $[L,L]_t^{(d)} = \sum_{0<s\leq t}(\Delta L_s)^2$ and

$$\sigma_t^2 = \sigma_0^2 + \beta t - \eta \int_0^t \sigma_s^2 ds + \varphi \sum_{0<s\leq t} \sigma_{s-}^2 (\Delta L_s)^2 \quad t \geq 0. \tag{2}$$

- $$\int_{\mathbb{R}} \log\left(1+\varphi x^2\right)\Pi_L(dx) < \eta \quad \Longleftrightarrow \quad EX_1 > 0$$

$$\Longleftrightarrow \quad \sigma_t^2 \xrightarrow{d} \sigma_\infty^2 \stackrel{d}{=} \beta \int_0^\infty e^{-X_t}\,dt.$$

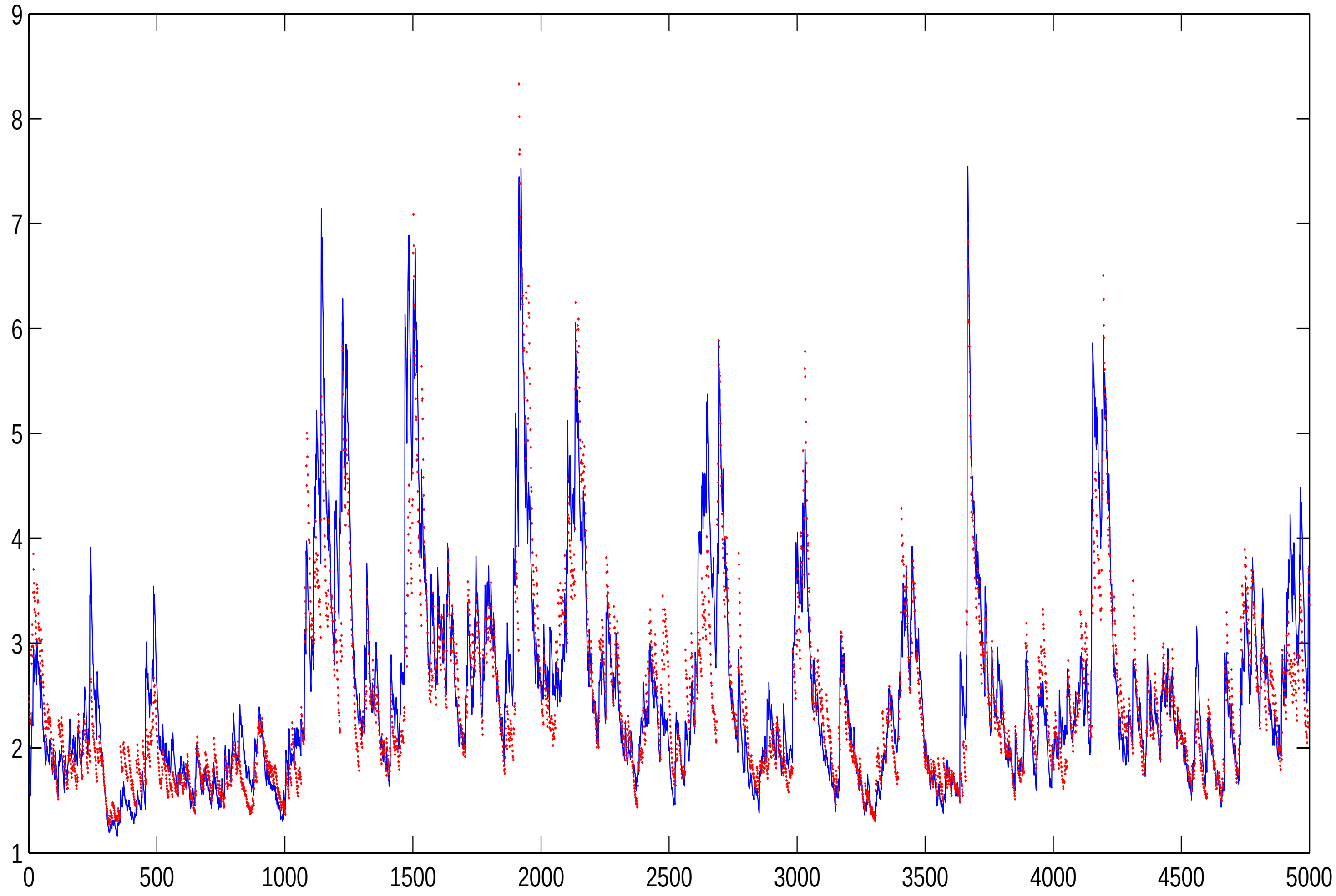
Sample path behaviour

- From (2) we know that σ_t^2 has only upwards jumps.

