On the Hughes' model for pedestrian flow: The one-dimensional case

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Abstract

In this paper we investigate the mathematical theory of Hughes' model for the flow of pedestrians (cf. [17]), consisting of a nonlinear conservation law for the density of pedestrians coupled with an eikonal equation for a potential modelling the common sense of the task. For such an approximated system we prove existence and uniqueness of entropy solutions (in one space dimension) in the sense of Kružkov [22], in which the boundary conditions are posed following the approach of Bardos et al. [7]. We use BV estimates on the density ρ and stability estimates on the potential ϕ in order to prove uniqueness. Furthermore, we analyse the evolution of characteristics for the original Hughes' model in one space dimension and study the behaviour of simple solutions, in order to reproduce interesting phenomena related to the formation of shocks and rarefaction waves. The characteristic calculus is supported by numerical simulations.

Keywords: Pedestrian flow; Scalar conservation laws; Eikonal equation; Elliptic coupling; Entropy solutions; Characteristics.

1. Introduction

The mathematical modelling of large human crowds has gained a lot of scientific interest in the last decades. This is due to various reasons. First of all, a very serious issue in this context is to shed a light on the dynamics in critical circumstances. A well known practical example is the Jamarat Bridge in Saudi Arabia: the huge number of pilgrims cramming the bridge on occasion of the pilgrimage to Mecca gave rise to serious pedestrian disasters in the nineties [15]. Moreover, the analytical and numerical study of the qualitative behaviour of human individuals in a crowd with high densities can improve traditional socio–biological investigation methods. The dynamics of a human crowd has also applications in structural engineering and architecture: the London Millennium Footbridge which had to be closed on the day of its opening due to unexpected anomalous synchronization, is a very evocative example in this sense. Other applications of pedestrian flow modelling arise in transport systems, spectator occasions, political demonstrations, panic situations such as earthquakes and fire escapes. More light-hearted examples are the simulation of pedestrian movement in computer games and animated movies, see [38].

Several models for the movement of crowds have been proposed in the past. One can distinguish between two general approaches: microscopic and macroscopic models. In the microscopic framework, people are treated as individual entities (particles). The evolution of the particles in time is determined by physical and social laws which describe the interaction among the particles as well as their interactions with the physical surrounding. Examples for microscopic methods are social-force models (see [14] and the references therein), cellular automata, e.g [12, 30], queuing models e.g. [40] or continuum dynamic approaches like [38]. For an extensive review on different microscopic approaches we refer to [13]. Note that the microscopic approach in [38] uses the eikonal equation to compute the pedestrians' optimal path. This is a common feature with the model we will analyse in this paper.

In contrast to microscopic models, macroscopic models treat the whole crowd as an entity without considering the movement of single individuals. Classical approaches use well known concepts from fluid and gas dynamics, see [16]. More recent models are based on optimal transportation methods [29], mean field games (see [24] for a general introduction) or non-linear conservation laws [8]. In [32], an approach based on time-evolving measures is presented. We finally note that crowd motion models share many features with traffic models [1].

In this paper we shall analyse a model introduced by R. L. Hughes in 2002 [17]. Hughes' model treats the crowds as a "thinking" fluid and has been applied to diverse scenarios like the Battle of Agincourt and the annual Muslim Hajji [18]. It is given by

$$\rho_t - \operatorname{div}(\rho f^2(\rho)\nabla\phi) = 0 \tag{1a}$$

$$|\nabla\phi| = \frac{1}{f(\rho)} \tag{1b}$$

Here x denotes the position variable with $x \in \Omega$, a bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega$, $t \geq 0$ is time and $\rho = \rho(x,t)$ is the crowd density. The function $f(\rho)$ is given by $f(\rho) = 1 - \rho$, modelling the existence of a maximal density of individuals which can be normalized to 1 by a simple scaling. System

(1) is supplemented with the following boundary conditions for ϕ

$$\phi(x,t) = 0, \qquad x \in \partial\Omega,\ t \geq 0 \tag{2}$$

and the initial condition

$$\rho(x,0) = \rho_I(x) \geq 0. \tag{3}$$

We shall be more precise about the boundary conditions for ρ and give a more detailed interpretation of the model in the next section.

Note that if the term $\frac{1}{f(\rho)}$ in (1b) is replaced by 1, the system decouples and (1a) reduces to a non-linear conservation law with discontinuous flux. This type of equation has been analysed and simulated in [19, 20]. Even though Hughes' system (1) shares some features with this class of equations it is methodologically much more challenging. This is due to the non-linearity of the eikonal equation (1b) as well as the implicit time dependence of the potential $\nabla\phi$ in (1a). In fact, for the unique viscosity solution ϕ of the eikonal equation, no more regularity than Lipschitz continuity can be expected. In this paper we present an existence and uniqueness theory for a regularized version of (1) in one space dimension. Additionally, we discuss the behaviour of simple solution for the original system (1) and validate these results numerically.

Numerical simulations are already available in literature, see Ling et al. [27]. Their approach does not cover the case of discontinuous flux inside the computational domain. Nevertheless we follow the iterative procedure presented in [27], i.e. first solve the eikonal equation (1b) then the conservation law (1a). Numerical methods for non-linear conservation laws with discontinuous flux can be found in literature, e.g. [37]. We will use the approach presented by J. Towers for our numerical simulations. Note that equation (1a) is similar to the Lighthill-Witham-Richards traffic flow model [26, 34], and similar numerical schemes can be used. Various approaches can be found in the literature, e.g. [6, 5, 41, 42]. These schemes are usually based on numerical methods for non-linear conservation laws, for a general introduction we refer to [25, 36] and the references therein.

This paper is organized as follows: In the remaining part of the introduction, we shall explain the model in more detail (subsec. 1.1), present regularized versions (subsec. 1.2) and state our main results (subsec. 1.3). In sec. 2, we prove existence and uniqueness of entropy solutions for a regularized model and in sec. 3 we will analyse some special cases for the non regularized problem and compare the results with our numerical simulations.

1.1. Hughes' model

We start with a brief motivation of Hughes' model (1) (for further details see [17]). The density of individuals $\rho = \rho(x,t)$ satisfies the continuity equation

$$\rho_t + \operatorname{div}(\rho V) = 0, \tag{4}$$

and we use the following 'polar decomposition' notation for the velocity field $V(x,t)$

$$V(x,t) = |V(x,t)|Z(x,t), \quad |Z(x,t)| = 1. \tag{5}$$

In order to prescribe a logistic dependency of $|V|$ with respect to ρ we choose the classical linear expression

$$|V(x,t)| = 1 - \rho.$$

As for the directional unit vector $Z(x,t)$, we assume it to be parallel to the gradient of the potential $\phi(x,t)$. Such potential is determined by solving the eikonal equation in (1). The potential ϕ rules the common sense of the task (the task is represented by the boundary $\partial\Omega$). More precisely, the pedestrians tend to minimize their estimated travel time to the target. In a very naive way, this could be modelled by prescribing the eikonal equation

$$|\nabla\phi| = 1, \qquad \phi|_{\partial\Omega} = 0,$$

which has the unique semi-concave solution $\phi(x) = \operatorname{dist}(x,\partial\Omega)$ at least in the case of a convex domain Ω. However, it is reasonable to assume that individuals temper their estimated travel time by avoiding extremely high densities, i. e.

$$|\nabla\phi| = \frac{1}{1-\rho}, \qquad \phi|_{\partial\Omega} = 0, \tag{6}$$

which implies a 'density driven' rearrangement of the level sets of ϕ. This leads to $Z(x,t) = \frac{\nabla\phi(x,t)}{|\nabla\phi(x,t)|} = (1-\rho)\nabla\phi$ and therefore the continuity equation in (1) is justified.

1.2. An attempt to a mathematical theory: approximations

A successful attempt to develop a mathematical theory for the model (1) has never been carried out so far. The non-linearity with respect to ρ in the continuity equation forces using the notion of entropy solution for scalar conservation laws, as it is well known that weak L^∞ solutions to such kind of equations are in general not unique. On the other hand, the vector field $\nabla\phi$ may clearly develop discontinuities in subsets of Ω which may vary in time.

In general, the subsets of discontinuity of $\nabla\phi$ depend on ρ non-linearly and non–locally. This may be seen by simple examples in one space dimension. Moreover, the presence of the term $1-\rho$ in the right-hand-side of the eikonal equation renders the problems even more difficult, because of the possible blow–up of $|\nabla\phi|$ as ρ approaches the over-crowding density $\rho = 1$.

A full understanding of the model is highly non-trivial, even in one space dimension, where the model can be decoupled by solving the eikonal equation by integration.

In order to overcome such difficulties, we propose reasonable approximations to the Hughes' model (1), basically consisting of a regularization of the potential to avoid the discontinuity of $|\nabla\phi|$. At a first glance, a very natural way to approximate the equation for the potential would be simply adding a small 'viscosity', i. e.

$$-\delta\Delta\phi + |\nabla\phi|^2 = \frac{1}{f(\rho)^2}, \qquad \delta > 0.$$

Such an approximation still has the drawback of (possibly) producing a blow up of the right hand side when the density approaches the overcrowding value $\rho = 1$. This problem can be bypassed considering instead

$$-\delta\Delta\phi + f(\rho)^2|\nabla\phi|^2 = 1, \qquad \delta > 0. \tag{7}$$

On the other hand, the development of a satisfactory existence and uniqueness theory by using the coupling (7) is seriously complicated by the presence of the density dependent coefficient multiplying the Hamilton-Jacobi term $|\nabla\phi|^2$.

The model for which we shall develop a full existence and uniqueness theory uses the following elliptic regularization of the eikonal equation in (1), namely

$$-\delta_1\Delta\phi + |\nabla\phi|^2 = \frac{1}{(f(\rho)+\delta_2)^2}, \quad \delta_1, \delta_2 > 0. \tag{8}$$

The sign in front of δ_1 (δ in the alternative equation (7)) is chosen such that we would recover the unique viscosity solution in a possible limit $\delta_1 \to 0$. The second order term in (8) is meant to smooth the potential ϕ in order to avoid discontinuities for $|\nabla\phi|$. The elliptic operator in (8) is a classical elliptic Hamilton-Jacobi operator, and it is therefore easier to deal with if compared to the one in (7). On the other hand equation (8) contains one further approximation on the right-hand-side which can be motivated as follows.

Without the elliptic regularization, the potential ϕ in (8) would satisfy

$$|\nabla\phi| = \frac{1}{(1-\rho+\delta_2)} \tag{9}$$

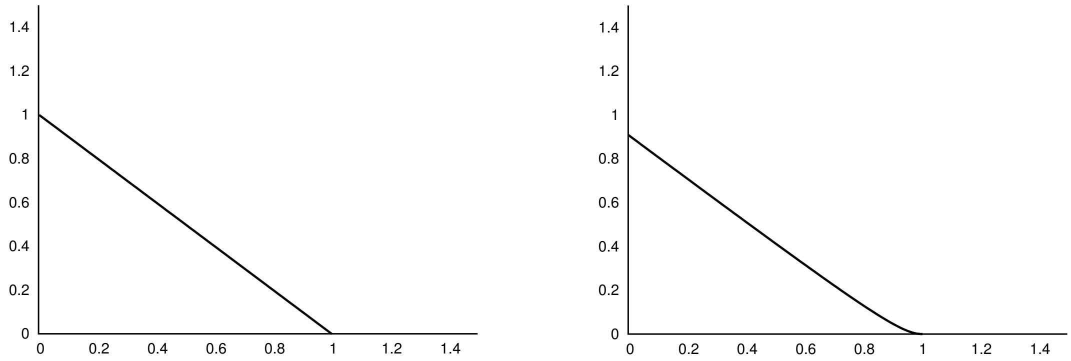


Figure 1: Comparison between the scalar ‘logistic’ speed $|V|$ of the pedestrian in Hughes’ model (1) (left) and the model with elliptic coupling (8)

Then, the polar decomposition of the velocity field introduced in (4) reads in this case

$$\begin{aligned} &V = |V|Z, \quad |Z| = 1 \\ &|V| = f(\rho)^2|\nabla\phi| = \frac{f(\rho)^2}{\delta_2 + f(\rho)} = \frac{(1-\rho)^2}{\delta_2 + (1-\rho)}, \quad Z = \frac{\nabla\phi}{|\nabla\phi|}. \end{aligned} \tag{10}$$

The profile of $|V|$ as a function of ρ in (10) has essentially the same properties of the logistic function $|V|(\rho) = 1 - \rho$ of the original Hughes’s model, except that the vacuum at $\rho = 1$ is achieved with a zero derivative and the maximal velocity is slightly penalized, i. e. $|V|_{max} = 1/1 + \delta_2$ instead of $|V|_{max} = 1$ of the original model (cf. Figure 1).

