

MEAN FIELD GAMES WITH NONLINEAR MOBILITIES IN PEDESTRIAN DYNAMICS

MARTIN BURGER

Institute for Computational and Applied Mathematics
University of Münster
Einsteinstrasse 62, 48149 Münster, Germany

MARCO DI FRANCESCO

Department of Mathematical Sciences
4W, 1.14, University of Bath
Claverton Down, Bath (UK), BA2 7AY

PETER A. MARKOWICH

King Abdullah University of Science and Technology
Thuwal 23955-6900
Kingdom of Saudia Arabia

MARIE-THERESE WOLFRAM

Department of Mathematics
University of Vienna
Nordbergstrasse 15, 1090 Vienna Austria

(Communicated by the associate editor name)

ABSTRACT. In this paper we present an optimal control approach modeling fast exit scenarios in pedestrian crowds. In particular we consider the case of a large human crowd trying to exit a room as fast as possible. The motion of every pedestrian is determined by minimizing a cost functional, which depends on his/her position, velocity, exit time and the overall density of people. This microscopic setup leads in the mean-field limit to a parabolic optimal control problem. We discuss the modeling of the macroscopic optimal control approach and show how the optimal conditions relate to the Hughes model for pedestrian flow. Furthermore we provide results on the existence and uniqueness of minimizers and illustrate the behavior of the model with various numerical results.

2010 *Mathematics Subject Classification.* 49J20, 35K55, 65K10.

Key words and phrases. Pedestrian dynamics, mean field limit, optimal control, calculus of variations, numerical simulations.

MTW acknowledges financial support of the Austrian Science Foundation FWF via the Hertha Firnberg Project T456-N23. MDF is supported by the FP7-People Marie Curie CIG (Career Integration Grant) Diffusive Partial Differential Equations with Nonlocal Interaction in Biology and Social Sciences (DifNonLoc), by the 'Ramon y Cajal' sub-programme (MICINN-RYC) of the Spanish Ministry of Science and Innovation, Ref. RYC-2010-06412, and by the by the Ministerio de Ciencia e Innovación, grant MTM2011-27739-C04-02. The authors thank the anonymous referees for useful comments to improve the manuscript.

1. Introduction. Mathematical modeling of human crowd motion such as street-way pedestrian flows or the evacuation of large buildings is a topic of high practical relevance. The complex behavior of human crowds poses a significant challenge in the modeling and a majority of questions still remains open. Consequently this topic receives more and more attention, and in particular in the last years a strong development in the mathematical literature becomes visible. A variety of different approaches has been proposed and partly analyzed, which can roughly be grouped as follows:

1. *Microscopic force-based models* such as the social force model (cf. [26, 25, 14]) and their continuum limits.
2. *Cellular automata* approaches (discrete random walks with exclusion principles [12, 6]) and their continuum limits (cf. [11]).
3. *Fluid-dynamical and related macroscopic models* (cf. [15, 33, 16, 3]), among which in particular the Hughes model and its variants received strong attention (cf. [28, 17, 1, 22]).
4. Microscopic *optimal control* and *game* approaches (cf. [27, 37]) and *mean field game* models (cf. [31, 18]), which can be considered as the appropriate continuum limit (cf. [32, 24]).

For evacuation scenarios an additional complication in the basic model arises, namely how the boundaries (i.e. doors, impermeable walls) and the goal of the pedestrians to leave the room as quickly as possible are modeled. A canonical approach is to use a potential force (or drift in random walk models) based on the distance function to the doors. This means that the potential ϕ solves some kind of Eikonal equation in the domain Ω with zero Dirichlet boundary conditions on the part $\Gamma_E \subset \partial\Omega$, which corresponds to the door. The appropriate statement of the Eikonal equation and its boundary conditions in a crowded situation is a delicate issue, which is mainly done in an ad-hoc fashion (even for detailed microscopic models) and has led to different directions. Some models simply use the distance function itself, i.e. the viscosity solution of

$$|\nabla\phi| = 1 \quad \text{in } \Omega, \quad (1)$$

which corresponds to an overall optimization of the evacuation path independent of other pedestrians. This might be questionable in crowds due to limited visibility or global changes of the evacuation path to avoid jamming regions. Other models use variants of the Eikonal equation that incorporate the density ρ of pedestrians. The celebrated Hughes model, cf. [28], uses an equation of the form

$$f(\rho)|\nabla\phi| = 1 \quad \text{in } \Omega, \quad (2)$$

where f is a function introducing saturation effects such as $f(\rho) = \rho_{\max} - \rho$ with ρ_{\max} denoting the maximal density.

In this paper we introduce a rather unifying mean-field game approach to the modeling of evacuation scenarios, which links several of the above mentioned models and provides a further understanding of different modeling issues. The presented mean field framework gives novel interpretations of different forms of the Eikonal equation and covers the modeling of ‘realistic’ boundary conditions. We provide a detailed mathematical analysis and numerical simulations of our approach and discuss interesting special cases and limits, in particular the connection to the Hughes’ model.

This paper is organized as follows: In Section 2 we present the modeling on the microscopic level and its mean-field limit. Furthermore we discuss its relation to the Hughes' model and the appropriate choice of boundary conditions. Section 3 is devoted to the analysis of the macroscopic optimal control approach. Finally we present a steepest descent approach to solve the parabolic optimal control problem in Section 4 and illustrate the behavior with several numerical examples. We conclude the paper by presenting a more extended idea of viewing the Hughes' model in a mean field game framework in Section 5.

2. Mathematical Modeling. In the following we discuss a model paradigm based on the idea of a fast exit scenario. At the single particle level, this yields a classical or stochastic optimal control problem, which we reformulate as PDE-constrained optimization model for the particle density. From the optimality conditions we then obtain a version of the Eikonal equation as the adjoint problem.

2.1. Fast Exit of Particles. Let us start with a deterministic particle (of unit mass) trying to leave the domain Ω as fast as possible. Throughout this paper Ω shall be a bounded domain in \mathbb{R}^d , $d = 1, 2$. Let $X = X(t)$ denote the particle trajectory and the exit time

$$T_{exit}(X) = \sup\{t > 0 \mid X(t) \in \Omega\}.$$

Then it makes sense to look for a weighted minimization of the exit time T_{exit} and the kinetic energy, i.e.

$$\frac{1}{2} \int_0^{T_{exit}} |V(t)|^2 dt + \frac{\alpha}{2} T_{exit}(X) \rightarrow \min_{X, V}, \quad (3)$$

subject to $\dot{X}(t) = V(t)$, $X(0) = X_0$, where $\alpha > 0$ encodes the weighting of the fast exit. We mention that the weighted optimization problem (3) can be interpreted also as a version of an inequality constrained problem, with α respectively its inverse taking the role of a Lagrange parameter. This idea applied to the particular problem of minimizing the kinetic energy under an upper bound on the exit time. On the other hand, $\lambda = \frac{1}{\alpha}$ is a Lagrange parameter for the even more natural problem of minimizing the exit time under an upper bound on the kinetic energy. Another alternative model that should be worth to be investigated is to minimize the exit time under an upper bound on the speed (or equivalently its square) at each time step. In that case one would end up with a time-dependent Lagrange-parameter, i.e. a minimization of a problem of the form

$$\frac{1}{2} \int_0^{T_{exit}} \lambda(t) |V(t)|^2 dt + \frac{1}{2} T_{exit}(X) \rightarrow \min_{X, V},$$

which is however beyond our interest in this paper.

Introducing the Dirac measure $\mu = \delta_{X(t)}$, and a final time T sufficiently large, we can rewrite the functional (3) as

$$I_T(\mu, v) = \frac{1}{2} \int_0^T \int_{\Omega} |v(x, t)|^2 d\mu dt + \frac{\alpha}{2} \int_0^T \int_{\Omega} d\mu dt, \quad (4a)$$

subject to

$$\partial_t \mu + \nabla \cdot (\mu v) = 0, \quad (4b)$$

and initial value $\mu|_{t=0} = \delta_{X_0}$. We mention that the minimization in (4) takes place over ρ and v . Furthermore note that for $T > T_{exit}$

$$T_{exit}(X) = \int_0^T \int_{\Omega} d\delta_{X(t)} dt$$

and

$$\int_0^T \int_{\Omega} |v(x, t)|^2 d\mu dt = \int_0^T \int_{\Omega} |v(x, t)|^2 d\delta_{X(t)} dt = \int_0^{T_{exit}} |v(X(t), t)|^2 dt,$$

which yields equivalence between the particle formulation (3) and the continuum version (4) by mapping the Eulerian to the Lagrangian coordinates, cf. [27, 37]. Note that due to linearity with respect to the measure μ , we can simultaneously minimize the trajectories for N particles by using the initial value $\mu(0) = \frac{1}{N} \sum_{i=1}^N \delta_{X_i(0)}$ or even a smoothed version via convolution with a kernel g , i.e., $\mu(0) = \frac{1}{N} \sum_{i=1}^N g * \delta_{X_i(0)}$

Next we consider a stochastic particle, which moves according to the Langevin equation. Instead of Newtons' laws of motion we consider the stochastic differential equation

$$dX(t) = V(t) dt + \sigma dW(t), \quad (5a)$$

where W is a Wiener process with diffusivity σ . The natural generalization of (4) is the stochastic optimal control problem

$$\mathbb{E}_{X_0} \left[\frac{1}{2} \int_0^{T_{exit}} |V(t)|^2 dt + \frac{\alpha}{2} T_{exit}(X) \right] \rightarrow \min_V \quad (5b)$$

where the random variable X satisfies (5a) with initial value X_0 . Reformulating (5) for the distribution already reveals the structure of a mean-field game for the particle density. That is, writing $d\mu = \rho dx$ we obtain the minimization functional

$$I_T(\rho, v) = \frac{1}{2} \int_0^T \int_{\Omega} \rho(x, t) |v(x, t)|^2 dx dt + \frac{\alpha}{2} \int_0^T \int_{\Omega} \rho(x, t) dx dt, \quad (6a)$$

subject to (cf. [30])

$$\partial_t \rho + \nabla \cdot (\rho v) = \frac{\sigma^2}{2} \Delta \rho, \quad \rho(x, 0) = \rho_0(x). \quad (6b)$$

System (6) is a parabolic optimal control problem, which has a strong connection to mean field games, as we will see in the next Subsection.

