A class of Levy process models with almost exact calibration to both barrier and vanilla fx options

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This is joint work with Peter Carr

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- This is joint work with Peter Carr of the Courant Institute, New York University and Bloomberg LP.
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FX and FX Options markets – stylised empirical observations

- Volatility (both implied and historical) changes randomly, fx options markets imply skews / smiles, jumps in underlying are observed.
- A key feature in fx options which is NOT observed in other markets is stochastic skew

  ie risk-reversals change in magnitude and actually also change sign on a moderately frequent basis (cf Equity options which are nearly always negatively skewed).

Vanilla options on spot fx rate

- Just as with other asset-classes, vanilla (standard European) options are very actively traded.
- Recall that the prices of vanilla options only depend upon the terminal distribution of the spot fx rate.

Barrier options on spot fx rate

- Unlike other asset-classes, there is a liquid and active market in barrier options. Many different types of barrier options are traded, eg double barrier knockout, knockin, window, with and without rebates, but the most liquid and actively traded are Double-No-Touch (DNT) options.
- DNT options pay one unit of domestic currency (at maturity) if neither a lower barrier nor an upper barrier are ever hit before maturity and they pay zero otherwise.

Joint calibration?

- It would be desirable to calibrate one’s model to both barrier and vanilla fx options. This is because:
- (a) Unlike vanilla options, barrier (eg DNT) options depend upon the full distribution of spot fx rates at all times up to and including maturity so they contain finer information about future spot fx rates (in the risk-neutral measure).
- (b) Traders would, of course, like a model which can match the market prices of barrier (especially DNT) options, since they are so actively traded.

Joint calibration?

- Therefore, it would be desirable to calibrate one’s model to both barrier and fx options.
- But this is easier said than done.
- This is because one could try and calibrate two different models to vanilla options eg
- (a) a Dupire (1994) local vol model
- (b) a Bates (1996) model (Heston-style stochastic vol plus jumps), perhaps making some of the parameters time-dependent.

Which model does one choose?

- Both these models could be calibrated very well to the vanilla options surface.
- Unfortunately, these two different models will give two different prices for DNT (or other barrier) options.
- Neither may coincide with the market price.
- One model might be better for, say, shorter-dated DNT options and the other better for longer-dated DNT options.
- The question is: Which model would one choose?
- Traders note that stochastic vol models tend to overprice DNT options while local vol models tend to underprice them.

Utopia

- WIBNI if we could have a model that can capture all the empirical features of the fx markets (eg jumps, stochastic vol, stochastic skew) and which also has something akin to a semi-automatic or semi-autonomous calibration to both vanilla fx options and barrier fx options.
- Lets see how we could do it.

Notation

- Today (the initial time) denoted by $t_0 \equiv 0$
- Calendar time denoted by t .
- We work only in the (non-unique) risk-neutral measure, which we denote by Q .
- Expectations, at time t , under Q denoted by $E_t^Q[\bullet]$
- Spot fx rate (number of units of domestic currency per unit of foreign currency) denoted by $S(t)$. Write $\log\left(S(t)/S(t_0)\right) \equiv X(t)$

Notation

- Domestic (respectively foreign) interest-rates are constant and denoted by r_d (respectively r_f).
- Price, at time t_0, of a domestic bond, maturing at time T, is denoted by $P_d(t_0,T)$ ie
  $P_d(t_0,T) = \exp(-r_d(T - t_0))$
- We introduce lower and upper barriers L and U which correspond to the barrier levels of the DNT (or other barrier) options to which we'll calibrate. We call (L,U) the "corridor". (Assume $L < S(t_0) < U$).

- We denote the first exit time of the spot fx rate from the corridor by τ ie

$\tau \equiv \inf\{t : S(t) \le L$ or $S(t) \ge U\}$.

- Set $\tau = \infty$ if spot fx rate has never exited from the corridor.

First key assumption

- We assume that, at time t_0 and while $\tau = \infty$, the dynamics of the spot fx rate, under Q , are such that we can compute the Laplace Transforms of certain quantities of interest (essentially the LT of joint probability distribution of τ and $X(\tau)$).

- This is a reasonably flexible assumption.

Risk-neutral dynamics of the spot fx rate

- Number of possibilities for the dynamics:
- The Kou (2002) Double Exponential Jump-Diffusion (DEJD) model.
- Or a jump-diffusion process with an arbitrary number of sums of double exponential jumps and where the diffusion volatility and the jump intensity rates are stochastic and driven by a Markov chain. We christen this the CEE2 process and this is the process we consider in the paper (can be used (see Asmussen et al. (2007)) to approximate a time-changed CGMY process or indeed a time-changed Levy process for any Levy process with a monotone Levy density eg CGMY, Generalised Hyperbolic, NIG (although some parameter restrictions may apply)).
- We conjecture (unproven) that it may possible to use the Markov chain regime-switching MMGBM model of Di Graziano and Rogers (2006).