As for the unit vector Z, which is parallel to $\nabla\phi$, the only difference with the original model is that individuals ‘sense’ the target as the density reaches the maximum value $\rho = 1$. In this case $|\nabla\phi| = 1/\delta_2$, i. e. the slope of $\nabla\phi$ is very high in absolute value (δ_2 is thought as a small parameter), but not infinite as in the original model. On the other hand, when $\rho = 1$, $|V|$ vanishes, and therefore the above mentioned difference is not effective (individuals do not move at all when $\rho = 1$!).

1.3. Results

We shall first cover the one dimensional existence and uniqueness theory for the regularized model with elliptic coupling (8) introduced in the previous subsection, more precisely we shall study the model system

$$\begin{cases} \rho_t - (\rho f^2(\rho)\phi_x)_x = 0 \\ -\delta_1\phi_{xx} + |\phi_x|^2 = \dfrac{1}{(f(\rho)+\delta_2)^2}. \end{cases} \tag{11}$$

As the continuity equation in (11) features non-linear convection, we shall address the existence and uniqueness theory in the framework of weak entropy solutions, cf. for instance [22]. The results are contained in Section 2. More precisely, the notion of solution is stated in Definition 2.1, the existence result is provided in Theorem 2.10, and the uniqueness result is proven in Theorem 2.12.

The problem (11) is posed on the bounded interval $x \in [-1, 1]$ with homogeneous Dirichlet boundary conditions. We shall follow the approach by Bardos et al. [7] (see also [9, 2, 28]) to recover suitable boundary conditions for a scalar conservation law. This aspect is explained at the beginning of the next section.

2. The regularized model: existence and uniqueness theory

In this section we establish our existence and uniqueness results for the regularized Hughes' model system (11) with $f(\rho) = (1 - \rho)$. For future use we denote

$$g(\rho) := \rho f(\rho)^2.$$

System (11) is coupled with the initial condition

$$\rho(x, 0) = \rho_I(x) \geq 0, \tag{12}$$

and with the Dirichlet boundary conditions

$$\min_{k \in [0, \mathrm{tr}\, \rho]} \{g(\mathrm{tr}\,(\rho)) - g(k)\} = 0, \tag{13}$$

$$\phi(\pm 1, t) = 0. \tag{14}$$

Here $\mathrm{tr}\, \rho$ denotes the trace of ρ on the boundary. More precisely,

$$\mathrm{tr}\, \rho(-1, t) = \lim_{x \to -1^+} \rho(x, t), \qquad \mathrm{tr}\, \rho(1, t) = \lim_{x \to 1^-} \rho(x, t).$$

It was originally proven in [7] that (13) is the correct way to pose Dirichlet boundary conditions for a scalar conservation law, mainly for two reasons: first, (13) comes as a natural condition from the vanishing viscosity limit of solutions with zero Dirichlet boundary data; second, (13) encloses the natural interplay between the boundary datum and the value of the solution which is transported via characteristics in the linear case (the boundary datum needs to be posed only if characteristics at the boundary are directed towards the interior of the domain). The boundary condition provided in [7] assumes the simplified form (13) since we shall deal with non-negative solutions and due to a trivial monotonicity property of the potential ϕ (cf. Lemma 2.11 below). We remark here that the boundary condition (13) reduces to

$$g(\mathrm{tr}\, \rho) \geq g(k) \qquad \text{on} \quad x = \pm 1, \quad \text{for all } k \in [0, \mathrm{tr}\, \rho],$$

which expresses the fact that the allowed densities on the boundary are those for which the function g is *non-decreasing*. A deeper understanding of the boundary conditions for nonlinear conservation laws in one space dimension can be also found in [9].

We shall prove that the system (11) has a unique solution (ρ, ϕ) in a sense made precise by the following definition. For the density component ρ we will use the classical notion of entropy solutions originally introduced by Kružkov in [22] and adapted to boundary value problems by Bardos et al. in [7].

Definition 2.1 (Entropy Solution). *Let $\rho_I \in BV([-1,1])$. A couple (ρ, ϕ) is a weak* entropy *solution to the system* (11) *if*

- $\rho \in BV([-1,1] \times [0,T)) \cap L^\infty([-1,1] \times [0,T))$
- $\phi \in W^{2,\infty}[-1,1]$
- *ρ and ϕ satisfy the inequality*

$$\iint\limits_{\Omega_T} |\rho - k|\psi_t \, dxdt + \int_{-\infty}^{\infty} \rho_I \psi_0 dx - \iint\limits_{\Omega_T} \operatorname{sgn}(\rho - k)[g(\rho) - g(k)]\psi_x \phi_x \, dxdt$$
$$+ \iint\limits_{\Omega_T} \operatorname{sgn}(\rho - k) g(k) \psi \phi_{xx} dxdt - \operatorname{sgn}(k) \int_0^T [g(\operatorname{tr}\rho) - g(k)]\phi_x \psi|_{x=\pm 1} dt \geq 0, \tag{15}$$

  for every Lipschitz continuous test function ψ on $[-1,1] \times [0,T)$ having compact support.

- *ϕ and ρ satisfy the second equation in* (11) *almost everywhere in x and t.*

As usual in the context of conservation laws, we shall approximate the targeted model (11) via a vanishing viscosity approach, namely we shall work on the system

$$\rho_t - (\rho f^2(\rho)\phi_x)_x = \varepsilon \rho_{xx} \tag{16a}$$

$$-\delta_1 \phi_{xx} + |\phi_x|^2 = \frac{1}{(f(\rho) + \delta_2)^2}, \tag{16b}$$

for a small $\varepsilon > 0$. System (16) is coupled with homogeneous boundary condition

$$\rho(x,t)|_{x=\pm 1} = 0 \qquad \phi(x,t)|_{x=\pm 1} = 0,$$

and the initial condition

$$\rho(x,0) = \rho_I(x).$$

Existence of unique (smooth) solutions to the above regularized problem follow from standard results. For the elliptic coupling see e.g. [23, Chapter 3, Lemma 1.1] and [23, Chapter 3, Thm. 1.2]. For the parabolic approximation we refer to [39, Section 5, Thm. 5.3 and Thm. 5.4]. The proof of this theorem is based on semi group techniques. The strategy is to first linearise the equation to an evolution equation with a linear but time depending operator. Under the given assumptions, it is known that there exists a solution to such an equation (see e.g. [35]). Then, the solution to the non-linear equation in obtained using a fixed-point argument.

In the next subsections we shall first derive suitable a-priori estimates on ϕ and ρ, then we shall recall our notion of entropy solution, and finally prove existence and uniqueness of the limit as $\varepsilon \to 0$.

2.1. A Priori Estimates on ϕ and ρ

We shall now derive some a-priori estimates for the elliptic coupling, i.e.

$$\begin{aligned} &-\delta_1 \phi_{xx} + \phi_x^2 = F_{\delta_2}(\rho) := \frac{1}{(\delta_2 + f(\rho))^2} \\ &\phi(\pm 1) = 0. \end{aligned} \tag{17}$$

Our strategy is the following: we shall first replace the term $f(\rho)$ by

$$\widetilde{f}(\rho) := \begin{cases} f(\rho) & \text{if} \quad \rho \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

in order to have the right-hand side $F_{\delta_2}(\rho)$ uniformly bounded and non-degenerate. The result is stated in the Lemmas 2.2 and 2.3. Then, we use the estimates on the elliptic coupling in order to prove that the density ρ satisfies $\rho \in [0,1]$, see Lemma 2.4. Since the solution to the $\widetilde{f}$-modified system coincides with the one to (11), by uniqueness of smooth solutions to the regularized problem (11) we conclude that the estimates for ρ and ϕ hold without replacing f by $\widetilde{f}$. In order to simplify the notation, we shall drop the tilde symbol above f.

Let us introduce the *Hopf–Cole* transformation

$$\psi(x,t) := e^{-\frac{\phi(x,t)}{\delta_1}}, \tag{18}$$

which implies

$$\psi_x = -\frac{\psi\phi_x}{\delta_1}, \quad \psi_t = -\frac{\psi\phi_t}{\delta_1}, \quad \phi_x = -\delta_1\frac{\psi_x}{\psi}, \quad \phi_t = -\delta_1\frac{\psi_t}{\psi} \tag{19}$$

$$\psi_{xx} = -\frac{\phi_{xx}\psi}{\delta_1} - \frac{\phi_x\psi_x}{\delta_1} = \frac{\psi}{\delta_1^2}\left(-\delta_1\phi_{xx} + \phi_x^2\right) = \frac{\psi}{\delta_1^2}F_{\delta_2}(\rho). \tag{20}$$

Therefore, ψ satisfies

$$\begin{cases} \delta_1^2 \psi_{xx} = \psi F_{\delta_2}(\rho) \\ \psi(\pm 1) = 1. \end{cases} \tag{21}$$

As a first estimate, we prove that ψ is uniformly bounded in H^1 and in L^∞.

Lemma 2.2. *There exists a constant $C > 0$ depending only on δ_1 and δ_2 such that*

$$\|\psi\|_{H^1([-1,1])} \leq C, \qquad \|\psi\|_{L^\infty([-1,1])} \leq C, \qquad \|\psi_{xx}\|_{L^\infty([-1,1])} \leq C. \tag{22}$$

Proof. Let us introduce the variable

$$\widetilde{\psi} := \psi - 1,$$

which satisfies

$$\begin{cases} \delta_1^2 \widetilde{\psi}_{xx} = \widetilde{\psi} F_{\delta_2}(\rho) + F_{\delta_2}(\rho) \\ \widetilde{\psi}(\pm 1) = 0. \end{cases} \tag{23}$$

Multiplication of (23) by $\widetilde{\psi}$ and integration over $[-1,1]$ leads to (after integration by parts)

$$-\delta_1^2 \int \widetilde{\psi}_x^2 dx = \int \widetilde{\psi}^2 F_{\delta_2}(\rho) dx + \int \widetilde{\psi} F_{\delta_2}(\rho) dx.$$

Since

$$\frac{1}{(1+\delta_2)^2} \leq F_{\delta_2}(\rho) \leq \frac{1}{\delta_2^2}, \tag{24}$$

by a trivial use of Young's inequality we get

$$\int \widetilde{\psi}_x^2 dx + \int \widetilde{\psi}^2 dx \leq C,$$

for a constant C depending on δ_1 and δ_2. Sobolev's inequality then implies

$$\|\psi\|_{L^\infty} \leq C.$$

The last assertion in (22) follows by the equation (21). $\square$

Next we prove that ψ is non-negative on $[-1,1]$ and uniformly bounded from below by a positive constant, which implies the desired estimates on the ϕ variable.

Lemma 2.3. *There exists a constant $C > 0$ such that*

$$\psi(x,t) \geq C \tag{25}$$

for all $(x,t) \in [-1,1] \times [0,+\infty)$. Moreover,

$$\|\phi\|_{H^1([-1,1])} \leq C, \qquad \|\phi\|_{L^\infty([-1,1])} \leq C, \qquad \|\phi_{xx}\|_{L^\infty([-1,1])} \leq C. \tag{26}$$

Proof. Let us consider the original equation (17) satisfied by ϕ. We have

$$\delta_1\phi_{xx} + \frac{1}{\delta_2^2} \geq \delta_1\phi_{xx} + F_{\delta_2}(\rho) = \phi_x^2 \geq 0,$$

which can be written as

$$\left(\delta_1\phi + \frac{|x|^2}{2\delta_2}\right)_{xx} \geq 0.$$

Therefore the function $\delta_1\phi + \frac{|x|^2}{2\delta_2}$ attains its maximum at the boundary, ϕ is bounded from above and $\psi = e^{-\phi/\delta_1}$ is bounded away from zero. The statements (26) follow as a consequence of (25) and of (19)-(20). □

We conclude by proving that ρ is always bounded above by the maximal density $\rho = 1$.

