Quadratic hedging and utility indifference pricing in stochastic volatility models with jumps


Jan Kallsen

CAU Kiel

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Outline

1. Affine stochastic volatility models
2. Quadratic hedging
3. Integral transform methods
4. Semiexplicit solutions
5. Numerical illustration
6. Exponential utility-based pricing and hedging
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Affine stochastic volatility models

- (discounted) asset price process

\[ S_t = S_0 \exp(X_t) \]

- \( X \) return process
- stylized facts: e.g. semi-heavy tails, volatility clustering
- generality vs. tractability?
- examples of affine stochastic volatility models
  - \( X \) Lévy process (e.g. BM, VG, NIG; volatility still constant)
  - Stein & Stein (1991)
  - Heston (1993)
  - Bates (1996)
  - Barndorff-Nielsen & Shephard (2001)
  - Carr, Geman, Madan, Yor (2003)
  - Carr & Wu (2003)
  - Carr & Wu (2004)
  - …
Affine stochastic volatility models (ct’d)
for $S_t = S_0 \exp(X_t)$

- Barndorff-Nielsen & Shephard (2001)
  \[
  dX_t = \delta v_t \, dt + \sqrt{v_t} \, dW_t \\
  dv_t = -\lambda v_t \, dt + dZ_t
  \]
  - $W$ Brownian motion
  - $Z$ increasing Lévy process

- Carr, Geman, Madan, Yor (2003)
  \[
  X_t = L V_t \\
  dV_t = v_t \, dt \\
  dv_t = -\lambda v_t \, dt + dZ_t
  \]
  - $L$ Lévy process
  - $Z$ increasing Lévy process
  - reduces to BNS for Brownian motion $L$
Consider bivariate process \((X, v)\).

\((b, c, K)\): (differential) characteristics of semimartingale \((X, v)\)

- interpretation: local Lévy-Khintchine triplet
- \(b\): local drift coefficient
- \(c\): local diffusion coefficient
- \(K\): local Lévy measure

\((X, v)\) affine semimartingale:

\[
\begin{align*}
b_t &= b + \tilde{b}v_{t^-} \\
c_t &= c + \tilde{c}v_{t^-} \\
K_t &= K + \tilde{K}v_{t^-}
\end{align*}
\]

- \((b, c, K), (\tilde{b}, \tilde{c}, \tilde{K})\) Lévy-Khintchine triplets on \(\mathbb{R}^2\)
- triplet of \((X, v)\) is affine function of current value \(v_{t^-}\)
Conditional characteristic function of \((X, \nu)\)

\[
E\left( \exp(iu_1 X_{s+t} + iu_2 \nu_{s+t}) \middle| \mathcal{F}_s \right) = \exp\left( \psi(t, iu) + \tilde{\psi}(t, iu) \nu_s \right),
\]

where

- \(\tilde{\psi}\) solution to generalized Riccati equation

\[
\tilde{\psi}(0, u) = u, \quad \frac{d}{dt} \tilde{\psi}(t, u) = -\tilde{\psi}(\tilde{\psi}(t, u))
\]

- \(\psi\) obtained by integration

\[
\psi(t, u) = -\int_0^t \psi((\tilde{\psi}(s, u))ds
\]

- \(\psi, \tilde{\psi}\) Lévy exponents of \((b, c, K), (\tilde{b}, \tilde{c}, \tilde{K})\)

\[
\psi(u) = u^\top b + \frac{1}{2} u^\top c u + \int (e^{u^\top x} - 1 - u^\top x)K(dx)
\]

\[
\tilde{\psi}(u) = u^\top \tilde{b} + \frac{1}{2} u^\top \tilde{c} u + \int (e^{u^\top x} - 1 - u^\top x)\tilde{K}(dx)
\]
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Quadratic hedging
The problem

- discounted asset price process $S_t = S_0 \exp(X_t)$
- contingent claim $H = f(S_T)$
- How to hedge the risk from selling the claim?

\[
\min_{v_0, \varphi} E \left( \left( v_0 + \int_0^T \varphi_t dS_t - H \right)^2 \right) =: \varepsilon^2
\]

- $v_0^*$ variance-optimal initial endowment
- $\varphi^*$ variance-optimal hedging strategy
- $\varepsilon^2$ expected squared hedging error
Quadratic hedging
The martingale case (Föllmer & Sondermann 1986)

- assume \( S \) to be martingale.
- define \( V_t := E(H | \mathcal{F}_t) \)
- variance-optimal initial endowment
  \[ v_0^* = V_0 \]

