How to generate all possible rational Wilf–Zeilberger forms?

S. Chen, C. Koutschan, Y. Wang

RICAM-Report 2024-13

How to generate all possible rational Wilf–Zeilberger forms?

Shaoshi Chen[a,b], Christoph Koutschan[c], Yisen Wang[a,b,c]

[a] *KLMM, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China*
[b] *School of Mathematical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*
[c] *Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, Altenberger Straße 69, 4040 Linz, Austria*

Abstract

Wilf–Zeilberger pairs are fundamental in the algorithmic theory of Wilf and Zeilberger for computer-generated proofs of combinatorial identities. Wilf–Zeilberger forms are their high-dimensional generalizations, which can be used for proving and discovering convergence acceleration formulas. This paper presents a structural description of all possible rational such forms, which can be viewed as an additive analog of the classical Ore–Sato theorem. Based on this analog, we show a structural decomposition of so-called multivariate hyperarithmetic terms, which extend multivariate hypergeometric terms to the additive setting.

Keywords: Additive Ore–Sato theorem, Hyperarithmetic term, Orbital decomposition, Wilf–Zeilberger form

1. Introduction

The definition of Wilf–Zeilberger forms was first introduced by Zeilberger [20]; they are a direct generalization of Wilf–Zeilberger pairs [18, 19, 16] to tuples with more than two entries. The interest in such pairs and forms originates from the algorithmic proof theory of hypergeometric summation identities. In this paper, we restrict our attention to forms with rational functions instead of hypergeometric entries. These forms can also be seen as a

Email addresses: `schen@amss.ac.cn` (Shaoshi Chen), `christoph.koutschan@ricam.oeaw.ac.at` (Christoph Koutschan), `wangyisen@amss.ac.cn` (Yisen Wang)

S. Chen was partially supported by the National Key R&D Program of China (No. 2023YFA1009401), the NSFC grant (No. 12271511), CAS Project for Young Scientists in Basic Research (Grant No. YSBR-034), and the CAS Fund of the Youth Innovation Promotion Association (No. Y2022001). C. Koutschan and Y. Wang were partially supported by the Austrian Science Fund (FWF): 10.55776/I6130.

difference version of differential closed 1-forms. It is challenging to extend the above structure theory to the hypergeometric case, which would be a useful tool to generate combinatorial identities automatically.

Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integers. Let K be an algebraically closed field of characteristic zero and $K(x_1, \ldots, x_n)$ be the field of rational functions in the variables $x_1, \ldots, x_n$ over K, which is also written as $K(\mathbf{x})$. We define the shift operators σ_i that act on elements $f \in K(\mathbf{x})$ as follows:

$$\sigma_i\big(f(x_1, \ldots, x_n)\big) := f(x_1, \ldots, x_i + 1, \ldots, x_n), \quad \forall i \in \{1, \ldots, n\}.$$

The action of operators on functions is also denoted by $\bullet$, e.g., $\sigma_i \bullet f = \sigma_i(f)$. Analogously, the forward difference operators are defined as

$$\Delta_i(f) := \sigma_i(f) - f, \quad \forall i \in \{1, \ldots, n\}.$$

Definition 1 (Hypergeometric term, hyperarithmetic term)**.** *A nonzero term H is said to be* hypergeometric *over $K(\mathbf{x})$ if there exist rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ such that*

$$\frac{\sigma_i(H)}{H} = f_i, \quad \forall i \in \{1, \ldots, n\}.$$

A nonzero term H is said to be hyperarithmetic *over $K(\mathbf{x})$ if there exist rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ such that*

$$\sigma_i(H) - H = f_i, \quad \forall i \in \{1, \ldots, n\}.$$

In both cases, the rational functions $f_1, \ldots, f_n$ are called the certificates *of H. Two hypergeometric (resp. hyperarithmetic) terms H_1 and H_2 are* conjugate*, denoted by $H_1 \simeq H_2$, if they have the same certificates.*

Since σ_i and σ_j commute, the certificates $f_1, \ldots, f_n$ of a hypergeometric term H satisfy the following compatibility conditions:

$$\frac{\sigma_i(f_j)}{f_j} = \frac{\sigma_j(f_i)}{f_i}, \quad \forall i, j \in \{1, \ldots, n\}. \tag{1}$$

The certificates $f_1, \ldots, f_n$ of a hyperarithmetic term H satisfy the following compatibility conditions:

$$\sigma_i(f_j) - f_j = \sigma_j(f_i) - f_i, \quad \forall i, j \in \{1, \ldots, n\}. \tag{2}$$

Definition 2. *An n-tuple $(f_1, \ldots, f_n) \in K(\mathbf{x})^n$ is called a* rational Wilf–Zeilberger form *with respect to $(\Delta_1, \ldots, \Delta_n)$ if $\Delta_i(f_j) = \Delta_j(f_i)$ for all $i, j \in \{1, \ldots, n\}$. We abbreviate "rational Wilf–Zeilberger form" as "WZ-form" in the rest of this paper. If $n = 2$, then we call it a* WZ-pair.

The classical Ore–Sato theorem plays an important role in the theory of multivariate hypergeometric terms, because it describes the multiplicative structure of nonzero rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ that satisfy the compatibility conditions (1). The bivariate case was proven by Ore [15] and the multivariate case by Sato [17]. According to this theorem, any multivariate hypergeometric term can be decomposed into a product of one rational function and several factorial terms (which are basically products of Gamma functions).

Theorem 3 (Ore–Sato theorem)**.** *Let $f_1, \ldots, f_n \in K(\mathbf{x})$ be nonzero rational functions satisfying the compatibility conditions* (1)*. Then there exist a rational function $a \in K(\mathbf{x})$, constants $\mu_1, \ldots, \mu_n \in K$, a finite set $V \subset \mathbb{Z}^n$, and for each $\mathbf{v} \in V$ a univariate monic rational function $r_{\mathbf{v}} \in K(z)$ such that*

$$f_j = \frac{\sigma_j(a)}{a}\mu_j \prod_{\mathbf{v}\in V} \prod\nolimits_{\ell}{}_{0}^{v_j} r_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{x}+\ell),$$

where $\mathbf{v}\cdot\mathbf{x} := v_1x_1 + \cdots + v_nx_n$ and where the product notation is defined as follows: for $s, t \in \mathbb{Z}$,

$$\prod\nolimits_{\ell}{}_{s}^{t} \alpha_\ell := \begin{cases} \alpha_s\alpha_{s+1}\cdots\alpha_{t-1}, & \text{if } t \geq s; \\ \dfrac{1}{\alpha_t\alpha_{t+1}\cdots\alpha_{s-1}}, & \text{if } t < s. \end{cases}$$

Christopher's theorem [9, 21] is an analog of the Ore–Sato theorem in the continuous case. Other analogs concern the q-discrete case [12] and the continuous-discrete case [6]. In this paper, we want to explore the additive structure of nonzero rational functions $f_1, \ldots, f_n \in K(\mathbf{x})$ satisfying the compatibility conditions (2), i.e., $(f_1, \ldots, f_n)$ is a WZ-form. Our main result, which is stated in the following theorem, reveals this additive structure and therefore implies an additive decomposition of hyperarithmetic terms.

Theorem 4 (Additive Ore–Sato theorem)**.** *Let $f_1, \ldots, f_n \in K(\mathbf{x})$ be nonzero rational functions satisfying the compatibility conditions* (2)*. Then there exist a rational function $a \in K(\mathbf{x})$, constants $\mu_1, \ldots, \mu_n \in K$, a finite set $V \subset \mathbb{Z}^n$, and for each $\mathbf{v} \in V$ a univariate monic rational function $r_{\mathbf{v}} \in K(z)$ such that*

$$f_j = \sigma_j(a) - a + \mu_j + \sum_{\mathbf{v}\in V} \sum\nolimits_{\ell}{}_{0}^{v_j} r_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{x}+\ell),$$

where $\mathbf{v}\cdot\mathbf{x} := v_1x_1 + \cdots + v_nx_n$ and where we use the sum notation (for $s, t \in \mathbb{Z}$)

$$\sum\nolimits_{i}{}_{s}^{t} \alpha_i := \begin{cases} \alpha_s + \alpha_{s+1} + \cdots + \alpha_{t-1}, & \text{if } t \geq s; \\ -(\alpha_t + \alpha_{t+1} + \cdots + \alpha_{s-1}), & \text{if } t < s. \end{cases}$$

In the proof of Ore–Sato theorem, the complete irreducible factorization was used as a key ingredient. When it comes to the additive case, we need another auxiliary tool, the so-called orbital decomposition, which compensates the missing of partial fraction decompositions of multivariate rational functions. Hence, our additive Ore–Sato theorem is not just a straight-forward analog of its multiplicative predecessor, but is significantly different in its structure and proof strategy.

2. WZ-forms and structure of WZ-pairs

The goal of this section is to introduce some notions that will help us to describe the proofs in the later sections more concisely.

