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Abstract

A large literature specifies conditions under which the information complexity for a sequence of numerical problems defined for dimensions $1, 2, \ldots$ grows at a moderate rate, i.e., the sequence of problems is *tractable*. Here, we focus on the situation where the space of available information consists of all linear functionals and the problems are defined as linear operator mappings between Hilbert spaces. We unify the proofs of known tractability results and generalize a number of existing results. These generalizations are expressed as five theorems that provide equivalent conditions for (strong) tractability in terms of sums of functions of the singular values of the solution operators.

1 Introduction

The information complexity of a problem is the number of function data required to solve the problem within the desired error tolerance. A problem is tractable if the information complexity does not grow too large as the dimension of the problem increases or the error threshold decreases. There are a number of tractability results for numerical problems, beginning in the 1990s with [1, 2], spawning numerous articles since then, and filling a series of three impressive volumes [3, 4, 5]. Research on tractability continues at a vigorous pace. This article aims to unify and generalize many existing tractability results for approximation problems where all linear functional data is available.

Let $\{\mathcal{F}_d\}_{d \in \mathbb{N}}$ and $\{\mathcal{G}_d\}_{d \in \mathbb{N}}$ be sequences of Hilbert spaces, and let $\{\mathrm{SOL}_d : \mathcal{F}_d \to \mathcal{G}_d\}_{d \in \mathbb{N}}$ be a sequence of linear solution operators with adjoints SOL_d^* such that $\mathrm{SOL}_d^* \mathrm{SOL}_d : \mathcal{F}_d \to \mathcal{F}_d$ has eigenvalues and orthonormal eigenvectors

$$\lambda_{1,d}^2 \geq \lambda_{2,d}^2 \geq \cdots, \qquad u_{1,d}, u_{2,d}, \ldots, \qquad d \in \mathbb{N}. \tag{1}$$

This means that the non-negative $\lambda_{i,d}$ are the singular values of the operator SOL_d. We remark that in the literature on Information-Based Complexity the singular values of SOL_d are often denoted by $\sqrt{\lambda_{i,d}}$ instead of $\lambda_{i,d}$. However, since we assume non-negativity of the $\lambda_{i,d}$, the notation used here is equivalent.

We make the technical assumption that SOL_d is a compact operator. This implies (see, e.g., [3]) that $\lambda_{i,d}$ converges to zero as i tends to infinity for every d, which in turn implies that the problem is solvable by suitable algorithms. To avoid trivial cases, we assume that there are an infinite number of positive singular values for every d.

The goal is to find a sequence of approximate solution operators, $\{\mathrm{APP}_d : \mathcal{B}_d \times (0, \infty) \to \mathcal{G}_d\}_{d \in \mathbb{N}}$, defined on the unit ball of functions in $\mathcal{F}_d$, denoted $\mathcal{B}_d$, which satisfy an absolute error criterion:

$$\|\mathrm{SOL}_d(f) - \mathrm{APP}_d(f, \varepsilon)\|_{\mathcal{G}_d} \leq \varepsilon \qquad \forall f \in \mathcal{B}_d, \ \varepsilon \in (0, \infty)^1. \tag{2}$$

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[1] We note that the radius of the $\mathcal{B}_d$ could be taken to be an arbitrary positive value, R, in which case the

Here, $\mathrm{APP}_d(f, \varepsilon)$ is allowed to depend on arbitrary linear functionals. Note that we also allow adaptive information, i.e., the linear functionals may sequentially depend on each other.

The information complexity of the problem SOL_d is given by the function

$$\mathrm{comp} : (0, \infty) \times \mathbb{N} \to \mathbb{N}_0,$$

where $\mathrm{comp}(\varepsilon, d)$ is the number of linear functionals required by the best admissible algorithm to satisfy the error criterion (2). By definition, the information complexity is non-decreasing as ε tends to zero. We also expect it to be non-decreasing as d tends to infinity since d is typically the number of variables for the functions in $\mathcal{F}_d$.

For this case where arbitrary linear functionals are allowed, the optimal approximate solution operator is known to be

$$\mathrm{APP}_d(f, \varepsilon) = \sum_{i=1}^{n} \mathrm{SOL}_d(u_{i,d}) \left\langle f, u_i \right\rangle_{\mathcal{F}_d},$$

and the information complexity of our linear problem is

$$\mathrm{comp}(\varepsilon, d) = \min\{n \in \mathbb{N}_0 : \lambda_{n+1,d} \le \varepsilon\}, \tag{3}$$

see [3].

Strictly speaking, we are considering the *absolute* error criterion in this paper. Alternatively, one could also consider the *normalized* error criterion, where the error in (2) is divided by the so-called *initial error*, which is the error without sampling the function, or, equivalently, the operator norm of SOL_d. In the latter case, it is known (see, e.g., again [3]) that the information complexity is given by

$$\mathrm{comp}^{\mathrm{norm}}(\varepsilon, d) = \min\{n \in \mathbb{N}_0 : \lambda_{n+1,d} \le \varepsilon \, \lambda_{1,d}\}.$$

This means that, essentially, in the tractability analysis below, all criteria would be normalized by the first singular value $\lambda_{1,d}$. For the sake of brevity and clarity of notation, we restrict ourselves to the absolute error criterion here, but presumably analogous results hold for the normalized setting.