- If $(L_t)_{t\geq 0}$ is compound Poisson with jump times $0 = T_0 < T_1 < \ldots$,

$$\sigma_t^2 = \frac{\beta}{\eta} + \left(\sigma_{T_j}^2 - \frac{\beta}{\eta}\right) e^{-(t-T_j)\eta}, \quad t \in (T_j, T_{j+1}).$$

- For the stationary process, we have $\sigma_\infty^2 \geq \dfrac{\beta}{\eta}$ a.s.



Sample paths of σ_t^2 (solid line) and $\widehat{\sigma}_t^2$ (dotted line) of one simulation of a VG process.

Theorem Suppose that $EL_1 = 0$, $\mathrm{var}(L_1) = 1$. Define $Ee^{-sX_t} = e^{t\Psi_X(s)}$. Assume that the volatility process is stationary, and define $G_i^{(1)} := \int_{i-r}^{i} \sigma_{s-} dL_s$. If $\Psi_X(1) < 0$, then

$$EG_i^{(1)} = 0,\ E(G_i^{(1)})^2 = \frac{r\beta}{-\Psi_X(1)} EL_1^2 \text{ and } \mathrm{corr}(G_i^{(1)}, G_{i+h}^{(1)}) = 0.$$

If $EL_1^4 < \infty$, $\Psi_X(2) < 0$ and $\int_{\mathbb{R}} x^3 \nu_L(dx) = 0$, then for $k, p > 0$

$$\mathrm{corr}((G_i^{(1)})^2, (G_{i+h}^{(1)})^2) = ke^{-hp}, \quad h \in \mathbb{N}.$$

□

Theorem Assume that L_1 is symmetric and that there exists $\kappa > 0$ such that

$$|L_1|^\kappa \log^+ |L_1| < \infty \quad \text{and} \quad \Psi_X(\kappa/2) = 1.$$

Then a stationary version of the volatility process exists with

$$P(\sigma_t > x) \sim cx^{-\kappa/2}, \quad x \to \infty.$$

□

Stylized facts of volatility:

(1) volatility is random;

(2) volatility has heavy-tailed marginals (higher moments do not exist: K., Lindner and Maller (2004), Fasen, K., Lindner (2004));

(3) volatility has skewed marginals (leverage effect introduced in Haug et al.)

(4) volatility is a stochastic process with long-range dependence effect (acf decreases geometrically: K., Lindner and Maller (2004));

(5) volatility is a stochastic process with clusters in the extremes: Fasen: Extremes of genOU processes (2006, 2007).

Question: Can we find a discrete time skeleton, which approximates the COGARCH(1,1) process, and is a GARCH(1,1) process.

The following approximation, called **first jump approximation** shows that (under some technical conditions) the solution of a Lévy-driven SDE can be approximated arbitrarily close, by replacing the Lévy process with its first jump approximation.

Theorem [Szimayer and Maller (2007), Haug and Stelzer (2007)]
Let L be a Lévy process in $\mathbb{R}^d$, which has no Brownian part, drift γ_L and Lévy measure Π_L and satisfies $EL^2(1) = 1$.
For $n \in \mathbb{N}$ let $1 > \varepsilon^{(n)} \downarrow 0$ and $0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} \cdots \uparrow \infty$.
Set $\delta^{(n)} := \sup_{i \in \mathbb{N}}(t_i^{(n)} - t_{i-1}^{(n)})$ and assume that $\lim_{n\to\infty} \delta^{(n)} = 0$. Assume that

$$\lim_{n\to\infty} \delta^{(n)}(\Pi(\{x \in \mathbb{R}^d \, : \, |x| > \varepsilon^{(n)}\})^2 = 0\,. \tag{3}$$

Define for all $n \in \mathbb{N}$

$$
\begin{aligned}
\gamma^{(n)} &:= \gamma_L - \int_{\varepsilon^{(n)}<|x|\leq 1} x\Pi_L(dx) \\
\tau_i^{(n)} &:= \inf\{t : t_{i-1}^{(n)} < t \leq t_i^{(n)}, |\Delta L_t| > \varepsilon^{(n)}\} \quad \forall i \in \mathbb{N} \\
\widetilde{L}_t^{(n)} &:= \gamma^{(n)} t + \sum_{\{i\in\mathbb{N}:\tau_i^{(n)}\leq t\}} \Delta L_{\tau_i^{(n)}} \quad \forall t \geq 0 \\
\overline{L}_t^{(n)} &:= \widetilde{L}^{(n)}_{t_{i-1}^{(n)}}.
\end{aligned}
$$

Then

$$\widetilde{L}^{(n)} \to L \quad \text{in ucp as } n \to \infty \quad \text{and} \quad d_S(\overline{L}^{(n)}, L) \xrightarrow{P} 0 \quad n \to \infty.$$

□

Remark (i) Whenever one of the sequences $(\delta^{(n)})$ or $(\varepsilon^{(n)})$ are given, one can always choose the other such that (3) holds.

