Quadratic hedging and utility indifference pricing in stochastic volatility models with jumps

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Outline

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Affine stochastic volatility models

- (discounted) asset price process

$$S_t = S_0 \exp(X_t)$$

  X return process

- stylized facts: e.g. semi-heavy tails, volatility clustering
- generality vs. tractability?
- examples of affine stochastic volatility models
  - X Lévy process (e.g. BM, VG, NIG; volatility still constant)
  - Stein & Stein (1991)
  - Heston (1993)
  - Bates (1996)
  - Barndorff-Nielsen & Shephard (2001)
  - Carr, Geman, Madan, Yor (2003)
  - Carr & Wu (2003)
  - Carr & Wu (2004)
  - ...

Affine stochastic volatility models (ct'd)

for $S_t = S_0 \exp(X_t)$

- Barndorff-Nielsen & Shephard (2001)

$$\begin{aligned} dX_t &= \delta v_{t-} dt + \sqrt{v_{t-}} dW_t \\ dv_t &= -\lambda v_{t-} dt + dZ_t \end{aligned}$$

  - W Brownian motion
  - Z increasing Lévy process

- Carr, Geman, Madan, Yor (2003)

$$\begin{aligned} X_t &= L_{V_t} \\ dV_t &= v_{t-} dt \\ dv_t &= -\lambda v_{t-} dt + dZ_t \end{aligned}$$

  - L Lévy process
  - Z increasing Lévy process
  - reduces to BNS for Brownian motion L

Affine semimartingales

Definition

- Consider bivariate process (X, v).
- (b, c, K): (differential) characteristics of semimartingale (X, v)
  - interpretation: local Lévy-Khintchine triplet
  - b: local drift coefficient
  - c: local diffusion coefficient
  - K: local Lévy measure
- (X, v) affine semimartingale:

$$
\begin{aligned}
b_t &= b + \widetilde{b} v_{t-} \\
c_t &= c + \widetilde{c} v_{t-} \\
K_t &= K + \widetilde{K} v_{t-}
\end{aligned}
$$

  - (b, c, K), $(\widetilde{b}, \widetilde{c}, \widetilde{K})$ Lévy-Khintchine triplets on $\mathbb{R}^2$
  - triplet of (X, v) is affine function of current value v_{t-}

Affine semimartingales

Characterization by Duffie, Filipovic, Schachermayer (2003)

- Conditional characteristic function of (X, v)

$$E\left(\exp(iu_1 X_{s+t} + iu_2 v_{s+t}) \middle| \mathscr{F}_s \right) = \exp\left(\Psi(t, iu) + \widetilde{\Psi}(t, iu) v_s \right),$$

  where

  - $\widetilde{\Psi}$ solution to generalized Riccati equation

$$\widetilde{\Psi}(0, u) = u, \qquad \frac{d}{dt}\widetilde{\Psi}(t, u) = -\widetilde{\psi}(\widetilde{\Psi}(t, u))$$

  - Ψ obtained by integration

$$\Psi(t, u) = -\int_0^t \psi((\widetilde{\Psi}(s, u))ds$$

  - $\psi, \widetilde{\psi}$ Lévy exponents of (b, c, K), $(\widetilde{b}, \widetilde{c}, \widetilde{K})$

$$\begin{aligned} \psi(u) &= u^\top b + \frac{1}{2} u^\top c u + \int (e^{u^\top x} - 1 - u^\top x) K(dx) \\ \widetilde{\psi}(u) &= u^\top \widetilde{b} + \frac{1}{2} u^\top \widetilde{c} u + \int (e^{u^\top x} - 1 - u^\top x) \widetilde{K}(dx) \end{aligned}$$

Outline

Quadratic hedging

The problem

- discounted asset price process $S_t = S_0 \exp(X_t)$
- contingent claim $H = f(S_T)$
- How to hedge the risk from selling the claim?

$$\min_{v_0,\varphi} E\left(\left(v_0 + \int_0^T \varphi_t dS_t - H\right)^2\right) =: \varepsilon^2$$

- $v_0^\star$ variance-optimal initial endowment
- $\varphi^\star$ variance-optimal hedging strategy
- ε^2 expected squared hedging error

Quadratic hedging

The martingale case (Föllmer & Sondermann 1986)

- assume S to be martingale.
- define $V_t := E(H|\mathscr{F}_t)$
- variance-optimal initial endowment

$$v_0^\star = V_0$$

- variance-optimal hedging strategy

$$\varphi_t^\star = \frac{d\langle V, S\rangle_t}{d\langle S, S\rangle_t}$$

  (i.e. $\langle V, S\rangle = \varphi^\star \bullet \langle S, S\rangle$)
- expected squared hedging error

$$\varepsilon^2 := E\Big(\langle V, V\rangle_T - (\varphi^\star)^2 \bullet \langle S, S\rangle_T\Big)$$

- How to compute $v_0^\star$, $\varphi^\star$, ε^2 more explicitly?