2.2. Optimality, Eikonal Equations, and Mean-Field Games. In the following we provide an explanatory derivation of the optimality conditions for the optimization problem

$$I_T(\rho, v) \rightarrow \min_{(\rho, v) \text{ satisfying (6b)}} \quad (7)$$

We shall derive the optimality condition at a formal level, without providing rigorous details on the functional setting and on the boundary conditions. The precise setting for a more general model will be discussed in Section 2.6.

We define the Lagrangian with dual variable $\phi = \phi(x, t)$ as

$$L_T(\rho, v, \phi) = I_T(\rho, v) + \int_0^T \int_{\Omega} (\partial_t \rho + \nabla \cdot (\rho v) - \frac{\sigma^2}{2} \Delta \rho) \phi dx dt + \int_{\Omega} \phi(\rho(x, 0) - \rho_0(x)) dx. \quad (8)$$

For the optimal solution we have, in addition to (6b), the following equations, i.e.

$$0 = \partial_v L_T(\rho, v, \phi) = \rho v - \rho \nabla \phi, \quad (9)$$

and

$$0 = \partial_\rho L_T(\rho, v, \phi) = \frac{1}{2}|v|^2 + \frac{\alpha}{2} - \partial_t \phi - v \cdot \nabla \phi - \frac{\sigma^2}{2} \Delta \phi, \quad (10)$$

with the additional terminal condition $\phi(x, T) = 0$ (obtained from the integration by parts in the term $\int_0^T \int_\Omega \partial_t \rho \phi \, dx dt$). Inserting $v = \nabla \phi$ we obtain the following system, which has the structure of a mean field game, i.e.

$$\partial_t \rho + \nabla \cdot (\rho \nabla \phi) - \frac{\sigma^2}{2} \Delta \rho = 0 \quad (11a)$$

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + \frac{\sigma^2}{2} \Delta \phi = \frac{\alpha}{2}. \quad (11b)$$

One observes the automatic emergence of a transient viscous Eikonal equation, which is to be solved backward in time with terminal condition. The connection to models using the distance function as a potential is quite apparent for a time interval $[0, S]$ with $S \ll T$ and $\sigma = 0$. Noticing that the Hamilton-Jacobi equation is solved backwards in time and that the backward time is large for $t \leq S$, we see that the solution ϕ is mainly determined by the large-time asymptotics (cf. [29]) solving

$$|\nabla \tilde{\phi}|^2 = c, \quad (12)$$

for some constant c . Hence the potential becomes approximately a multiple of the distance function.

2.3. Mean Field Games and Crowding. We now turn our attention to a more general mean field game, respectively its optimal control formulation. We generalize the terms depending on the density for reasons to be explained in detail below to obtain

$$I_T(\rho, v) = \frac{1}{2} \int_0^T \int_\Omega F(\rho) |v(x, t)|^2 dx dt + \frac{1}{2} \int_0^T \int_\Omega E(\rho) dx dt, \quad (13a)$$

subject to

$$\partial_t \rho + \nabla \cdot (G(\rho)v) = \frac{\sigma^2}{2} \Delta \rho, \quad (13b)$$

and a given initial value $\rho(x, 0) = \rho_0(x)$. The motivation for those terms is as follows:

- The above generalization is purely empirical, but motivated by recent results in modeling crowd motion. The function $G = G(\rho)$ corresponds to nonlinear mobilities, which are frequently used in crowding models, e.g. $G(\rho) = \rho(\rho_{\max} - \rho)$ derived from microscopic lattice exclusion processes (cf. [11] for a pedestrian case). Nonlinear mobilities have been used in different setting and applications of crowded motion, e.g. ion channels [10, 9] or cell biology [36, 34, 19]. A typical form of G is close to linear and increasing for small densities, while saturating and possibly decreasing to zero for larger densities.
- The function $F = F(\rho)$ corresponds to transport costs created by large densities. In particular one may think of F tending to infinity as ρ tends towards a maximal density. As we shall see below however, the changes in F can also be related equivalently to changes in the mobility.
- A nonlinear function $E = E(\rho)$ may model active avoidance of jams in the exit strategy, in particular by penalizing large density regions.

The optimality conditions of (13) can be (again formally) derived via the Lagrange functional

$$L_T(\rho, v, \phi) = I_T(\rho, v) + \int_0^T \int_{\Omega} (\partial_t \rho + \nabla \cdot (G(\rho)v) - \frac{\sigma^2}{2} \Delta \rho) \phi \, dx \, dt + IC, \quad (14)$$

where IC corresponds to the last term in (8) (accounting for the initial condition for ρ). For the optimal solution we have

$$0 = \partial_v L_T(\rho, v, \phi) = F(\rho)v - G(\rho)\nabla\phi, \quad (15)$$

and

$$0 = \partial_{\rho} L_T(\rho, v, \phi) = \frac{1}{2} F'(\rho) |v|^2 + \frac{1}{2} E'(\rho) - \partial_t \phi - G'(\rho)v \cdot \nabla \phi - \frac{\sigma^2}{2} \Delta \phi, \quad (16)$$

with terminal condition $\phi(x, T) = 0$. Inserting $v = \frac{G}{F} \nabla \phi$ we obtain the optimality system

$$\partial_t \rho + \nabla \cdot \left(\frac{G(\rho)^2}{F(\rho)} \nabla \phi \right) - \frac{\sigma^2}{2} \Delta \rho = 0 \quad (17a)$$

$$\partial_t \phi + \frac{1}{2} \left(2 \frac{GG'}{F} - \frac{F'G^2}{F^2} \right) |\nabla \phi|^2 + \frac{\sigma^2}{2} \Delta \phi = \frac{1}{2} E'(\rho), \quad (17b)$$

again a mean field game.

2.4. Momentum Formulation. In order to reduce ambiguities in the definition of a velocity and also to simplify the constraint PDE, we use a momentum or flux-based formulation in the following, cf. [2, 5]. The flux density (or momentum) is given by $j = G(\rho)v$, hence the functional to be minimized in (13) becomes

$$\tilde{I}_T(\rho, j) = \frac{1}{2} \int_0^T \int_{\Omega} \frac{|j|^2}{H(\rho)} \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Omega} E(\rho) \, dx \, dt, \quad (18a)$$

with $H := \frac{G^2}{F}$, subject to

$$\partial_t \rho + \nabla \cdot j = \frac{\sigma^2}{2} \Delta \rho. \quad (18b)$$

We see the redundancy of F and G in the above formulation, effectively only the fraction $H := \frac{G^2}{F}$ determines the different cases. We mention that in order to obtain a rigorous formulation, we replace $\frac{|j|^2}{H(\rho)}$ in (18a) by

$$K(j, \rho) := \begin{cases} \frac{|j|^2}{H(\rho)} & \text{if } H(\rho) \neq 0 \\ 0 & \text{if } j = 0, H(\rho) = 0 \\ +\infty & \text{if } j \neq 0, H(\rho) = 0. \end{cases} \quad (19)$$

The Lagrange functional for this problem is given by

$$\tilde{L}_T(\rho, v, \phi) = \tilde{I}_T(\rho, j) + \int_0^T \int_{\Omega} (\partial_t \rho + \nabla \cdot j - \frac{\sigma^2}{2} \Delta \rho) \phi \, dx \, dt + IC, \quad (20)$$

and the optimality conditions are

$$\partial_t \rho + \nabla \cdot (H(\rho)\nabla\phi) - \frac{\sigma^2}{2} \Delta \rho = 0 \quad (21a)$$

$$\partial_t \phi + \frac{H'(\rho)}{2} |\nabla \phi|^2 + \frac{\sigma^2}{2} \Delta \phi = \frac{1}{2} E'(\rho). \quad (21b)$$

We recall that the rigorous formulation of the problem in terms of the boundary conditions will be performed in subsection 2.6.

2.5. Relation to the Hughes Model. For vanishing viscosity $\sigma = 0$ the optimality system (17) respectively (21) has a similar structure as the Hughes model for pedestrian flow, see [28], which reads as

$$\partial_t \rho - \nabla \cdot (\rho f^2(\rho) \nabla \phi) = 0 \quad (22a)$$

$$|\nabla \phi| = \frac{1}{f(\rho)}. \quad (22b)$$

Hughes proposed that pedestrians seek the fastest path to the exit, but at the same time try to avoid congested areas. Let ρ_{\max} denote the maximum density, then the function $f(\rho)$ models how pedestrians change their direction and velocity due to the overall density. A common choice for example is $f(\rho) = \rho_{\max} - \rho$.

To obtain the connection with the Hughes model we choose $H(\rho) = \rho f(\rho)^2$, $E(\rho) = c\rho$, then (21) reads as

$$\partial_t \rho + \nabla \cdot (\rho f(\rho)^2 \nabla \phi) = 0 \quad (23a)$$

$$\partial_t \phi + \frac{f(\rho)}{2} (f(\rho) + 2\rho f'(\rho)) |\nabla \phi|^2 = \frac{c}{2}. \quad (23b)$$

For large T we expect equilibration of ϕ backward in time. Then for time t of order one the limiting model (very formal) becomes

$$\partial_t \rho + \nabla \cdot (\rho f(\rho)^2 \nabla \phi) = 0 \quad (24a)$$

$$(f(\rho) + 2\rho f'(\rho)) |\nabla \phi|^2 = \frac{c}{f(\rho)}, \quad (24b)$$

which is almost equivalent to the Hughes model (22), if we set $c = 1$. Note that the opposite sign in (22) and (24) is not an actual difference, since due to the signs in the backward equation we shall obtain ϕ as the negative of the distance function used in the Hughes model. The major difference results from the term $2f'(\rho)$, which is however severe. For $f(\rho) = \rho_{\max} - \rho$ we have

$$f(\rho) + 2\rho f'(\rho) = \rho_{\max} - 3\rho.$$

Thus for small densities the behavior is similar, but the singular point is $\rho = \frac{\rho_{\max}}{3}$. A possible conclusion is that the coupling between forward and backward equations in the optimality system is too strong if large densities appear and hence the equilibration of ϕ may not hold. The latter can only hold for a stronger decoupling by a mean-field game, which we shall discuss in Section 5.

A (so far partial) attempt to a rigorous mathematical theory for the original Hughes model (22) has been provided in [1]. Previous results [17, 15] considered smoothed version of that model. El-Khatib et al. first considered a variant with a cost in the Eikonal equation not related to $f(\rho)$, cf. [21], and finally Gomes and Saude provided a priori estimates for (23) in [23].

