Liquidation Strategies for Infinitely Divisble Portfolios

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The Model

- Agent holds θ units of American-style claim, payoff per-unit claim $C(Y)$ (or $C(Y,\theta)$) where Y is asset value
- Perpetual case, can exercise over infinite horizon
- Risk averse agent cannot trade Y so incomplete market
- In complete market, standard perpetual American option problem (Samuelson/McKean (1965)/Dixit and Pindyck (1994)) - exercise threshold independent of quantity
- How you can divide up the claim important in incomplete market - we assume claim is infinitely divisible

- Assume Y is transient to zero with scale function $\mathcal{S}$, chosen such that $\mathcal{S}(0) = 0$
- Denote by Θ_t the number of options remaining at time t, $\Theta_0 = \theta$
- The agent with initial wealth x solves

$$\max_{(\Theta_t)\in\mathcal{M},\Theta_0=\theta} \mathbb{E}U\left(x + \int_{t=0}^{\infty} C(Y_t, \Theta_t)|d\Theta_t|\right)$$

where $\mathcal{M}$ is the set of positive decreasing processes $(\Theta_t)_{t\geq 0}$.
Rewrite as

$$\max_{(\tau_\phi)_{0\leq\phi\leq\theta},\tau_\phi\in\mathcal{T}} \mathbb{E}U\left(x + \int_{\phi=0}^{\theta} C(Y_{\tau_\phi}, \phi)d\phi\right)$$

where $\mathcal{T}$ is the family of decreasing stopping times parameterised by quantity ϕ which represents the number of unexercised claims, here $\tau_\phi = \inf\{t : \Theta_t \leq \phi\}$.

The canonical example

- Consider American call option so $C(Y) = (Y - K)^+$
- Asset value Y follows

$$\frac{dY}{Y} = \nu dt + \eta dW$$

for constants ν, η where $\nu \leq \eta^2/2$. Then $\mathcal{S}(y) = y^\beta$ where $\beta = 1 - 2\nu/\eta^2$.

- Work with discounted quantities so K is constant with respect to the bond numeraire
- Agent has exponential utility, $U(x) = -e^{-\gamma x}/\gamma$ or power utility $U(x) = x^{1-\alpha}/(1-\alpha)$.

Applications and Literature

Applications

- Real options - Y not financial asset
- Executive stock options - Y is stock, but executive restricted from trading it

Literature

- Henderson (2004) - perfectly indivisible
- Grasselli and Henderson (2006) - finitely divisible

- Jain and Subramanian (2004)
- Grasselli (2005)
- Rogers and Scheinkman (2007)
- Leung and Sircar (2007)

- Bank and Becherer
- Schied and Schöneborn (2008)

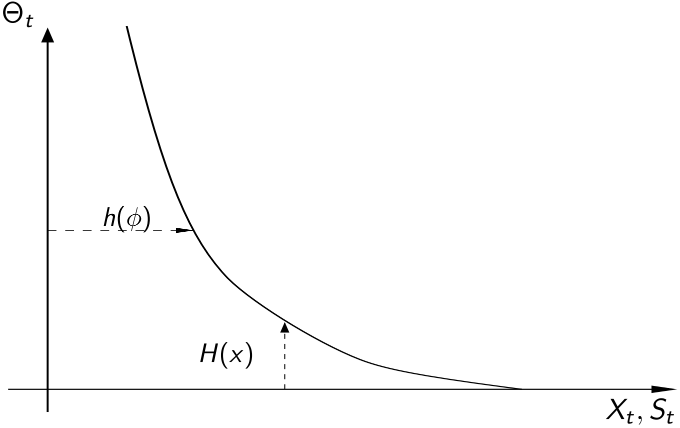

Figure: *A generic threshold $h(\phi)$.*

The total revenue

We solve for the value function for an arbitrary boundary and use calculus of variations to determine the optimal boundary

Consider exercising the infinitesimal θth (to go) unit of option, the first time, if ever, Y exceeds $h(\theta)$, where h decreasing, continuous, differentiable and $h(\theta) \geq K$. ie. let $\Theta_t = h^{-1}(\max_{0\leq s\leq t} Y_s)$

In region $\theta < h^{-1}(y)$ we have that total exercise revenue is

$$R = -\int_0^\infty C(Y_s, \Theta_s) d\Theta_s = \int_0^\theta d\phi C(h(\phi), \phi) i_{(S\geq h(\phi))}$$

where $S = \max_{0\leq t\leq\infty} Y_t$.

In the region $\theta > h^{-1}(y)$ we have

$$R = \int_{h^{-1}(y)}^\theta C(Y_0, \phi) d\phi + \int_0^{h^{-1}(y)} d\phi C(h(\phi), \phi) i_{(S\geq h(\phi))}$$

Note that conditional on S, R is non-random.