(ii) Note that the time grid is not necessarily equidistant. The construction allows for discrete sampling of a continuous-time Lévy-driven model. This is useful for high-frequency data.

(iii) The construction allows also the embedding of a discrete-time model into a continuous-time jump model. □

Example [COGARCH(1,1) and its GARCH(1,1) approximation]

Maller, Müller and Szimayer (2007) specify this approach and apply it to:

(1) Parameter estimation by pseudo MLE.

(2) Option pricing using the approach of Ritchken and Trevor (1999).

For an alternative approach, see Kallsen and Vesenmayer (2007).

Example [COGARCH(1,1) and its GARCH(1,1) approximation, Maller, Müller and Szimayer (2007)]

We use the notation as in the theorem and assume that all assumptions hold. For $n \in \mathbb{N}$ set $\Delta t_i(n) := t_i^{(n)} - t_{i-1}^{(n)}$ and define $\Delta L_{\tau_i^{(n)}}$ as the first jump of size larger than $\varepsilon^{(n)}$ in $(t_{i-1}^{(n)}, t_i^{(n)}]$. Define

$$Z_{i,n} = \frac{\mathbf{1}_{\{\tau_i^{(n)}\}<\infty} \Delta L_{\tau_i^{(n)}} - \nu_i^{(n)}}{\xi_i^{(n)}}, \quad i \in \mathbb{N}.$$

By the strong Markov property $(\mathbf{1}_{\{\tau_i^{(n)}<\infty\}} \Delta L_{\tau_i^{(n)}})_{i\in\mathbb{N}}$ is an iid sequence with distribution

$$\frac{\Pi(dx)\mathbf{1}_{\{|x|>\varepsilon^{(n)}\}}}{\Pi(\{x \in \mathbb{R}^d : |x| > \varepsilon^{(n)}\}} \Big(1 - e^{-\eta \Delta t_i(n) \Pi(\{x\in\mathbb{R}^d : |x|>\varepsilon^{(n)}\})}\Big), \quad x \in \mathbb{R} \setminus \{0\}.$$

Then $(Z_{i,n})_{i\in\mathbb{N}}$ is an iid sequence with mean 0 and variance 1.

Now recall

$$d\sigma_t^2 = (\beta - \eta\sigma_{t-}^2)\,dt + \varphi\,\sigma_{t-}^2 d[L,L]_t^{(d)} \quad \text{and} \quad G_t = \int_{(0,t]} \sigma_{t-} dL_t \quad t > 0.$$

We discretise as follows: for $G_{0,n} = G_0 = 0$ set

$$G_{i,n} - G_{i-1,n} = \sigma_{i-1,n}\sqrt{\Delta t_i(n)}Z_{i,n}, \quad i \in \mathbb{N},$$

and

$$\sigma_{i,n}^2 = \beta\Delta t_i(n) + \big(1 + \varphi\Delta t_i(n)Z_{i,n}^2\big)e^{-\eta\Delta t_i(n)}\sigma_{i-1,n}^2, \quad i \in \mathbb{N}.$$

This defines a discrete time GARCH(1,1) random recurrence equation; cf. p. 4.

Follow the construction as before and introduce continuous-time versions (piecewise constant) of the auxiliary process $X_{i,n}$, $\sigma^2_{i,n}$ and $G_{i,n}$. Then with the usual technical efforts, it is shown that

$$d_S((G_n, \sigma_n^2), (G, \sigma^2)) \xrightarrow{P} 0 \quad n \to \infty. \qquad \square$$

Question: Can we define a reasonable multivariate COGARCH model.

Definition [Multivariate COGARCH(1,1) model, Stelzer (2007)]

Let L be a d-dimensional Lévy process and

$A, B \in M_d(\mathbb{R})$ (the $d \times d$ matrices),

$C \in \mathbb{S}_d^+$ (the $d \times d$ positive semi-definite matrices) and set

$[L, L]_t^{(d)} := \sum_{0<s\leq t} \Delta L_s (\Delta L_s)^*$.