2.6. Boundary Conditions. We finally turn to the modeling of the boundary conditions, which we have neglected in the computations above. To do so we consider (18) on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 1, 2$, corresponding to a room with one or several exits. We assume that the boundary $\partial\Omega$ is split into a Neumann part $\Gamma_N \subseteq \partial\Omega$, modeling walls, and the exits $\Gamma_E \subseteq \partial\Omega$, $\partial\Omega = \Gamma_N \cup \Gamma_E$ and $\Gamma_N \cap \Gamma_E = \emptyset$. We now formulate natural boundary conditions on the density, which shall lead to adjoint boundary conditions for ϕ .

On the Neumann boundary Γ_N we clearly have no outflux, hence naturally

$$\left(-\frac{\sigma^2}{2}\nabla\rho + j\right) \cdot n = 0,$$

where n denotes the unit outer normal vector. The exit part is more difficult. Here the outflux depends on how fast people can leave the room. If we denote the rate of passing the exit by β , then we have an outflow proportional to $\beta\rho$. Hence, we arrive at the Robin boundary condition

$$\left(-\frac{\sigma^2}{2}\nabla\rho + j\right) \cdot n = \beta\rho. \quad (25)$$

Then the boundary conditions for the adjoint variable ϕ can be calculated again from the optimality conditions. From the corresponding Lagrangian we obtain:

$$\begin{aligned} L_T(\rho, j, \phi) &= I_T(\rho, j) + \int_0^T \int_{\Omega} (\partial_t \rho + \nabla \cdot j - \frac{\sigma^2}{2} \Delta \rho) \phi \, dx \, dt \\ &= I_T(\rho, j) + \int_0^T \int_{\Omega} [\rho(-\partial_t \phi - \frac{\sigma^2}{2} \Delta \phi) - j \cdot \nabla \phi] \, dx \, dt \\ &\quad + \int_0^T \int_{\partial\Gamma_E} \left(-\frac{\sigma^2}{2} \nabla \rho \cdot n \phi + \frac{\sigma^2}{2} \rho \nabla \phi \cdot n + j \cdot n \phi\right) \, ds \, dt \\ &\quad + \int_0^T \int_{\partial\Gamma_N} \frac{\sigma^2}{2} \rho \nabla \phi \cdot n \, ds \, dt \\ &= I_T(\rho, j) + \int_0^T \int_{\Omega} \rho(-\partial_t \phi - \frac{\sigma^2}{2} \Delta \phi - j \cdot \nabla \phi) \, dx \, dt \\ &\quad + \int_0^T \int_{\partial\Gamma_E} (\beta \rho \phi + \frac{\sigma^2}{2} \rho \nabla \phi \cdot n) \, ds \, dt + \int_0^T \int_{\partial\Gamma_N} \frac{\sigma^2}{2} \rho \nabla \phi \cdot n \, ds \, dt. \end{aligned} \quad (26)$$

In this form we see that the optimality condition with respect to ρ , i.e. $\partial_{\rho} L_T = 0$, yields the following boundary conditions for the adjoint variable ϕ in addition to the adjoint PDE:

$$\frac{\sigma^2}{2} \nabla \phi \cdot n + \beta \phi = 0 \text{ on } \Gamma_E \quad \text{and} \quad \frac{\sigma^2}{2} \nabla \phi \cdot n = 0 \text{ on } \Gamma_N. \quad (27)$$

One observes that our model provides meaningful and easy to interpret boundary conditions for the density ρ as well as for the adjoint variable ϕ . A homogeneous Dirichlet boundary condition for ϕ on doors arises only in the asymptotic limit $\beta \rightarrow \infty$. In this case one also needs to specify a homogeneous Dirichlet boundary condition for ρ for consistence. This is in contrast to previous models, which frequently used Dirichlet boundary conditions for ϕ , but arbitrary Dirichlet values for ρ or an outflux boundary condition as above with finite β .

An important focus of this paper is to understand how solutions of the proposed mean field model differ from the Hughes' model (22). Therefore we discuss appropriate choices of boundary conditions for either model, which allow for a comparison. In the case of the Hughes model (22) the potential ϕ is assumed to satisfy homogeneous Dirichlet boundary conditions, i.e. $\phi = 0$, while the density ρ either satisfies a Dirichlet boundary conditions of type $\rho = \rho_D$ or an outflux condition as in (25). There are two possibilities to compare the behavior of solutions of (22) and (23):

- Either by prescribing a density at the exit, i.e. $\rho = \rho_D$ and homogeneous Dirichlet boundary conditions for the adjoint, i.e. $\phi = 0$. This, however, is not realistic in practical applications.

- Or by choosing outflux boundary conditions for the density ρ , as in (25) with a large proportionality constant β , and the corresponding no-flux boundary conditions (27) for ϕ . The latter choice for the potential ϕ is not compatible with (22), but approximates the homogeneous Dirichlet boundary condition (typically chosen in the Hughes model) for large values of β .

Note that no flux boundary conditions for the density ρ , i.e. $\beta = 0$ in (25) would result in a homogeneous Neumann boundary conditions (27) for ϕ . Such boundary conditions are not compatible with the Eikonal equation (22b) in the Hughes model and also contradict the initial modeling assumptions of a fast exit scenario.

3. Analysis of the Optimal Control Model.

3.1. Existence of Minimizers. In this Section we discuss the existence and uniqueness of minimizers for the optimization problem (13), respectively the reformulation (18). In order to avoid any ambiguity with the formulations we can simply set

$$F = G = H,$$

which will remain as a standing assumption throughout the rest of this section. Let us fix the maximal density to be $\rho_{\max} > 0$, and $\Upsilon = [0, \rho_{\max}]$. Typically $\rho_{\max} = 1$. We make the following basic assumption on F and E :

- (A1) $F = F(\rho) \in C^1(\mathbb{R})$, F bounded, $E = E(\rho) \in C^1(\mathbb{R})$ and $F(\rho) \geq 0$, $E(\rho) \geq 0$ for $\rho \in \Upsilon$.

Existence of minimizers is guaranteed if

- (A2) $E = E(\rho)$ is convex.

As recalled in subsection 2.5, models with maximal density are of great importance in the applied setting, therefore we shall consider the special class of mobilities F , which yields a density uniformly bounded by a maximal density. In order to force the minimizers of the problem to satisfy $\rho \in \Upsilon$, we extend F to $F(\rho) = 0$ on $\rho \in \mathbb{R} \setminus \Upsilon$. This defines the next assumption on F , i.e.

- (A3) $F(0) > 0$ if $\rho \in \Upsilon$ and $F = 0$ otherwise.

Note that assumption (A3) is satisfied in particular by $F(\rho) = \rho f(\rho)$ in the Hughes model in subsection 2.5. To show uniqueness we need the additional assumption that

- (A4) $F = F(\rho)$ is concave.

We consider the optimization problem on the set $V \times Q$, i.e. $I_T(\rho, v) : V \times Q \rightarrow \mathbb{R}$, where V and Q are defined as follows

$$V = L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)) \text{ and } Q = L^2(\Omega \times (0, T)). \quad (28)$$

Hence the optimization problem (13) reads as

$$\min_{(\rho, v) \in V \times Q} I_T(\rho, v) \text{ such that } \partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \nabla \cdot (F(\rho)v), \quad (29)$$

respectively in momentum formulation

$$\min_{(\rho, j) \in V \times Q} \tilde{I}_T(\rho, j) \text{ such that } \partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \nabla \cdot (j). \quad (30)$$

For the existence proof in the case of non-concave F , we introduce another formulation based on the rather nonphysical variable

$$w = \sqrt{F(\rho)}v,$$

which has been used already in the case of linear F in [8]. Then the functional I_T can be re-written as

$$J(\rho, w) = \frac{1}{2} \int_0^T \int_{\Omega} (|w|^2 + E(\rho)) \, dx \, dt, \quad (31)$$

and the optimization problem formally becomes

$$\min_{(\rho, w) \in V \times Q} J(\rho, w) \text{ such that } \partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \nabla \cdot (\sqrt{F(\rho)} w). \quad (32)$$

In the following we shall provide a more rigorous definition of the minimization problem to ensure that the well definedness of the differential constraint is satisfied and the Robin boundary conditions are incorporated in the functional setting considered.

In order to make the relation among the problems (29), (30), and (32) rigorous, we need to extend the domain of the velocity v to

$$\tilde{Q}_\rho := \{v \text{ measurable} \mid \sqrt{F(\rho)} v \in Q\}. \quad (33)$$

Moreover, for given ρ we define an extension mapping $w \in Q$ to $v \in \tilde{Q}_\rho$ via

$$R_\rho(w)(x) := \begin{cases} \frac{w(x)}{\sqrt{F(\rho(x))}} & \text{if } F(\rho(x)) \neq 0, \\ 0 & \text{else.} \end{cases} \quad (34)$$

With this notation, we easily get (we omit the details)

Lemma 3.1. *Let (A1) be satisfied and let $\sigma > 0$. Then the following relations hold:*

- *If $(\rho, v) \in V \times \tilde{Q}_\rho$ solves (29), then $(\rho, \sqrt{F(\rho)} v) \in V \times Q$ solves (32). If $(\rho, w) \in V \times Q$ solves (32), then $(\rho, R_\rho(w)) \in V \times \tilde{Q}_\rho$ solves (29).*
- *If $(\rho, w) \in V \times \tilde{Q}_\rho$ solves (32), then $(\rho, \sqrt{F(\rho)} w) \in V \times Q$ solves (30). If $(\rho, j) \in V \times Q$ solves (30), then $(\rho, R_\rho(j)) \in V \times \tilde{Q}_\rho$ solves (32).*

As a consequence of the above Lemma, we see that if one of the models has a minimizer with $F(\rho)$ uniformly bounded away from zero, the minimizer is in $V \times Q$ and minimizes all three models. We are now ready to state the minimization problem (29) and its reformulation (32) rigorously, which incorporates the Robin boundary conditions and defines the problem in the proper functional setting.