Lemma 2.4 (Boundedness of ρ)**.** *Assume that $\rho_I \leq 1$. Then the solution to* (16a) *with $f(\rho) = (1-\rho)$ satisfies $\rho(x,t) \leq 1$ for all $(x,t) \in [-1,1] \times [0,+\infty)$.*

Proof. We first define the function

$$\eta(\rho) = \begin{cases} 0 & \rho \leq 0, \\ \frac{\rho^2}{4\gamma} & 0 < \rho \leq 2\gamma, \\ \rho - \gamma & \rho > 2\gamma. \end{cases} \tag{27}$$

and use it to approximate $(\rho-1)_+$ (the positive part of $(\rho - 1)$). Here $\gamma > 0$ is a small parameter. Our goal is to show that this positive part, being zero at $t = 0$, does not increase. We consider

$$\begin{aligned}
\frac{d}{dt}\int \eta(\rho-1)\,dx &= \int \eta'(\rho-1)(\varepsilon\rho_x + (\rho(1-\rho)^2\phi_x)_x\,dx \\
&= -\varepsilon\int \eta''(\rho-1)\rho_x^2\,dx + \varepsilon\eta'(\rho-1)\rho_x|_{x=\pm1} \\
&\quad - \int_{0\leq(\rho-1)\leq\gamma} \eta''(\rho-1)\rho(1-\rho)^2\rho_x\phi_x\,dx + \eta'(\rho-1)\rho(1-\rho)\phi_x|_{x=\pm1} \\
&\leq -2\varepsilon\int \eta''(\rho-1)\rho_x^2\,dx - C_\varepsilon\int_{0\leq(\rho-1)\leq\gamma} \eta''(\rho-1)\rho^2(1-\rho)^4|\phi_x|^2\,dx \\
&\leq -2\varepsilon\int \eta''(\rho-1)\rho_x^2\,dx - C_{\varepsilon,\delta}\gamma^3(1+\gamma)^2,
\end{aligned}$$

where $C_{\varepsilon,\delta}$ depends on ε, δ_1 and δ_2. Here, we employed Young's inequality and the Dirichlet boundary conditions. Furthermore we used the ε-independent L^∞ bound on ϕ_x we obtained in Lemma 2.3. Letting $\gamma \to 0$, we infer

$$\frac{d}{dt}\int (\rho-1)_+\,dx = -\varepsilon\int \eta''(\rho-1)|\rho_x|^2\,dx \leq 0,$$

and thus the integral is decreasing in time. As $(\rho-1)_+$ is a positive function and zero at $t=0$, we conclude that is stays zero for all times and thus that ρ is always bounded by 1. $\square$

Note that using the same technique, but approximating the negative part of ρ we also obtain that the solution is almost everywhere non-negative (since $\rho_I \geq 0$).

2.2. BV estimate on ρ

We are now ready to prove the crucial BV estimate on ρ which serves as a tool to get compactness in the limit as $\varepsilon \to 0$. Furthermore, it will guarantee the existence of $\operatorname{tr}\rho$, see [7, Lemma 1]. Let us start with estimating the L^1 norm of ρ_x.

Lemma 2.5. *Suppose $\rho_I \in W^{1,1}([-1,1])$. Then, there exists a constant $C>0$ independent on ε such that*

$$\|\rho_x(t)\|_{L^1(\Omega)} \leq (\|(\rho_I)_x\|_{L^1(\Omega)} + C)e^{Ct}$$

for all $t \geq 0$.

Before we start the proof let us define an approximation $\eta_\gamma(z)$ of the function $|z|$ as $\gamma \to 0$ such that

$$\begin{aligned}
&\eta_\gamma(z) \to |z|, \quad \eta_\gamma'(z) \to \operatorname{sign}(z), \quad \eta_\gamma'(z)z \to |z| \qquad \text{as } \ \gamma \to 0\\
&\eta_\gamma'(z)z \geq 0, \quad \eta_\gamma''(z) \geq 0\\
&\eta_\gamma''(z) \leq \mathbf{1}_{[-\gamma,\gamma]}(z)\frac{C}{\gamma}
\end{aligned} \tag{28}$$

for some constant $C>0$.

Remark 2.6 (Properties of η)**.** *We remark that the definition of η implies the following properties, which shall be often used in the sequel:*

- *All integrals of the form, with $f \in L^2(\Omega)$,*

$$\int_\Omega \eta_\gamma''(f(x))f(x)^2\,dx \leq \frac{C}{\gamma}\int_{|f(x)|\leq\gamma} f(x)^2\,dx \leq C\gamma|\Omega|$$

  tend to zero as $\gamma \to 0$.

- *Furthermore, with $g \in \mathcal{C}^1(\mathbb{R}_+)$, $f,h \in L^1(\Omega)$, $k \in \mathbb{R} > 0$ we have*

$$\begin{aligned}
&\int_\Omega \eta_\gamma''(f(x)-k)(g(f(x))-g(k))h(x)\,dx\\
&\leq \frac{C}{\gamma}\int_{0<|f(x)-k|\leq\gamma} \|g'\|_{L^\infty(\mathbb{R}_+)}|f(x)-k||h(x)|\,dx\\
&\leq \frac{C}{\gamma}\gamma\|g'\|_{L^\infty(\mathbb{R}_+)}\int_{0<|f(x)-k|\leq\gamma}|h(x)|\,dx \to 0,
\end{aligned}$$

as $\gamma \to 0$.

Proof of Lemma 2.5. We deduce that

$$\frac{d}{dt}\int \eta_\gamma(\rho_x)dx = \int \eta'_\gamma(\rho_x)\rho_{xt}dx = \int \eta'_\gamma(\rho_x)(g(\rho)\phi_x)_{xx}dx + \varepsilon \int \eta'_\gamma(\rho_x)\rho_{xxx}dx$$
$$= \int \eta'_\gamma(\rho_x)(g'(\rho)\rho_x\phi_x)_x + \int \eta'_\gamma(\rho_x)(g(\rho)\phi_{xx})_x - \varepsilon \int \eta''_\gamma(\rho_x)\rho_{xx}^2 dx$$
$$= -\int \eta''_\gamma(\rho_x)\rho_{xx}g'(\rho)\rho_x\phi_x dx + \int \eta'_\gamma(\rho_x)g'(\rho)\rho_x\phi_{xx}dx$$
$$+ \int \eta'_\gamma(\rho_x)g(\rho)\phi_{xxx}dx - \varepsilon \int \eta''_\gamma(\rho_x)\rho_{xx}^2 dx$$
$$\leq -\frac{\varepsilon}{2}\int \eta''_\gamma(\rho_x)\rho_{xx}^2 dx + C(\varepsilon)\int \eta''_\gamma(\rho_x)\phi_x^2\rho_x^2 dx + C\int |\rho_x|dx + C. \tag{29}$$

Here the last step is justified by the identities (19) and (20), by (25), and by

$$\|\psi_{xxx}(t)\|_{L^1(\Omega)} \leq C\|\rho_x(t)\|_{L^1(\Omega)} + C, \text{ since } \psi_{xxx} = F_{\delta_2}(\rho)\psi_x + \psi F'_{\delta_2}(\rho)\rho_x.$$

The sum of the boundary terms

$$\int \eta'_\gamma(\rho_x)(\varepsilon\rho_{xx} + g'(\rho)\rho_x\phi_x + g(\rho)\phi_{xx})\, d\sigma_x = \int \eta'_\gamma(\rho_x)\rho_t\, d\sigma_x$$

vanishes, as ρ_t is constant along the boundary. Due to Rem. 2.6, the second term on the right hand side of (29) vanishes as $\gamma \to 0$, therefore we obtain the desired assertion in the limit (after integration with respect to time). □

Before estimating the L^1 norm of ρ_t we have the following technical lemma.

Lemma 2.7. *There exists a constant $C > 0$ independent of ε and of t such that*

$$\|\psi_t(t)\|_{L^\infty(\Omega)} \leq C\|\rho_t(t)\|_{L^1(\Omega)} \tag{30}$$
$$\|\psi_{xxt}(t)\|_{L^1(\Omega)} \leq C\|\rho_t(t)\|_{L^1(\Omega)} \tag{31}$$
$$\|\psi_{xt}(t)\|_{L^\infty(\Omega)} \leq C\|\rho_t(t)\|_{L^1(\Omega)}. \tag{32}$$

Proof. We start with the proof of estimate (30). Differentiation of (21) with respect to time yields

$$\psi_{xxt} = \frac{1}{\delta_1^2}\left(\psi_t F_{\delta_2}(\rho) + \psi F'_{\delta_2}(\rho)\rho_t\right). \tag{33}$$

Next we multiply (33) by ψ_t and integrate over $[-1,1]$. Using the fact that $\psi_t = 0$ at the boundary, we integrate by parts to obtain

$$-\delta_1^2 \int \psi_{xt}^2 dx = \int F_{\delta_2}(\rho)\psi_t^2 dx + \int F'_{\delta_2}(\rho)\rho_t\psi\psi_t dx.$$

In view of (24) and Lemma 2.3 we can find a constant $C = C(\delta_1, \delta_2) > 0$ such that

$$\|\psi_t(t)\|^2_{H^1(\Omega)} \leq C\|\psi_t(t)\|_{L^\infty(\Omega)}\|\rho_t(t)\|_{L^1(\Omega)},$$

and the Sobolev inequality $\|\psi_t(t)\|_{L^\infty(\Omega)} \leq \|\psi_t(t)\|_{H^1(\Omega)}$ implies the assertion.

The inequality (31) follows by a direct use of the equation (21) and by (30). Finally, the last statement (32) follows from the inequality

$$\|\psi_{xt}(t)\|_{L^\infty(\Omega)} \leq \|\psi_{xxt}(t)\|_{L^1(\Omega)},$$

which is a consequence of the fact that $\int \psi_{xt} dx = \psi_t(1,t) - \psi_t(-1,t) = 0$ and that every $W^{1,1}$ function in one space dimension admits an absolutely continuous representative. □

We are now ready to estimate the L^1 norm of the time derivative.

Lemma 2.8. *Assuming $\rho_I \in W^{2,1}([-1,1])$ and $\varepsilon > 0$, there exists a constant $C > 0$ independent on ε such that*

$$\|\rho_t(t)\|_{L^1(\Omega)} \leq Ce^{Ct},$$

for all $t \geq 0$.

Proof. Again we consider the approximation η_γ of the absolute value, given by (28). We deduce that

$$\begin{aligned}
\frac{d}{dt}\int \eta_\gamma(\rho_t)dx &= \int \eta'_\gamma(\rho_t)\rho_{tt}dx = \int \eta'_\gamma(\rho_t)(g(\rho)\phi_x)_{tx}dx + \varepsilon\int \eta'_\gamma(\rho_t)\rho_{xxt}dx \\
&= \int \eta'_\gamma(\rho_t)(g'(\rho)\rho_t\phi_x)_x + \int \eta'_\gamma(\rho_t)(g(\rho)\phi_{xt})_x - \varepsilon\int \eta''_\gamma(\rho_t)\rho^2_{xt}dx \\
&= -\int \eta''_\gamma(\rho_t)\rho_{xt}g'(\rho)\rho_t\phi_x dx + \int \eta'_\gamma(\rho_t)g'(\rho)\rho_x\phi_{xt}dx \\
&\quad + \int \eta'_\gamma(\rho_t)g(\rho)\phi_{xxt}dx - \varepsilon\int \eta''_\gamma(\rho_t)\rho^2_{xt}dx \\
&\leq -\frac{\varepsilon}{2}\int \eta''_\gamma(\rho_t)\rho^2_{xt}dx + C(\varepsilon)\int \eta''_\gamma(\rho_t)\phi^2_x\rho^2_t dx \\
&\quad + C\|\phi_{xt}(t)\|_{L^\infty(\Omega)}\int |\rho_x|dx + C\int |\phi_{xxt}|dx.
\end{aligned}$$

All boundary terms in the above calculation are zero as ρ_t and thus $\eta'_\gamma(\rho_t)$ is zero on the boundary. The second term on the r.h.s. above vanishes as $\gamma \to 0$. As for the other terms, we can differentiate (18) to easily obtain

$$\|\phi_{xt}(t)\|_{L^\infty(\Omega)} \leq C\|\psi_{xt}(t)\|_{L^\infty(\Omega)} + C\|\psi_t(t)\|_{L^\infty(\Omega)} \leq C\|\rho_t(t)\|_{L^1(\Omega)}$$

and

$$\|\phi_{xxt}(t)\|_{L^1(\Omega)} \leq C\|\psi_{xxt}(t)\|_{L^1(\Omega)} + C\|\psi_{xt}(t)\|_{L^1(\Omega)} + C\|\psi_t(t)\|_{L^1(\Omega)} \leq C\|\rho_t(t)\|_{L^1(\Omega)}.$$

Therefore, integration with respect to time and Lemma 2.5 results in

$$\|\rho_t(t)\|_{L^1(\Omega)} \leq (\|(\rho(0)_t\|_{L^1(\Omega)} + C)e^{Ct},$$

for all $t \geq 0$. Using the fact that ρ_I is in $W^{2,1}(\Omega)$ and that ε is bounded, we can use equation (16a) to estimate

$$\|\rho_t(0)\|_{L^1(\Omega)} \leq \|g'(\rho_I)\phi_x(t)\|_{L^\infty(\Omega)}\|(\rho_I)_x\|_{L^1(\Omega)} + \varepsilon\|(\rho_I)_{xx}\|_{L^1(\Omega)}.$$

We thus conclude that $\|\rho_t(0)\|_{L^1(\Omega)}$ is bounded as well completing the proof. □

2.3. Stability estimates on ϕ

Next, we prove some stability estimates for the elliptic equation (17) with respect to the variable ρ. These estimates will be useful later on to prove uniqueness of an entropy solution ρ in the limit.