Then the process $G = (G_t)_{t\in\mathbb{R}^+}$ solving

$$
\begin{aligned}
dG_t &= V_{t-}^{1/2} dL_t \\
dV_t &= (B(V_{t-} - C) + (Y_{t-} - C)B^*)dt + AV_{t-}^{1/2} d[L, L]_t^{(d)} V_{t-}^{1/2} A^*
\end{aligned}
$$

with $G_0 \in \mathbb{R}^d$ and $Y_0 \in \mathbb{S}_d^+$ is a **multivariate COGARCH(1,1) process**.

Note: This definition agrees for $d = 1$ with the COGARCH(1,1) process.

For details see work by Robert Stelzer.

Question: Can we find a class of models, where the COGARCH(1,1) and the Barndorff-Nielsen and Shephard model belong to.

Recall the COGARCH(1,1) volatility process:

$$\sigma_t^2 = e^{-X_{t-}} \left(\beta \int_0^t e^{X_s} ds + \sigma_0^2 \right) \quad t \geq 0$$

where $(X_t)_{t\geq 0}$ is a spectrally negative Lévy process with positive drift.

Compare to the Barndorff-Nielsen and Shephard OU process

$$\widetilde{\sigma}_t^2 = e^{-\alpha t} \left(\int_0^t e^{\alpha s} dL_{\alpha s} + \widetilde{\sigma}_0^2 \right) \quad t \geq 0$$

where $(L_t)_{t\geq 0}$ is a subordinator.

This motivates the definition of the **generalised Ornstein-Uhlenbeck (genOU) process**

$$V_t = e^{-\xi_t} \left(\int_0^t e^{\xi_{s-}} d\eta_s + V_0 \right) \quad t \geq 0 \tag{4}$$

where $(\xi_t, \eta_t)_{t \geq 0}$ is a bivariate Lévy process and V_0 is an independent starting random variable.

This process has global properties concerning stationarity and second order behaviour, which explains the similarity between the COGARCH model and the BN-S OU model.

The similarity breaks down for the extremal behaviour.
See work by Vicky Fasen and Alexander Lindner.

References:

- Klüppelberg, C., Lindner, A. and Maller, R. (2004)
  A continuous time GARCH(1,1) process driven by a Lévy process: stationarity and second order behaviour. *J. Appl. Prob.* **41**, 1-22.
- Fasen, V., Klüppelberg, C. and Lindner, A. (2004)
  Extremal behavior of stochastic volatility models. In: Shiryaev, A., Grossihno, M.d.R., Oliviera, P. and Esquivel, M. (Eds.) *Stochastic Finance,* pp. 107-155. Springer, New York.
- Klüppelberg, C., Lindner, A. and Maller, R. (2004)
  Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models. In: Kabanov, Y., Lipster, R. and Stoyanov, J. (Eds.) *From Stochastic Calculus to Mathematical Finance: the Shiryaev Festschrift*, pp. 393-419. Springer, Berlin.
- Lindner, A.M. (2007) Continuous time approximations to GARCH and stochastic volatility models. In: Andersen, T.G., Davis, R.A., Kreiß, J.-P. and Mikosch, Th. (Eds.) *Handbook of Financial Time Series.* Springer, to appear.
- Kallsen, J. and Vesenmayer, B. (2007)
  COGARCH as a continuous-time limit of GARCH(1,1). *Stoch. Proc. Appl.* to appear.

- Haug, S., Klüppelberg, C., Lindner, A., Zapp, M. (2007)
  Method of moment estimation in the COGARCH(1,1) model.
  The Econometrics Journal **10**, 320-341.

- Fasen, V. (2008)
  Extremes of continous-time processes. In: Andersen, T.G., Davis, R.A., Kreiss, J.-P. and Mikosch, T. (Eds.) *Handbook of Financial Time Series.* Springer, Heidelberg, 2007, to appear.

- Fasen, V. (2007)
  Asymptotic results for sample autocovariance functions and extremes of integrated generalized Ornstein-Uhlenbeck processes. Submitted.

- Maller, R., Müller, G. and Szimayer, A. (2007)
  GARCH modelling in continuous time for irregularly spaced time series data.
  Bernoulli, accepted for publication.

- Müller, G., Durand, R., Maller, R. and Klüppelberg, C. (2008)
  Analysis of stock market volatility by continuous-time GARCH models. In: Gregoriou, G.N. (2009) *Stock Market Volatility.* Chapman Hall/Taylor and Francis, London. To appear.