Definition 3.2 (Minimisation problem, 1st formulation). Let $\rho_0 \in L^2(\Omega)$. A pair $(\rho, v) \in V \times \tilde{Q}_\rho$ is a *weak solution* to the minimization problem (29) with initial condition ρ_0 , if $\rho(0) = \rho_0$ and

$$\langle \partial_t \rho, \psi \rangle_{H^{-1}, H^1} + \int_{\Omega} \left(\frac{\sigma^2}{2} \nabla \rho - F(\rho) v \right) \cdot \nabla \psi \, dx = - \int_{\Gamma_E} \beta \rho \psi \, ds, \quad (35)$$

for all $\psi \in H^1(\Omega)$, and if

$$I_T(\rho, v) = \min \{ I_T(\bar{\rho}, \bar{v}), : (\bar{\rho}, \bar{v}) \in V \times Q, (\bar{\rho}, \bar{v}) \text{ satisfy (35)} \}.$$

A similar re-formulation of (30) in terms of the momentum variable j replacing v can be also deduced.

Definition 3.3 (Minimisation problem, 2nd formulation). Let $\rho_0 \in L^2(\Omega)$. A pair $(\rho, j) \in V \times \tilde{Q}_\rho$ is a *weak solution* to the minimization problem (30) with initial condition ρ_0 , if $\rho(0) = \rho_0$ and

$$\langle \partial_t \rho, \psi \rangle_{H^{-1}, H^1} + \int_{\Omega} \left(\frac{\sigma^2}{2} \nabla \rho - j \right) \cdot \nabla \psi \, dx = - \int_{\Gamma_E} \beta \rho \psi \, ds, \quad (36)$$

for all $\psi \in H^1(\Omega)$, and if

$$\tilde{I}_T(\rho, j) = \min \left\{ \tilde{I}_T(\bar{\rho}, \bar{j}), : (\bar{\rho}, \bar{j}) \in V \times Q, (\bar{\rho}, \bar{j}) \text{ satisfy (36)} \right\},$$

where

$$\tilde{I}_T(\rho, j) = \frac{1}{2} \int_0^T \int_{\Omega} K(j, \rho) dx dt + \frac{1}{2} \int_0^T \int_{\Omega} E(\rho) dx dt,$$

and K is defined as in (19).

Similarly, on the variables setting (ρ, w) we get:

Definition 3.4 (Minimisation problem, 3rd formulation). Let $\rho_0 \in L^2(\Omega)$. A pair $(\rho, w) \in V \times Q$ is a *weak solution* to the minimization problem (32) with initial condition ρ_0 , if $\rho(0) = \rho_0$ and

$$\langle \partial_t \rho, \psi \rangle_{H^{-1}, H^1} + \int_{\Omega} \left(\frac{\sigma^2}{2} \nabla \rho - \sqrt{F(\rho)} w \right) \cdot \nabla \psi dx = - \int_{\Gamma_E} \beta \rho \psi ds, \quad (37)$$

for all $\psi \in H^1(\Omega)$, and if

$$J_T(\rho, w) = \min \{ J_T(\bar{\rho}, \bar{w}), : (\bar{\rho}, \bar{w}) \in V \times Q, (\bar{\rho}, \bar{w}) \text{ satisfy (37)} \}.$$

We now prove our main existence result, which is based on the variable w . The result is valid for non-concave F , but relies on a positive viscosity σ . First, we provide an a-priori estimate:

Lemma 3.5. *Let $\rho_0 \in L^2(\Omega)$. Let (A1) and (A2) be satisfied and let $\sigma > 0$, $\beta \geq 0$. Let $w \in Q$ and let $\rho \in V$ be a weak solution of*

$$\langle \partial_t \rho, \psi \rangle_{H^{-1}, H^1} + \int_{\Omega} \left(\frac{\sigma^2}{2} \nabla \rho - \sqrt{F(\rho)} w \right) \cdot \nabla \psi dx = - \int_{\Gamma_E} \beta \rho \psi ds, \quad (38)$$

for all $\psi \in H^1(\Omega)$. Then there exist constants $C_1, C_2 > 0$ depending on F , σ , Ω and T only, such that

$$\|\rho\|_V \leq C_1 \|w\|_Q + C_2. \quad (39)$$

Proof. We use the test function $\psi = \rho(t)$ and obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^2 dx = - \frac{\sigma^2}{2} \int_{\Omega} |\nabla \rho|^2 dx + \int_{\Omega} \sqrt{F(\rho)} w \nabla \rho dx - \int_{\Gamma_E} \beta \rho^2 ds.$$

Since $F = F(\rho)$ is bounded, $w \in L^2(\Omega)$, and $\beta \geq 0$, by weighed Cauchy-Schwarz inequality integration in time we get

$$\frac{1}{2} \|\rho(t)\|_{L^2}^2 + \frac{\sigma^2}{4} \int_0^t \|\rho\|_{H^1}^2 d\tau \leq c_1 \int_0^t \|w\|_{L^2}^2 d\tau + \frac{1}{2} \|\rho_0\|_{L^2(\Omega)}^2,$$

with a constant $c_1 \in \mathbb{R}^+$ depending only on σ and F . For the time derivative we deduce that

$$\|\partial_t \rho\|_{H^{-1}(\Omega)} = \sup_{\|\psi\|_{H^1} \leq 1} |\langle \rho_t, \psi \rangle| \leq \sup_{\|\psi\|_{H^1} \leq 1} [c_2 \|\rho\|_{H^1} \|\psi\|_{H^1} + c_3 \|w\|_{L^2} \|\nabla \psi\|_{L^2}],$$

where the first term on the right-hand side includes the gradient terms in the domain as well as the boundary term via a trace theorem. Squaring and integrating in time finally yields the desired estimate for $\|\rho\|_V$. \square

The following technical Lemma is needed to ensure that assumption (A3) gives $\rho \in \Upsilon$. We provide the details for the sake of completeness, although the result is quite standard.

Lemma 3.6. *Let ρ and w satisfy the same assumptions as in Lemma 3.5, and assume further that F satisfies (A3). Then, $\rho(\cdot, t) \in \Upsilon$ for all $t \in (0, T]$ if $\rho_0(x) \in \Upsilon$.*

Proof. Assume first $w \in C^1$, and let $\eta_\delta(\rho)$ be a $C^{1,1}$ regularization of the positive part $(\rho)_+ = \max\{\rho, 0\}$ as $\delta \searrow 0$, such that $\eta'_\delta(\rho), \eta''_\delta(\rho) \geq 0$, and $\eta''_\delta(\rho) = \frac{1}{\delta} \chi_{0 < \rho < \delta}$. We can use the test function $\eta'(\rho - \rho_{\max})$ in Definition 3.4 to get

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \eta_\delta(\rho - \rho_{\max}) \, dx &= -\frac{\sigma^2}{2} \int_{\Omega} \eta''_\delta(\rho - \rho_{\max}) |\nabla \rho|^2 \, dx \\ &+ \int_{\Omega} \sqrt{F(\rho)} w \eta''_\delta(\rho - \rho_{\max}) \cdot \nabla \rho \, dx - \beta \int_{\Gamma_E} \rho \eta'_\delta(\rho - \rho_{\max}) \\ &\leq \int_{\Omega} \sqrt{F(\rho)} w \eta''_\delta(\rho - \rho_{\max}) \cdot \nabla \rho \, dx \\ &= - \int_{\Omega} \xi_\delta(\rho) \operatorname{div} w \, dx, \end{aligned} \tag{40}$$

where

$$\xi_\delta(\rho) := \int_{\rho_{\max}}^{\rho} \sqrt{F(z)} \eta''_\delta(z - \rho_{\max}) \, dz.$$

Now, it is clear that $\xi_\delta(\rho) = 0$ if $\rho < \rho_{\max}$. Moreover, for $\rho_{\max} < \rho < \rho_{\max} + \delta$ we have

$$\xi_\delta(\rho) = \frac{1}{\delta} \int_{\rho_{\max}}^{\rho} \sqrt{F(z)} \, dz \leq C,$$

since F is uniformly bounded. Finally, in the case $\rho > \rho_{\max} + \delta$, we have

$$\xi_\delta(\rho) = \frac{1}{\delta} \int_{\rho_{\max}}^{\rho_{\max} + \delta} \sqrt{F(z)} \, dz \rightarrow \sqrt{F(\rho_{\max})} = 0,$$

as $\delta \searrow 0$. Combining all these assumptions, we get $\xi_\delta(\rho)$ uniformly bounded and such that $\xi_\delta(\rho) \rightarrow 0$ almost everywhere on $\Omega \times [0, T]$. Since $w \in C^1$, we can integrate in time in (40) and send $\delta \searrow 0$ to get

$$\int_{\Omega} (\rho(x, t) - \rho_{\max})_+ \, dx \leq \int_{\Omega} (\rho_0(x) - \rho_{\max})_+ \, dx = 0,$$

which gives the assertion. The general case $w \in Q$ can be recovered by a standard approximation argument, we omit the details. \square

Theorem 3.7 (Existence in the general case). *Let $\rho_0 \in L^2(\Omega)$. Let (A1) and (A2) be satisfied and let $\sigma > 0$. Then the variational problem (32) has at least a weak solution $(\rho, w) \in V \times Q$ with initial condition ρ_0 in the sense of Definition 3.3. If in addition (A3) is satisfied, then $\rho \in \Upsilon$.*

Proof. We show the existence of a minimizer by the direct method. Let $(\rho_k, w_k) \in V \times Q$ be a minimizing sequence. Then we can assume w.l.o.g. that there exists a constant c , such that

$$\int_0^T \int_{\Omega} (|w_k|^2 + E(\rho_k)) \, dx \, dt \leq c.$$

Clearly, w_k is bounded in $L^2(\Omega \times (0, T))$ and hence has a weakly convergent subsequence with limit \hat{w} , again denoted by w_k . Now we use Lemma 3.5 to obtain boundedness of $\rho_k \in V$, from which we can extract another weakly convergent sub-sequence. From the Lemma of Aubin and Lions, cf. [35], we have a compact embedding of $\rho_k \in L^2(\Omega \times (0, T))$. Then the continuity and boundedness of F

implies that $\sqrt{F(\rho_k)}$ converges strongly in $L^2(\Omega \times (0, T))$ and the limit $\sqrt{F(\hat{\rho})}$ is bounded. Hence, the product $\sqrt{F(\rho_k)}w_k$ converges weakly in $L^1(\Omega \times (0, T))$ and the limit $\sqrt{F(\hat{\rho})}\hat{w}$ is in $L^2(\Omega \times (0, T))$. This allows us to pass to the limit in the weak formulation of the constraint equation. Thus, $(\hat{\rho}, \hat{w})$ is admissible, and the weak lower semicontinuity of the objective functional implies that it is indeed a minimizer. Note that the convexity of E is crucial in this case, as it provides the lower semi continuity of $\int_{\Omega} E(\rho_k)dx$. Assume now that (A3) is satisfied. Then, due to Lemma 3.6 for all k we have $\rho_k \in \Upsilon$. Then we can pass to the limit (possibly by extracting a further subsequence converging almost everywhere) to obtain the desired result. \square

We now provide an alternative argument, which can be extended to the non-viscous case for zero-flux flux boundary conditions, i. e. for $\beta = 0$. The argument below provides less regularity for the minimizer compared to the previous result, but the proof is much shorter.