The utility of total revenue

Proposition

For $y \leq h(\theta)$

$$\mathbb{E}^{y,\theta}[U(x+R)] = U(x) + \mathcal{S}(y)\int_0^\theta d\phi (\mathcal{S}(h(\phi)))^{-1} C(h(\phi),\phi) U'\left(x + \int_\phi^\theta d\psi C(h(\psi),\psi)\right)$$

Sketch of Proof:

$R(s)$ denote the revenue conditional on $S = s$:

$$R(s) = \int_0^\theta d\phi C(h(\phi), \phi) i_{(s \geq h(\phi))} = \int_{h^{-1}(s)}^\theta d\phi C(h(\phi), \phi).$$

$$\begin{aligned}
&\mathbb{E}^{y,\theta}[U(x+R)] \\
&= \int_y^\infty \mathbb{P}(S \in ds) U(x + R(s)) \\
&= -\mathbb{P}^y(S \geq s) U(x+R(s))\big|_y^\infty + \int_y^\infty \mathbb{P}^y(S \geq s) R'(s) U'(x+R(s)) ds \\
&= U(x) + \int_{h(\theta)}^{h(0)} \mathbb{P}^y(S \geq s) \left| \frac{d}{ds} h^{-1}(s) \right| C(s, h^{-1}(s)) U'(x + R(s)) ds \\
&= U(x) + \int_0^\theta \mathbb{P}^y(S \geq h(\phi)) d\phi C(h(\phi), \phi) U'(x + \int_\phi^\theta d\psi C(h(\psi), \psi))
\end{aligned}$$

Theorem

Let $c(\cdot,\theta) = C^{-1}(\cdot,\theta)$*. The optimal* h *satisfies*

$$h'(\phi) = -\frac{\left[c_\phi - A(h,\phi;w_0,\theta_0)C^2c_z + 2C_\phi c_z + CC_\phi c_{zz} + Cc_{z\phi}\right]}{\left[2C_xc_z + B(\mathcal{S},h(\phi))Cc_z + CC_xc_{zz}\right]} \tag{1}$$

where (1) *is evaluated at* $x = h(\phi)$ *and* $z = C(h(\phi),\phi)$ *and*

$$\begin{aligned} A(h,\phi;w,\theta) &= \frac{U''(w+\int_\phi^\theta C(h(\psi),\psi)d\psi)}{U'(w+\int_\phi^\theta C(h(\psi),\psi)d\psi)} \\ B(\mathcal{S},h(\phi)) &= \frac{\mathcal{S}''(h(\phi))}{\mathcal{S}'(h(\phi))} - 2\frac{\mathcal{S}'(h(\phi))}{\mathcal{S}(h(\phi))} \end{aligned}$$

For exponential utility and call options

$$\max_{h\geq K}\mathbb{E}U(x+R) = -\frac{1}{\gamma}e^{-\gamma x}\min_{h\geq K}\mathbb{E}e^{-\gamma R} = -\frac{1}{\gamma}e^{-\gamma x}[1-y^{\beta}\max_{h\geq K}D_h(\theta)]$$

where

$$D_h(\theta) = \gamma\int_0^{\theta} d\phi\ \ h(\phi)^{-\beta}(h(\phi)-K)e^{-\gamma\int_{\phi}^{\theta}d\psi(h(\psi)-K)}.$$

Rescale problem with $\alpha = \gamma\theta K$ and $h(\psi) = Kf(\psi/\theta) = Kf(x)$. Define

$$A(\alpha) = \max_{f\geq 1}\int_0^1 dx f(x)^{-\beta}(f(x)-1)e^{-\alpha\int_x^1 dz(f(z)-1)}$$

Suppose $\alpha = 0$. Provided $\beta > 1$ the max is $F(x) = \frac{\beta}{\beta-1}$, or $F(x) = \infty$ if $\beta \leq 1$.

Dixit and Pindyck (1994)/McDonald and Siegel (1986)

Let $g(x) = \int_x^1 (f(z) - 1) dz$. Maximise

$$-\int_0^1 dx (1 - g'(x))^{-\beta} g'(x) e^{-\alpha g(x)}.$$

By calculus of variations, the maximiser $\tilde{g}$ satisfies

$$(1-\tilde{g}'(x))^{-\beta} \tilde{g}'(x) e^{-\alpha \tilde{g}(x)} - \tilde{g}' \frac{\partial}{\partial \tilde{g}'} \left[(1 - \tilde{g}'(x))^{-\beta} \tilde{g}'(x) e^{-\alpha \tilde{g}(x)} \right] = \text{constant}$$