Given two densities ρ and $\bar{\rho}$, let ϕ and $\bar{\phi}$ solve

$$\begin{aligned} -\delta_1\phi_{xx} + \phi_x^2 &= F_{\delta_2}(\rho), \\ -\delta_1\bar{\phi}_{xx} + \bar{\phi}_x^2 &= F_{\delta_2}(\bar{\rho}), \end{aligned}$$

with boundary conditions $\phi(\pm 1) = \bar{\phi}(\pm 1) = 0$. For both solutions we consider the corresponding Hopf–Cole transformation

$$\psi(x,t) := e^{-\frac{\phi(x,t)}{\delta_1}} \qquad \bar{\psi}(x,t) := e^{-\frac{\bar{\phi}(x,t)}{\delta_1}}.$$

Then we can deduce the following lemma:

Lemma 2.9. *There exists a constant $C > 0$ independent on ε and on t such that*

$$\|\phi(t) - \bar{\phi}(t)\|_{L^1(\Omega)} \leq C\|\rho(t) - \bar{\rho}(t)\|_{L^1(\Omega)} \tag{34}$$

$$\|\phi_{xx}(t) - \bar{\phi}_{xx}(t)\|_{L^1(\Omega)} \leq C\|\rho(t) - \bar{\rho}(t)\|_{L^1(\Omega)} \tag{35}$$

$$\|\phi_x(t) - \bar{\phi}_x(t)\|_{L^\infty(\Omega)} \leq C\|\rho(t) - \bar{\rho}(t)\|_{L^1(\Omega)}. \tag{36}$$

Proof. Let us multiply equation

$$\delta_1^2(\psi_{xx} - \bar{\psi}_{xx}) = (\psi - \bar{\psi})F_{\delta_2}(\rho) + \bar{\psi}(F_{\delta_2}(\rho) - F_{\delta_2}(\bar{\rho})) \tag{37}$$

by $\eta'_\gamma(\psi - \bar{\psi})$, with η_γ given by (28) and integrate over $[-1,1]$. Integration by parts implies

$$\begin{aligned} &-\delta_1^2 \int (\psi_x - \bar{\psi}_x)^2 \eta''_\gamma(\psi - \bar{\psi})dx \\ &\qquad = \int (\psi - \bar{\psi})\eta'_\gamma(\psi - \bar{\psi})F_{\delta_2}(\rho)dx + \int \bar{\psi}\eta'_\gamma(\psi - \bar{\psi})[F_{\delta_2}(\rho) - F_{\delta_2}(\bar{\rho})]dx. \end{aligned}$$

We use the properties of η_γ and (24) to obtain, as $\gamma \to 0$

$$C(\delta) \int |\psi - \bar{\psi}| dx \leq \int F_\delta(\rho) |\psi - \bar{\psi}| dx \leq \int \bar{\psi} |F_\delta(\rho) - F_\delta(\bar{\rho})| dx \leq C \int |\rho - \bar{\rho}| dx.$$

Next we can deduce (34) by using the Hopf–Cole transformation as usual. To prove (35), multiply (37) by $\operatorname{sign}(\psi_{xx} - \bar{\psi}_{xx})$ and integrate over $[-1,1]$ to obtain

$$\delta_1^2 \int |\psi_{xx} - \bar{\psi}_{xx}| dx \leq C \int |\psi - \bar{\psi}| dx + C \int |\rho - \bar{\rho}| dx.$$

Next we obtain (35) by using (34) and passing to the variable ϕ. Inequality (36) follows by the Sobolev inequality as at the end of the proof of Lemma 2.7. $\square$

2.4. The limit as $\varepsilon \to 0$

Our next goal is to study the behaviour of the solution $(\rho^\varepsilon, \phi^\varepsilon)$ to the system (16) as the parameter ε tends to zero. Using Lemma 2.5 and Lemma 2.8 we know that ρ^ε is in the space of functions having bounded variation $BV(\Omega)$. Therefore, we can employ the classical Helly's theorem on strong L^1–compactness of functions with bounded BV–norm, cf. [11] for instance. Thus, ρ^ε has a strong limit in L^1 up to subsequences. As for the ϕ variable, since ρ_x is uniformly estimated in L^1, differentiating the elliptic equation with respect to x implies that ϕ^ε_{xxx} is uniformly bounded in L^1 and therefore ϕ^ε_{xx} is strongly compact in L^1. Denoting by (ρ, ϕ) the limit $\varepsilon \to 0$ of $(\rho^\varepsilon, \phi^\varepsilon)$, as the convergence is strong in L^1 and due to the estimates on ϕ proven in subsection 2.1, it is immediately clear that ϕ solves the second equation in (11) and ρ is a weak solution of

$$\rho_t - (\rho f^2(\rho) \phi_x)_x = 0. \tag{38}$$

In the remainder of this section, we will show that (ρ, ϕ) is in fact the unique entropy solution to the system (11) in the sense of Definition 2.1. First we shall state the existence theorem.

Theorem 2.10 (Existence of entropy solutions)**.** *There exists an entropy solution (ρ, ϕ) to system* (11) *with initial condition* (12) *and boundary conditions* (13)-(14) *in the sense of Definition 2.1. Such solution is the limit as $\varepsilon \to 0$ of the solution ρ^ε to* (16a)-(16b).

Proof. To recover the notion of entropy solutions, we consider again the regularized equation

$$\rho_t = (\rho f^2(\rho) \phi_x)_x + \varepsilon \rho_{xx}. \tag{39}$$

We multiply this equation by $\eta'(\rho-k)\psi$ (with η' defined in (28)) and integrate over $\Omega_T=[-1,1]\times[0,T]$

$$\iint\limits_{\Omega_T}\eta'(\rho-k)\rho_t\psi\;dxdt=\iint\limits_{\Omega_T}\eta'(\rho-k)(g(\rho)\phi_x)_x\psi\;dxdt$$
$$+\,\varepsilon\iint\limits_{\Omega_T}\eta'(\rho-k)\rho_{xx}\psi\;dxdt.$$

Adding

$$0=\iint\limits_{\Omega_T}\eta'(\rho-k)g(k)\phi_x\psi_x\;dxdt-\iint\limits_{\Omega_T}\eta'(\rho-k)g(k)\phi_x\psi_x\;dxdt$$

and integrating by parts leads to

$$\begin{aligned}
&\iint\limits_{\Omega_T}\eta'(\rho-k)\rho_t\psi\;dxdt=-\iint\limits_{\Omega_T}\eta'(\rho-k)[g(\rho)-g(k)]\psi_x\phi_x\;dxdt\\
&+\iint\limits_{\Omega_T}\eta'(\rho-k)g(k)\phi_{xx}\psi\;dxdt-\iint\limits_{\Omega_T}\eta''(\rho-k)[g(\rho)-g(k)]\phi_x\rho_x\psi\;dxdt\\
&-\int_0^T\eta'(k)(g(0)-g(k))\phi_x\psi|_{x=\pm1}dt-\varepsilon\iint\limits_{\Omega_T}\eta''(\rho-k)\rho_x^2\psi\;dxdt\\
&-\varepsilon\iint\limits_{\Omega_T}\eta'(\rho-k)\rho_x\psi_x\;dxdt+\int_0^T\varepsilon\eta'(\rho-k)\rho_x\psi|_{x=\pm1}dt\\
\le&-\iint\limits_{\Omega_T}\eta'(\rho-k)[g(\rho)-g(k)]\psi_x\phi_x\;dxdt\\
&+\iint\limits_{\Omega_T}\eta'(\rho-k)g(k)\psi\phi_{xx}\;dxdt-\eta'(k)\int_0^T[g(0)-g(k)]\phi_x\psi|_{x=\pm1}dt\\
&-\varepsilon\iint\limits_{\Omega_T}\eta(\rho-k)\rho_x\psi_x\;dxdt-\eta'(k)\int_0^T\varepsilon\rho_x\psi|_{x=\pm1}dt\\
&-\iint\limits_{\Omega_T}\eta''(\rho-k)[g(\rho)-g(k)]\phi_x\rho_x\psi\;dxdt
\end{aligned}$$

Next we integrate the first term by parts and multiply it by -1. Taking the limit as $\gamma\to0$ the last term on the right hand side vanishes (due to the continuity of g

and the boundedness of ϕ_x and ψ, cf. Remark 2.6) and we obtain

$$
\begin{aligned}
\iint\limits_{\Omega_T} |\rho - k|\psi_t \, dxdt + \int_{-1}^{1} \rho_I(x)\psi(x,0)\, dx& \\
\geq \iint\limits_{\Omega_T} \operatorname{sgn}(\rho - k)[g(\rho) - g(k)]\psi_x\phi_x \, dxdt& \\
- \iint\limits_{\Omega_T} \operatorname{sgn}(\rho - k) g(k)\psi\phi_{xx}\, dxdt + \operatorname{sgn}(k)\int_0^T [g(0) - g(k)]\phi_x\psi|_{x=\pm 1} dt& \\
+ \varepsilon \iint\limits_{\Omega_T} |\rho - k|\rho_x\psi_x \, dxdt + \operatorname{sgn}(k)\int_0^T \varepsilon\rho_x\psi|_{x=\pm 1} dt.&
\end{aligned}
\tag{40}
$$

Next we consider the limit $\varepsilon \to 0$. Using Lemma 2.5, the fourth term on the right hand side can be estimated by

$$
\left| \varepsilon \iint\limits_{\Omega_T} |\rho - k|\rho_x\psi_x \, dxdt \right| \leq \varepsilon C \|\psi_x(t)\|_{L^\infty(\Omega)}, \tag{41}
$$

and thus tends to zero. To compute the limit for the last term, i.e.

$$
\lim_{\varepsilon\to 0} \varepsilon \int_0^T \rho_x\psi|_{x=\pm 1} dt,
$$

we introduce (following [7]), for some $\kappa > 0$ the function $\xi_\kappa \in C^2([-1,1])$ with the following properties

$$
\begin{cases}
\xi_\kappa(x) = 1 & \text{on } x = \pm 1, \\
\xi_\kappa(x) = 0 & \text{on } \{x \in [-1,1]\,;\, \operatorname{dist}(x, \partial[-1,1]) \geq \kappa\}, \\
0 \leq \xi_\kappa(x) \leq 1 & \text{on } (-1,1).
\end{cases}
\tag{42}
$$

Furthermore, defining $\mathcal{M}([-1,1])$ as the space of Radon measures on $[-1,1]$, we choose ξ_κ such that

$$
\partial_x \xi_\kappa \to \mu|_{\{-1,1\}} \in \mathcal{M}([-1,1]) \text{ as } \kappa \to 0,
$$

defined as

$$
\mu = \delta_{x=1} - \delta_{x=-1}.
$$

Now we obtain

$$
\varepsilon \iint\limits_{\Omega_T} \rho_{xx}\psi\xi_\kappa \, dxdt = -\varepsilon \iint\limits_{\Omega_T} \rho_x(\psi\xi_\kappa)_x \, dxdt + \varepsilon \int_0^T \rho_x\psi|_{x=\pm 1} dt.
$$

The second term in this equation

$$-\varepsilon \iint\limits_{\Omega_T} \rho_x (\psi \xi_\kappa)_x \, dxdt = -\varepsilon \iint\limits_{\Omega_T} \rho_x (\psi_x \xi_\kappa + \phi (\xi_\kappa)_x) \, dxdt$$

vanishes in the limit $\epsilon \to 0$ due to the L^∞ bounds on ψ, ψ_x, ξ_κ, $(\xi_\kappa)_x$ (given for $\kappa > 0$ since $\xi_\kappa \in C^2([-1,1])$) and the L^1-boundedness of ρ_x. Using (39) we therefore obtain

$$\lim_{\varepsilon \to 0} \left(\varepsilon \int_0^T \rho_x \psi|_{x=\pm 1} dt \right) = - \iint\limits_{\Omega_T} (\rho \psi_t - g(\rho)\phi_x \psi_x) \, \xi_\kappa \, dxdt + \int_\Omega \rho \psi \xi_k \, dx \Big|_{t=0}$$
$$+ \iint\limits_{\Omega_T} g(\rho)\phi_x \psi (\xi_\kappa)_x \, dxdt - \int_0^T g(0)\phi_x \psi|_{x=\pm 1} dt.$$

Finally letting $\kappa \to 0$, the first term on the right hand side tends to zero while the second tends to an evaluation on the boundary. Due to the continuity of ρ and ψ the boundary term resulting from the integration by parts in time vanishes as the support of ξ_κ converges to a set of Lebesgue measure zero (i.e. $\{-1, 1\}$). Thus we have

$$\lim_{\varepsilon \to 0} \varepsilon \int_0^T \rho_x \psi|_{x=\pm 1} dt = \int_0^T (g(\operatorname{tr} \rho) - g(0))\phi_x(s,t)\psi|_{x=\pm 1} dt.$$

Combining this result with (40) we finally obtain the entropy formulation as in Definition 2.1 and this completes the proof. $\square$

Next we prove that the boundary condition (13) can be recovered by the definition of entropy solution.