Theorem 3.8 (Existence for concave mobility). *Let (A1), (A2), (A3), and (A4) be satisfied and let $\sigma > 0$. Then the variational problem (30) has at least one minimizer $(\rho, j) \in L^\infty(\Omega \times (0, T)) \times Q$ such that $\rho(x) \in \Upsilon$ for almost every $x \in \Omega$. If E is strictly convex, then the minimizer is unique.*

Proof. Let (ρ_k, j_k) be a minimizing sequence. Since $F(\rho_k)$ is bounded, j_k is uniformly bounded in $L^2(\Omega \times (0, T))$ and we can extract a weakly convergent subsequence with limit \hat{j} . By the boundedness of the objective functional, we have $F(\rho_k) > 0$ almost everywhere, which gives $\rho_k(x, t) \in \Upsilon$ for all k and for all $(x, t) \in \Omega \times [0, T]$. From the latter we can extract a sub sequence, again denoted by ρ_k such that $\rho_k \rightharpoonup^* \hat{\rho}$ in $L^\infty(\Omega \times (0, T))$, respectively weakly in L^p for $p \in (1, \infty)$. Since the constraint PDE is linear we can easily pass to the limit in a weak formulation and see that

$$\partial_t \hat{\rho} + \nabla \cdot (\hat{j}) = \frac{\sigma^2}{2} \Delta \hat{\rho},$$

holds in a weak sense. Moreover, the convexity of the objective functional guaranteed by (A2), (A3), and (A4) (recall that $1/F(\rho)$ is extended to $+\infty$ when $F(\rho) = 0$) implies weak semi-continuity and hence $(\hat{\rho}, \hat{j})$ is a minimizer. The uniqueness follows from a standard strict convexity argument. \square

Remark 1. The result in Theorem 3.8 can be extended for $\sigma = 0$ in the case of no-flux boundary conditions, i.e. $\beta = 0$. For this choice we can trivially pass to the limit $k \rightarrow +\infty$, a strategy not possible for $\beta > 0$. Then a more refined analysis, based on the theory of nonlinear conservation laws with boundary conditions, cf. [4], is necessary.

Remark 2. We mention that under the above assumptions one might expect Gamma-convergence of the optimal control problems as $\sigma \rightarrow 0$. The lower semi-continuity arguments can be used to verify the lower bound inequality, however it is so far unclear how the upper bound inequality can be verified.

Note that the convexity constraint (A2) can be weakened, i.e. the function E has to be weakly lower semicontinuous in suitable function spaces to guarantee existence of minimizers. Since we only consider convex functions E , we did not state this more general result.

3.2. Adjoint Equations and Optimality System. In the following we investigate the existence of adjoints, i.e. the optimality condition with respect to the state variable ρ :

Proposition 1. *Let assumption (A1) and (A2) be satisfied and let ρ be such that $H(\rho) \geq \gamma$ for some $\gamma > 0$. Then the adjoint equation of problem (30), i.e.*

$$\partial_t \phi + \frac{\sigma^2}{2} \Delta \phi = \frac{1}{2} E'(\rho) - \frac{1}{2} |j|^2 \frac{F'}{F^2} \quad (41a)$$

$$\phi(x, T) = 0 \quad (41b)$$

with boundary conditions (27) has a unique solution $\phi \in L^q(0, T; W^{1,q}(\Omega))$ with $q < \frac{N+2}{N+1}$.

Note that the right hand side of (41) is only in $L^1(\Omega \times (0, T))$. Existence and uniqueness of solutions ϕ follows from the results of Boccardo and Gallouët, cf. [7]. However this result does not provide the necessary regularity for ϕ to define the Lagrange functional L_T properly.

Finally we investigate the full optimality system, where we eliminate $j = F(\rho)\nabla\phi$ and obtain

$$\partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \nabla \cdot (F(\rho)\nabla\phi) \quad (42a)$$

$$\partial_t \phi = -\frac{\sigma^2}{2} \Delta \phi + \frac{1}{2} E'(\rho) - \frac{1}{2} F'(\rho) |\nabla\phi|^2. \quad (42b)$$

with boundary conditions

$$\left(-\frac{\sigma^2}{2} \nabla \rho + F(\rho)\nabla\phi \right) \cdot n = \begin{cases} 0 & \text{on } \Gamma_N \\ \beta\rho & \text{on } \Gamma_E \end{cases} \quad \frac{\sigma^2}{2} \nabla\phi \cdot n = \begin{cases} 0 & \text{on } \Gamma_N \\ -\beta\phi & \text{on } \Gamma_E. \end{cases} \quad (42c)$$

To show uniqueness of the forward backward system (42) we follow an approach proposed by Lasry and Lions for mean field games, cf. [32].

Theorem 3.9 (Uniqueness for the optimality system). *For a fixed initial condition $\rho_0 \in L^2(\Omega)$, there exists a unique weak solution*

$$(\rho, \phi) \in L^2(0, T; H^1(\Omega)) \times L^2(0, T; H^1(\Omega))$$

to the optimality system (42) with boundary conditions (42c).

Proof. Assume there exists two classical solutions (ϕ_1, ρ_1) and (ϕ_2, ρ_2) to the optimality system (42). Let $\bar{\rho}$ and $\bar{\phi}$ denote the difference between the solutions, i.e. $\bar{\rho} = \rho_1 - \rho_2$ and $\bar{\phi} = \phi_1 - \phi_2$. Then $\bar{\rho}$ and $\bar{\phi}$ satisfy

$$\partial_t \bar{\rho} = \frac{\sigma^2}{2} \Delta \bar{\rho} - \nabla \cdot (F(\rho_1)\nabla\phi_1 - F(\rho_2)\nabla\phi_2) \quad (43a)$$

$$\partial_t \bar{\phi} = -\frac{\sigma^2}{2} \Delta \bar{\phi} + \frac{1}{2} E'(\rho_1) - \frac{1}{2} E'(\rho_2) - \frac{1}{2} F'(\rho_1) |\nabla\phi_1|^2 + \frac{1}{2} F'(\rho_2) |\nabla\phi_2|^2. \quad (43b)$$

Note that the boundary conditions for the differences $\bar{\rho}$ and $\bar{\phi}$ on Γ_E read as:

$$\left(-\frac{\sigma^2}{2} \nabla \bar{\rho} + (F(\rho_1)\nabla\phi_1 - F(\rho_2)\nabla\phi_2) \right) \cdot n = \beta\bar{\rho} \quad \text{and} \quad \frac{\sigma^2}{2} \nabla \bar{\phi} + \beta\bar{\phi} = 0. \quad (44)$$

Using (43) we calculate (integrating by parts):

$$\begin{aligned}
 \frac{d}{dt} \int_{\Omega} \bar{\phi} \bar{\rho} \, dx &= \int_{\Omega} (\bar{\phi} \partial_t \bar{\rho} + \bar{\rho} \partial_t \bar{\phi}) \, dx \\
 &= \int_{\Omega} ((F(\rho_1) \nabla \phi_1 - F(\rho_2) \nabla \phi_2) \nabla \bar{\phi}) \, dx + \frac{1}{2} \int_{\Omega} ((E'(\rho_1) - E'(\rho_2)) \bar{\rho}) \, dx \\
 &\quad - \int_{\Omega} \frac{1}{2} (F'(\rho_1) |\nabla \phi_1|^2 - F'(\rho_2) |\nabla \phi_2|^2) \bar{\rho} \, dx \\
 &= \int_{\Omega} [(F(\rho_1) \nabla \phi_1 - F(\rho_2) \nabla \phi_2) \nabla \bar{\phi} - \frac{1}{2} (F'(\rho_1) |\nabla \phi_1|^2 - F'(\rho_2) |\nabla \phi_2|^2) \bar{\rho}] \, dx \\
 &\quad + \frac{1}{2} \int_{\Omega} (E'(\rho_1) - E'(\rho_2)) \bar{\rho} \, dx.
 \end{aligned}$$

Note that due to the boundary conditions (44), all boundary terms that result from the integration by parts in the previous calculation vanish. Due to the concavity of F we deduce that

$$\begin{aligned}
 (F(\rho_1) \nabla \phi_1 - F(\rho_2) \nabla \phi_2) \nabla \bar{\phi} - \frac{1}{2} (F'(\rho_1) |\nabla \phi_1|^2 - F'(\rho_2) |\nabla \phi_2|^2) \bar{\rho} &= \\
 = (F(\rho_1) - \frac{1}{2} F'(\rho_1) \bar{\rho}) |\nabla \phi_1|^2 + (F(\rho_2) + \frac{1}{2} F'(\rho_2) \bar{\rho}) |\nabla \phi_2|^2 - (F(\rho_1) + F(\rho_2)) \nabla \phi_1 \nabla \phi_2 & \\
 \geq \frac{1}{2} (F(\rho_1) + F(\rho_2)) (\nabla \phi_1 - \nabla \phi_2)^2. &
 \end{aligned}$$

Since E is in $C^1(\Upsilon)$ and convex, E' is monotone and we obtain

$$\frac{d}{dt} \int_{\Omega} \bar{\phi} \bar{\rho} \, dx \geq \int_{\Omega} \frac{1}{2} (F(\rho_1) + F(\rho_2)) (\nabla \phi_1 - \nabla \phi_2)^2 \, dx.$$

We integrate over the interval $[0, T]$ and obtain (since $\bar{\rho}(x, 0) = 0$ and $\bar{\phi}(x, T) = 0$) that:

$$\int_{\Omega} \frac{1}{2} (F(\rho_1) + F(\rho_2)) (\nabla \phi_1 - \nabla \phi_2)^2 \, dx \leq 0. \tag{45}$$

The function F is positive, therefore (45) implies that

$$\nabla \phi_1 = \nabla \phi_2 \text{ in } \{\rho_1 > 0\} \cup \{\rho_2 > 0\}.$$

Then ρ_1 is a solution of $\partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \nabla \cdot (F(\rho) \nabla \phi_2) = 0$ and we conclude (by uniqueness) that $\rho_1 = \rho_2$. \square

4. Numerical simulations. In this Section we present a steepest descent approach for the solution of the parabolic optimal control problem (13). This method can be used for convex optimization problems and defines an iterative scheme to determine the optimal velocity v .