Definition

Let $\beta = 1 - 2\nu/\eta^2$ *and suppose* $\beta > 0$*. For* $\beta > 1$ *define* $E(\beta) = \beta/(\beta - 1)$*, and set* $E(\beta) = \infty$ *otherwise. For* $1 < y < E(\beta)$ *define*

$$I(y) = \frac{2}{(y-1)} - (1+\beta)\ln\left(\frac{y}{y-1}\right) + i_{(\beta>1)}\left[(1+\beta)\ln\beta - 2(\beta-1)\right],$$

and for $\beta > 1$ *and* $y \geq E(\beta)$ *set* $I(y) = 0$*. Finally, let* J *be the inverse to* I *with* $J(0) = E(\beta)$ *for* $\beta > 1$ *and* $J(0) = \infty$ *otherwise.*

Theorem

Suppose $\beta > 0$*. For* $0 < y < \infty$ *and* $0 \leq \theta < \infty$ *define* $\Lambda(y, \theta; \gamma, K) = \Lambda(y, \theta)$ *by*

$$\begin{cases} 1 - y^{\beta} J(\gamma\theta K)^{-(\beta+1)} K^{-\beta}(\beta - (\beta-1)J(\gamma\theta K)) & y \leq KJ(\gamma\theta K) \\ \beta e^{-(y/K-1)(\gamma\theta K - I(y/K))}(1 - K/y) & KJ(\gamma\theta K) < y < KE(\beta) \\ e^{-\gamma(y-K)\theta} & KE(\beta) \leq y \ (\text{if } \beta > 1). \end{cases}$$

Then

$$V = V(x, y, \theta) = -\frac{1}{\gamma} e^{-\gamma x} \Lambda(y, \theta)$$

and the optimal strategy is to take

$$\Theta_t = \frac{1}{\gamma K} I\left(\frac{1}{K} \max_{0 \leq s \leq t} Y_s\right)$$

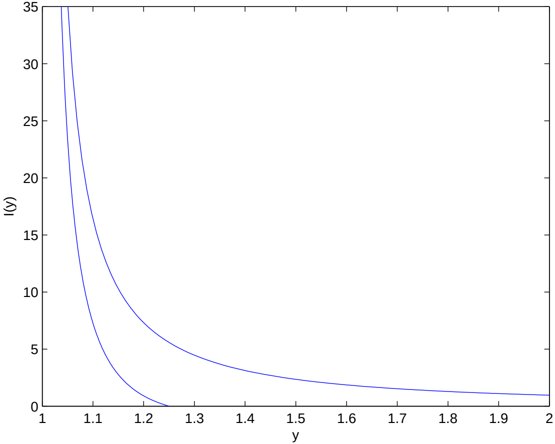

Figure: *Plots of $I(y)$ in the two cases $\beta > 1$, and $0 < \beta \leq 1$. The lower line corresponds to $\beta = 5$ and the upper line $\beta = 0.5$.*

Example: Power utility

$U(x) = x^{1-\alpha}/(1-\alpha)$, lognormal dynamics and stock $C(x) = x$.

The problem becomes to maximise

$$\int_0^\theta h(\theta)^{1-\beta} \left(x + \int_\phi^\theta h(\psi) d\psi \right)^{-\alpha}$$

If $\nu < 0$ so that $\beta > 1$ then the problem is degenerate and all stock is sold instantly.

So suppose $\nu > 0$. Set $\chi = (\alpha + \beta - 1)/\alpha < 1$. We will need $\chi > 0$ else the problem is degenerate.

So suppose $\chi > 0$. Suppose $y \le h(\theta; x) = h(\theta)$. From calculus of variations we deduce

$$h(\phi) = x \left(\frac{1}{\chi} - 1 \right) \frac{1}{\theta} \left(\frac{\theta}{\phi} \right)^{1/\chi}$$

What if $x = 0$?

More generally, if $y > h(\theta; x)$ then sell an initial tranche to reduce holdings until $h(\psi; x + (\theta - \psi)y) = y$. Then proceed as before.

If $y > h(\theta; x)$ then the agent should reduce holdings to ψ where

$$\psi = \frac{(x + \theta y)}{x}(1 - \eta)$$

Example: Exponential utility, price impact and portfolios of options

Suppose the payoff of the option depends on the number of options remaining: $C = C(Y_t, \Theta_t)$. This could be because

- the agent has a portfolio of options, and the order in which she sells them is prescribed,
- the agent has a portfolio of call options, in which case she sells the low strike options first,
- the act of selling options impacts upon the price.

The optimal strategy is again of threshold form.

Suppose $C(y, \theta) = (ye^{-p(\theta - \Theta_0)} - K(\theta))^+$ for $K(\theta)$ decreasing.
p is the parameter representing (permanent) price impact.
$K(\theta)$ is the strike of the θ^{th}-to-go option, if they are sold in order of increasing strike.