Lemma 2.11. *Let ρ be an entropy solution given by Definition 2.1. Then, the following inequality holds for all $k \in [0, \operatorname{tr} \rho]$*

$$g(\operatorname{tr} \rho) \geq g(k) \quad \text{at } x = \pm 1. \tag{43}$$

Proof. In (15), we choose the special test function $\psi = \nu(t)\omega_\kappa$ with $\nu \in \mathcal{C}^2(]0,T[)$ positive and $\omega_\kappa \in C^2([-1,1])$ with the following properties:

$$\begin{cases} \omega_\kappa(x) = 1 & \text{on } x = -1, \\ \omega_\kappa(x) = 0 & \text{on } \{x \in [-1,1]\,;\, |x+1| \geq \kappa\}, \\ 0 \leq \omega_\kappa(x) \leq 1 & \text{on } (-1,1). \end{cases} \tag{44}$$

Similarly as before for ξ_κ, we choose ω_κ such that

$$\partial_x \omega_\kappa \to -\delta_{-1} \text{ as } \kappa \to 0,$$

where δ_{-1} denotes the Dirac delta measure centered at -1. Then, in the limit $\kappa \to 0$ (15) converges to

$$\int_0^T \operatorname{sgn}(\operatorname{tr}\rho - k)[g(\operatorname{tr}\rho) - g(k)]\phi_x|_{x=-1}\nu(t)dt \\ + \operatorname{sgn}(k)\int_0^T [g(\operatorname{tr}\rho) - g(k)]\phi_x|_{x=-1}\nu(t)dt \geq 0,$$

for all $k \in \mathbb{R}$. Thus, almost everywhere in $\{-1\} \times (0,T)$ we have

$$(\operatorname{sgn}(\operatorname{tr}\rho - k) + \operatorname{sgn}(k))[g(\operatorname{tr}\rho) - g(k)]\phi_x \geq 0.$$

To conclude the proof we note that ϕ_x is always (i.e. independently of the given ρ) non-negative at $x = -1$. This is a consequence of the fact that $\phi = 0$ at $x = \pm 1$ (boundary conditions) and positive on the whole domain, due to a trivial minimum principle for the equation (16b). Employing Hopf's Lemma we therefore conclude strict positivity of ϕ_x at $x = -1$. In a similar way, one can construct a function ω_k concentrating on $x = 1$ with a derivative converging to a Dirac delta at $x = 1$. The same inequality is obtained since the change of sign in the derivative of concentrator ω_k is balanced by the change of sign in ϕ_x (non-increasing at $x = 1$). To conclude, we note that $(\operatorname{sgn}(\operatorname{tr}\rho - k) + \operatorname{sgn}(k)) = 0$ for all $k \notin [0, \operatorname{tr}\rho]$ (as $\operatorname{tr}\rho \geq 0$) and equal to 2 otherwise. □

2.5. Uniqueness

Next we shall prove that the entropy solution in the sense of Definition 2.1 is unique.

Theorem 2.12 (Uniqueness of entropy solutions)**.** *There exists at most one entropy solution* (ρ, ϕ) *to the system* (11) *with initial condition* (12) *and boundary conditions* (13)-(14) *in the sense of Definition 2.1.*

The above stated result is a consequence of the following stability theorem, which follows the same technique developed in [19]. Here the authors use the variables doubling technique originally introduced in [22]. A similar strategy was also used e.g. [3, 4].

We state the following useful result:

Lemma 2.13. *([19]) Consider a function* $z = z(x)$ *belonging to* $L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ *and let* h *be Lipschitz on the interval* $I_z := [-\|z\|_{L^\infty}, \|z\|_{L^\infty}]$*. Then* $h(z)$ *belongs to* $L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)$ *and*

$$\left|\frac{\partial}{\partial x_j} h(z)\right| \leq \|h\|_{Lip(I_z)} \left|\frac{\partial}{\partial x_j} z\right|$$

in the sense of measures for $j = 1, \ldots, d$*.*

Uniqueness can be deduced from the following theorem:

Theorem 2.14. *Let* (ρ,ϕ)*,* $(\bar\rho,\bar\phi)$ *be the two entropy solutions to system* (11) *according to Definition 2.1 with initial data* ρ_I*,* $\bar\rho_I \in L^\infty([-1,1]) \cap BV([-1,1])$ *respectively. Then for almost all* $t \in (0,T)$*,*

$$\begin{aligned}\|\rho(t)-\bar\rho(t)\|_{L^1(\Omega)} \le \|\rho_I-\bar\rho_I\|_{L^1(\Omega)} &+ t\|g\|_{L^\infty(\Omega)}\|\phi_{xx}(t)-\bar\phi_{xx}(t)\|_{L^\infty((0,T);L^1(\Omega))}\\ &+ t\|g\|_{Lip(\Omega)}\|\rho_x(t)\|_{L^1(\Omega)}\|\phi_x(t)-\bar\phi_x(t)\|_{L^\infty((0,T);L^\infty(\Omega))}\end{aligned}$$

holds.

Combining this result with (35) and (36) from Lemma 2.9 we obtain

$$\|\rho(t)-\bar\rho(t)\|_{L^1(\Omega)} \le \|\rho_I-\bar\rho_I\|_{L^1(\Omega)} + tC\|\rho(t)-\bar\rho(t)\|_{L^1(\Omega)}, \tag{45}$$

for some positive constant C. Choosing t small enough this inequality contradicts the existence of two different solutions ρ and $\bar\rho$ having the same initial datum and thus implies uniqueness. It remains to prove Theorem 2.14.

Proof. We first note that in this proof there will sometimes, after integration by parts, be terms which insolve derivatives of the sgn. To be precise, the sgn needs to be approximated in these situations, as in the proof of Lemma 2.5. However, to increase the readability of this proof, we will omit this detail here. Consider a nonnegative, compactly supported, Lipschitz continuous function $\psi(x,t,\bar x,\bar t)$, defined on $[-1,1]\times[0,T[\times[-1,1]\times[0,T[$. Furthermore, let ψ be zero on $\{-1,1\}\times[0,T)$. Next, we take two admissible solutions $\rho(x,t)$, $\bar\rho(\bar x,\bar t)$ and write (15) as

$$\begin{aligned}&\iint\limits_{\Omega_T} |\rho-\bar\rho|\psi_t\,dxdt - \iint\limits_{\Omega_T} \operatorname{sgn}(\rho-\bar\rho)[g(\rho)-g(\bar\rho)]\psi_x\phi_x(x,t)\,dxdt \;+\\ &\iint\limits_{\Omega_T} \operatorname{sgn}(\rho-\bar\rho)g(\bar\rho)\psi\phi_{xx}(x,t)dxdt - \operatorname{sgn}(\bar\rho)\int_0^T [g(\operatorname{tr}\rho)-g(\bar\rho)]\phi_x(x,t)\psi\mid_{x=\pm1}dt \ge 0.\end{aligned}$$

and

$$\begin{aligned}&\iint\limits_{\Omega_T} |\bar\rho-\rho|\psi_{\bar t}\,d\bar xd\bar t - \iint\limits_{\Omega_T} \operatorname{sgn}(\bar\rho-\rho)[g(\bar\rho)-g(\rho)]\psi_{\bar x}\bar\phi_{\bar x}(\bar x,\bar t)\,d\bar xd\bar t \;+\\ &\iint\limits_{\Omega_T} \operatorname{sgn}(\bar\rho-\rho)g(\rho)\psi\bar\phi_{\bar x\bar x}(\bar x,\bar t)\,d\bar xd\bar t - \operatorname{sgn}(\rho)\int_0^T [g(\operatorname{tr}\bar\rho)-g(\rho)]\bar\phi_{\bar x}(\bar x,\bar t)\psi\mid_{\bar x=\pm1}d\bar t \ge 0.\end{aligned}$$

Integrating both the above inequalities over $\Omega_T := \Omega\times[0,T[$, the first with respect to $\bar x$, $\bar t$ and the second with respect to x, t and adding the resulting equations leads to

$$
\begin{aligned}
&\iiiint\limits_{\Omega_T\times\Omega_T} |\rho-\bar\rho|(\psi_t+\psi_{\bar t})\,dzd\bar z \\
&-\iiiint\limits_{\Omega_T\times\Omega_T} \underbrace{\left[\operatorname{sgn}(\rho-\bar\rho)\left(g(\rho)\phi_x(x,t)-g(\bar\rho)\bar\phi_x(\bar x,\bar t)\right)(\psi_x+\psi_{\bar x})\right]}_{:=I_1} dzd\bar z \\
&-\iiiint\limits_{\Omega_T\times\Omega_T} \underbrace{\left[\operatorname{sgn}(\rho-\bar\rho)\left(g(\bar\rho)\psi_x\left(\bar\phi_{\bar x}(\bar x,\bar t)-\phi_x(x,t)\right)+g(\rho)\psi_{\bar x}\left(\bar\phi_{\bar x}(\bar x,\bar t)-\phi_x(x,t)\right)\right)\right]}_{:=I_{2,1}} dzd\bar z \\
&+\iiiint\limits_{\Omega_T\times\Omega_T} \underbrace{\left[\operatorname{sgn}(\rho-\bar\rho)(g(\bar\rho)\phi_{xx}(x,t)-g(\rho)\bar\phi_{\bar x\bar x}(\bar x,\bar t))\psi\right]}_{:=I_{2,2}} dzd\bar z \\
&=\iiiint\limits_{\Omega_T\times\Omega_T} \left(|\rho-\bar\rho|(\psi_t+\psi_{\bar t})+I_1+I_{2,1}+I_{2,2}\right)dzd\bar z\ge 0.
\end{aligned}
$$

Here $z=(x,t)$ and $\bar z=(\bar x,\bar t)$. We take a symmetric function $\delta\in\mathcal{C}^\infty(\mathbb{R})$ with total mass one and $\operatorname{Supp}(\delta)\subset(-1,1)$. We define

$$
\delta_h(\cdot):=\frac{1}{h}\delta\left(\frac{\cdot}{h}\right)
$$

and choose the following test function

$$
\psi=\nu\left(\frac{t+\bar t}{2},\frac{x+y}{2}\right)\delta_h\left(\frac{t-\bar t}{2}\right)\delta_h\left(\frac{x-\bar x}{2}\right).
$$

From this definition we conclude

$$
\begin{aligned}
&\iiiint\limits_{\Omega_T\times\Omega_T} \left(|\rho-\bar\rho|(\psi_t+\psi_{\bar t})+I_1\right)\,dxdtd\bar xd\bar t \\
&=\iiiint\limits_{\Omega_T\times\Omega_T} \left(|\rho-\bar\rho|\nu_t+\operatorname{sgn}(\rho-\bar\rho)\left(g(\rho)\phi_x(x,t)-g(\bar\rho)\bar\phi_{\bar x}\right)\nu_x\right)\times \\
&\qquad\times\delta_h\left(\frac{t-\bar t}{2}\right)\delta_h\left(\frac{x-\bar x}{2}\right)\,dxdtd\bar xd\bar t.
\end{aligned}
$$