The presented numerical simulations focus on the different behavior of the classical model of the Hughes (22) and the corresponding general mean field modification (23). This comparison requires the simulation of (22), which is done using a finite volume scheme for the nonlinear conservation law and a fast sweeping method for the Eikonal equation, as presented by Di Francesco et al. in [17].

The steepest descent scheme for (13) can be written as:

Scheme 1. Let $\rho_0 = \rho_0(x)$, $\phi(x, t_k) = \phi_H(x)$, where ϕ_H is the solution of the Eikonal equation in the classical Hughes model (22b) with $\rho(x, t_k) = \rho_0(x)$ for all

discrete times $t_k \in [0, T]$ and $v = \frac{G(\rho)}{F(\rho)} \nabla \phi$, be the given initial data. Then the steepest descent scheme reads as:

1. Solve the nonlinear convection diffusion equation

$$\partial_t \rho = \frac{\sigma^2}{2} \Delta \rho - \nabla \cdot (F(\rho)v) \quad (46a)$$

$$\left(-\frac{\sigma^2}{2} \nabla \rho + j\right) \cdot n = \beta \rho \text{ on } \Gamma_E \quad \text{and} \quad \left(-\frac{\sigma^2}{2} \nabla \rho + j\right) \cdot n = 0 \text{ on } \Gamma_N, \quad (46b)$$

forward in time. System (46) is solved implicitly in time. The resulting nonlinear equation $A(\rho, v) = 0$ is discretized using a mixed hybrid discontinuous Galerkin (MHDG) method, cf. [20], and solved using Newton's method.

2. Calculate the backward evolution of the adjoint variable $\phi = \phi(x, t)$ using the previously calculated density $\rho = \rho(x, t)$ in

$$\begin{aligned} -\partial_t \phi - \frac{\sigma^2}{2} \Delta \phi - G'(\rho)v \cdot \nabla \phi &= -\frac{1}{2} F'(\rho)|v|^2 - \frac{1}{2} E'(\rho), \\ -\frac{\sigma^2}{2} \nabla \phi \cdot n - \beta \phi &= 0 \text{ on } \Gamma_E \quad \text{and} \quad -\frac{\sigma^2}{2} \nabla \phi \cdot n = 0 \text{ on } \Gamma_N, \end{aligned}$$

with an implicit in time discretization and a MHDG method for the spatial discretization.

3. Update the velocity via $v = v - \tau(F(\rho)v - G(\rho)\nabla \phi)$, where τ is a suitably chosen step size.
4. Go to (1) until convergence of the functional (13a).

Egger and Schöberl presented a MHDG method for linear convection dominated problems, which can be adapted for the nonlinear problems considered. We use the following basis functions for the discretization, i.e.

$$\rho, \phi, v \in P^0(T) \text{ and } \nabla \rho, \nabla \phi \in RT^0(T),$$

where P^0 denotes piecewise constant basis function on the interval T and RT^0 lowest order Raviart-Thomas basis functions. The Newton iteration in (46) is terminated, if $\|A(\rho, v)\|_{L^2(\Omega)} \leq 10^{-6}$. The line search parameter τ was set to 1 in all presented numerical simulations, unless stated otherwise.

Throughout this section we consider the domain $\Omega = [-1, 1]$ with exits located at $x = \pm 1$. The interval Ω is divided into a set of equidistant intervals of length h . Furthermore the maximum density ρ_{\max} is set to 1.

4.1. Comparison of the classical and the mean field type Hughes model.

We have seen in Section 2.5 that the proposed mean field model has a similar structure as the classical model by Hughes. Hence we want illustrate the behavior of both models with various experiments. Throughout this section we set the weighting parameter α in (3) equal to one. In the first example we focus on the influence of the boundary conditions as discussed in Section 2.6. In the second example we illustrate the basic difference of the mean field game approach and the classical model of Hughes.

4.1.1. *Behavior for different values of β .* We consider the time interval $t \in [0, 3]$. In the classical Hughes model (22) the boundary conditions are set to

$$\phi(x, t) = 0 \text{ and } \left(-\frac{\sigma^2}{2} \Delta \rho + \rho f(\rho) \nabla \phi\right) \cdot n = \beta \rho \quad x \in \{-1, 1\},$$

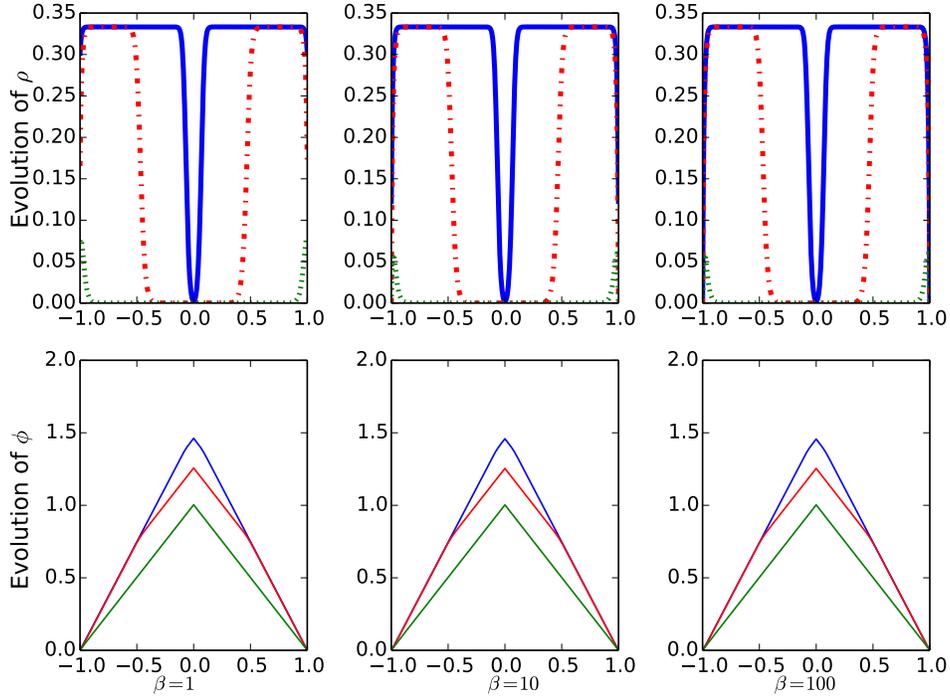


FIGURE 1. Solution of the classical Hughes model (22) with initial datum $\rho_0(x) = \frac{1}{3}$ at times $t = 0.1, 0.7, 1.5$ for different values of α

with $f(\rho) = 1 - \rho$. For the corresponding mean field model (23) the functions ρ and ϕ satisfy (25) and (27) at $x = \pm 1$. In both cases we use the same diffusion coefficient, i.e $\sigma = 0.1$. The spatial discretization is set to $h = 5 \times 10^{-2}$ in (22) and $h = 10^{-3}$ in (23), the time steps to $\Delta t = 10^{-5}$ and $\Delta t = 10^{-1}$ respectively. The different magnitudes of the time stepping can be explained by the explicit in time discretization of (22) and the implicit time discretization of (23). Figures 1 and 2 show the evolution in time of the solutions of (22) and (23) for different values of β . Although the models have a very similar structure, their behavior is different. In the mean field model a small congestion at the boundary is visible in the case of $\beta = 1$. Furthermore we do not observe the immediate vacuum formation at $x = 0$ as in the Hughes model (22). People rather tend to “wait for a little while” at the center and then start to move at a higher speed. The expected equilibration of ϕ in (23) is clearly visible in Figure 2 for all values of β .

4.1.2. *Fast exit of several groups.* In this example we consider three groups, which want to leave the domain as fast as possible. The particular initial datum is given by

$$\rho_0(x) = \begin{cases} 0.8 & \text{if } -0.8 \leq x \leq -0.6 \\ 0.6 & \text{if } -0.3 \leq x \leq 0.3 \\ 0.95 & \text{if } 0.4 \leq x \leq 0.8 \\ 0 & \text{otherwise.} \end{cases}$$

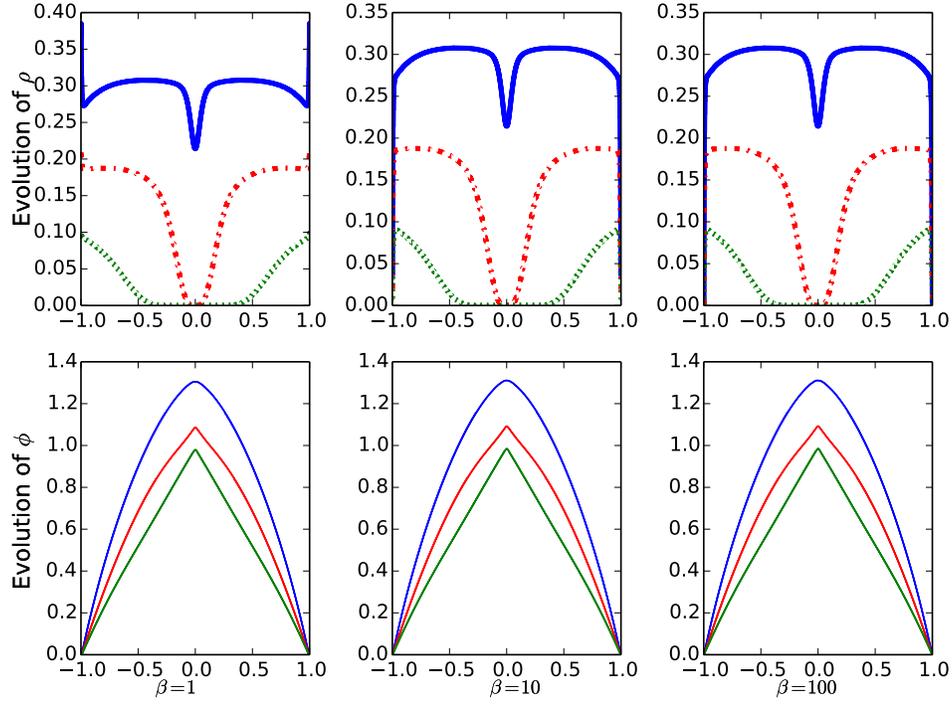


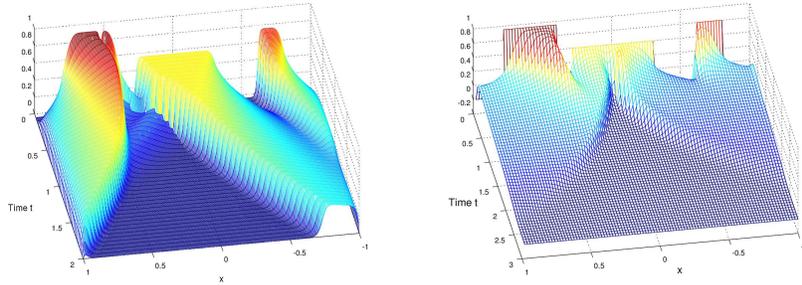
FIGURE 2. Solution of the mean field type Hughes model (23) with initial datum $\rho_0(x) = \frac{1}{3}$ at times $t = 0.1, 0.7, 1.5$ for different values of α