Definition 5 ((Pairwise) shift-invariant)**.** *A rational function $f \in K(\mathbf{x})$ is called* shift-invariant *if there exists a nonzero integer vector $\mathbf{v} \in \mathbb{Z}^n$ such that $f(\mathbf{x}+\mathbf{v}) = f(\mathbf{x})$. It is called* pairwise shift-invariant *if for each pair $\sigma, \tau \in \{\sigma_1, \ldots, \sigma_n\}$, there are $s, t \in \mathbb{Z}$, not both zero, such that $\sigma^s(f) = \tau^t(f)$.*

Definition 6 (Integer-linearity)**.** *An irreducible polynomial $p \in K[\mathbf{x}]$ is called* integer-linear *over K if there exist a univariate polynomial $P \in K[z]$ and a nonzero integer vector $\mathbf{v} \in \mathbb{Z}^n$ such that*

$$p(\mathbf{x}) = P(\mathbf{v} \cdot \mathbf{x}).$$

We can always assume that $\gcd(v_1, \ldots, v_n) = 1$ because a common factor can be extracted and absorbed by P. Such a vector $\mathbf{v}$ is called the integer-linear type *of p. We say that $f \in K(\mathbf{x})$ is integer-linear of type $\mathbf{v}$ if all the irreducible factors of its numerator and its denominator are of the common integer-linear type $\mathbf{v}$.*

There is an efficient algorithm for the computation of the integer-linear decomposition of multivariate polynomials [13], which will be used for computing additive decompositions in Section 6. The next lemma reveals the equivalence between the pairwise shift-invariant and the integer-linearity of a rational function.

Lemma 7. *[3, Proposition 7] A rational function $f \in K(\mathbf{x})$ is pairwise shift-invariant if and only if there exist a nonzero integer vector $\mathbf{v} \in \mathbb{Z}^n$ and a univariate rational function $r \in K(z)$ such that*

$$f(\mathbf{x}) = r(\mathbf{v} \cdot \mathbf{x}),$$

i.e., f is integer-linear of type $\mathbf{v}$.

Given the integer-linear type of f, one can easily see that f is pairwise shift-invariant. In contrast, the opposite direction of Lemma 7 is not that obvious. However, it follows, by using an inductive argument, from the bivariate case that is illustrated in the following remark.

Remark 8. *Let $f \in K(x, y)$ be such that $\sigma_x^s \sigma_y^t(f) = f$ with $s, t \in \mathbb{Z}$ not both zero. If $s = 0$, then f is free of y, which implies that f is integer-linear of type $(1, 0)$. Similarly if $t = 0$, then f is integer-linear of type $(0, 1)$. If both of them are nonzero, then f is integer-linear of type $(\bar{t}, \bar{s})$, where $\bar{t} = t/\gcd(s, t)$ and $\bar{s} = s/\gcd(s, t)$.*

According to Definition 6, an element in K can be viewed as having any integer-linear type. But for a non-constant rational function whose factors are of the same integer-linear type, its type is unique. Such a type remains unchanged under addition and under application of shift operators.

We now introduce two kinds of special WZ-forms, i.e., exact WZ-forms and uniform WZ-forms, which will play an important role in describing the structure of general WZ-forms (see Theorem 4).

Definition 9 (Exact WZ-form). *A WZ-form* $(f_1, \ldots, f_n)$ *with respect to* $(\Delta_1, \ldots, \Delta_n)$ *is said to be* exact *if there exists* $g \in K(\mathbf{x})$ *such that* $f_i = \Delta_i(g)$, *for all* $i \in \{1, \ldots, n\}$.

Definition 10 (Uniform WZ-form). *A WZ-form* $(f_1, \ldots, f_n)$ *with respect to* $(\Delta_1, \ldots, \Delta_n)$ *is called a* uniform WZ-form *if there exists an integer vector* $\mathbf{v}$ *such that each* f_i *is integer-linear of type* $\mathbf{v}$.

Remark 11. *A WZ-form can be both exact and uniform, for example,* $\big(\Delta_x(\frac{1}{x+y}), \Delta_y(\frac{1}{x+y})\big)$ *is an exact WZ-pair where each component is integer-linear of type* $(1, 1)$.

In the remaining part of this section we recall the structure theorem on WZ-pairs in [5] that is described in terms of exact and cyclic pairs.

Definition 12 (Cyclic operator). *Let* $G = \langle \sigma_1, \ldots, \sigma_n \rangle$. *For any* $m \in \mathbb{Z}$ *and* $\theta \in G$, *define*

$$\frac{\theta^m - 1}{\theta - 1} := \begin{cases} 1 + \theta + \cdots + \theta^{m-1}, & \text{if } m > 0; \\ 0, & \text{if } m = 0; \\ -(\theta^m + \cdots + \theta^{-1}), & \text{if } m < 0. \end{cases}$$

Definition 13 (Cyclic pair). *A WZ-pair* (f, g) *w.r.t.* (Δ_x, Δ_y) *is called a* cyclic pair *if there exists* $h \in K(x, y)$ *that satisfies* $\sigma_x^s(h) = \sigma_y^t(h)$ *for some* $s, t \in \mathbb{Z}$, *not both zero, such that*

$$f = \frac{\sigma_y^t - 1}{\sigma_y - 1} \bullet h \quad \text{and} \quad g = \frac{\sigma_x^s - 1}{\sigma_x - 1} \bullet h.$$

Note that any cyclic pair is a uniform WZ-pair by Remark 8. The following theorem shows that each WZ-pair can be decomposed into one exact WZ-pair plus several cyclic pairs.

Theorem 14 (Structure of WZ-pairs). *Any WZ-pair can be decomposed into one exact WZ-pair plus several cyclic WZ-pairs.*

When it comes to a multivariate generalization of Theorem 14, cyclic pairs will be replaced by uniform WZ-forms, see Theorem 20. For this purpose, we define orbital decompositions and orbital residues of rational functions in the next section.

3. Orbital decompositions and orbital residues

In this section, we recall the notion of orbital decompositions of rational functions that was first used in studying the existence problem of telescopers in [7] and present a modified definition of discrete residues, which were originally introduced in [8] with polynomial and elliptic analogs in [14, 11].

Definition 15 (Shift-equivalence)**.** *Let F be a subgroup of $\langle \sigma_1, \ldots, \sigma_n \rangle$. For $a, b \in K(\mathbf{x})$, we say a and b are* F-equivalent *if there exists $\tau \in F$ such that $\tau(a) = b$, denoted by $a \sim_F b$. We call the set*

$$[a]_F := \{\tau(a) \mid \tau \in F\}$$

the F-orbit of a. *Note that if $a \sim_F b$ then $[a]_F = [b]_F$.*

The orbital decomposition of a rational function $f = P/Q \in K(\mathbf{x})$ depends on the variable x_1 and a subgroup F. In order to define it, we first focus on its denominator as a polynomial in x_1, that is, $Q \in K(\widehat{\mathbf{x}})[x_1]$ with $\widehat{\mathbf{x}} := x_2, \ldots, x_n$. The first step consists in factoring the polynomial Q completely over $K(\widehat{\mathbf{x}})$. We sort all of its irreducible factors into distinct F-orbits as follows:

$$Q = c \cdot \prod_{i=1}^{I} \prod_{j=1}^{J} \prod_{\tau \in \Lambda_{i,j}} \tau(b_i^j),$$

where $c \in K(\widehat{\mathbf{x}})$, $\Lambda_{i,j}$ are finite subsets of F, and the $b_i \in K(\widehat{\mathbf{x}})[x_1]$ are monic irreducible polynomials in distinct F-orbits. Note that this factorization is unique up to the choice of the representative b_i in each F-orbit. Moreover, we impose on the sets $\Lambda_{i,j}$ the condition that $\tau(b_i) \neq \tau'(b_i)$ for $\tau, \tau' \in \Lambda_{i,j}$ with $\tau \neq \tau'$. In the second step, we compute the unique irreducible partial fraction decomposition of f with respect to the above factorization:

$$f = p + \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{\tau \in \Lambda_{i,j}} \frac{a_{i,j,\tau}}{\tau(b_i^j)}, \tag{3}$$

where $p, a_{i,j,\tau} \in K(\widehat{\mathbf{x}})[x_1]$ with $\deg_{x_1}(a_{i,j,\tau}) < \deg_{x_1}(b_i)$ for all i, j, τ. For a polynomial $b \in K(\widehat{\mathbf{x}})[x_1]$, a subgroup $F \leq G$, and $j > 0$, we define the following linear $K(\widehat{\mathbf{x}})$-subspace:

$$U_{b,j}^F := \mathrm{Span}_{K(\widehat{\mathbf{x}})} \left\{ \frac{a}{\tau(b^j)} \;\middle|\; \tau \in F,\ a \in K(\widehat{\mathbf{x}})[x_1],\ \deg_{x_1}(a) < \deg_{x_1}(b) \right\}. \tag{4}$$

In Equation (3), we have each sum $\sum_\tau \frac{a_{i,j,\tau}}{\tau(b_i^j)} \in U_{b_i,j}^F$. Since the decomposition (3) exists for any $f \in K(\mathbf{x})$, and since the orbits $[b]_F$ do not overlap, we obtain the following direct sum decomposition:

$$K(\mathbf{x}) = K(\widehat{\mathbf{x}})[x_1] \oplus \left(\bigoplus_{j>0} \bigoplus_{[b]_F} U_{b,j}^F \right), \tag{5}$$

where $[b]_F$ runs over all orbits in $K(\widehat{\mathbf{x}})[x_1]/\sim_F$. Such a direct sum decomposition is called [7] *the orbital decomposition of* $K(\mathbf{x})$ *with respect to the variable* x_1 *and the group* F.

According to the definition of $U_{b,j}^F$, it is easy to check that this linear subspace is closed under the application of any operator in $K(\widehat{\mathbf{x}})[F]$, that is, any operator of the form $\sum_{\tau\in F} c_\tau \tau$ with $c_\tau \in K(\widehat{\mathbf{x}})$. The following lemma is a direct generalization of Lemma 5.1 in [7].

Lemma 16. *If* $f \in U_{b,j}^F$ *and* $\theta \in K(\widehat{\mathbf{x}})[F]$*, then* $\theta(f) \in U_{b,j}^F$.