Quadratic hedging

The general case (Černý & K. 2007)

- find opportunity process L and adjustment process $\widetilde{a}$ (both defined by some characteristic equation)
- define $N_t := \mathscr{L}(L) - \widetilde{a} \bullet S - [\widetilde{a} \bullet S, \mathscr{L}(L)]$
- define $V_t := E(H\mathscr{E}(N - N^t)_T | \mathscr{F}_t)$
- define opportunity-neutral measure $P^\star$ with density $\frac{dP^\star}{dP} := \frac{1}{E(L_0)\mathscr{E}(A^{\mathscr{L}(L)})_T}$
- define $\xi_t := \frac{d\langle V,S\rangle_t^{P^\star}}{d\langle S,S\rangle_t^{P^\star}}$
- variance-optimal initial endowment $v_0^\star = V_0$
- variance-optimal hedging strategy $\varphi_t^\star = \xi_t - (v_0 + \varphi^\star \bullet S_{t-} - V_{t-})\widetilde{a}_t$
- expected squared hedging error $\varepsilon^2 := E\Big(L \bullet (\langle V, V\rangle^{P^\star} - (\varphi^\star)^2 \bullet \langle S, S\rangle^{P^\star})_T\Big)$
- How to compute $v_0^\star$, $\varphi^\star$, ε^2 more explicitly?

Outline

Concrete calculations in affine models

Problems

- need to compute
  - $V, \varphi^\star, \varepsilon^2$ (martingale case) or
  - $L, \widetilde{a}, N, V, \xi, \varphi^\star, \varepsilon^2$ (general case)
- luckily:
  closed form expression for $L, \widetilde{a}, N$ in many affine models
- problem:
  no closed form expression for V (and hence $\xi, \varphi^\star, \varepsilon^2$)
- way out:
  integral transform representation

Integral representation of options

cf. Hubalek & Krawczyk (1998), Carr & Madan (1999), Raible (2000)

- consider option of the form

$$H = \int_{R-i\infty}^{R+i\infty} S_T^z \ell(z) dz$$

- e.g. $R > 1$,

$$\ell(z) = \frac{1}{2\pi i} \frac{K^{1-u}}{u(u-1)}$$

  for $H = (S_T - K)^+$

- compute $V, \xi, \varphi^\star, \varepsilon^2$ for $H = S_T^z$ (explicit solutions exist)
- then use linearity of $V, \xi, \varphi^\star$ in H (resp. biliniarity of ε^2)

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Solution in the Barndorff-Nielsen & Shephard model

non-martingale case

- denote by ψ^Z characteristic exponent of Lévy process Z
- define $V(z)_t = e^{\Psi_0(T-t,z)+\Psi_1(T-t,z)v_t+zX_t}$ with
  $\alpha_2(t) = \frac{(\delta+\frac{1}{2})^2}{\lambda}\left(e^{-\lambda t}-1\right),$
  $\alpha_1(t) = \int_0^t \psi^Z(\alpha_2(\tau))d\tau,$
  $\Psi_1(t,z) = \frac{-z(z-1)}{2\lambda}\left(e^{\lambda(t)}-1\right),$
  $\Psi_0(t,z) = \int_0^t \left(\psi^Z(\Psi_1(\tau,z)+\alpha_2(\tau)) - \psi^Z(\alpha_2(\tau))\right)d\tau$
- define $V_t = \int_{R-i\infty}^{R+i\infty} V(z)_t \ell(z)dz$
- define $\xi_t = \int_{R-i\infty}^{R+i\infty} \frac{V(z)_{t-}}{S_t} z\ell(z)dz$
- variance-optimal initial endowment: $v_0^\star = V_0$
- variance-optimal hedging strategy:
  $\varphi_t^\star = \xi_t - (v_0 + \varphi^\star \bullet S_{t-} - V_{t-})\frac{\delta+\frac{1}{2}}{S_t}$

Solution in the Barndorff-Nielsen & Shephard model

Hedging error

$$\varepsilon^2 = \int_{R-i\infty}^{R+i\infty} \int_{R-i\infty}^{R+i\infty} \vartheta(z_1, z_2)\ell(z_1)\ell(z_2)dz_1 dz_2$$