A similar example was already considered in [17], where the simulations showed that a small part of the group located around $x = 0.6$ initially splits to move to the exit at $x = -1$, but later on turns around to take the closer exit at $x = 1$. This behavior can be explained by the fact that the pedestrians in the Hughes model (22) adapt their velocity in every time step (depending on the overall density at that time). Due to the initially high density of people located in front of exit $x = 1$, parts of the group start to move towards $x = -1$, but turn around when the density is decreasing as more and more people exit.

We do not expect to observe this behavior for the modified Hughes model (23). One of the underlying features of the proposed optimal control approach, is the fact that each pedestrian knows the distribution of all other people at all times. Therefore he/she is anticipating the behavior of the group in the future and will in this case rather wait than move to the more distant exit. This expected behavior can be observed in Figure 3. Here the mean field model (23) was solved using the steepest descent approach detailed before. The parameters were set to:

$$h = 10^{-3}, \Delta t = 5 \times 10^{-2}, T = 3 \text{ and } \beta = 1.$$

Note that the optimal control formulation of (23) is not a convex problem (the function $H(\rho) = \rho(1 - \rho)^2$ is only concave for $\rho \in [0, \frac{2}{3}]$), but the steepest descent approach converged without any problems. For the Hughes model we chose the



(a) Solution of the classical Hughes model (22) (b) Solution of the mean field optimal control approach (23)

FIGURE 3. Fast exit scenario for three groups

following parameters:

$$h = 5 \times 10^{-2}, \Delta t = 5 \times 10^{-5} \text{ and } \beta = 1.$$

4.2. **Linear vs. nonlinear E .** In the final example we illustrate the behavior for different functions $E = E(\rho)$. In particular we set

$$F(\rho) = G(\rho) = \rho \quad \text{and} \quad E(\rho) = \begin{cases} 3\rho \\ e^{3\rho} \end{cases}.$$

The later choice of E actively penalizes regions of high density and urges the group to get out more efficiently. In this example the group is initially located at

$$\rho_0(x) = \begin{cases} 0.5 & \text{if } -0.25 \leq x \leq 0.4 \\ 0 & \text{otherwise.} \end{cases}$$

The parameters are given by $h = 10^{-3}, \Delta t = 0.1, T = 3$ and $\beta = 1$. In this example we set the line search parameter τ to $\frac{1}{2}$ for the nonlinear choice of E to ensure a stable descent at the beginning. Figure 4 shows the evolution of the density ρ for the different choices of E in time. As expected we observe that the group exits much faster and spreads out more evenly for $E = e^{3\rho}$ than in the linear case.

5. Understanding the Hughes model from a mean field game perspective.

In this last section we give a final interpretation of the Hughes model (22) in terms of mean field game approaches. We already discussed that the agents in the model of Hughes adapt their velocity in every time step according to the actual observed density at that time, see example 4.1.2. In the mean field game approach all agents anticipate the future, therefore the velocity is determined by a Hamilton-Jacobi equation. To account for this difference we propose a modified mean field approach by extrapolating the current density into the future, i.e. $\rho = \rho(x, t)$ is assumed to be the density of the system for all times $s > t$.

Let us consider N particles with position $X_k = X_k(t)$ and the empirical density

$$\rho^N(t) = \frac{1}{N} \sum_{k=1}^N \delta(x - X_k(t)).$$

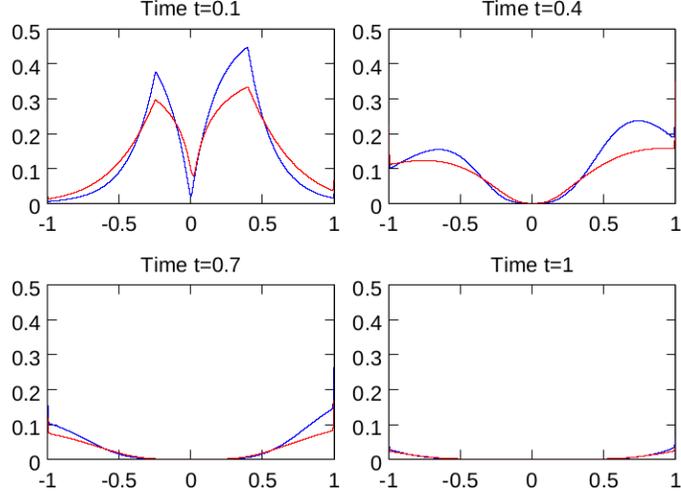


FIGURE 4. Evolution of the density ρ for different energies in time. The blue line corresponds to $E = 3\rho$, the red one to $E = e^{3\rho}$.

To define the cost functional in a proper way we introduce the smoothed approximation ρ_g^N by

$$\rho_g^N(t) = (\rho^N * g)(x, t) = \frac{1}{N} \sum_{k=1}^N g(x - X_k(t)),$$

where g is a sufficiently smooth positive kernel.

Following the ideas of Section 2 we state that the optimal velocity of a particle at location $X = X(t)$ is determined by minimizing the following cost functional:

$$C(X; \rho(t)) = \min_{(X, V)} \frac{1}{2} \int_t^{T+t} \frac{|V(s)|^2}{G(\rho_g^N(\xi(s; t), t))} ds + \frac{1}{2} T_{exit}(X, V), \quad (47)$$

subject to $\frac{d\xi}{ds} = V(s)$ and $\xi(0) = X(t)$. Note that the minimizer depends on X via the initial condition for ξ . This means that the particle looks for its optimal trajectory in the extrapolated configuration.

The above problem can be reformulated in the space of probability measures, since we effectively transport a δ -distribution, noticing that for sufficiently large $T > T_{exit}$:

$$T_{exit}(X, V) - t = \int_t^{T+t} \int_{\Omega} \delta_{\xi(s)} dx ds.$$

Moreover, since a macroscopically rescaled version of $\rho^N(t)$ converges to the mean-field $\rho(t)$ (at least in the weak-* sense of measures as $N \rightarrow \infty$), we follow the mean-field game paradigm and replace $\rho_g^N(t)$ by $\rho(t)$. Consequently

$$C(X; \rho(t)) = \min_{(\mu, w)} J(\mu, w) = \frac{1}{2} \int_t^{T+t} \int_{\Omega} \left(\frac{w^2(x, s)}{G(\rho(\xi(s; t), t))} + 1 \right) d\mu ds, \quad (48)$$

for the velocity field w and the probability measure μ satisfying $\mu(t=0) = \delta_X$ and

$$\partial_s \mu + \nabla \cdot (\mu w) = 0. \quad (49)$$

In the case of stochastic dynamics ($\nu \neq 0$) we can simply add a diffusion term to this equation. Then the cost can be interpreted as the expected value, a standard problem in stochastic optimal control.

Now let us formally compute the optimality conditions and $\nabla_X C$. For this sake we introduce the Lagrange functional

$$L_X(\mu, w, \phi, \psi, \rho(t)) = J(\mu, w) + \int_{\Omega} \phi \, d\mu(0) - \phi(X) - \int_t^{T+t} \int_{\Omega} (\partial_s \psi + w \cdot \nabla \psi) \, d\mu ds, \quad (50)$$

and use the fact that for a minimizer

$$\nabla_X C(X; \rho(t)) = \partial_X L_T(\mu, w, \phi, \psi, \rho^N(t)) = -\nabla \phi(X). \quad (51)$$

From $\partial_w L_X = 0$ we conclude that

$$w = G(\rho) \nabla \psi. \quad (52)$$

The derivative with respect to μ , i.e. $\partial_{\mu} L_X = 0$, gives $\psi(T+t) = 0$, $\psi(t) = -\phi$, and

$$\partial_s \psi + \frac{1}{2} G(\rho(t)) |\nabla \psi|^2 = \frac{1}{2}. \quad (53)$$

For $T \rightarrow 0$ the behavior at $s = t$ represents the long-time behavior of the Hamilton-Jacobi equation. Note that the Hamiltonian is independent of s . Hence we expect $\phi = -\psi(t)$ to solve

$$G(\rho(t)) |\nabla \psi|^2 = 1. \quad (54)$$

Then we recover the Hughes model by choosing $G(\rho) = f(\rho)^2$ i.e.

$$\partial_t \rho + \nabla \cdot (\rho f(\rho)^2 \nabla \phi) = 0, \quad (55a)$$

$$|\nabla \phi| = \frac{1}{f(\rho)}. \quad (55b)$$

Note that there is a different sign in ϕ as in the original model. This becomes apparent from the way the Eikonal equation is to be understood from the backward equation (also potential sign of diffusive regularization), we effectively compute the negative distance function.

Conclusion. In this paper we presented a novel mean field game approach for evacuation and fast exit situations in crowd motion. We motivated the model on the microscopic level and discussed its generalization in the mean field limit. The optimality system of the resulting parabolic optimal control problem, is a mean field game and establishes interesting links to well known models, like the Hughes model for pedestrian flow. Furthermore the proposed model gives new insights into the mathematical modeling of boundary conditions for pedestrian crowds. We present first existence and uniqueness results for the proposed optimal control approach and illustrate the behavior of solutions with various numerical simulations.