Theorem 17. *Let* $f = p+\sum_{i=1}^I \sum_{j=1}^J f_{i,j}$ *with* $p \in K(\widehat{\mathbf{x}})[x_1]$ *and* $f_{i,j} \in U_{b_i,j}^F$ *be an orbital decomposition of* f *with respect to* x_1 *and* F*, and let* $\theta_1, \theta_2 \in K(\widehat{\mathbf{x}})[F]$*. We have* $\theta_1(f) = \theta_2(g)$ *for some* $g \in K(\mathbf{x})$*, if and only if* $\theta_1(p) = \theta_2(q)$ *for some* $q \in K(\widehat{\mathbf{x}})[x_1]$ *and for each* i, j*, there exists* $g_{i,j} \in U_{b_i,j}^F$ *such that* $\theta_1(f_{i,j}) = \theta_2(g_{i,j})$.

Proof. The sufficiency is due to the linearity of the operators in $K(\widehat{\mathbf{x}})[F]$. For the necessity, suppose $g = q + \sum_{i=1}^I \sum_{j=1}^J g_{i,j}$, where $q \in K(\widehat{\mathbf{x}})[x_1]$ and each $g_{i,j} \in U_{b_i,j}^F$. By Lemma 16, the orbital decomposition of $\theta_1(f)$ with respect to x_1 and F is

$$\theta_1(f) = \theta_1(p) + \sum_{i=1}^I \sum_{j=1}^J \theta_1(f_{i,j}).$$

Similarly, we get

$$\theta_2(g) = \theta_2(q) + \sum_{i=1}^I \sum_{j=1}^J \theta_2(g_{i,j}).$$

By the uniqueness of the direct sum decomposition (5), we have $\theta_1(p) = \theta_2(q)$ and $\theta_1(f_{i,j}) = \theta_2(g_{i,j})$ for each i, j. $\square$

For $f \in K(\mathbf{x})$, we say that f is σ_i*-summable* if there exists $g \in K(\mathbf{x})$ such that $f = \Delta_i(g)$. Let $(f_1, \ldots, f_n)$ be a WZ-form w.r.t. $(\Delta_1, \ldots, \Delta_n)$. Then $\Delta_i(f_1)$ is σ_1-summable, because we have $\Delta_i(f_1) = \Delta_1(f_i)$. The first part in our proof of Theorem 4 is to decompose f_1 and find the shift-invariance of each part.

Next, for the definition of orbital residues, let us look at the orbital decomposition of $f \in K(\mathbf{x})$ with respect to x_1 and the subgroup $F = \langle\sigma_1\rangle$. In this case, the decomposition (3) can be written as

$$f = p + \sum_{i=1}^I \sum_{j=1}^J \sum_{\ell=0}^L \frac{a_{i,j,\ell}}{\sigma_1^\ell(d_i^j)}, \tag{6}$$

where the d_i are irreducible polynomials in distinct $\langle\sigma_1\rangle$-orbits.

Definition 18 (Orbital residue)**.** *Let* f *be given in the form* (6)*, let* $d \in K(\widehat{\mathbf{x}})[x_1]$ *be irreducible, and let* $j \in \{1, \ldots, J\}$*. If there is* $i \in \{1, \ldots, I\}$ *such that* $d_i \in [d]_{\langle\sigma_1\rangle}$ *(by the properties of the orbital decomposition, such* i *is uniquely* *determined), then the* orbital residue *of f at d of multiplicity j, denoted by* $\mathrm{res}_{\sigma_1}(f, d, j)$*, is defined to be the $\langle\sigma_1\rangle$-orbit $[r]_{\langle\sigma_1\rangle}$ with*

$$r := \sum_{\ell=0}^{L} \sigma_1^{-\ell}(a_{i,j,\ell}).$$

If no such i exists, we define $\mathrm{res}_{\sigma_1}(f, d, j) = 0$*. If it is clear from the context, we will abbreviate $[r]_{\langle\sigma_1\rangle}$ by $[r]$.*

Note that the definition of orbital residue does not depend on the representation (3) of f: if instead of d_i some other representative of $[d_i]_{\langle\sigma_1\rangle}$ is used, at the cost of changing the range of ℓ, then also the polynomial r in Definition 18 changes, but it will stay in the same $\langle\sigma_1\rangle$-orbit. This is the reason why the residue is defined to be an orbit, instead of a single polynomial. Similarly, we have $\mathrm{res}_{\sigma_1}(f, d, j) = \mathrm{res}_{\sigma_1}(f, d', j)$ whenever $d \sim_{\langle\sigma_1\rangle} d'$.

Example 19. *Let $b := 4x + 6y + 5z$ and if*

$$f = \frac{x}{b^2} + \frac{x+y}{(b+1)^2} + \frac{2x}{(b-3)^2} + \frac{2x+3}{(b+3)^2},$$

then we observe that $b + 1 = \sigma_x(b - 3)$ and $\{b, b - 3, b + 3\}$ are in distinct $\langle\sigma_1\rangle$-orbits. By Definition 18, we have

$$\mathrm{res}_{\sigma_x}(f, b, 2) = [x], \quad \mathrm{res}_{\sigma_x}(f, b-3, 2) = [3x+y-1], \quad \mathrm{res}_{\sigma_x}(f, b+3, 2) = [2x+3].$$

4. Additive decompositions of WZ-forms

Exact and uniform WZ-forms are special kinds of WZ-forms. Conversely, the following theorem shows that these two forms are the only basic building blocks of all possible WZ-forms. This section is dedicated to proving the following theorem, which is a generalization of Theorem 14 to the multivariate setting.

Theorem 20. *Any WZ-form can be decomposed into one exact WZ-form plus several uniform WZ-forms.*

First we recall the following notion of isotropy groups first introduced by Sato [17] in order to prove the classical Ore–Sato theorem.

Definition 21 (Isotropy group)**.** *Let $p \in K[\mathbf{x}]$. The set*

$$G_p = \{\tau \in G \mid \tau(p) = p\}$$

is a subgroup of G, called the isotropy group *of p in G.*

This definition can be easily extended to rational functions. The next lemma shows that shift-equivalent elements have the same isotropy group.

Lemma 22. *Let $f, g \in K(\mathbf{x})$. If $f \sim_G g$, then $G_f = G_g$.*

Proof. Let $\sigma \in G$ such that $f = \sigma(g)$. For $\tau \in G_g$ we have $\tau(g) = g$. Applying σ to both sides of the equation yields $\sigma\big(\tau(g)\big) = \sigma(g)$. Since σ and τ commute, we have $\tau\big(\sigma(g)\big) = \sigma(g)$, i.e., $\tau(f) = f$. Thus $\tau \in G_f$, which implies that $G_g \subseteq G_f$. Since $\sigma^{-1} \in G$ such that $g = \sigma^{-1}(f)$, similarly we have $G_f \subseteq G_g$. Hence $G_f = G_g$. $\square$

We recall the crucial lemma that leads to the structure theorem of WZ-pairs, which will be used to conduct the induction step in the proof of Theorem 20.

Lemma 23. *[5, Lemma 6] Let $f \in K(x, y)$ be a rational function of the form*

$$f = \frac{a_0}{b^m} + \frac{a_1}{\sigma_y(b^m)} + \cdots + \frac{a_n}{\sigma_y^n(b^m)},$$

where $m, n \in \mathbb{N}$ with $m > 0$, $a_0, \ldots, a_n, b \in K(y)[x]$ with $a_n \neq 0$. Moreover, we assume that $\deg(a_i) < \deg(b)$, b is irreducible and monic, and that $\sigma_y^i(b) \not\sim_{\langle\sigma_x\rangle} \sigma_y^j(b)$ for all $i, j \in \{0, \ldots, n\}$ with $i \neq j$. If for some $g \in K(x, y)$ we have $\Delta_y(f) = \Delta_x(g)$, then there exists $t \in \mathbb{Z}$ such that $\sigma_y^{n+1}(a_0) = \sigma_x^t(a_0)$, $\sigma_y^{n+1}(b) = \sigma_x^t(b)$, and $a_\ell = \sigma_y^\ell(a_0)$ for all $\ell \in \{0, \ldots, n\}$. Furthermore, for some $g_0 \in K(y)$ we get

$$f = \frac{\sigma_y^{n+1} - 1}{\sigma_y - 1} \bullet \frac{a_0}{b^m} \quad \text{and} \quad g = \frac{\sigma_x^t - 1}{\sigma_x - 1} \bullet \frac{a_0}{b^m} + g_0.$$

According to Remark 8, the bivariate function f as above is of a certain integer-linear type. We will use Lemma 23 to reduce the problem from the multivariate case to the bivariate one in Lemma 27.

Recall that $G = \langle \sigma_1, \ldots, \sigma_n \rangle$ and $\widehat{\mathbf{x}} = x_2, \ldots, x_n$. Let $\omega := (f_1, \ldots, f_n) \in K(\mathbf{x})^n$ be a WZ-form w.r.t. $(\Delta_1, \ldots, \Delta_n)$. Then we apply the orbital decomposition (3) with respect to x_1 and G to f_1, yielding

$$f_1 = p + \sum_{i=1}^{I} \sum_{j=1}^{J} \sum_{\tau \in \Lambda_{i,j}} \frac{a_{i,j,\tau}}{\tau(b_i^j)}, \tag{7}$$

where for all i, j, τ we have $p, a_{i,j,\tau} \in K(\widehat{\mathbf{x}})[x_1]$ with $\deg_{x_1}(a_{i,j,\tau}) < \deg_{x_1}(b_i)$ and $\Lambda_{i,j} \subset G$. The following reduction formula is crucial in Abramov's algorithm for rational summation [1, 2].