We assume Kou (2002) DEJD dynamics in this talk

- However to fix ideas, we'll assume that the dynamics, while $\tau = \infty$ ie while the spot fx rate has never exited the corridor, are those of the Kou (2002) Double Exponential Jump-Diffusion (DEJD) model in this talk.

Hence, we assume, under Q :

$$dX(t) \equiv d(\log(S(t)/S(t_0))) = \left(r_d - r_f - \frac{1}{2}\sigma^2 - \sum_{i=1}^{M} \log E_{t_0}^{Q}[\exp(\rho_i N_i(1))] \right) dt$$

$$+ \sigma dz(t) + \sum_{i=1}^{M} \rho_i dN_i(t)$$

where $M=2$, $\rho_1 = 1$ (gives up jumps), $\rho_2 = -1$ (down jumps), and $N_i(t)$ for each $i=1,2$ is a compound Poisson process with exponentially distributed jump amplitudes with mean $1/b_i$. Furthermore $dz(t)$ denotes Brownian increments and σ denotes a constant volatility term.

Second key assumption

- At the instant that the spot fx rate first exits from the corridor, ie at $t = \tau$, we assume the dynamics (under Q) can change. The only practical requirement is that these dynamics are such that we can compute the Laplace Transform of the characteristic function (preferably in closed form).
- => We could use any Levy process model and some time-changed Levy processes.

- We want to price DNT and vanilla options rapidly to allow fast calibration.
- Pricing DNT options (or double barrier knockout options) with barriers at L and U is now straightforward using Kou and Wang (2003) and Sepp (2004) since the change of dynamics at the first exit time from the corridor is irrelevant as then the option expires worthless.
- Now we are done for pricing these barrier options !

Pricing vanilla options

- Now we want to price vanilla options.
- By “in-out” parity, the price of a vanilla equals the price of a double barrier knockout plus the price of a double barrier knockin.
- But, as on the last slide, we can price double barrier knockout options using Kou and Wang (2003) and Sepp (2004).
- All we have to do is price a double barrier knockin option with the same strike and maturity as the vanilla and we are done !

Remember the spot fx rate can first exit the corridor in one of 4 ways:

(1)Diffuse through upper barrier => $S(\tau)=U$

(2)Jump (overshoot) through the upper barrier

=> $S(\tau)=U\exp(x)\equiv S(t_0)\exp(u+x)$ for some $x>0$

(3)Diffuse through lower barrier => $S(\tau)=L$

(4)Jump (overshoot) through the lower barrier

=> $S(\tau)=L\exp(x)\equiv S(t_0)\exp(l+x)$ for some $x<0$

Let us denote by $P_d(t,T)V(S(t),K,T-t)$ the price, at time t, of a vanilla option with strike K and remaining time to maturity equal to $T-t$ ie $V(S(t),K,T-t)$ is the UNDISCOUNTED vanilla price. We can think of a double barrier knockin option as paying a vanilla option at time τ, the first exit time from the corridor (L,U).

- **So we can write the price $C\left(S(t_0), K, L, U, T-t_0\right)$ at time t_0 of a double barrier knock-in as:**

$$C\left(S(t_0), K, L, U, T-t_0\right) = P_d(t_0, T)\int_{t_0}^{T} g(s)ds$$ **where**

$$g(s) \equiv \Pr\left(X(\tau) = u, \tau \in ds\right)V\left(S(t_0)\exp(u), K, T-\tau\right)$$

$$+\int_{0+}^{\infty} \Pr\left(X(\tau) = u + x, \tau \in ds\right)V\left(S(t_0)\exp(u+x), K, T-\tau\right)dx$$

$$+\Pr\left(X(\tau) = l, \tau \in ds\right)V\left(S(t_0)\exp(l), K, T-\tau\right)$$

$$+\int_{-\infty}^{0-} \Pr\left(X(\tau) = l + x, \tau \in ds\right)V\left(S(t_0)\exp(l+x), K, T-\tau\right)dx$$

($0+$ and $0-$ remind us that the barrier was overshot)

- From Kou and Wang (2003), we can show:
- $\Pr(X(\tau) = u + x \mid X(\tau) > u) = b_1 \exp(-\rho_1 b_1 x)$ (here $x > 0$ and $\rho_1 = 1$)

  $\Pr(X(\tau) = l + x \mid X(\tau) < l) = b_2 \exp(-\rho_2 b_2 x)$ (here $x < 0$ and $\rho_2 = -1$)
- This is a key result for us. It is not true for all jump processes but holds in the DEJD model (more or less, follows from the memory-less property of the exponential distribution). We derive a somewhat analogous result for our CEE2 process in the paper.