- variance-optimal hedging strategy
  \[ \varphi_t^* = \frac{d\langle V, S \rangle_t}{d\langle S, S \rangle_t} \]
  (i.e. \( \langle V, S \rangle = \varphi^* \cdot \langle S, S \rangle \))
- expected squared hedging error
  \[ \varepsilon^2 := E \left( \langle V, V \rangle_T - (\varphi^*)^2 \cdot \langle S, S \rangle_T \right) \]

How to compute \( v_0^*, \varphi^*, \varepsilon^2 \) more explicitly?
Quadratic hedging

The general case (Černý & K. 2007)

- find opportunity process \( L \) and adjustment process \( \tilde{a} \) (both defined by some characteristic equation)
- define \( N_t := \mathcal{L}(L) - \tilde{a} \cdot S - [\tilde{a} \cdot S, \mathcal{L}(L)] \)
- define \( V_t := E(HE(N - N^t)_T | \mathcal{F}_t) \)
- define opportunity-neutral measure \( P^* \) with density
  \[
  \frac{dP^*}{dP} := \frac{1}{E(L_0)E(A\mathcal{L}(L))_T}
  \]
- define \( \xi_t := \frac{d\langle V, S\rangle_t^{P^*}}{d\langle S, S\rangle_t^{P^*}} \)
- variance-optimal initial endowment \( v_0^* = V_0 \)
- variance-optimal hedging strategy
  \[
  \varphi_t^* = \xi_t - (v_0 + \varphi^* \cdot S_t - V_t)\tilde{a}_t
  \]
- expected squared hedging error
  \[
  \varepsilon^2 := E\left( L \cdot (\langle V, V\rangle^{P^*} - (\varphi^*)^2 \cdot \langle S, S\rangle^{P^*})_T \right)
  \]
- How to compute \( v_0^*, \varphi^*, \varepsilon^2 \) more explicitly?
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Concrete calculations in affine models

Problems

- need to compute
  - $V, \varphi^*, \varepsilon^2$ (martingale case) or
  - $L, \tilde{a}, N, V, \xi, \varphi^*, \varepsilon^2$ (general case)

- luckily:
  closed form expression for $L, \tilde{a}, N$ in many affine models

- problem:
  no closed form expression for $V$ (and hence $\xi, \varphi^*, \varepsilon^2$)

- way out:
  integral transform representation
Integral representation of options

- consider option of the form

\[ H = \int_{R-i\infty}^{R+i\infty} S_T^z \ell(z) dz \]

- e.g. \( R > 1, \)

\[ \ell(z) = \frac{1}{2\pi i} \frac{K^{1-u}}{u(u-1)} \]

for \( H = (S_T - K)^+ \)

- compute \( V, \xi, \varphi^*, \varepsilon^2 \) for \( H = S_T^z \)
  (explicit solutions exist)

- then use linearity of \( V, \xi, \varphi^* \) in \( H \)
  (resp. biliniarity of \( \varepsilon^2 \))
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Solution in the Barndorff-Nielsen & Shephard model
non-martingale case

denote by $\psi^Z$ characteristic exponent of Lévy process $Z$

define $V(z)_t = e^{\psi_0(T-t,z) + \psi_1(T-t,z)v_t + zX_t}$ with

$\alpha_2(t) = \frac{(\delta + 1)^2}{\lambda} (e^{-\lambda t} - 1)$,

$\alpha_1(t) = \int_0^t \psi^Z(\alpha_2(\tau)) d\tau$,

$\Psi_1(t, z) = \frac{-z(z-1)}{2\lambda} (e^{\lambda(t)} - 1)$,

$\Psi_0(t, z) = \int_0^t \left( \psi^Z(\Psi_1(\tau, z) + \alpha_2(\tau)) - \psi^Z(\alpha_2(\tau)) \right) d\tau$

define $V_t = \int_{R-i\infty}^{R+i\infty} V(z)_t \ell(z) dz$

define $\xi_t = \int_{R-i\infty}^{R+i\infty} \frac{V(z)_t}{S_t} z\ell(z) dz$

variance-optimal initial endowment: $v^*_0 = V_0$

variance-optimal hedging strategy:

$\varphi^*_t = \xi_t - (v_0 + \varphi^* \cdot S_{t-} - V_{t-}) \frac{\delta + 1/2}{S_t}$
Solution in the Barndorff-Nielsen & Shephard model