We now consider the term $I_{2,1}$

$$
\begin{aligned}
I_{2,1}&=-\operatorname{sgn}(\rho-\bar\rho)\left[\bar\phi_x(\bar x,\bar t)\left(g(\bar\rho)+g(\rho)\right)-\phi_x(x,t)\left(g(\bar\rho)+g(\rho)\right)\right]\frac{1}{2}\nu_x\delta_h\delta_h \\
&\quad-\operatorname{sgn}(\rho-\bar\rho)\left[\bar\phi_x(\bar x,\bar t)\left(g(\bar\rho)-g(\rho)\right)-\phi_x(x,t)\left(g(\bar\rho)-g(\rho)\right)\right]\nu(\delta_h\delta_h)_x \\
&=:I_{2,1,1}+I_{2,1,2}.
\end{aligned}
$$

Here, we used that by definition we have $\nu_{\bar{x}} = \frac{1}{2}\nu_x$ and $(\delta_h\delta_h)_{\bar{x}} = -(\delta_h\delta_h)_x$. Integrating by parts in $I_{2,1,2}$ leads to

$$
\begin{aligned}
&-\iiiint\limits_{\Omega_T\times\Omega_T} \operatorname{sgn}(\rho-\bar{\rho})\left[\bar{\phi}_x(\bar{x},\bar{t})\left(g(\bar{\rho})-g(\rho)\right)-\phi_x(x,t)\left(g(\bar{\rho})-g(\rho)\right)\right]\times\\
&\qquad\times\nu(\delta_h\delta_h)_x\; dxdtd\bar{x}d\bar{t}\\
&=\iiiint\limits_{\Omega_T\times\Omega_T} \operatorname{sgn}(\rho-\bar{\rho})\left[\bar{\phi}_x(\bar{x},\bar{t})\left(g(\bar{\rho})-g(\rho)\right)-\phi_x(x,t)\left(g(\bar{\rho})-g(\rho)\right)\right]\times\\
&\qquad\times\frac{1}{2}\nu_x\delta_h\delta_h\; dxdtd\bar{x}d\bar{t}\\
&+\iiiint\limits_{\Omega_T\times\Omega_T} \bar{\phi}_{\bar{x}}[(\operatorname{sgn}(\rho-\bar{\rho})(g(\bar{\rho})-g(\rho)))_x-\phi_{xx}(x,t)\operatorname{sgn}(\rho-\bar{\rho})\left(g(\bar{\rho})-g(\rho)\right)]\times\\
&\qquad\times\nu\delta_h\delta_h\; dxdtd\bar{x}d\bar{t}\\
&+\iiiint\limits_{\Omega_T\times\Omega_T} -\phi_x(x,t)(\operatorname{sgn}(\rho-\bar{\rho})\left(g(\bar{\rho})-g(\rho)\right))_x\nu\delta_h\delta_h\; dxdtd\bar{x}d\bar{t}.
\end{aligned}
$$

Noticing that

$$
\begin{aligned}
&-\phi_{xx}(x,t)\operatorname{sgn}(\rho-\bar{\rho})\left(g(\bar{\rho})-g(\rho)\right)+I_{2,2}\\
&\qquad\qquad=-\operatorname{sgn}(\rho-\bar{\rho})(\bar{\phi}_{\bar{x}\bar{x}}(\bar{x},\bar{t}))-\phi_{xx}(x,t))g(\rho)\nu\delta_h\delta_h
\end{aligned}
$$

and adding again $I_{2,1,1}$ we obtain

$$
\begin{aligned}
&\iiiint\limits_{\Omega_T\times\Omega_T}\left(I_{2,2}+I_{2,1,2}+I_{2,1,1}\right)\; dxdtd\bar{x}d\bar{t}\\
&\qquad=\iiiint\limits_{\Omega_T\times\Omega_T} -\operatorname{sgn}(\rho-\bar{\rho})(\bar{\phi}_{\bar{x}\bar{x}}(\bar{x},\bar{t})-\phi_{xx}(x,t))g(\rho)\nu\delta_h\delta_h\; dxdtd\bar{x}d\bar{t}\\
&\qquad+\iiiint\limits_{\Omega_T\times\Omega_T}(\bar{\phi}_{\bar{x}}-\phi_x)(\operatorname{sgn}(\rho-\bar{\rho})(g(\bar{\rho})+g(\rho)))_x\nu\delta_h\delta_h\; dxdtd\bar{x}d\bar{t}\\
&\qquad+\iiiint\limits_{\Omega_T\times\Omega_T}\underbrace{\operatorname{sgn}(\rho-\bar{\rho})\left[\bar{\phi}_x(\bar{x},\bar{t})g(\rho)-\phi_x(x,t)g(\rho)\right]\nu_x\delta_h\delta_h)}_{=:J}\; dxdtd\bar{x}d\bar{t}.
\end{aligned}
$$

As there are no more derivatives in the terms involving $\delta_h\delta_h$, we consider the limit $h\to 0$, remove two integrals and set $x=\bar{x}$, $t=\bar{t}$. This is a rather technical point which is explained in great detail in [21]. We choose the new test function

$$\nu(x,t)=\nu_{\kappa,\tilde{h}}(x,t)=(1-\xi_\kappa(x))\chi_{\tilde{h}}(t),$$

with for some $0 < t_1 < t_2 < T$ fixed

$$\chi_h(t) = \int_{-\infty}^{t} (\delta_h(\tau - t_1) - \delta_h(\tau - t_2)) \ d\tau,$$

and ξ_κ as defined in (42). We observe that all terms which are bounded in L^1 and multiplied by $(\nu_{\kappa,\tilde{h}}(x,t))_x$ converge to a boundary term in the limit $\kappa \to 0$. We thus have

$$\lim_{\substack{\tilde{h}\to 0\\ \kappa\to 0}} \iint\limits_{\Omega_T} (I_1 + J) \ dxdt = -\int_{t_1}^{t_2} \int_{\partial\Omega} \operatorname{sgn}(\operatorname{tr}\rho - \operatorname{tr}\bar{\rho})\bar{\phi}_x[g(\operatorname{tr}\rho) - g(\operatorname{tr}\bar{\rho})] \ dsdt,$$

and therefore

$$\begin{aligned}
&-\lim_{\substack{\tilde{h}\to 0,\\ \kappa\to 0}} \iint\limits_{\Omega_T} (|\rho - \bar{\rho}|\nu_t + I_1 + I_{2,1} + I_{2,2}) \ dxdt \\
&= -\int_{-1}^{1} (|\rho - \bar{\rho}|) \ dx\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\int_{-1}^{1} -\operatorname{sgn}(\rho - \bar{\rho})(\bar{\phi}_{xx}(x,t)) - \phi_{xx}(x,t))g(\rho) \ dxdt \\
&\quad + \int_{t_1}^{t_2}\int_{-1}^{1} (\bar{\phi}_x - \phi_x)(\operatorname{sgn}(\rho - \bar{\rho})(g(\bar{\rho}) + g(\rho)))_x \ dxdt \\
&\quad + \int_{t_1}^{t_2} \operatorname{sgn}(\operatorname{tr}\rho - \operatorname{tr}\bar{\rho})\bar{\phi}_x[g(\operatorname{tr}\rho) - g(\operatorname{tr}\bar{\rho})]|_{x=\pm 1} dt \geq 0.
\end{aligned}$$

Using Lemma 2.13, we have

$$|(\operatorname{sgn}(\operatorname{tr}\rho - \operatorname{tr}\bar{\rho})(g(\operatorname{tr}\bar{\rho}) - g(\operatorname{tr}\rho)))_x| \leq \|g\|_{Lip(I)}|\rho_x|. \tag{46}$$

Collecting all the above terms we obtain

$$\begin{aligned}
\|\rho(t) - \bar{\rho}(t)\|_{L^1(\Omega)}\Big|_{t_1}^{t_2} \leq \int_{t_1}^{t_2}\int_{\Omega} \big[&|\phi_{xx}(x,t) - \bar{\phi}_{xx}(x,t)|\|g\|_{L^\infty(\Omega)} \\
&+\|\phi_x(t) - \bar{\phi}_x(t)\|_{L^\infty(\Omega)}\|g\|_{Lip(I)}|\rho_x|\big] \ dxdt \\
+ \int_{t_1}^{t_2}\int_{\partial\Omega} &\operatorname{sgn}(\operatorname{tr}\rho - \operatorname{tr}\bar{\rho})\bar{\phi}_x[g(\operatorname{tr}\rho) - g(\operatorname{tr}\bar{\rho})] \ dsdt.
\end{aligned} \tag{47}$$

Following [7], we define

$$k(x,t) = \begin{cases} \operatorname{tr}\rho & \text{if} \quad \operatorname{tr}\rho < \operatorname{tr}\bar{\rho}, \\ 0 & \text{if} \quad \operatorname{tr}\rho = \operatorname{tr}\bar{\rho}, \\ \operatorname{tr}\bar{\rho} & \text{if} \quad \operatorname{tr}\rho > \operatorname{tr}\bar{\rho}. \end{cases}$$

This allows us to write, at $x=-1$

$$\begin{aligned}\operatorname{sgn}(\operatorname{tr}\rho-\operatorname{tr}\bar\rho)\bar\phi_x(-1,t)[g(\operatorname{tr}\rho)-g(\operatorname{tr}\bar\rho)] &= \operatorname{sgn}(\operatorname{tr}\rho-k)\bar\phi_x(-1,t)[g(\operatorname{tr}\rho)-g(k)]\\ &\quad+\operatorname{sgn}(\operatorname{tr}\bar\rho-k)\bar\phi_x(-1,t)[g(\operatorname{tr}\bar\rho)-g(k)].\end{aligned}$$

Note that $\bar\phi_x(-1,t)>0$. At $x=1$, the same holds true. Using Lemma 2.11 we conclude that the last term on the right hand side of (47) is negative and can therefore be omitted. Thus letting $t_1\to 0$ we arrive at the desired inequality and this completes the proof. $\square$

Remark 2.15 (Alternative Regularization). *We also considered the alternative regularisation as introduced in Sec. 1.2, namely system* (16) *with* $\delta_2=0$. *We were able to show the existence of weak solutions. However, it was not possible to obtain a uniqueness result. Even though we still obtained a bound on* $\|\rho_x(t)\|_{L^1(\Omega)}$ *(using a Aubin-Lions like argument) we couldn't obtain this bound for* ρ_t. *The main reason for this is that the Hopf-Cole transform cannot be used for this regularisation and it was not possible to control terms of the form* ϕ_{xt} *and* ϕ_{xxt}. *Detailed results will be part of the PhD thesis [33].*

3. Numerics and Examples for the Hughes' model

In this section we discuss the behaviour of solutions for the non regularized one-dimensional problem with simple initial data. Already these examples show quite interesting features which can be reproduced by numerical simulations. The content of this section is formal as we don't provide any existence and uniqueness theory. However, the characteristic calculus provides a useful tool to understand qualitatively the behaviour of the solution in the simple examples considered and is in complete agreement with the numerical results.

3.1. Characteristic Calculus

We consider the non-regularized problem

$$\rho_t-(\rho f^2(\rho)\phi_x)_x=0, \tag{48a}$$

$$|\phi_x|=\frac{1}{f(\rho)}. \tag{48b}$$

In the following, we always consider the unique viscosity solution ϕ to (48b). We use (in a non rigorous way) the notion of a viscosity solution to be able to interpret ϕ as a biased shortest distance to the exit. Note that thus this solution has a unique turning point $x_0(t)$ (i.e. point, where ϕ_x changes sign) given by the implicit relation

$$\int_{-1}^{x_0(t)}\frac{1}{f(\rho)}\,dx=\int_{x_0(t)}^{1}\frac{1}{f(\rho)}\,dx.$$

Thus, (48a) can be written as (using that $|\phi_x| = \phi_x \operatorname{sgn} \phi_x$)

$$\rho_t - (\rho f(\rho) \operatorname{sgn} \phi_x)_x = 0. \tag{49}$$

The natural boundary conditions (in the spirit of [7, 9]) are given by

$$f(\operatorname{tr} \rho) \geq f(k) \qquad \text{on } x = \pm 1, \quad \text{for all } k \in [0, \operatorname{tr} \rho], \tag{50}$$

which is satisfied if and only if $\operatorname{tr} \rho$ belongs to the interval of densities corresponding to outgoing characteristics, i. e. $\operatorname{tr} \rho \in [0, 1/2]$. As shown in [9], the boundary condition in case of incoming characteristics is determined by solving a Riemann problem between the boundary datum (i.e. zero in this case) and the trace of the density next to the boundary.