- The last slide plus the linear homogeneity property of vanilla option prices =>

$$g(s) = \Pr(X(\tau) = u, \tau \in ds) V(S(t_0)\exp(u), K, T-\tau)$$

$$+\int_{0+}^{\infty} \Pr(X(\tau) > u, \tau \in ds) b_1 \exp((1-\rho_1 b_1)x) V(U, K\exp(-x), T-\tau)\,dx$$

$$+\Pr(X(\tau) = l, \tau \in ds) V(S(t_0)\exp(l), K, T-\tau)$$

$$+\int_{-\infty}^{0-} \Pr(X(\tau) < l, \tau \in ds) b_2 \exp((1-\rho_2 b_2)x) V(L, K\exp(-x), T-\tau)\,dx$$

- But now we can make progress in evaluating $C(S(t_0),K,L,U,T-t_0)/P_d(t_0,T)=\int_{t_0}^{T} g(s)ds$ because we have 4 terms each of which is form of a convolution.

Hence:

$$L[C(S(t_0),K,L,U,T-t_0)/P_d(t_0,T)]=$$
$$L[\Pr(X(\tau)=u,\tau\in ds)]L[V(S(t_0)\exp(u),K,\tau)]$$
$$+L[\Pr(X(\tau)>u,\tau\in ds)]L\left[\int_{0+}^{\infty} b_1\exp((1-\rho_1 b_1)x)V(U,K\exp(-x),\tau)dx\right]$$
$$+L[\Pr(X(\tau)=l,\tau\in ds)]L[V(S(t_0)\exp(l),K,\tau)]$$
$$+L[\Pr(X(\tau)<l,\tau\in ds)]L\left[\int_{-\infty}^{0-} b_2\exp((1-\rho_2 b_2)x)V(L,K\exp(-x),\tau)dx\right]$$

“Probability-like” terms

- Our last equation involves eight Laplace Transforms.
- The four on the LHS of each line are:

$$L\left[\Pr\left(X\left(\tau\right)=u,\tau\in ds\right)\right] \quad L\left[\Pr\left(X\left(\tau\right)=l,\tau\in ds\right)\right]$$

$$L\left[\Pr\left(X\left(\tau\right)<l,\tau\in ds\right)\right] \quad L\left[\Pr\left(X\left(\tau\right)>u,\tau\in ds\right)\right]$$

- Need to be computable in closed form (this was our First Key Assumption).
- Furthermore, in some models they are indeed computable:
- eg Kou (2002) DEJD model.
- eg The CEE2 process which we consider in the paper (ie a jump-diffusion model with an essentially arbitrary number of sums of double exponential jumps and with stochastic diffusion volatility and jump intensity rates driven by a continuous-time Markov chain).

"Option price-like" terms

- Using Fourier methods, we can write vanilla option price as essentially an integral involving the Characteristic Function.
- Hence the Laplace Transform of vanilla option price is essentially an integral involving the Laplace Transform of the Characteristic Function.
- The integrals over x in the terms

$$L\left[\int_{0+}^{\infty} b_1 \exp\left(\left(1-\rho_1 b_1\right)x\right) V\left(U, K\exp\left(-x\right), \tau\right) dx\right] \qquad L\left[\int_{-\infty}^{0-} b_2 \exp\left(\left(1-\rho_2 b_2\right)x\right) V\left(L, K\exp\left(-x\right), \tau\right) dx\right]$$

can be done analytically.

- Hence we can compute the Laplace Transform of

  $L\left[C\left(S\left(t_0\right), K, L, U, T-t_0\right)/P_d\left(t_0, T\right)\right]$

  with 4 Fourier inversions and hence compute

  $C\left(S\left(t_0\right), K, L, U, T-t_0\right)/P_d\left(t_0, T\right)$

  and hence $C\left(S\left(t_0\right), K, L, U, T-t_0\right)$ by Laplace inversion.

  This gives us the price of a double barrier knockin option.

  We already know how to price double barrier knockout prices.

  Hence by "in-out" parity we can compute vanilla option prices.