No price impact; tranches of options

Suppose $K(\theta) = k_1$ for $\theta \leq \theta_1$; $K(\theta) = k_2$ for $\theta_1 \leq \theta \leq \theta_2$.

By the main Theorem, for $\phi < \theta_1$ the optimal h solves

$$h'(\phi) = -\frac{\gamma(h(\phi) - k_1)^2 h(\phi)}{k_1(1+\beta) + (1-\beta)h(\phi)} \tag{2}$$

which can be solved as before.

Set $\bar{x} = h(\theta_1 -)$.

Let $\hat{x}$ solve

$$\frac{\beta k_1 + (1-\beta)\bar{x}}{\bar{x}^{1+\beta}} = \frac{\beta k_2 + (1-\beta)\hat{x}}{\hat{x}^{1+\beta}}$$

Then, for $\theta \in (\theta_1, \theta_2)$ the optimal h is given by the inverse to H where

$$H(x) = \theta_1 + \frac{2}{\gamma(x-k_2)} - \frac{2}{\gamma(\hat{x}-k_2)} + \frac{(1+\beta)}{\gamma k_2} \ln\left(\frac{(x-k_2)\hat{x}}{x(\hat{x}-k_2)}\right).$$

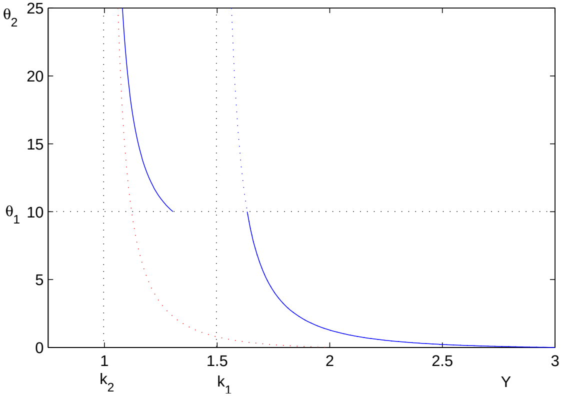

Figure: *The solid lines are the thresholds for agent with $\theta_1 = 10$ options wit strike $k_1 = 1.5$ and $\theta_2 - \theta_1 = 15$ options with strike $k_2 = 1$. Computations give $\bar{x} = 1.63$ and $\hat{x} = 1.3$. Also shown (dotted lines) are h_1 and h_2 which satisfy $h_2 \leq h \leq h_1$. Other parameters are $\beta = 2$, $\gamma = 1$.*

Price impact; identical options with strike k

Write $g(\psi) = e^{-p(\theta_0-\psi)} h(\psi)$ and abbreviate p/γ to ξ, so that ξ measures the relative importance of the price impact and the risk aversion.

Then the optimal g satisfies

$$g'(\theta) = \frac{-\gamma g \left(g^2 + g(\xi(\beta-1) - 2k) + k(k - \beta\xi)\right)}{g(1-\beta) + (\beta+1)k} \tag{3}$$

with $g(0) = e^{-p\theta_0}\bar{h}$ where $\bar{h} = \operatorname{argmax} h^{-\beta}(he^{-p\theta_0} - k)$.

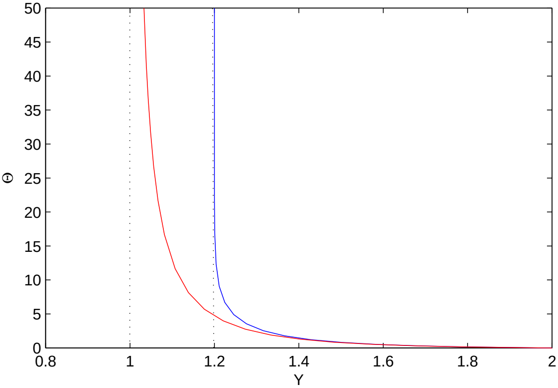

Figure: *Exercise boundaries for options with strike $k = 1$. Other parameter are $\beta = 2$ and $\gamma = 1$. The rightmost boundary uses price impact parameter $p = 0.05$ and for these parameters, $g(\infty) = 1.2$. The leftmost boundary has no price impact and hence $g(\infty) = k = 1$. Both boundaries have $g(0) = k\beta/(\beta - 1) = 2$.*

Final Remarks

- We have a method for generating the candidate optimal threshold/strategy. A verification lemma is required to finish the analysis.
- The advantage is that we decouple the problems of finding the value function and the optimal threshold, a more traditional approach solves for both simultaneously.
- The ideas can apply to incorporate more features: can include partial hedging in a correlated asset (which increases the continuation region, which in turn reduces the effective risk aversion), or Principal/Agent problems with effort.