Hedging error

\[ \varepsilon^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \vartheta(z_1, z_2) \ell(z_1) \ell(z_2) \, dz_1 \, dz_2 \]

with

- \[ \vartheta(z_1, z_2) = \int_0^T J(T - t, z_1, z_2) e^{\alpha_1(T - t)} + \tilde{\psi}_0(T - t, z_1, z_2) + \gamma_0(t, t, z_1, z_2) + \gamma_1(t, z_1, z_2) \nu_0 + (z_1 + z_2) \lambda_0 \, dt \]
- \[ \tilde{\psi}_0(t, z_1, z_2) = \psi_0(t, z_1) + \psi_0(t, z_2) \]
- \[ \tilde{\psi}_1(t, z_1, z_2) = \psi_1(t, z_1) + \psi_1(t, z_2) + \alpha_2(t) \]
- \[ \xi(z_1, z_2) = \frac{\delta(z_1 + z_2) + \frac{1}{2}(z_1 + z_2)^2}{\lambda} \]
- \[ \gamma_1(s; t, z_1, z_2) = \xi(z_1, z_2) + \left( \tilde{\psi}_1(t, z_1, z_2) - \xi(z_1, z_2) \right) e^{-\lambda s} \]
- \[ \gamma_0(t, z_1, z_2) = \int_0^t \psi^Z(\gamma_1(\tau; t, z_1, z_2)) \, d\tau \]
- \[ J(t, z_1, z_2) = \psi^Z(\alpha_2(t) + \psi_1(t, z_1) + \psi_1(t, z_2)) + \psi^Z(\alpha_2(t)) - \psi^Z(\alpha_2(t) + \psi_1(t, z_1)) - \psi^Z(\alpha_2(t) + \psi_1(T - t, z_2)) \]
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Variance-optimal initial endowment
Barndorff-Nielsen & Shephard vs. Black-Scholes model

Mean value (Strike = 100, T = 63)
Variance-optimal initial hedge
Barndorff-Nielsen & Shephard vs. Black-Scholes model

Hedging strategies (Strike = 100, T = 63)
Variance-optimal hedging error
Barndorff-Nielsen & Shephard vs. Black-Scholes model

Hedging error (Strike = 100, T = 63)
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Utility indifference pricing and hedging

Definition

- Consider $\varepsilon$ options that are to be sold.
- Determine $\pi = \pi(\varepsilon)$ such that

$$\max_{\varphi} E(1 - \exp(v + \varphi \cdot S_T)) = \max_{\varphi} E(1 - \exp(v + \varphi \cdot S_T + \varepsilon \pi - \varepsilon H))$$

- Notation: $\varphi(0)$, $\varphi(\varepsilon)$ maximizer on the left resp. right
- $\pi(\varepsilon)$ utility indifference price per unit of $H$ for $\varepsilon$ options
- $\frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon}$ utility-based hedging strategy per unit of $H$ for $\varepsilon$ options
Approximate indifference pricing and hedging
as a first-order approximation

- **Goal:** approximate $\pi(\varepsilon)$, $\varphi(\varepsilon)$ for small $\varepsilon$
- **Expansion:**

  \[
  \pi(\varepsilon) = \pi(0) + \varepsilon \gamma + o(\varepsilon)
  \]

  \[
  \varphi(\varepsilon) = \vartheta^* + \varepsilon \xi + o(\varepsilon)
  \]

- **Interpretation:**
  - $\pi(0)$: limiting price for very small number of options (Davis 1997, Karatzas and Kou 1996)
  - $\gamma$: risk premium per option that is to be sold
  - $\vartheta^*$: optimal strategy for pure investment problem without options
  - $\xi$: hedging strategy per option

- **How to determine** $\pi(0)$, $\vartheta^*$, $\gamma$, $\xi$?
Approximate indifference pricing and hedging


- Solve pure investment problem for exponential utility.
- Solve quadratic hedging problem under the minimal entropy martingale measure.
- Obtain $\pi(0)$, $\vartheta^*$, $\gamma$, $\xi$. 
Numerical illustration
Barndorff-Nielsen & Shephard model

Indifference prices (Strike = 100, T = 63, −2 ≤ n ≤ 2, β = 1)

π(2)
π(1)
π(0)
π(−1)
π(−2)
Payoff

stock price
Indifference price
Numerical illustration

Barndorff-Nielsen & Shephard model (ct’d)

Trading and hedging strategies (Strike = 100, T = 63, n=1, β=1)

θ * pure investment strategy
φ(1) optimal trading strategy
ξ hedging strategy under Q