Model Recipe Stage 1

- What we have is a very flexible framework.
- While $\tau = \infty$ ie before the first exit time from the corridor, we assume that the dynamics are such that we can compute the “probability-like” terms eg the Kou (2002) DEJD model (but, as already indicated, other richer models such as our CEE2 process are possible with only modest changes to the equations we have derived).

Model Recipe Stage 1 continued

- Calibrate the model parameters by fitting to the market prices of DNT (or other double barrier) options with barriers at L and U .
- Then price double barrier knockout options (with the same strikes and maturities as the vanilla options) using these same estimated parameters. Subtract these latter prices from the vanilla option (market) prices to give the prices of double barrier knockin options which we will use in Stage 2.

Model Recipe Stage 2

- Pick any model for which we know the Laplace Transform of the Characteristic Function in closed form eg any Levy process and some time-changed Levy processes.
- Eg Simplest specification would be to use the Kou (2002) DEJD model or our CEE2 process but with different parameters.
- Or eg CGMY process.
- Or eg time-changed Tempered Stable processes.

Model Recipe Stage 2 continued

- Take the parameters from Stage 1 as given, calibrate the model parameters, for the dynamics of the spot fx rate after the first exit time from the corridor, to the market prices of vanilla options (by using the results we have derived and the prices of the double barrier knockin options we obtained in Stage 1).

- By design, we have separated the two stages which will give us two good calibrations:
- Firstly to DNT options (or other types of barrier options).
- Secondly to vanilla options.
- An additional benefit is that we have lowered the dimensionality of the least squares fit calibration.

- We performed such a fit to DNT and vanilla options on cable (USD/STG) as of 06/07/2007.
- For stage 1 (ie for the dynamics before the first exit time from the corridor), we assumed the dynamics were our CEE2 process with 4 Poisson processes and two states in the Markov chain (eight parameters to fit).
- For stage 2, we used two different specifications. They were:

(1) Our CEE2 process (but with different parameters after the first exit time from the corridor compared to before the first exit time), again with 4 Poisson processes and two states (eight parameters to fit).

(2) We time-changed two independent Tempered Stable processes (one produces up jumps and the second produces down jumps), with the time-change generated by two independent two-state continuous-time Markov chains.

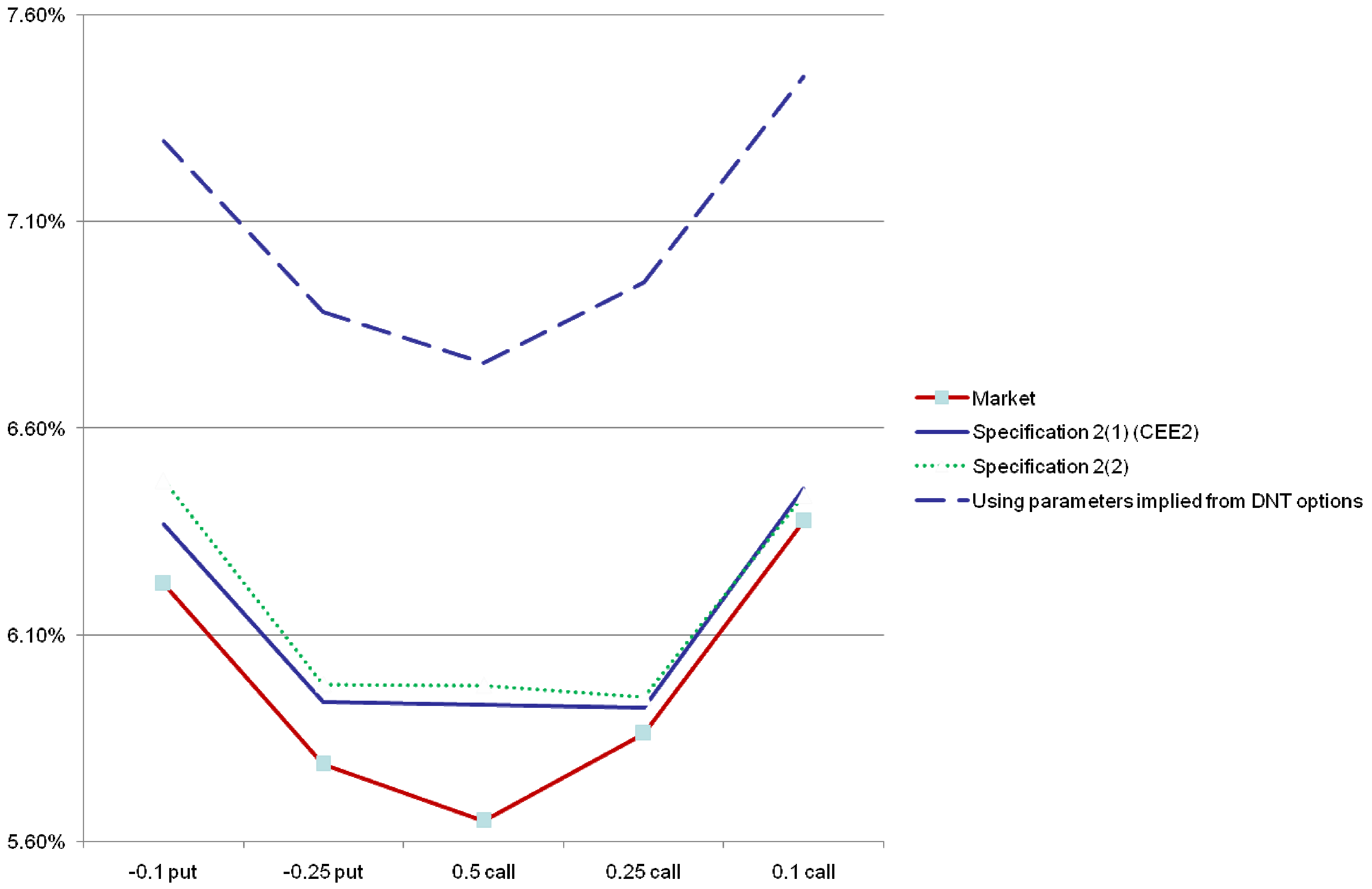
- Specification (2) also has eight parameters to fit.
- Note that Specification (2) can generate stochastic skew.