The general formulation of the proposed model poses interesting challenges for future research. In general the derivation of mean-field limit with nonlinear mobilities is of significant importance and raised a lot of interest in the scientific community, cf. e.g. [13]. In addition challenging questions and problems arise in the mathematical analysis of the optimal control approach for convex as well as non-convex problems. Another important direction of research will focus on more realistic modeling assumptions. We have seen in the second example of Section 4 that people anticipate the density of all others in time, which is questionable in many situations.

Therefore it would be interesting to study the behavior of (13) with a temporal discount factor, as proposed in [31]. Finally the development of a numerical solver for 2D problems and its generalization to non-convex problems would allow for more realistic simulations.

REFERENCES

- [1] D. Amadori and M. Di Francesco, The one-dimensional Hughes model for pedestrian flow: Riemann-type solutions, *Acta Math. Sci. Ser. B Engl. Ed.*, **32** (2012), 259–280, URL [http://dx.doi.org/10.1016/S0252-9602\(12\)60016-2](http://dx.doi.org/10.1016/S0252-9602(12)60016-2).
- [2] L. Ambrosio, S. Lisini and G. Savaré, Stability of flows associated to gradient vector fields and convergence of iterated transport maps, *Manuscripta Math.*, **121** (2006), 1–50, URL <http://dx.doi.org/10.1007/s00229-006-0003-0>.
- [3] C. Appert-Rolland, P. Degond and S. Motsch, Two-way multi-lane traffic model for pedestrians in corridors, *Netw. Heterog. Media*, **6** (2011), 351–381, URL <http://dx.doi.org/10.3934/nhm.2011.6.351>.
- [4] C. Bardos, A. Y. le Roux and J.-C. Nédélec, First order quasilinear equations with boundary conditions, *Comm. Partial Differential Equations*, **4** (1979), 1017–1034, URL <http://dx.doi.org/10.1080/03605307908820117>.
- [5] J.-D. Benamou and Y. Brenier, A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem, *Numer. Math.*, **84** (2000), 375–393, URL <http://dx.doi.org/10.1007/s002110050002>.
- [6] V. J. Blue and J. L. Adler, Cellular automata microsimulation for modeling bi-directional pedestrian walkways, *Transportation Research Part B: Methodological*, **35** (2001), 293 – 312, URL <http://www.sciencedirect.com/science/article/pii/S0191261599000521>.
- [7] L. Boccardo and T. Gallouët, Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.*, **87** (1989), 149–169, URL [http://dx.doi.org/10.1016/0022-1236\(89\)90005-0](http://dx.doi.org/10.1016/0022-1236(89)90005-0).
- [8] C. Brune, *4D Imaging in Tomography and Optimal Nanoscopy*, PhD thesis, University of Münster, 2010.
- [9] M. Burger, B. Schlake and M.-T. Wolfram, Nonlinear Poisson-Nernst-Planck equations for ion flux through confined geometries, *Nonlinearity*, **25** (2012), 961–990, URL <http://dx.doi.org/10.1088/0951-7715/25/4/961>.
- [10] M. Burger, M. Di Francesco, J.-F. Pietschmann and B. Schlake, Nonlinear cross-diffusion with size exclusion, *SIAM J. Math. Anal.*, **42** (2010), 2842–2871, URL <http://dx.doi.org/10.1137/100783674>.
- [11] M. Burger, P. A. Markowich and J.-F. Pietschmann, Continuous limit of a crowd motion and herding model: analysis and numerical simulations, *Kinet. Relat. Models*, **4** (2011), 1025–1047, URL <http://dx.doi.org/10.3934/krm.2011.4.1025>.
- [12] C. Burstedde, K. Klauck, A. Schadschneider and J. Zittartz, Simulation of pedestrian dynamics using a two-dimensional cellular automaton, *Physica A: Statistical Mechanics and its Applications*, **295** (2001), 507 – 525, URL <http://www.sciencedirect.com/science/article/pii/S0378437101001418>.
- [13] R. Carmona and F. Delarue, Probabilistic analysis of mean-field games, *SIAM J. Control Optim.*, **51** (2013), 2705–2734, URL <http://dx.doi.org/10.1137/120883499>.
- [14] M. Chraïbi, U. Kemloh, A. Schadschneider and A. Seyfried, Force-based models of pedestrian dynamics, *Netw. Heterog. Media*, **6** (2011), 425–442, URL <http://dx.doi.org/10.3934/nhm.2011.6.425>.
- [15] R. M. Colombo, M. Garavello and M. Lécureux-Mercier, A class of nonlocal models for pedestrian traffic, *Math. Models Methods Appl. Sci.*, **22** (2012), 1150023, 34, URL <http://dx.doi.org/10.1142/S0218202511500230>.
- [16] R. M. Colombo, P. Goatin and M. D. Rosini, A macroscopic model for pedestrian flows in panic situations, in *Current advances in nonlinear analysis and related topics*, vol. 32 of GAKUTO Internat. Ser. Math. Sci. Appl., Gakkōtoshō, Tokyo, 2010, 255–272.
- [17] M. Di Francesco, P. A. Markowich, J.-F. Pietschmann and M.-T. Wolfram, On the Hughes’ model for pedestrian flow: the one-dimensional case, *J. Differential Equations*, **250** (2011), 1334–1362, URL <http://dx.doi.org/10.1016/j.jde.2010.10.015>.
- [18] C. Dogbé, Modeling crowd dynamics by the mean-field limit approach, *Math. Comput. Modelling*, **52** (2010), 1506–1520, URL <http://dx.doi.org/10.1016/j.mcm.2010.06.012>.

- [19] L. Dyson, P. Maini and R. Baker, Macroscopic limits of individual-based models for motile cell populations with volume exclusion., *Phys. Rev. E Stat. Nonlin. Soft Matter Phys.*, **86** (2012), 031903.
- [20] H. Egger and J. Schöberl, A hybrid mixed discontinuous Galerkin finite-element method for convection-diffusion problems, *IMA J. Numer. Anal.*, **30** (2010), 1206–1234, URL <http://dx.doi.org/10.1093/imanum/drn083>.
- [21] N. El-Khatib, P. Goatin and M. D. Rosini, On entropy weak solutions of Hughes’ model for pedestrian motion, *Z. Angew. Math. Phys.*, **64** (2013), 223–251, URL <http://dx.doi.org/10.1007/s00033-012-0232-x>.
- [22] P. Goatin and M. Mimault, The wave-front tracking algorithm for Hughes’ model of pedestrian motion, *SIAM J. Sci. Comput.*, **35** (2013), B606–B622, URL <http://dx.doi.org/10.1137/120898863>.
- [23] D. Gomes and J. Saúde, *Mean field games - a brief survey*, Technical report, submitted, 2013.
- [24] O. Guéant, J.-M. Lasry and P.-L. Lions, Mean field games and applications, in *Paris-Princeton Lectures on Mathematical Finance 2010*, vol. 2003 of Lecture Notes in Math., Springer, Berlin, 2011, 205–266, URL http://dx.doi.org/10.1007/978-3-642-14660-2_3.
- [25] D. Helbing, I. Farkas and T. Vicsek, Simulating dynamical features of escape panic, *Nature*, **407** (2000), 487–490, URL <http://arxiv.org/abs/cond-mat/0009448>.
- [26] D. Helbing and P. Molnar, Social force model for pedestrian dynamics, *Physical Review E*, **51** (1998), 4282–4286, URL <http://arxiv.org/abs/cond-mat/9805244>.
- [27] S. P. Hoogendoorn and P. H. L. Bovy, Pedestrian route-choice and activity scheduling theory and models, *Transportation Research Part B: Methodological*, **38** (2004), 169–190.
- [28] R. L. Hughes, A continuum theory for the flow of pedestrians, *Transportation Research Part B: Methodological*, **36** (2002), 507 – 535, URL <http://www.sciencedirect.com/science/article/pii/S0191261501000157>.
- [29] H. Ishii, Asymptotic solutions for large time of hamilton–jacobi equations in euclidean n space, *Annales de l’Institut Henri Poincaré (C) Non Linear Analysis*, **25** (2008), 231 – 266, URL <http://www.sciencedirect.com/science/article/pii/S0294144907000054>.
- [30] I. Karatzas and S. E. Shreve, *Brownian Motion and Stochastic Calculus (Graduate Texts in Mathematics)*, 2nd edition, Springer, 1991, URL <http://www.amazon.com/exec/obidos/redirect?tag=citeulike07-20&path=ASIN/0387976558>.
- [31] A. Lachapelle and M.-T. Wolfram, On a mean field game approach modeling congestion and aversion in pedestrian crowds, *Transportation Research Part B: Methodological*, **45** (2011), 1572 – 1589, URL <http://www.sciencedirect.com/science/article/pii/S0191261511001111>.
- [32] J.-M. Lasry and P.-L. Lions, Mean field games, *Jpn. J. Math.*, **2** (2007), 229–260, URL <http://dx.doi.org/10.1007/s11537-007-0657-8>.
- [33] M. Moussad, E. G. Guilloit, M. Moreau, J. Fehrenbach, O. Chabiron, S. Lemerrier, J. Pettr, C. Appert-Rolland, P. Degond and G. Theraulaz, Traffic instabilities in self-organized pedestrian crowds, *PLoS Comput. Biol.*, **8** (2012), e1002442, URL <http://dx.doi.org/10.1371/journal.pcbi.1002442>.
- [34] K. J. Painter and T. Hillen, Volume-filling and quorum-sensing in models for chemosensitive movement, *Can. Appl. Math. Q.*, **10** (2002), 501–543.
- [35] R. E. Showalter, *Monotone operators in Banach space and nonlinear partial differential equations*, vol. 49 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 1997.
- [36] M. Simpson, B. Hughes and K. Landman, Diffusing populations: Ghosts or folks, *Australasian Journal of Engineering Education*, **15** (2009), 59–68, URL <http://eprints.qut.edu.au/44917/>.
- [37] J. van den Berg, S. Patil, J. Sewall, D. Manocha and M. Lin, Interactive navigation of multiple agents in crowded environments, in *Proceedings of the 2008 symposium on Interactive 3D graphics and games*, ACM, 2008, 139–147.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: martin.burger@wwu.de

E-mail address: m.difrancesco@bath.ac.uk

E-mail address: P.A.Markowich@damtp.cam.ac.uk

E-mail address: marie-therese.wolfram@univie.ac.at