Fact 24. *For all $a, u \in K[\mathbf{x}]$ with $u \neq 0$ and automorphism ϕ of $K(\mathbf{x})$, we have*

$$\frac{a}{\phi^m(u)} = \phi(g) - g + \frac{\phi^{-m}(a)}{u}, \tag{8}$$

where

$$g = \begin{cases} \displaystyle\sum_{i=0}^{m-1} \frac{\phi^{i-m}(a)}{\phi^i(u)}, & \text{if } m \geq 0; \\ \displaystyle -\sum_{i=m}^{-1} \frac{\phi^{i-m}(a)}{\phi^i(u)}, & \text{if } m < 0. \end{cases} \tag{9}$$

Let $E := \langle \sigma_2, \ldots, \sigma_n \rangle$. Then each $\tau \in G$ can be written as $\sigma_1^m \lambda$ for some $m \in \mathbb{Z}$ and $\lambda \in E$. By taking $\phi = \sigma_1$ and $u = \lambda(b)$ in Formula (8), we get

$$\frac{a}{\tau(b)} = \frac{a}{\sigma_1^m(u)} = \Delta_1(g) + \frac{\sigma_1^{-m}(a)}{u} = \Delta_1(g) + \frac{\sigma_1^{-m}(a)}{\lambda(b)}, \tag{10}$$

for some $g \in K(\mathbf{x})$ of the form (9). Applying the above reduction (10) to each summand $a_{i,j,\tau}/\tau(b_i^j)$ in Equation (7) yields

$$f_1 = \Delta_1(g_0) + \sum_{i=1}^{I}\sum_{j=1}^{J} \widetilde{f}_{1,i,j} \quad \text{with} \quad \widetilde{f}_{1,i,j} = \sum_{\lambda \in \widetilde{\Lambda}_{i,j}} \frac{\widetilde{a}_{i,j,\lambda}}{\lambda(b_i^j)}, \tag{11}$$

where $g_0 \in K(\mathbf{x})$, $\widetilde{\Lambda}_{i,j} \subseteq E$, and $\lambda(b_i) \not\sim_{\langle \sigma_1 \rangle} \lambda'(b_i)$ whenever λ, λ' are two distinct elements from $\widetilde{\Lambda}_{i,j}$. Since the shift operators σ_1^{-m} preserve the degrees of the polynomials $a_{i,j,\lambda}$, we have for all i,j that $\widetilde{f}_{1,i,j} \in U_{b_i,j}^G$. In fact,

$$[\widetilde{a}_{i,j,\lambda}] = \operatorname{res}_{\sigma_1}\big(f_1, \lambda(b_i), j\big).$$

We give an illustrative example to show how we can immediately obtain the orbital residue via the reduction (11). Note that the result is the same as specified in Definition 18.

Example 25 (Continuing Example 19)**.** *Rewrite f as*

$$f = \frac{x}{b^2} + \frac{x+y}{\sigma_x^{-1}\sigma_z(b^2)} + \frac{2x}{\sigma_x\sigma_y^{-2}\sigma_z(b^2)} + \frac{2x+3}{\sigma_x^{-3}\sigma_z^3(b^2)}.$$

First we get rid of the operator σ_x among all the denominators,

$$\begin{aligned} f = \Delta_x\Bigg(&-\frac{x+y}{\sigma_x^{-1}\sigma_z(b^2)} + \frac{2x-2}{\sigma_y^{-2}\sigma_z(b^2)} - \frac{2x+3}{\sigma_x^{-3}\sigma_z^3(b^2)} - \frac{2x+5}{\sigma_x^{-2}\sigma_z^3(b^2)} - \frac{2x+7}{\sigma_x^{-1}\sigma_z^3(b^2)}\Bigg) \\ &+ \frac{x}{b^2} + \frac{x+y+1}{\sigma_z(b^2)} + \frac{2x-2}{\sigma_y^{-2}\sigma_z(b^2)} + \frac{2x+9}{\sigma_z^3(b^2)}. \end{aligned}$$

Note that $\sigma_y^{-2}\sigma_z(b^2) = \sigma_x^{-3}\sigma_z(b^2)$, so we continue the reduction as follows:

$$\frac{2x-2}{\sigma_y^{-2}\sigma_z(b^2)} = \Delta_x\Bigg(-\frac{2x-2}{\sigma_x^{-3}\sigma_z(b^2)} - \frac{2x}{\sigma_x^{-2}\sigma_z(b^2)} - \frac{2x+2}{\sigma_x^{-1}\sigma_z(b^2)}\Bigg) + \frac{2x+4}{\sigma_z(b^2)}.$$

Hence

$$f = \Delta_x(g) + \frac{x}{b^2} + \frac{3x+y+5}{\sigma_z(b^2)} + \frac{2x+9}{\sigma_z^3(b^2)},$$

for some $g \in K(\mathbf{x})$. We observe that $\{b^2, \sigma_z(b^2), \sigma_z^3(b^2)\} = \{b^2, (b+5)^2, (b+15)^2\}$ are pairwise $\langle \sigma_x \rangle$-inequivalent, hence the reduction is done. We have

$$\operatorname{res}_{\sigma_x}(f, b, 2) = [x], \ \operatorname{res}_{\sigma_x}\big(f, \sigma_z(b), 2\big) = [3x+y+5], \ \operatorname{res}_{\sigma_x}\big(f, \sigma_z^3(b), 2\big) = [2x+9].$$

Using the g_0 that was obtained by Abramov's reduction (8), we define an exact WZ-form $\omega_0 := \big(\Delta_1(g_0), \ldots, \Delta_n(g_0)\big)$, which we remove from the given WZ-form ω. To this end, we let $\widetilde{f}_i := f_i - \Delta_i(g_0)$ and observe that $\big(\widetilde{f}_1, \ldots, \widetilde{f}_n\big)$ is still a WZ-form, which implies that for each $k \in \{2, \ldots, n\}$, $\Delta_k(\widetilde{f}_1)$ is σ_1-summable. Note that $\sum_{i=1}^{I} \sum_{j=1}^{J} \widetilde{f}_{1,i,j}$ is the orbital decomposition of $\widetilde{f}_1$ with respect to x_1 and G. By Theorem 17, for each i, j, we have $\Delta_k\big(\widetilde{f}_{1,i,j}\big)$ is σ_1-summable. Then we can focus on each orbital component of $\widetilde{f}_1$ in a linear $K(\widehat{\mathbf{x}})$-subspace $U^G_{b,m}$.

Remark 26. *We claim that $a \in K(\mathbf{x}) \setminus K(\widehat{\mathbf{x}})$ is pairwise shift-invariant if and only if for each $k \in \{2, \ldots, n\}$, there exist $L_k, N_k \in \mathbb{Z}$ with $L_k \neq 0$, such that $\sigma_k^{L_k}(a) = \sigma_1^{N_k}(a)$. The necessity follows from Definition 5. For the sufficiency, we combine for any $k, s \in \{2, \ldots, n\}$ the N_s-fold application of $\sigma_k^{L_k}(a) = \sigma_1^{N_k}(a)$ with the N_k-fold application of $\sigma_s^{L_s}(a) = \sigma_1^{N_s}(a)$ to obtain*

$$\sigma_k^{L_k N_s}(a) = \sigma_1^{N_k N_s}(a) = \sigma_s^{L_s N_k}(a).$$

If $N_k = N_s = 0$, then a is free of x_k and x_s which implies that $\sigma_k^1(a) = \sigma_s^1(a)$.

Lemma 27. *Let $f_1 = \sum_{\lambda \in \Lambda} a_\lambda / \lambda(b^m) \in U^G_{b,m}$ with $\Lambda \subset E$ and the $\lambda(b)$ being in distinct $\langle \sigma_1 \rangle$-orbits. If $\Delta_k(f_1)$ is σ_1-summable for each $k \in \{2, \ldots, n\}$, then all of the a_λ and b are integer-linear of the same type.*

Proof. By Remark 26 and Lemma 7, it is sufficient to show that for each $k \in \{2, \ldots, n\}$, there exist $L_k, N_k \in \mathbb{Z}$ with L_k nonzero such that $\sigma_k^{L_k}(b) = \sigma_1^{N_k}(b)$ and $\sigma_k^{L_k}(a_\lambda) = \sigma_1^{N_k}(a_\lambda)$ for all $\lambda \in \Lambda$. Let $E_k := \langle \sigma_2, \ldots, \sigma_{k-1}, \sigma_{k+1}, \ldots, \sigma_n \rangle$. For each $\lambda \in \Lambda \subset E$, there exist $t_\lambda \in \mathbb{Z}$, $\eta_\lambda \in E_k$ such that $\lambda = \sigma_k^{t_\lambda} \eta_\lambda$, and therefore

$$f_1 = \sum_{\lambda \in \Lambda} \frac{a_\lambda}{\sigma_k^{t_\lambda} \eta_\lambda(b^m)}.$$