Calibration to market DNT option prices

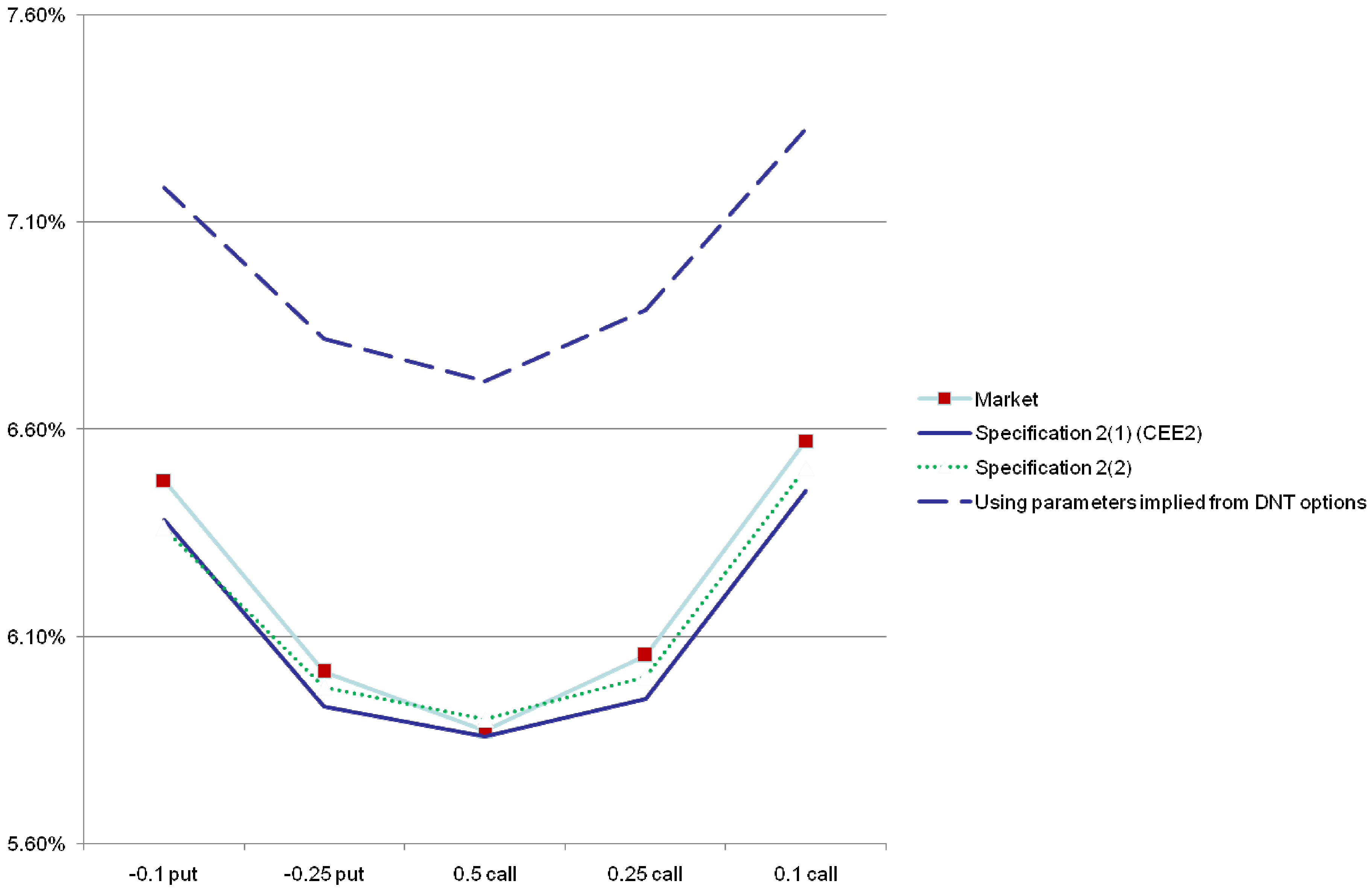
- Calibration to the market prices of the DNT options are the same in each case: Spot fx rate = 2.0060 , $L=1.95$, $U=2.05$