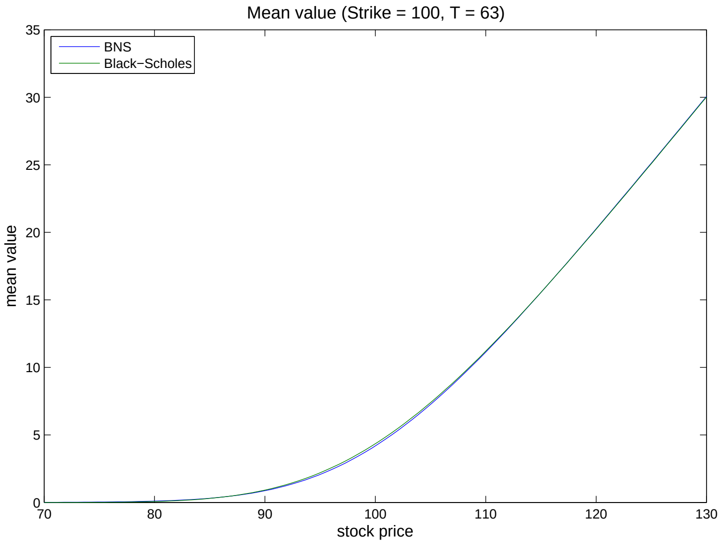
with

- $\vartheta(z_1, z_2) =$
  $\int_0^T J(T-t, z_1, z_2) e^{\alpha_1(T-t)+\tilde{\Psi}_0(T-t,z_1,z_2)+\Upsilon_0(t,t,z_1,z_2)+\Upsilon_1(t,z_1,z_2)v_0+(z_1+z_2)X_0} dt$
- $\tilde{\Psi}_0(t, z_1, z_2) = \Psi_0(t, z_1) + \Psi_0(t, z_2)$
- $\tilde{\Psi}_1(t, z_1, z_2) = \Psi_1(t, z_1) + \Psi_1(t, z_2) + \alpha_2(t)$
- $\xi(z_1, z_2) = \frac{\delta(z_1+z_2)+\frac{1}{2}(z_1+z_2)^2}{\lambda}$
- $\Upsilon_1(s; t, z_1, z_2) = \xi(z_1, z_2) + \left(\tilde{\Psi}_1(t, z_1, z_2) - \xi(z_1, z_2)\right) e^{-\lambda s}$
- $\Upsilon_0(t, z_1, z_2) = \int_0^t \psi^Z(\Upsilon_1(\tau; t, z_1, z_2))d\tau$
- $J(t, z_1, z_2) = \psi^Z(\alpha_2(t) + \Psi_1(t, z_1) + \Psi_1(t, z_2)) + \psi^Z(\alpha_2(t))$
  $- \psi^Z(\alpha_2(t) + \Psi_1(t, z_1)) - \psi^Z(\alpha_2(t) + \Psi_1(T-t, z_2))$