By applying the reduction formula (8) once again, we can rewrite f_1 in the form

$$f_1 = \Delta_1(f_{1,k}) + \sum_{\eta \in \Lambda_k} \sum_{\ell=0}^{T_\eta} \frac{\widetilde{a}_{\eta,\ell}}{\sigma_k^\ell \eta(b^m)}, \tag{12}$$

where $\Lambda_k \subset E_k$, $\eta(b) \not\sim_{\langle \sigma_1, \sigma_k \rangle} \eta'(b)$ if $\eta \neq \eta'$, $\sigma_k^\ell(b) \not\sim_{\langle \sigma_1 \rangle} \sigma_k^{\ell'}(b)$ if $\ell \neq \ell'$, and $\widetilde{a}_{\eta,T_\eta} \neq 0$ for each η. Furthermore, we assume that this representation is such that $T_\eta \geq 0$ is as small as possible. Note that $\sum_{\ell=0}^{T_\eta} \widetilde{a}_{\eta,\ell} / \sigma_k^\ell \eta(b^m) \in U^{\langle \sigma_1, \sigma_k \rangle}_{\eta(b),m}$. Recall that by our assumption $\Delta_k(f_1)$ is σ_1-summable. Then by Theorem 17, we have that $\Delta_k\Big(\sum_{\ell=0}^{T_\eta} \widetilde{a}_{\eta,\ell} / \sigma_k^\ell \eta(b^m)\Big)$ is σ_1-summable for each η. Now Lemma 23 implies that there exist integers S_η such that

$$\sigma_k^{T_\eta+1}\big(\eta(b)\big) = \sigma_1^{S_\eta}\big(\eta(b)\big), \tag{13}$$

$$\sigma_k^{T_\eta+1}(\widetilde{a}_{\eta,0}) = \sigma_1^{S_\eta}(\widetilde{a}_{\eta,0}), \tag{14}$$

$$\widetilde{a}_{\eta,\ell} = \sigma_k^\ell(\widetilde{a}_{\eta,0}), \text{ for all } \ell \in \{0, \ldots, T_\eta\}. \tag{15}$$

Applying η^{-1} to both sides of Equation (13) yields $\sigma_k^{T_\eta+1}(b) = \sigma_1^{S_\eta}(b)$ since G is commutative. Since the $\sigma_k^\ell(b)$ are in distinct $\langle\sigma_1\rangle$-orbits, we have $T_\eta = T_{\eta'}$ and $S_\eta = S_{\eta'}$ for any two $\eta, \eta' \in \Lambda_k$. Let $L_k := T_\eta + 1$ and $N_k := S_\eta$, then L_k is the minimal positive integer such that $\sigma_k^{L_k}(b) \sim_{\langle\sigma_1\rangle} b$ and $\sigma_k^{L_k}(b) = \sigma_1^{N_k}(b)$. According to Equation (14) and (15), for each η, ℓ, $\sigma_k^{L_k}(\widetilde{a}_{\eta,\ell}) = \sigma_1^{N_k}(\widetilde{a}_{\eta,\ell})$. We observe that

$$\operatorname{res}_{\sigma_1}\big(f_1, \lambda(b), m\big) = [a_\lambda] \quad \text{and} \quad \operatorname{res}_{\sigma_1}\big(f_1, \sigma_k^\ell\eta(b), m\big) = [\widetilde{a}_{\eta,\ell}].$$

For each $\lambda \in \Lambda$, there exists a unique pair (η, ℓ) where $\eta \in \Lambda_k, \ell \in \{0, \ldots, T_\eta\}$ such that $\lambda(b) \sim_{\langle\sigma_1\rangle} \sigma_k^\ell\eta(b)$. By Definition 18 we have $a_\lambda \sim_{\langle\sigma_1\rangle} \widetilde{a}_{\eta,\ell}$. Now Lemma 22 implies that $\sigma_k^{L_k}(a_\lambda) = \sigma_1^{N_k}(a_\lambda)$. □

Now we are ready to give the proof of Theorem 20.

Proof. We proceed by induction on n. For the base case when $n = 1$, the theorem follows from the fact that any univariate rational function is a uniform WZ-form. Suppose now that $n \geq 2$ and the theorem holds for any WZ-forms in $(n-1)$ variables. As in Lemma 27, we focus on each component of the orbital decomposition of f_1 and rewrite it as in (12). Next we use the cyclic operator to describe f_1 in a more precise way as

$$f_1 = \Delta_1(f_{1,k}) + \frac{\sigma_k^{L_k} - 1}{\sigma_k - 1} \bullet \sum_{\eta\in\Lambda_k} \frac{\widetilde{a}_{\eta,0}}{\eta(b^m)}.$$

Suppose that $L_k, N_k \in \mathbb{Z}$ with L_k nonzero such that

$$\sigma_k^{L_k}\left(\frac{\widetilde{a}_{\eta,0}}{\eta(b^m)}\right) = \sigma_1^{N_k}\left(\frac{\widetilde{a}_{\eta,0}}{\eta(b^m)}\right).$$

For each $k \in \{2, \ldots, n\}$, let

$$f_k' = \Delta_k(f_{1,k}) + \frac{\sigma_1^{N_k} - 1}{\sigma_1 - 1} \bullet \sum_{\eta\in\Lambda_k} \frac{\widetilde{a}_{\eta,0}}{\eta(b^m)},$$

then one can easily check that $\Delta_k(f_1) = \Delta_1(f_k')$ with f_k' and f_1 being integer-linear of the same type. For $k, \ell \in \{2, \ldots, n\}$ with $k \neq \ell$, we have $\Delta_k(f_1) = \Delta_1(f_k')$ and $\Delta_\ell(f_1) = \Delta_1(f_\ell')$, from which it follows that

$$\Delta_\ell\Delta_1(f_k') = \Delta_\ell\Delta_k(f_1) = \Delta_k\Delta_1(f_\ell').$$

Thus $\Delta_1\big(\Delta_\ell(f_k') - \Delta_k(f_\ell')\big) = 0$, i.e., $\Delta_\ell(f_k') - \Delta_k(f_\ell') \in K(\widehat{\mathbf{x}})$. By construction, we have $f_{1,k} \in U_{b,m}^G$ and $f_2', \ldots, f_n' \in U_{b,m}^G$. By Lemma 16, also $\Delta_\ell(f_k') - \Delta_k(f_\ell')$ is an element of $U_{b,m}^G$. According to the definition of $U_{b,m}^G$ (4),

$$U_{b,m}^G \cap K(\widehat{\mathbf{x}}) = \{0\}.$$

Thus $\Delta_\ell(f'_k) - \Delta_k(f'_\ell) = 0$. By Definition 10, $(f_1, f'_2, \ldots, f'_n)$ is a uniform WZ-form in $U^G_{b,m}$, denoted by $\omega_{i,j}$ for some i, j.

In conclusion, from the orbital decomposition of f_1, we can obtain a WZ-form $(f_1, f'_2, \ldots, f'_n)$ which is one exact WZ-form ω_0 plus several uniform WZ-forms $\omega_{i,j}$. Note that there may remain a WZ-form: $(0, f_2 - f'_2, \ldots, f_n - f'_n)$. From the compatibility conditions (2), we have for each $k \in \{2, \ldots, n\}$, $\Delta_1(f_k - f'_k) = \Delta_k(0) = 0$, so $f_k - f'_k \in K(\widehat{\mathbf{x}})$. So the remaining can be viewed as an $(n-1)$-variable WZ-form w.r.t. $(\Delta_2, \ldots, \Delta_n)$. By the induction hypothesis we can complete the proof. □

Note that this decomposition is not unique in two aspects. Referring to Remark 11, when a WZ-form is both exact and uniform, we will put it into the exact part, which minimizes the uniform part. It is decided by the operators in G we choose in the orbital decomposition. Next we give an example to illustrate how the decomposition works.

Example 28. *Let $\omega = (f, g, h) \in K(x, y, z)^3$ be a WZ-form with*

$$\begin{aligned} f &= \sum_{\ell=0}^{3} \frac{1}{4x + 6y + 5z + \ell}, \\ g &= \sum_{\ell=0}^{5} \frac{1}{4x + 6y + 5z + \ell} + \sum_{\ell=0}^{2} \frac{1}{3y + 2z + \ell}, \\ h &= \sum_{\ell=0}^{4} \frac{1}{4x + 6y + 5z + \ell} + \sum_{\ell=0}^{1} \frac{1}{3y + 2z + \ell}. \end{aligned}$$

It is easy to check that (f, g, h) satisfy the following compatibility conditions:

$$\big\{\Delta_y(f) = \Delta_x(g),\ \Delta_z(f) = \Delta_x(h),\ \Delta_z(g) = \Delta_y(h)\big\}.$$

Here we let $b := 4x + 6y + 5z$ and rewrite f as

$$f = \frac{1}{b} + \frac{1}{\sigma_x^{-1}\sigma_y(b)} + \frac{1}{\sigma_x^{-1}\sigma_y(b)} + \frac{1}{\sigma_x^{-3}\sigma_z^3(b)}.$$

Note that this representation is not unique. Let $c := 3y + 2z$, then rewrite

$$\sum_{\ell=0}^{2} \frac{1}{3y + 2z + \ell} = \frac{1}{c} + \frac{1}{\sigma_y^{-1}\sigma_z^2(c)} + \frac{1}{\sigma_z(c)}.$$