Maturity	Model price	Mid-market	Bid/Offer
Barrier levels 1.95 / 2.05			
1 m	0.75156	0.765	0.75 / 0.78
2 m	0.48389	0.5	0.485 / 0.515
3 m	0.31510	0.325	0.31 / 0.34
4 m	0.21993	0.22	0.205 / 0.235
5 m	0.15559	0.15	0.135 / 0.165
6 m	0.11013	0.1	0.085 / 0.115
9 m	0.04496	0.05	0.035 / 0.065
12 m	0.01954	0.03	0.015 / 0.045
Barrier levels 1.97 / 2.04			
1 m	0.51154	0.515	0.5 / 0.53
3 m	0.10707	0.115	0.1 / 0.13
Barrier levels 1.98 / 2.03			
1 w	0.84506	0.85	0.825 / 0.875
1 m	0.24514	0.245	0.23 / 0.26

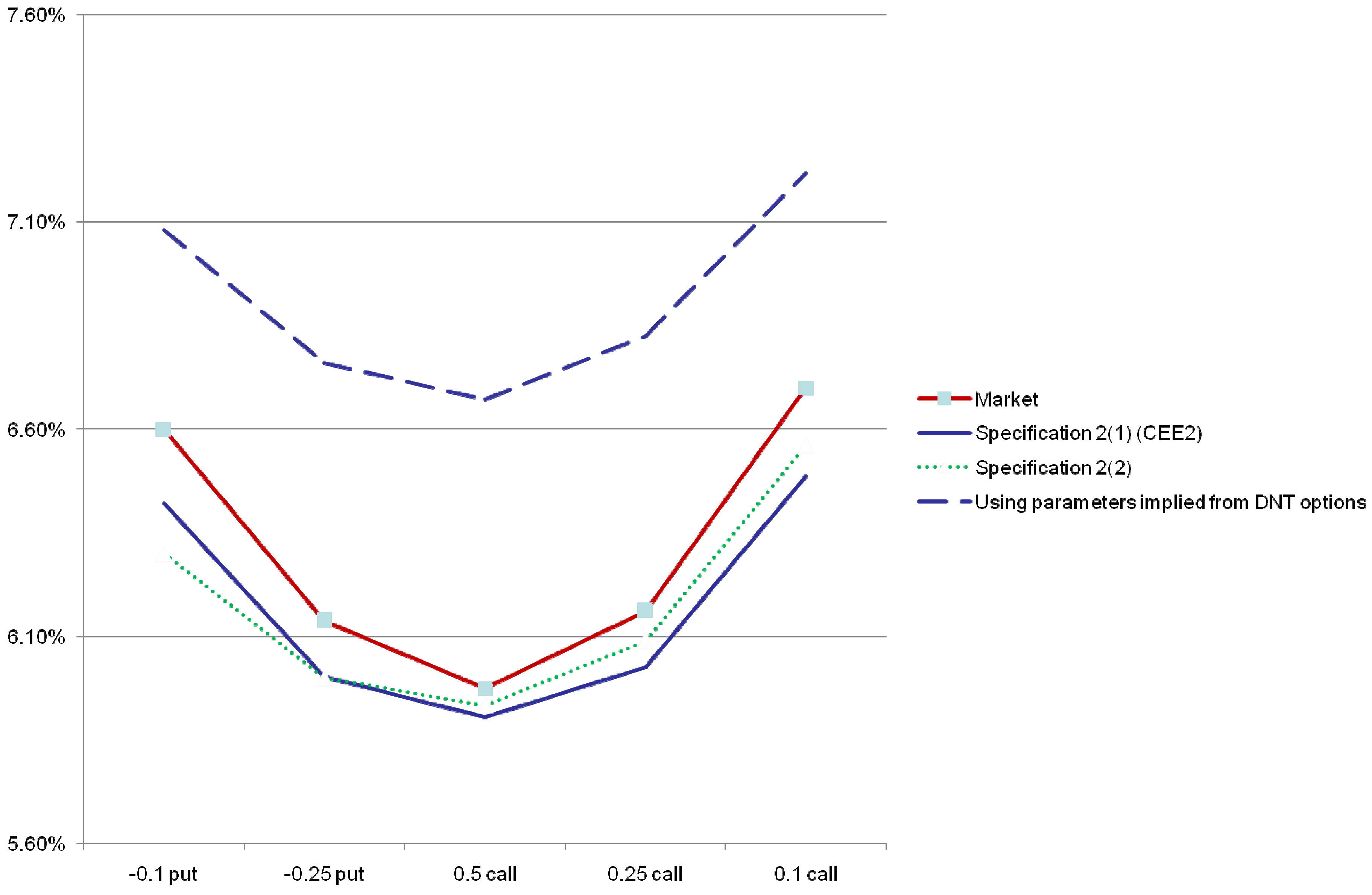
Calibration to market vanilla prices (6 months)