Outline

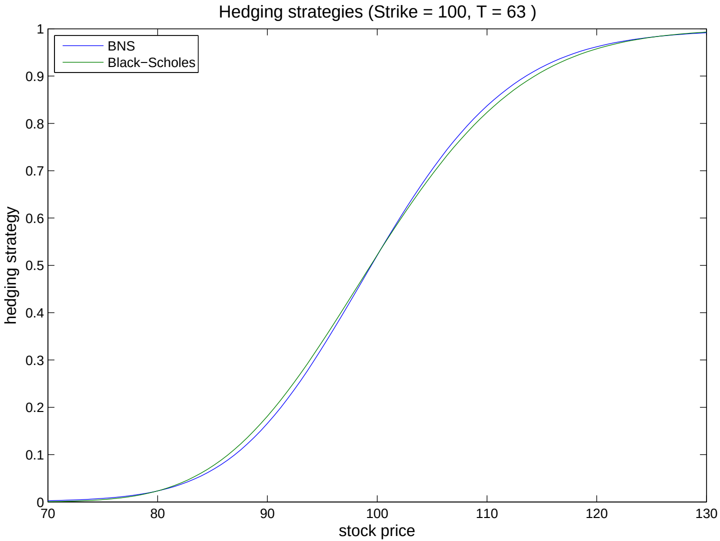
Variance-optimal initial endowment

Barndorff-Nielsen & Shephard vs. Black-Scholes model



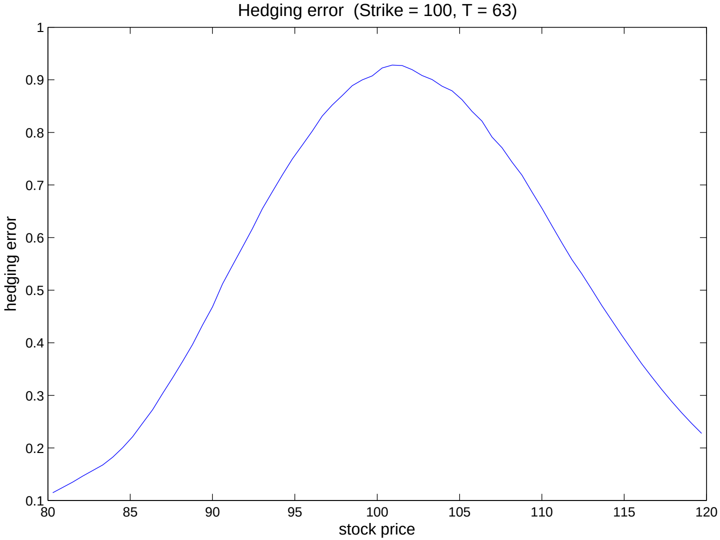
Variance-optimal initial hedge

Barndorff-Nielsen & Shephard vs. Black-Scholes model



Variance-optimal hedging error

Barndorff-Nielsen & Shephard vs. Black-Scholes model



Outline

Utility indifference pricing and hedging

Definition

- Consider ε options that are to be sold.
- Determine $\pi = \pi(\varepsilon)$ such that

$$\max_{\varphi} E(1 - \exp(v + \varphi \bullet S_T)) = \max_{\varphi} E(1 - \exp(v + \varphi \bullet S_T + \varepsilon\pi - \varepsilon H))$$

- Notation: $\varphi(0)$, $\varphi(\varepsilon)$ maximizer on the left resp. right
- $\pi(\varepsilon)$ utility indifference price per unit of H for ε options
- $\frac{\varphi(\varepsilon)-\varphi(0)}{\varepsilon}$ utility-based hedging strategy per unit of H for ε options

Approximate indifference pricing and hedging

as a first-order approximation

- Goal: approximate $\pi(\varepsilon)$, $\varphi(\varepsilon)$ for small ε
- Expansion:

$$\begin{aligned}\pi(\varepsilon) &= \pi(0) + \varepsilon\gamma + o(\varepsilon)\\ \varphi(\varepsilon) &= \vartheta^\star + \varepsilon\xi + o(\varepsilon)\end{aligned}$$

- Interpretation:
  - $\pi(0)$: limiting price for very small number of options (Davis 1997, Karatzas and Kou 1996)
  - γ: risk premium per option that is to be sold
  - $\vartheta^\star$: optimal strategy for pure investment problem without options
  - ξ: hedging strategy per option
- How to determine $\pi(0)$, $\vartheta^\star$, γ, ξ?

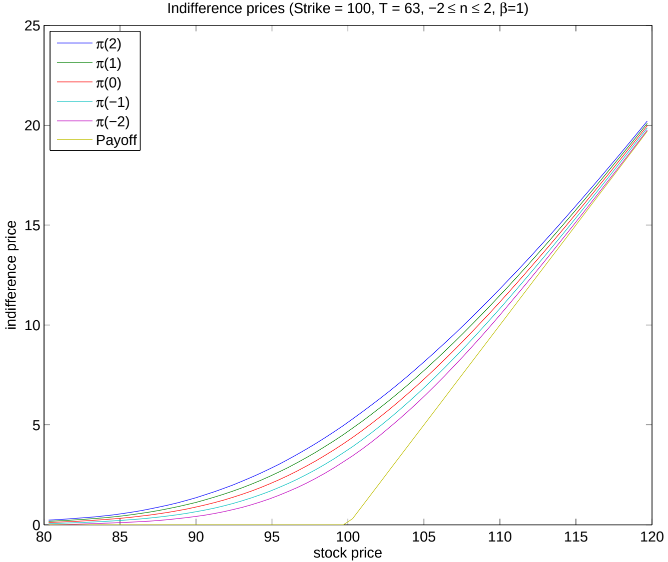
Approximate indifference pricing and hedging

Structure of the solution (Mania and Schweizer 2005, Becherer 2006, Rheinländer & K. 2008, for utilities on $\mathbb{R}_+$: Kramkov and Sîrbu 2006)

- Solve pure investment problem for exponential utility.
- Solve quadratic hedging problem under the minimal entropy martingale measure.
- Obtain $\pi(0)$, $\vartheta^\star$, γ, ξ.

Numerical illustration

Barndorff-Nielsen & Shephard model



Numerical illustration

Barndorff-Nielsen & Shephard model (ct'd)