Then we can decompose it into an exact WZ-form plus two uniform WZ-forms:

$$
\begin{aligned}
f &= \Delta_x(a+\bar{a}) + \left(\Delta_x(a_2) + \frac{\sigma_y^2-1}{\sigma_y-1}\cdot\frac{\sigma_z^2-1}{\sigma_z-1}\bullet\frac{1}{b}\right) + \frac{\sigma_y^0-1}{\sigma_y-1}\cdot\frac{\sigma_z^3-1}{\sigma_z-1}\bullet\frac{1}{c}\\
&= \Delta_x(a+\bar{a}) + \left(\Delta_x(a_3) + \frac{\sigma_z^4-1}{\sigma_z-1}\cdot\frac{\sigma_y-1}{\sigma_y-1}\bullet\frac{1}{b}\right) + \frac{\sigma_z^3-1}{\sigma_z-1}\cdot\frac{\sigma_y^0-1}{\sigma_y-1}\bullet\frac{1}{c},\\
g &= \Delta_y(a+\bar{a}) + \left(\Delta_y(a_2) + \frac{\sigma_x^3-1}{\sigma_x-1}\cdot\frac{\sigma_z^2-1}{\sigma_z-1}\bullet\frac{1}{b}\right) + \frac{\sigma_x-1}{\sigma_x-1}\cdot\frac{\sigma_z^3-1}{\sigma_z-1}\bullet\frac{1}{c},\\
h &= \Delta_z(a+\bar{a}) + \left(\Delta_z(a_3) + \frac{\sigma_x^5-1}{\sigma_x-1}\cdot\frac{\sigma_y-1}{\sigma_y-1}\bullet\frac{1}{b}\right) + \frac{\sigma_x-1}{\sigma_x-1}\cdot\frac{\sigma_y^2-1}{\sigma_y-1}\bullet\frac{1}{c}.
\end{aligned}
$$

where

$$
\begin{aligned}
&a = -\frac{1}{\sigma_x^{-1}\sigma_z(b)} - \frac{1}{\sigma_x^{-1}\sigma_y(b)} - \frac{1}{\sigma_x^{-3}\sigma_z^3(b)} - \frac{1}{\sigma_x^{-2}\sigma_z^3(b)} - \frac{1}{\sigma_x^{-1}\sigma_z^3(b)},\\
&a_2 = \frac{1}{\sigma_y\sigma_z(b)}, \quad a_3 = -\frac{1}{\sigma_x^{-1}\sigma_z^2(b)}, \quad \bar{a} = -\frac{1}{\sigma_y^{-1}\sigma_z^2(c)}.
\end{aligned}
$$

As we can see, the first uniform WZ-form is of the type $(4,6,5)$ *and the second is* $(0,3,2)$.

5. Structure of uniform WZ-forms

Theorem 20 tells us how every WZ-form can be decomposed into exact and uniform WZ-forms. While exact WZ-forms are easy to describe and to construct, Definition 10 only allows us to check whether a given tuple is a uniform WZ-form, but this characterization is not explicit enough to construct such forms. In this section, we use a difference homomorphism in order to write a uniform WZ-form in terms of its integer-linear type and a single univariate rational function. Then we finish our proof of the additive Ore–Sato theorem.

Let $(A,\boldsymbol{\sigma})$ and $(A,\boldsymbol{\tau})$ be two difference rings, where $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_n)$ and $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_n)$. A homomorphism (resp. isomorphism) $\phi\colon A\to A$ is called a difference homomorphism (resp. isomorphism) from $(A,\boldsymbol{\sigma})$ to $(A,\boldsymbol{\tau})$ if $\phi\circ\sigma_i=\tau_i\circ\phi$, for each $i\in\{1,\ldots,n\}$. That is to say for each i there is a commutative diagram:

$$
\begin{array}{ccc}
A & \xrightarrow{\sigma_i} & A\\
\big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\phi}\\
A & \xrightarrow{\tau_i} & A
\end{array}
$$

Lemma 29. *Given a unimodular matrix* $\mathbf{D}\in\mathbb{Z}^{n\times n}$, *i.e.,* $\mathbf{D}^{-1}\in\mathbb{Z}^{n\times n}$, *we define a ring isomorphism* $\phi\colon K(\mathbf{x})\to K(\mathbf{x})$ *by* $\phi(\mathbf{x})=\mathbf{D}\cdot\mathbf{x}$. *Furthermore, we let the* σ_i *act on vectors as* $\sigma_i(\mathbf{x})=\mathbf{x}+\mathbf{e}_i$, *where* $\mathbf{e}_i$ *denotes the* i*-th unit vector. If we define* $\tau_i(\mathbf{x})=\mathbf{x}+\mathbf{D}^{-1}\cdot\mathbf{e}_i$, *for all* $i\in\{1,\ldots,n\}$, *then* ϕ *is a difference isomorphism from* $\big(K(\mathbf{x}),\boldsymbol{\sigma}\big)$ *to* $\big(K(\mathbf{x}),\boldsymbol{\tau}\big)$.

Proof. We have to check that $\phi \circ \sigma_i = \tau_i \circ \phi$. For the left-hand side we get

$$\phi(\sigma_i(f(\mathbf{x}))) = \phi(f(\mathbf{x} + \mathbf{e}_i)) = f(\mathbf{D} \cdot \mathbf{x} + \mathbf{e}_i),$$

and the right-hand side gives

$$\tau_i(\phi(f(\mathbf{x}))) = \tau_i\big(f(\mathbf{D} \cdot \mathbf{x})\big) = f\big(\mathbf{D} \cdot (\mathbf{x} + \mathbf{D}^{-1} \cdot \mathbf{e}_i)\big) = f\big(\mathbf{D} \cdot \mathbf{x} + \mathbf{e}_i\big). \qquad \square$$

Given $f_1, \ldots, f_n \in K(\mathbf{x})$ satisfying the compatibility conditions (2), Theorem 2 in [4] shows that there exists a difference ring extension $\big(K(\mathbf{x})[H], \boldsymbol{\sigma}\big)$ of $\big(K(\mathbf{x}), \boldsymbol{\sigma}\big)$, where H is a hyperarithmetic term with the certificates $f_1, \ldots, f_n$. A difference homomorphism from $\big(K(\mathbf{x}), \boldsymbol{\sigma}\big)$ to $\big(K(\mathbf{x}), \boldsymbol{\tau}\big)$ can naturally be extended to the corresponding difference ring extensions.

Lemma 30. *[3, Proposition 9] For every integer vector $\mathbf{v} = (v_1, \ldots, v_n)$ there is an integer matrix $\mathbf{D} \in \mathbb{Z}^{n \times n}$ with the first row $\mathbf{v}$ and $\det(\mathbf{D}) = \gcd(v_1, \ldots, v_n)$.*

Next we use such a matrix $\mathbf{D}$ to construct the difference homomorphism.

Theorem 31. *Let $\big(f_1(\mathbf{v}\cdot\mathbf{x}), \ldots, f_n(\mathbf{v}\cdot\mathbf{x})\big)$ be a uniform WZ-form of the type $\mathbf{v}$, then there exist constants $\mu_1, \ldots, \mu_n \in K$ and a univariate rational function $r \in K(z)$ such that for each $i \in \{1, \ldots, n\}$,*

$$f_i(\mathbf{v} \cdot \mathbf{x}) = \mu_i + \sum\nolimits_{\ell}{}_{0}^{v_i} r(\mathbf{v} \cdot \mathbf{x} + \ell).$$

Proof. Let $H(\mathbf{x})$ be a hyperarithmetic term, and let $f_1(\mathbf{v} \cdot \mathbf{x}), \ldots, f_n(\mathbf{v} \cdot \mathbf{x})$ be its certificates. That is to say for each i,

$$\sigma_i\big(H(\mathbf{x})\big) = H(\mathbf{x}) + f_i(\mathbf{v} \cdot \mathbf{x}). \tag{16}$$

Without loss of generality, we can assume that $\gcd(v_1, \ldots, v_n) = 1$. By Lemma 30, there exists an integer matrix $\mathbf{D} = (d_{ij}) \in \mathbb{Z}^{n \times n}$ with $\det(\mathbf{D}) = 1$ whose first row is $\mathbf{v}$. Let $\phi \colon K(\mathbf{x}) \to K(\mathbf{x})$ such that

$$\phi\big(f(\mathbf{x})\big) = f\big(\mathbf{D}^{-1} \cdot \mathbf{x}\big), \text{ for all } f(\mathbf{x}) \in K(\mathbf{x}).$$

By Lemma 29, ϕ is a difference isomorphism from $\big(K(\mathbf{x})[H], \boldsymbol{\sigma}\big)$ to $\big(K(\mathbf{x})[H], \boldsymbol{\tau}\big)$, where $\tau_i(\mathbf{x}) = \mathbf{x} + \mathbf{D} \cdot \mathbf{e}_i$ for all i in $\{1, \ldots, n\}$. Applying the operator ϕ to Equation (16) yields

$$\begin{aligned} \phi\big(\sigma_i\big(H(\mathbf{x})\big)\big) &= \phi\big(H(\mathbf{x})\big) + \phi\big(f_i(\mathbf{v} \cdot \mathbf{x})\big), \\ \tau_i\big(\phi\big(H(\mathbf{x})\big) &= \phi\big(H(\mathbf{x})\big) + f_i(x_1). \end{aligned}$$

Let $H'(\mathbf{x}) = \phi\big(H(\mathbf{x})\big)$, then it follows that $\tau_i\big(H'(\mathbf{x})\big) = H'(\mathbf{x}) + f_i(x_1)$. For any integer $m > 0$ and $i \in \{1, \ldots, n\}$ we have

$$\tau_i^m\big(H'(\mathbf{x})\big) = H'(\mathbf{x}) + \sum_{j=0}^{m-1} f_i\big(x_1 + jd_{1i}\big) =: H'(\mathbf{x}) + f_{i,m}(x_1), \tag{17}$$

$$\tau_i^{-m}\big(H'(\mathbf{x})\big) = H'(\mathbf{x}) - \sum_{j=1}^{m} f_i\big(x_1 - jd_{1i}\big) =: H'(\mathbf{x}) + f_{i,-m}(x_1). \tag{18}$$