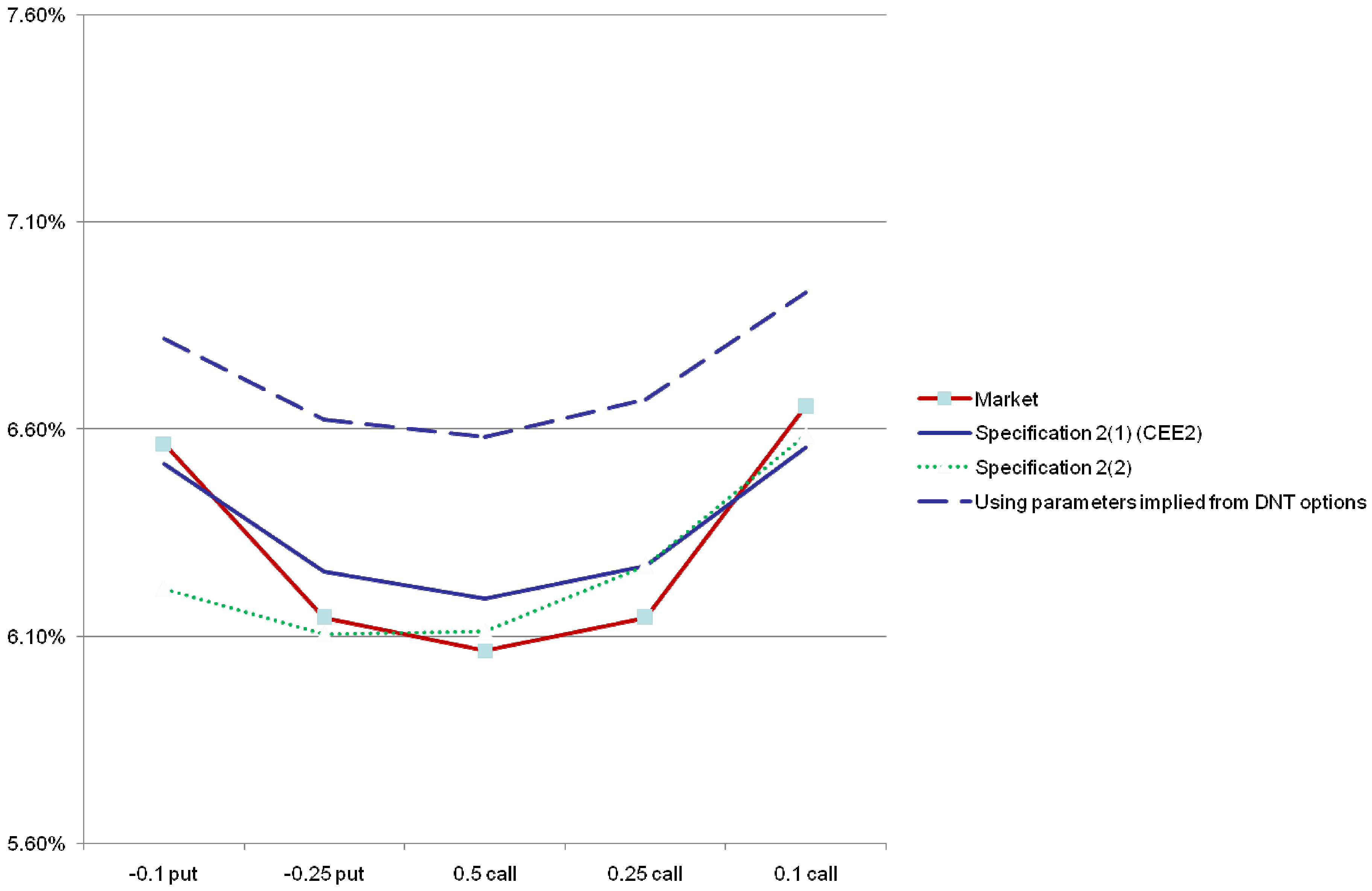
Calibration to market vanilla prices (9 months)