Away from the time dependent interface $x = x_0(t)$ (where ϕ_x is discontinuous) we can give sense to characteristics. They are defined by

$$\dot{x} = -(1 - 2\rho) \operatorname{sgn}(\phi_x).$$

Note that the Rankine-Hugoniot condition for a hyperbolic conservation law with flux F, i.e. $\rho_t + F(\rho)_x = 0$ is given by

$$[[F(\rho)]] = \dot{x}_0(t) \, [[\rho]] \, . \tag{51}$$

Here, $[[\cdot]]$ denotes the jump at the discontinuity x_0.

3.1.1. Constant initial data

We would like to understand the behaviour of the solution in the very simple case of constant initial data. Here we are particularly interested in the three cases which correspond to different characteristic speeds, i.e. ρ_I less, equal or greater than $1/2$. In particular we consider the cases $\rho_I = 1/4$, $\rho_I = 1/2$ and $\rho_I = 3/4$. In the case of constant initial data, the interface is constant in time, i.e. $\dot{x}_0 = 0$ and located at $x = 0$. Thus $\operatorname{sgn} \phi_x = -\operatorname{sgn} x$ and (48a) can be written as

$$\rho_t + (\rho f(\rho) \operatorname{sgn} x)_x = 0. \tag{52}$$

The RH condition (51) for this flux $F(\rho) = \rho f(\rho) \operatorname{sgn} x$ reads

$$f(\rho^+) + f(\rho^-) = 0,$$

where $\rho^\pm$ denote the right and left limit of ρ at the interface $x = 0$. An immediate consequence of this is that constant functions $\rho(x, t) = c$ with $c \in (0, 1)$ do not satisfy the RH condition (51) and are not weak solutions. If we start with a constant initial datum we expect the equation to "correct" this by forcing $\rho(0, t) = 0$ in arbitrary small time ($\rho(x, t) = c$ would also create a solution, which however

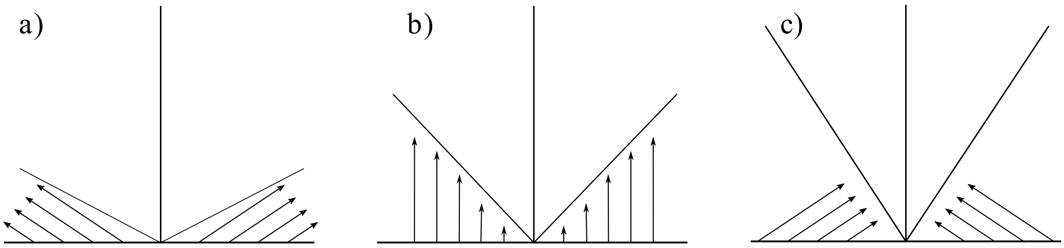


Figure 2: a) $\rho = 0.25$ b) $\rho = 0.5$ c) $\rho = 0.75$

does not fulfil the entropy condition). Then two shocks originate between $\rho(0,t) = 0$ and $\rho(x,t) = c$ for $x \neq 0$, which move towards the boundary. The slope of these shocks is determined by the RH condition (51). In the three cases considered we obtain

$$\dot{x} = \begin{cases} \pm\frac{3}{4} & \rho_I(x) = \frac{1}{4} \\ \pm\frac{1}{2} & \rho_I(x) = \frac{1}{2} \\ \pm\frac{1}{4} & \rho_I(x) = \frac{3}{4}. \end{cases}$$

This situation, locally around $x = 0$, is sketched in Fig. 2. Around the center $x = 0$ where no information is transported to, we expect the solution to be either zero or a rarefaction wave. In case of a rarefaction wave we make the ansatz $\rho(x,t) = u\left(\frac{x}{t}\right)$ and deduce from (52) that

$$u_{\mathrm{RF}}(x,t) = \frac{x+t}{2t}.$$

This solution continuously connects the two outgoing shocks but creates the constant value $1/2$ at $x = 0$ and is thus not admissible. Therefore, the we expect formation of a vacuum in between the two shocks in all three cases. In the case $\rho = 3/4$, we encounter an additional phenomenon at the boundaries. Here the characteristics point inwards, therefore we need to prescribe boundary conditions at $x = \pm 1$. We choose the following Dirichlet boundary conditions $\rho(\pm 1, t) = 1/2$ (maximal flux). Such condition is easily recovered by solving the Riemann problem between $\operatorname{tr}\rho = 3/4$ and the boundary value zero (cf. [9]).

This implies that the characteristics at the boundary are vertical while characteristics of slope $1/2$ transport the value $3/4$ into the domain. Hence we obtain two wedges (one at each boundary) in which no information is transported by characteristics. If we make again the ansatz $\rho(x,t) = u\left(\frac{x+1}{t}\right)$ (shifted to the left boundary), we obtain the following rarefaction wave

$$\rho(x,t) = \frac{x+1+t}{2t},$$

which is an admissible solution. Thus we expect rarefaction waves at both boundaries. At time $t = 4/3$, these rarefaction waves will hit the shocks coming from the interface (at $x = \pm 1/3$, respectively). To calculate the new slope of the shock we use the RH condition (51) which results in the following ODE

$$\dot{s}(t) = -\frac{s(t)}{2t} + \frac{t-1}{2t}, \quad s\left(\frac{4}{3}\right) = -\frac{1}{3}.$$

Using standard techniques we obtain the solution

$$s(t) = -\sqrt{t}\left(\frac{1+t}{\sqrt{t}} - \sqrt{3}\right).$$

A complete picture of the case $\rho_I(x) = 3/4$ is given in Fig. 3. In the next section we will see that all these phenomena can be observed in numerical simulations.

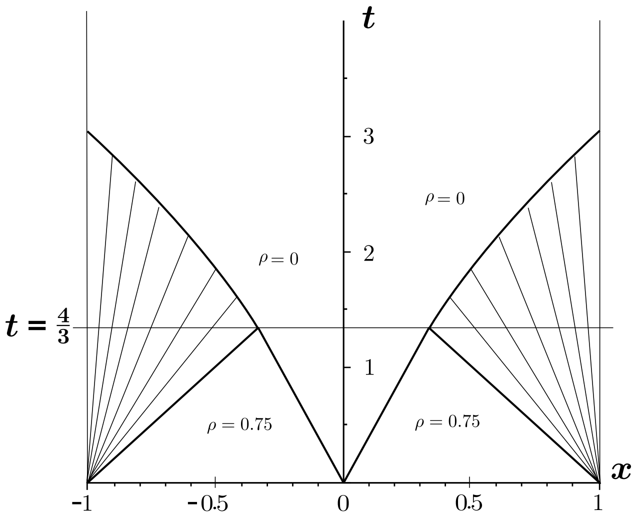


Figure 3: Details for the case $\rho_I(x) = 3/4$

Remark 3.1 (Boundary conditions in the regularized and non regularized case)**.** At a first glance there is a clear discrepancy between the boundary conditions in the regularized case (13) and the ones prescribed above for the non regularized model. In the latter case, the set of admissible boundary data is determined via the monotonicity of f at the boundary, whereas in the former case this set is determined via the function g. Hence, there is the possibility of a boundary layer in a possible limit as $\delta_1 \to 0$. However, the regularized problem has a source term $g(\rho)\phi_{xx}$, and this fact could possibly imply some compensation phenomena at the boundary which can avoid the boundary layer. This issue will be the topic of future study.

3.2. Numerical simulations

Next we present numerical simulations of (48) relating the results to the previous discussion in Section 3.1. We consider the regularized system on the domain $\Omega = [-1, 1]$

$$\rho_t - \operatorname{div}(\rho f(\rho) \operatorname{sgn} \phi_x) = \varepsilon \rho_{xx} \tag{53a}$$

$$|\phi_x| = \frac{1}{f(\rho)} \tag{53b}$$

with a regularization parameter $\varepsilon \geq 0$. The system is supplemented with the initial condition $\rho(x, 0) = \rho_I(x)$ and inhomogeneous Dirichlet boundary conditions $\rho(\pm 1, t) = \rho_D$. We use these boundary conditions to be consistent with the characteristic calculus presented in Sec. 3.1. This allows us to compare the numerical results with these computations. We solve (53) in an iterative manner, i.e.

1. Given ρ solve the eikonal equation (53b) with fast sweeping method.
2. Solve the non-linear conservation law (53a) for a given ϕ using an ENO scheme or resp. a Godunov scheme.

We choose the following discretisation. The domain $\mathbb{R}$ is divided into cells $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ with centers at points $x_j = j\Delta x$ for $j \in \mathbb{Z}$. The time domain $(0, \infty)$ is discretised in the same manner via $t^n = n\Delta t$ resulting in time strips $I^n = [t^n, t^{n+1}]$.

We used two different schemes to compare and understand the behaviour of solutions. In the first approach we use an ENO scheme with small diffusion on the whole domain $\Omega = [-1, 1]$. In the second approach we split the domain into two parts $\Omega = \Omega_1 \cup \Omega_2$ where $\Omega_1 = [0, x(t)]$ and $\Omega_2 = [x(t), 1]$, solve equation (53a) with a Godunov scheme (and no diffusion, i.e. $\varepsilon = 0$) on Ω_1 and Ω_2 and concatenate both solutions.

3.2.1. ENO scheme

J. Towers presented convergence results for an ENO scheme for conservation laws with discontinuous flux in [37]. This ansatz can be used in Step (2) to solve (53a) with small diffusion on the whole domain $\Omega = [-1, 1]$. Let χ_j^n denote the characteristic function on the rectangle $R_j^n = I_j \times I^n$. The finite difference scheme then generates for every mesh size Δx and Δt a piecewise constant solution ρ^Δ given by

$$\rho^\Delta(x, t) = \sum_{n \geq 0} \sum_{-\infty}^{\infty} \chi_j^n \rho_j^n.$$

The approximations ρ_j^n are generated by an explicit algorithm

$$\rho_j^{n+1} = \rho_j^n - \lambda_1(k_{j+\frac{1}{2}}h_{j+\frac{1}{2}} - k_{j-\frac{1}{2}}h_{j-\frac{1}{2}}) + \lambda_2(d_{j+\frac{1}{2}} - d_{j-\frac{1}{2}}). \tag{54}$$

Here $\lambda_1 = \frac{\Delta t}{\Delta x}$, $\lambda_2 = \frac{\varepsilon \Delta t}{\Delta x^2}$ and $k_{j\pm\frac{1}{2}} = \operatorname{sgn} \phi_x(x_{j\pm\frac{1}{2}})$. The diffusive flux is given by $d^n_{j+\frac{1}{2}} := \rho^n_{j+1} - \rho^n_j$, the convective one $\mathrm{h}_{\mathrm{j}+\frac{1}{2}} := h(v,u)$ is chosen such that it is consistent with the actual flux, i.e. $h(\rho,\rho) = g(\rho) = \rho f(\rho)$. To guarantee monotonicity the flux is transposed when $k_{j+\frac{1}{2}}$ changes sign, i.e.

$$\mathrm{h}_{\mathrm{j}+\frac{1}{2}} = \begin{cases} h(\rho_{j+1},\rho_j) & \text{if } k_{j+\frac{1}{2}} \geq 0 \\ h(\rho_j,\rho_{j+1}) & \text{if } k_{j+\frac{1}{2}} < 0. \end{cases}$$

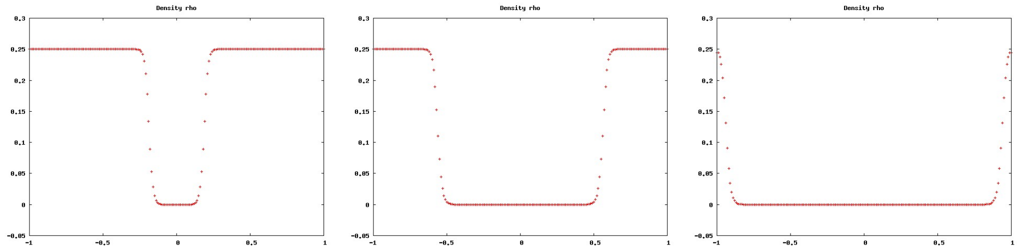
We choose the ENO flux [10] which is given by

$$h(v,u) = \frac{1}{2}\left(g(u) + g(v)\right) + \frac{1}{2}\int_u^v |g_u| du. \tag{55}$$

Godunov scheme. The Godunov scheme is derived by using the exact solution operator for $\rho_t + (F(\rho))_x = 0$ with piecewise constant initial data. The resulting numerical flux is $h(v,u) = F(u^G(v,u))$, where $u^G(v,u)$ is the similarity solution of the resulting Riemann problem with right and left state v and u evaluated anywhere on the vertical half-line $t > 0$ where the jump in the initial data occurs. The Godunov flux [31] is given by