Let $\mathbf{D}^{-1} =: \big(\widetilde{d}_{ij}\big)_{n\times n}$, then for all $i \in \{1,\dots,n\}$ we can rewrite σ_i in terms of $\tau_1,\dots,\tau_n$:

$$\sigma_i = \prod_{j=1}^{n} \tau_j^{\widetilde{d}_{ji}}.$$

By applying (17) and (18) repeatedly, we obtain $\sigma_i\big(H'(\mathbf{x})\big) = H'(\mathbf{x}) + f_i'(x_1)$ for some univariate rational function f_i'. That is to say, $\Delta_i\big(H'(\mathbf{x})\big) = f_i'(x_1)$. By the compatibility conditions (2) we have that $\Delta_1\big(f_k'(x_1)\big) = \Delta_k\big(f_1'(x_1)\big) = 0$, and thus $f_k' \in K$ for all $k \in \{2,\dots,n\}$. Then an easy induction shows that

$$H'(\mathbf{x}) \simeq F'(x_1) + \sum_{k=2}^{n} f_k' x_k,$$

where $F'(x_1)$ is a solution of the difference equation $y(x_1+1) - y(x_1) = f_1'(x_1)$. Next, we can recover $H(\mathbf{x})$ as follows,

$$\begin{aligned} H(\mathbf{x}) &= \phi^{-1}\big(H'(\mathbf{x})\big) \\ &= H'\big(\mathbf{D}\cdot\mathbf{x}\big) \\ &\simeq F'(\mathbf{v}\cdot\mathbf{x}) + \sum_{k=2}^{n} f_k'\left(\sum_{i=1}^{n} d_{ki}x_i\right) \\ &= F'(\mathbf{v}\cdot\mathbf{x}) + \sum_{i=1}^{n}\left(\sum_{k=2}^{n} f_k' d_{ki}\right)x_i, \end{aligned}$$

where $F'(\mathbf{v}\cdot\mathbf{x}+1) - F'(\mathbf{v}) = f_1'(\mathbf{v}\cdot\mathbf{x})$. Write that $\mu_i := \sum_{k=2}^{n} f_k' d_{ki}$. Then for each $i \in \{1,\dots,n\}$,

$$f_i(\mathbf{v}\cdot\mathbf{x}) = \Delta_i\big(H(\mathbf{x})\big) = \begin{cases} \mu_i + \displaystyle\sum_{\ell=0}^{v_i-1} f_1'(\mathbf{v}\cdot\mathbf{x}+\ell), & \text{if } v_i > 0, \\ \mu_i, & \text{if } v_i = 0, \\ \mu_i - \displaystyle\sum_{\ell=v_i}^{-1} f_1'(\mathbf{v}\cdot\mathbf{x}+\ell), & \text{if } v_i < 0. \end{cases}$$

Finally we let the univariate rational function r be defined as f_1'. □

Now we obtain Theorem 4 by combining Theorem 20 and Theorem 31. Note that we can save the $\{\mu_i\}_{i=1}^{n}$ since the constant tuple $(\mu_1,\dots,\mu_n)$ can be viewed as an exact WZ-form. Now we show that any hyperarithmetic term can be described up to conjugation in terms of a rational function plus a K-linear combination of polygamma functions. First we employ the partial fraction decomposition on the univariate function r over K:

$$r(z) = \sum_{s}\sum_{t} \frac{\beta_{s,t}}{(z+\alpha_s)^t},$$

where $s,t \in \mathbb{N}$ and $\alpha_s, \beta_{s,t} \in K$, both with the finite support set.

According to the recurrence formula of polygamma functions in [10, (5.15)]:

$$\psi^{(t)}(z+1) - \psi^{(t)}(z) = \frac{(-1)^t t!}{z^{t+1}}, \quad t = 0, 1, \ldots$$

we have

$$\psi^{(t)}(z+\alpha_s+1) - \psi^{(t)}(z+\alpha_s) = \frac{(-1)^t t!}{(z+\alpha_s)^{t+1}}.$$

Then the hyperarithmetic term H' with certificates

$$\Big(\sum\nolimits_{\ell}^{v_1}{}_0\, r(\mathbf{v}\cdot\mathbf{x}+\ell), \ldots, \sum\nolimits_{\ell}^{v_n}{}_0\, r(\mathbf{v}\cdot\mathbf{x}+\ell)\Big)$$

is conjugate to

$$\sum_s \sum_t \frac{\beta_{s,t+1}}{(-1)^t t!}\psi^{(t)}(\mathbf{v}\cdot\mathbf{x}+\alpha_s).$$

Corollary 32. *Any hyperarithmetic term is conjugate to*

$$a + \sum_{\mathbf{v}\in V}\sum_s\sum_t \beta_{\mathbf{v},s,t}\psi^{(t)}(\mathbf{v}\cdot\mathbf{x}+\alpha_{\mathbf{v},s}),$$

where a is a rational function, $V \subset \mathbb{Z}^n$, $s,t \in \mathbb{N}$, and for each $\mathbf{v}$, we have $\beta_{\mathbf{v},s,t}, \alpha_{\mathbf{v},s} \in K$.

Example 33. *Let H be a hyperarithmetic term with certificates (f,g,h) as in Example 28. Then H is conjugate to $\psi^{(0)}(4x+6y+5z)+\psi^{(0)}(3y+2z)$.*

6. Algorithms and implementation

Now we will present an algorithm for computing additive representations of WZ-forms based on the recursive idea in the proof of Theorem 4.

Definition 34 (Additive representation)**.** *Given a WZ-form $\omega = (f_1, \ldots, f_n)$, there is a decomposition of the form*

$$\omega = \big(\Delta_1(a), \ldots, \Delta_n(a)\big) + \sum_{\mathbf{v}\in V}\Big(\sum\nolimits_{\ell}^{v_1}{}_0\, r_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{x}+\ell), \ldots, \sum\nolimits_{\ell}^{v_n}{}_0\, r_{\mathbf{v}}(\mathbf{v}\cdot\mathbf{x}+\ell)\Big),$$

We call the list $\big(a, V, \{r_{\mathbf{v}}\}_{\mathbf{v}\in V}\big)$ an additive representation *of ω.*

Let $\omega = (f_1, \ldots, f_n) \in K(\mathbf{x})^n$ be a WZ-form. Firstly, we apply Abramov's reduction [1] with respect to the variable x_1 to decompose f_1 into

$$f_1 = \Delta_1(g_0) + \sum_{i=1}^{I}\sum_{j=1}^{J}\frac{a_{i,j}}{b_i^j},$$

where $g_0 \in K(\widehat{\mathbf{x}})[x_1]$, $a_{i,j}, b_i \in K[\widehat{\mathbf{x}}][x_1]$ with $\deg_{x_1}(a_{i,j}) < \deg_{x_1}(b_i)$, and the b_i are in distinct $\langle \sigma_1 \rangle$-orbits.

By Lemma 27, each $a_{i,j}/b_i^j$ are integer-linear of some type $\mathbf{v}_i$. In order to compute the type of each simple fraction in the above decomposition, we are reduced to the following problem.

Problem 35 (Integer-linear testing)**.** *Given a polynomial $p \in K[\mathbf{x}]$, decide whether there exist $u \in K[z]$ and $\mathbf{v} \in \mathbb{Z}^n$ such that $p = u(\mathbf{v} \cdot \mathbf{x})$.*

The above problem has been solved in [13]. Applying the algorithm `IntegerLinearDecomp` in [13] to the numerator and the denominator of each simple fraction $a_{i,j}/b_i^j$ yields

$$\frac{a_{i,j}}{b_i^j} = u_{i,j}(\mathbf{v}_i \cdot \mathbf{x}),$$

where $u_{i,j} \in K(z)$ and $\mathbf{v}_i \in \mathbb{Z}^n$ with the first entry $v_{i,1}$ being nonzero. By collecting the simple fractions of the same type, we obtain

$$f_1 = \Delta_1(g_0) + \sum_{\mathbf{v} \in V} u_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x}),$$

where $V \subset \mathbb{Z}^n$ is a finite set and $u_{\mathbf{v}} \in K(z)$ for each $\mathbf{v} \in V$. The next step is to write the rational function $u_{\mathbf{v}}$ into the form

$$u_{\mathbf{v}}(z) = \sum\nolimits_{\ell}{}_{0}^{v_1} r_{\mathbf{v}}(z + \ell),$$

where $r_{\mathbf{v}} \in K(z)$. Note that $r_{\mathbf{v}}$ must be a rational solution of the difference equation

$$y(z + v_1) - y(z) = u_{\mathbf{v}}(z + 1) - u_{\mathbf{v}}(z),$$

which can also be solved by Abramov's reduction.

Let $\omega_0 := \big(\Delta_1(g_0), \ldots, \Delta_n(g_0)\big)$ and $\omega_{\mathbf{v}} := \big(f_{1,\mathbf{v}}, \ldots, f_{n,\mathbf{v}}\big)$, where for each $k \in \{1, \ldots, n\}$,

$$f_{k,\mathbf{v}} := \sum\nolimits_{\ell}{}_{0}^{v_k} r_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{x} + \ell).$$

Then ω can be written as a summation of one exact WZ-form, several uniform WZ-forms and a "degenerate" WZ-form:

$$\omega = \omega_0 + \sum_{\mathbf{v} \in V} \omega_{\mathbf{v}} + \widetilde{\omega}.$$

We now proceed with the induction step by repeating the above process for $\widetilde{\omega}$ which only involves $(n-1)$-variables. The above process for computing additive representations of WZ-forms is summarized in Algorithm 1 and is illustrated in Example 36. Our Maple code for implementing Algorithm 1 is available at

`http://www.mmrc.iss.ac.cn/~schen/AddOreSato.html`

Algorithm 1 WZ-form decomposition algorithm

Function: `WZFormDecomp`$\big((f_1,\dots,f_n),\mathbf{x},Z\big)$
Input: WZ-form $(f_1,\dots,f_n)\in K(\mathbf{x})^n$, $\mathbf{x}=(x_1,\dots,x_n)$, and a new variable Z
Output: Its additive representation: $\big(a,V,R=\{r_{\mathbf{v}}\}_{\mathbf{v}\in V}\big)$