Calibration to market vanilla prices (12 months)



Calibration to market vanilla prices (2 years)



- When we use a CEE2 process, with the parameters the same for both before and after the first exit time from the corridor, but with the parameters obtained from the calibration to DNT prices, then we get a very poor fit to vanilla prices.
- What about the other way round? As an experiment, we fitted (a) a CEE2 process to the vanilla options (with the parameters the same for both before and after the first exit time from the corridor) and (b) a CGMY process to the vanilla options, then re-priced the DNT options.

Calibration to market DNT option prices

- Calibration to the market prices of the DNT options are the same in each case: Spot fx rate = 2.0060 , $L = 1.95$, $U = 2.05$

Maturity	Model price	Mid-market	Using parameters implied from vanillas (CEE2 process)	Using parameters implied from vanillas (CGMY)
Barrier levels 1.95 / 2.05				
1 m	0.75156	0.765	0.87750	0.8755
2 m	0.48389	0.5	0.68817	0.7266
3 m	0.31510	0.325	0.51895	0.5814
4 m	0.21993	0.22	0.40033	0.4687
5 m	0.15559	0.15	0.30634	0.3726
6 m	0.11013	0.1	0.23070	0.2908
9 m	0.04496	0.05	0.10390	0.1425
12 m	0.01954	0.03	0.04715	0.0695
Barrier levels 1.97 / 2.04				
1 m	0.51154	0.515	0.74133	0.7875
3 m	0.10707	0.115	0.26720	0.3815
Barrier levels 1.98 / 2.03				
1 w	0.84506	0.85	0.94998	0.9340
1 m	0.24514	0.245	0.50202	0.6644

- Conclusion: The CEE2 process (with no change in the dynamics at the first exit time from the corridor), the Kou (2002) DEJD model and the CGMY model (and possibly lots of other models) can’t simultaneously be calibrated to the market prices of both barrier and vanilla fx options with any great degree of success.

- When we allow the parameters (or the stochastic process) to change at the first exit time from the corridor, as we have done, then we get a very good fit to both barrier and vanilla fx options.
- This may support our assumption that the risk-neutral dynamics of the spot fx rate change at the first exit time from the corridor (or at least that traders, (perhaps unknowingly or based on heuristics) price fx options as if the risk-neutral dynamics do change).

The second key assumption revisited

Possible explanations for why the risk-neutral dynamics of the spot fx rate do change.

(a) Jumps (observed in the real-world physical measure P but they may also be present in the risk-neutral measure Q) due to traders rebalancing delta-hedges in large volumes.

(b) Risk-reversals change magnitude and sign.

- dQ/dP may change even if P does not because:

(c) Different degree of risk-aversion.

(d) Change in Arrow-Debreu state space.

Conclusions

- Our model is a very flexible framework. Our CEE2 process can approximate a time-changed Levy process for any Levy process with a monotone Levy density eg CGMY, Generalised Hyperbolic, NIG.
- Our CEE2 process can capture features such as jumps and stochastic vol (we can also capture stochastic skew).
- Main assumption is that the risk-neutral dynamics change when the spot fx rate first exits from the corridor.
- With this assumption, we can get a very good fit to both barrier and vanilla fx options.
- Have demonstrated that (at least within the Kou (2002) DEJD model or with our CEE2 process), there is evidence that the risk-neutral dynamics DO change when the spot fx rate first exits from the corridor.

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- We have put the paper we have written on the internet, see:

  www.john-crosby.co.uk