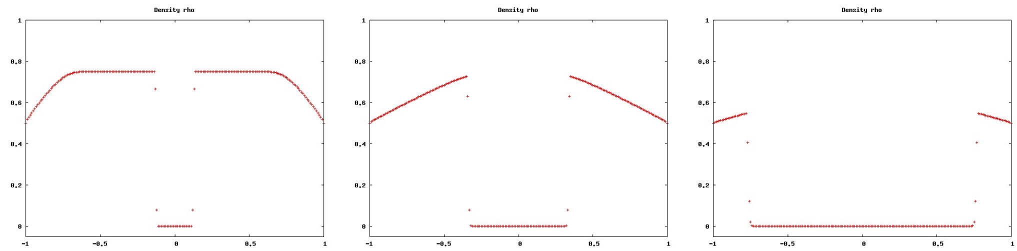
$$h(v,u) = \begin{cases} \min_{[u,v]} F(w) & \text{if } u \leq v \\ \max_{[u,v]} F(w) & \text{if } u \geq v. \end{cases} \tag{56}$$

Constant initial data. First we would like to validate the characteristic calculus presented in section 3.1. We choose constant initial data $\rho_I(x)$ that is smaller or larger than 1/2. The time discretisation is set to $\Delta t = 10^{-4}$, the spatial discretisation to $\Delta x = 10^{-2}$. Here we solved the non regularized problem with $\varepsilon = 0$ using Godunovs' method. First we choose $\rho_I(x) = 1/4$, the evolution is depicted in Figure 4. In this case the characteristics point outward, therefore we prescribe numerical boundary conditions instead of physical ones. In our second example we set $\rho_I(x) = 3/4$. Here we observe a good agreement of the numerical simulation with the theoretical results, see Figure 5. Note that the shock hits the rarefaction waves at $t = 4/3$ and that the subsequent shock hits the boundary at $t = 3$ (as predicted by our characteristic calculus).



(a) $t = \frac{1}{4}$ (b) $t = \frac{3}{4}$ (c) $t = \frac{5}{4}$

Figure 4: Evolution of ρ with initial datum $\rho_I(x) = 0.25$



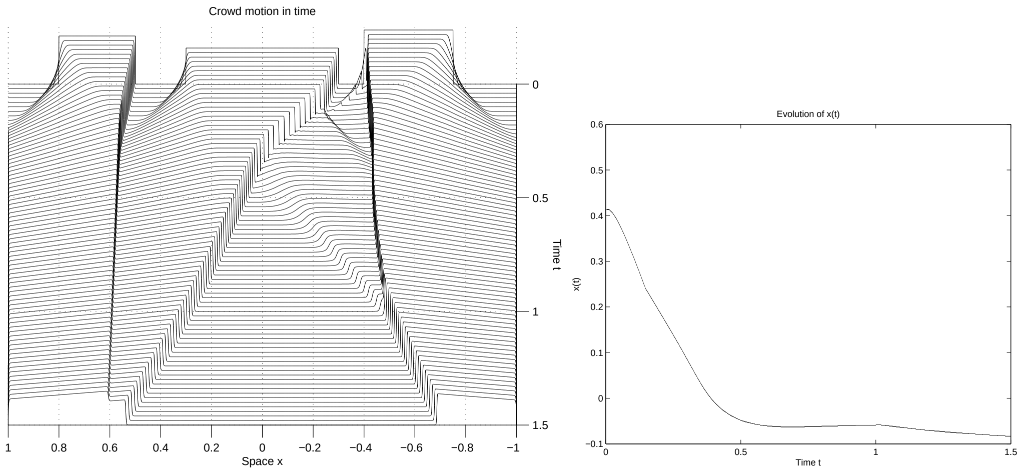
(a) $t = \frac{1}{2}$ (b) $t = \frac{4}{3}$ (c) $t = \frac{5}{2}$

Figure 5: Evolution of ρ with initial datum $\rho_I(x) = 0.75$ and Dirichlet boundary conditions $\rho(\pm 1) = 0.5$

Other examples. Finally we would like to illustrate the behavior with other examples. We choose the following initial guess

$$\rho_I(x) = \begin{cases} 0.8 & \text{if } -0.8 \leq x \leq -0.5 \\ 0.6 & \text{if } -0.3 \leq x \leq 0.3 \\ 0.9 & \text{if } \quad 0.4 \leq x \leq 0.75, \end{cases}$$

representing three groups which would like to exit at $x = 1$ or $x = -1$. We set the spatial discretisation to $\Delta x = 10^{-3}$, the discretisation in time to $\Delta t = 10^{-4}$. Here we solve (53a) on the whole domain using an ENO flux and $\varepsilon = 10^{-4}$. The evolution of the densities is illustrated in Figure 6. Here the y axis corresponds to time, running from 0 (top) to 1.5 (bottom). The right group (located between $0.4 \leq x \leq 0.75$) splits at the beginning, a small part moves to the left while the rest moves towards the right exit. We observe that the part of the group which was moving to the left changes direction and moves towards the right.



(a) $\rho(x,t)$ (b) $x(t)$

Figure 6: Evolution of ρ and $x(t)$

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References

[1] A. Aw and M. Rascle. Resurrection of "second order" models of traffic flow. *SIAM J. Appl. Math.*, 60(3):916–938 (electronic), 2000.

[2] F. Brezzi and P. A. Markowich. A convection-diffusion problem with small diffusion coefficient arising in semiconductor physics. *Boll. Un. Mat. Ital. B (7)*, 2(4):903–930, 1988.

[3] M. Burger, M. Di Francesco, and Y. Dolak-Struss. The Keller-Segel model for chemotaxis: linear vs. nonlinear diffusion. *SIAM J. Math. Anal.*, 38:1288–1315., 2006.

[4] M. Burger, Y. Dolak-Struss, and C. Schmeiser. Asymptotic analysis of an advection-dominated chemotaxis model in multiple spatial dimensions. *Commun. Math. Sci.*, 6:1–28, 2008.

[5] R. Bürger, , K. Karlsen, and J. Towers. A conservation law with discontinuous flux modelling traffic flow with abruptly changing road surface conditions. In *Hyperbolic Problems: Theory, Numerics and Applications*, volume 67 of *Proceedings of Symposia in Applied Mathematics*, pages 455–464, 2009.

[6] R. Bürger, G. A., K. Karlsen, and J. Towers. Difference schemes, entropy solutions and speedup impulse for an inhomogeneous kinetamtic traffic flow model. *Networks and heterogeneous media*, 3(2):1–41, 2008.

[7] J. C. N. C. Bardos, A. Y. Leroux. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4:1017–1034, 1979.

[8] R. M. Colombo and M. D. Rosini. Pedestrian flows and non-classical shocks. *Math. Methods Appl. Sci.*, 28(13):1553–1567, 2005.

[9] F. Dubois and P. LeFloch. Boundary conditions for nonlinear hyperbolic systems of conservation laws. *J. Differential Equations*, 71(1):93–122, 1988.

[10] B. Engquist and S. Osher. One-sided difference schemes and transonic flow. *Proc. Nat. Acad. Sci. U.S.A.*, 77(6):3071–3074, 1980.

[11] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions*. Crc Pr Inc, 1991.

[12] M. Fukui and Y. Ishibashi. Self-organized phase transitions in CA-models for pedestrians. *J. Phys. Soc. Japan*, 8:2861–2863, 1999.

[13] D. Helbing. Traffic and related self-driven many-particle systems. *Rev. Mod. Phys.*, 73(4):1067–1141, Dec 2001.

[14] D. Helbing, I. J. Farkas, P. Molnar, and T. Vicsek. Simulation of pedestrian crowds in normal and evacuation situations. *in: M. Schreckenberg and S. D. Sharma (eds.)* Pedestrian and Evacuation Dynamics *(Springer, Berlin).*, pages 21–58, 2002.

[15] D. Helbing, A. Johansson, and H. Z. Al-Abideen. The dynamics of crowd disasters: An empirical study. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, 75(4), 2007.

[16] L. F. Henderson. The statistics of crowd fluids. *Nature*, 229:381–383, 1971.

[17] R. L. Hughes. A continuum theory for the flow of pedestrians. *Transportation Research Part B: Methodological*, 36(6):507 – 535, 2002.

[18] R. L. Hughes. The flow of human crowds. *Annual Review of Fluid Mechanics*, 35:169–182, 2003.

[19] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete Contin. Dyn. Syst.*, 9(5):1081–1104, 2003.

[20] R. A. Klausen and N. H. Risebro. Stability of conservation laws with discontinuous coefficients. *J. Differential Equations*, 157(1):41–60, 1999.

[21] C. Klingenberg and N. H. Risebro. Convex conservation laws with discontinuous coefficients. Existence, uniqueness and asymptotic behavior. *Comm. Partial Differential Equations*, 20(11-12):1959–1990, 1995.

[22] S. N. Kružkov. First order quasilinear equations in serveral independent variables. *Math. USSR Sb*, 10:217–243, 1970.

[23] O. A. Ladyzhenskaia and N. N. Uraltseva. *Linear and quasilinear elliptic equations [by] Olga A. Ladyzhenskaya and Nina N. Uraltseva. Translated by Scripta Technica. Translation editor: Leon Ehrenpreis.* Academic Press, New York,, 1968.

[24] J.-M. Lasry and P.-L. Lions. Mean field games. *Jpn. J. Math.*, 2(1):229–260, 2007.

[25] R. J. LeVeque. *Numerical methods for conservation laws.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, second edition, 1992.

[26] M. J. Lighthill and G. B. Whitham. On kinematic waves. ii. a theory of traffic flow on long crowded roads. *Royal Society of London Proceedings Series A*, 229:317–345, May 1955.

[27] H. Ling, S. C. Wong, M. Zhang, C.-H. Shu, and W. H. K. Lam. Revisiting Hughes' dynamic continuum model for pedestrian flow and the development of an efficient solution algorithm. *Transportation Research Part B: Methodological*, 43(1):127 – 141, 2009.

[28] P. A. Markowich and P. Szmolyan. A system of convection-diffusion equations with small diffusion coefficient arising in semiconductor physics. *J. Differential Equations*, 81(2):234–254, 1989.

[29] B. Maury, A. Roudneff-Chupin, and F. Santambrogio. A macroscopic crowd motion model of the gradient-flow type. *M3AS*, 2009. accepted for publication.

[30] M. Muramatsu and T. Nagatani. Jamming transition in two-dimensional pedestrian traffic. *Physica A*, 275:281 – 291, 2000.

[31] S. Osher. Riemann solvers, the entropy condition, and difference approximations. *SIAM J. Numer. Anal.*, 21(2):217–235, 1984.

[32] B. Piccoli and A. Tosin. Pedestrian flows in bounded domains with obstacles. *Contin. Mech. Thermodyn.*, 21(2):85–107, 2009.

[33] J.-F. Pietschmann. Phd thesis. In preparation.

[34] P. I. Richards. Shock waves on the highway. *OPERATIONS RESEARCH*, 4(1):42–51, 1956.

[35] H. Tanabe. *Equations of evolution*, volume 6. Pitman (Advanced Publishing Program), 1979.

[36] J. W. Thomas. *Numerical partial differential equations*, volume 33 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1999. Conservation laws and elliptic equations.

[37] J. D. Towers. Convergence of a difference scheme for conservation laws with a discontinuous flux. *SIAM J. Numer. Anal.*, 38(2):681–698 (electronic), 2000.

[38] A. Treuille, S. Cooper, and Z. Popovic. Continuum crowds. In *ACM Transaction on Graphics*, volume 25, pages 1160–1168, 2006. Proceedings of SCM SIGGRAPH 2006.

[39] A. Yagi. Abstract quasilinear evolution equations of parabolic type in Banach spaces. *Boll. Un. Mat. Ital. B (7)*, 5(2):341–368, 1991.

[40] S. J. Yuhaski, Jr. and J. M. Smith. Modeling circulation systems in buildings using state dependent queueing models. *Queueing Systems Theory Appl.*, 4(4):319–338, 1989.

[41] M. Zhang, C.-W. Shu, G. C. K. Wong, and S. C. Wong. A weighted essentially non-oscillatory numerical scheme for a multi-class lighthill-whitham-richards traffic flow model. *Journal of Computational Physics*, 191(2):639 – 659, 2003.

[42] P. Zhang, S. Wong, and C.-W. Shu. A weighted essentially non-oscillatory numerical scheme for a multi-class traffic flow model on an inhomogeneous highway. *Journal of Computational Physics*, 212(2):739 – 756, 2006.