> **if** $f_1=0$ **then**
>> $(a,V,R)\leftarrow$ `WZFormDecomp`$\big((f_2,\dots,f_n),(x_2,\dots,x_n),Z\big)$
>> **for** $\mathbf{v}=(v_2,\dots,v_n)$ in V **do**
>>> $\mathbf{v}\leftarrow(0,v_2,\dots,v_n)$
>>
>> **end for**
>> **return** (a,V,R)
>
> **end if**
> Call `AbramovReduction`: $f_1=\Delta_1(g_0)+\sum_{i=1}^{I}\sum_{j=1}^{J}a_{i,j}/b_i^j$
> **if** $n=1$ **then**
>> **return** $\big(g_0,\big((1)\big),\big(f_1-\Delta_1(g_0)\big)\big)$
>
> **end if**
> **for** $1\le i\le I$ **do**
>> Call `IntegerLinearDecomp`: $b_i=q_i(\mathbf{w}_i\cdot\mathbf{x})$ with $q_i\in K[Z]$
>
> **end for**
> $V\leftarrow(\mathbf{v}_1,\dots,\mathbf{v}_m)$ with $\{\mathbf{v}_1,\dots,\mathbf{v}_m\}=\{\mathbf{w}_1,\dots,\mathbf{w}_I\}$ and $\mathbf{v}_i\neq\mathbf{v}_j$ for $i\neq j$
> **for** $1\le k\le m$ **do**
>> $u_k\leftarrow 0$
>> **for** $1\le i\le I$ **do**
>>> **if** the integer-linear type of b_i is $\mathbf{v}_k=(v_{k,1},\dots,v_{k,n})$ **then**
>>>> **for** $1\le j\le J$ **do**
>>>>> Perform the substitution $\mathbf{v}_k\cdot\mathbf{x}\to Z$ in $a_{i,j}$ so that $a_{i,j}\in K[Z]$
>>>>> $u_k\leftarrow u_k+a_{i,j}/q_i^j$
>>>>
>>>> **end for**
>>>
>>> **end if**
>>
>> **end for**
>> Call `AbramovReduction`: $\sigma_z(h_k)-h_k=u_k(v_{k,1}z+1)-u_k(v_{k,1}z)$
>> $r_k\leftarrow h_k(1/v_{k,1}Z)$
>
> **end for**
> $a\leftarrow g_0$, $R\leftarrow(r_1,\dots,r_m)$
> **for** $2\le k\le n$ **do**
>> $$f_k'\leftarrow f_k-\sum_{i=1}^{m}\sum_{\ell}^{v_{i,k}}{}_0\, r_i(\mathbf{v}_i\cdot\mathbf{x}+\ell)$$
>
> **end for**
> **if** $f_k'\neq 0$ for some k **then**
>> $\big(a',V',R'\big)\leftarrow$ `WZFormDecomp`$\big((f_2',\dots,f_n'),(x_2,\dots,x_n),Z\big)$
>> **for** $\mathbf{v}'=(v_2,\dots,v_n)$ in V' **do**
>>> $\mathbf{v}'\leftarrow(0,v_2,\dots,v_n)$
>>
>> **end for**
>> $a\leftarrow a+a'$, $V\leftarrow$ `Join`(V,V'), $R\leftarrow$ `Join`(R,R')
>
> **end if**
> **return** (a,V,R)

Example 36. *Set* $\omega := (f, g, h) \in K(x, y, z)^3$ *be a WZ-form with respect to* $(\Delta_x, \Delta_y, \Delta_z)$*, specifically,*

$$\begin{aligned} f &= \frac{xyz - y^2z - yz^2 + yz - 1}{x - y - z + 1}, \\ g &= \frac{x^2z - xyz - xz^2 + xy - y^2 - yz - 1}{x - y - z}, \\ h &= \frac{x^2y - xy^2 - xyz + xz - yz - z^2 - 1}{x - y - z}. \end{aligned}$$

Employing Abramov's reduction on f *yields*

$$f = \Delta_x(xyz) + \frac{1}{-x + y + z - 1}.$$

Then we record the following exact WZ-form as a part of ω*:*

$$\omega_0 := \big(\Delta_x(xyz), \Delta_y(xyz), \Delta_z(xyz)\big).$$

Obviously from the decomposition of f *there is only one integer-linear type* $\mathbf{v} = (-1, 1, 1)$ *and the corresponding univariate rational function is* $r_{\mathbf{v}} = 1/Z$*. Then a uniform WZ-form shows up as a part of* ω*:*

$$\omega_{\mathbf{v}} = \Big(\frac{1}{-x + y + z - 1}, \frac{1}{-x + y + z}, \frac{1}{-x + y + z}\Big).$$

Then we can update ω *by subtracting* ω_0 *and* $\omega_{\mathbf{v}}$*:* $\widetilde{\omega} = (0, y, z)$*, which is equivalent to the WZ-pair* (y, z) *with respect to* (Δ_y, Δ_z)*. By simple manipulation we can see it is an exact WZ-pair:*

$$\Big(\Delta_y\big(\tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big), \Delta_z\big(\tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big)\Big).$$

Combining this exact WZ-form with the previous one we can update ω_0 *as:*

$$\omega_0 = \Big(\Delta_x\big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big), \Delta_y\big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big), \Delta_z\big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2\big)\Big).$$

Finally the decomposition works as $\omega = \omega_0 + \omega_{\mathbf{v}}$*, i.e., the additive representation of* ω *is*

$$\Big(xyz + \tfrac{1}{2}y^2 + \tfrac{1}{2}z^2, \{(-1, 1, 1)\}, \{1/Z\}\Big).$$

Acknowledgment

We would like to thank Hui Huang for providing the Maple code of the integer-linear decomposition and Jing Guo for helpful discussions.

References

[1] Abramov, S. A., 1971. The summation of rational functions. Ž. Vyčisl. Mat i Mat. Fiz. 11, 1071–1075.

[2] Abramov, S. A., 1975. The rational component of the solution of a first order linear recurrence relation with rational right hand side. Ž. Vyčisl. Mat. i Mat. Fiz. 15 (4), 1035–1039, 1090.

[3] Abramov, S. A., Petkovšek, M., 2002. On the structure of multivariate hypergeometric terms. Adv. Appl. Math. 29 (3), 386–411.

[4] Bronstein, M., Li, Z., Wu, M., 2005. Picard–Vessiot extensions for linear functional systems. In: ISSAC '05: Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation. ACM, New York, USA, pp. 68–75.

[5] Chen, S., 2019. How to generate all possible rational Wilf–Zeilberger pairs? In: Algorithms and complexity in mathematics, epistemology, and science. Vol. 82 of Fields Inst. Commun. Springer, New York, pp. 17–34.

[6] Chen, S., Feng, R., Fu, G., Li, Z., 2011. On the structure of compatible rational functions. In: ISSAC '11: Proceedings of the 2011 International Symposium on Symbolic and Algebraic Computation. ACM, New York, NY, USA, pp. 91–98.

[7] Chen, S., Hou, Q.-H., Labahn, G., Wang, R.-H., 2016. Existence problem of telescopers: beyond the bivariate case. In: Proceedings of the 2016 ACM International Symposium on Symbolic and Algebraic Computation. ACM, New York, pp. 167–174.

[8] Chen, S., Singer, M. F., 2014. On the summability of bivariate rational functions. J. Algebra 409, 320–343.

[9] Christopher, C., 1999. Liouvillian first integrals of second order polynomial differential equations. Electron. J. Differential Equations 49, 1–7.

[10] DLMF, 2024. *NIST Digital Library of Mathematical Functions*. `https://dlmf.nist.gov/`, Release 1.2.0 of 2024-03-15, F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[11] Dreyfus, T., Hardouin, C., Roques, J., Singer, M. F., 2018. On the nature of the generating series of walks in the quarter plane. Invent. Math. 213 (1), 139–203.

[12] Du, H., Li, Z., 2019. The Ore–Sato theorem and shift exponents in the q-difference case. J. Syst. Sci. Complex. 32 (1), 271–286.

[13] Giesbrecht, M., Huang, H., Labahn, G., Zima, E., 2019. Efficient integer-linear decomposition of multivariate polynomials. In: ISSAC'19—Proceedings of the 2019 ACM International Symposium on Symbolic and Algebraic Computation. ACM, New York, pp. 171–178.

[14] Hou, Q.-H., Wang, R.-H., 2015. An algorithm for deciding the summability of bivariate rational functions. Advances in Applied Mathematics 64, 31–49.

[15] Ore, O., 1930. Sur la forme des fonctions hypergéométriques de plusieurs variables. J. Math. Pures Appl. (9) 9 (4), 311–326.

[16] Petkovšek, M., Wilf, H. S., Zeilberger, D., 1996. $A = B$. A K Peters Ltd., Wellesley, MA, with a foreword by Donald E. Knuth.

[17] Sato, M., 1990. Theory of prehomogeneous vector spaces (algebraic part)—the English translation of Sato's lecture from Shintani's note. Nagoya Math. J. 120, 1–34, notes by Takuro Shintani, Translated from the Japanese by Masakazu Muro.

[18] Wilf, H. S., Zeilberger, D., 1990. Rational functions certify combinatorial identities. J. Amer. Math. Soc. 3 (1), 147–158.

[19] Zeilberger, D., 1990. A fast algorithm for proving terminating hypergeometric identities. Discrete Math. 80 (2), 207–211.

[20] Zeilberger, D., 1993. Closed form (pun intended!). In: A tribute to Emil Grosswald: number theory and related analysis. Vol. 143 of Contemp. Math. Amer. Math. Soc., Providence, RI, pp. 579–607.

[21] Zoladek, H., 1998. The extended monodromy group and Liouvillian first integrals. J. Dynam. Control Systems 4 (1), 1–28.