The question of tractability is one of determining how fast the information complexity increases as ε tends to zero and/or as d tends to infinity. Although (3) has a relatively simple form, its dependence on the ordering of the singular values means that it is not obvious what the behavior of $\mathrm{comp}(\varepsilon, d)$ is as ε^{-1} and/or d tend to infinity.

We want to bound the information complexity in terms of a simple function, T, of ε^{-1} and d, and identify conditions on the singular values that are easier to verify that are equivalent to this bound on the information complexity holding. We note that generalized tractability specified by a function T has been considered before in the literature. In particular, there are several papers by Gnewuch and Woźniakowski (see [6, 7, 8, 9]), and also [3, Chapter 8] is devoted to this topic. To our understanding, however, these references mainly focus on the special case of tensor product problems, a restriction we do not make. Moreover, the conditions for tractability derived there focus mostly on relating the properties of the function T to the different varieties of tractability, and not in terms of sums of (functions of) the singular values of a problem. Hence, the present paper adds to what is presently known on this subject. In this article, we investigate a *generalized tractability function*, denoted by T:

$$T : (0, \infty) \times \mathbb{N} \times [0, \infty)^s \to (0, \infty). \tag{4}$$

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problem becomes equivalent to the original one with the tolerance ε replaced by ε/R since the problem and the optimal approximate soluion are both linear (see below).

The conditions we require T to satisfy are described in Section 2.1, however, the basic idea is that we define our approximation problem to be tractable if $\mathrm{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p})$ for some constant, $C_{\boldsymbol{p}}$, depending only on the parameter $\boldsymbol{p}$. The parameter $\boldsymbol{p}$ is an s-dimensional vector with $s \geq 1$, where $\boldsymbol{p} \geq \mathbf{0}$ implies that each component of $\boldsymbol{p}$ is non-negative. In practice, we expect $T(\varepsilon^{-1}, d, \boldsymbol{p})$ to increase sub-exponentially with increasing ε^{-1} and/or d, however, most of our theorems do not require such an assumption.

Theorems 1, 2, 4 and 5 below all take the following form: for a given $\mathscr{D} \in \{1, d\}$ and $\Omega \subseteq (0, \infty) \times \mathbb{N}$,

$$\exists C_{\boldsymbol{p}} > 0 \text{ with } \mathrm{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p}) \quad \forall (\varepsilon^{-1}, d) \in \Omega$$

$$\Longleftrightarrow \exists L_{\boldsymbol{p}} > 0 \text{ with } \sup_{d \in \mathbb{N}} \sum_{i = \lceil L_{\boldsymbol{p}} T(0, \mathscr{D}, \boldsymbol{p}) \rceil}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})} < \infty,$$

$$\text{where } \mathscr{E}(d) := \inf\{\varepsilon : (\varepsilon^{-1}, d) \in \Omega\}, \ \mathscr{U}(d) := \begin{cases} \min\{n \in \mathbb{N}_0 : \lambda_{n+1,d} \leq \mathscr{E}(d)\}, & \mathscr{E}(d) > 0, \\ \infty, & \mathscr{E}(d) = 0. \end{cases} \quad (5)$$

Although some of these theorems are special cases of others, we prove each separately to allow the reader to become familiar with the arguments used as additional layers of complexity are added. Theorem 3 has a somewhat different form than above.

In the following, we will denote the set of positive integers by $\mathbb{N}$ and the set of non-negative integers by $\mathbb{N}_0$. We write $\mathbf{0}$ to denote a vector with all components equal to zero, and $\boldsymbol{\infty}$ to denote a vector with all components equal to ∞. Furthermore, for two vectors $\boldsymbol{p} = (p_1, \ldots, p_s)$ and $\boldsymbol{p}' = (p_1', \ldots, p_s')$ in $[0, \infty)^s$, we write $\boldsymbol{p} \geq \boldsymbol{p}'$ if $p_j \geq p_j'$ for all $j \in \{1, \ldots, s\}$. Furthermore, $\boldsymbol{p} > \boldsymbol{p}'$ if $p_j > p_j'$ for all $j \in \{1, \ldots, s\}$. The expression $\boldsymbol{p} \in (\boldsymbol{p}', \boldsymbol{p}'')$ is to be interpreted as $\boldsymbol{p}' < \boldsymbol{p} < \boldsymbol{p}''$ in that sense, and analogously for half-closed or closed intervals.

The remainder of this paper is organized as follows. In Section 2, we give the definitions of the considered tractability notions, in particular (strong) tractability. We consider strong tractability when there is an upper bound on the information complexity that is independent of the dimension, d. We then turn to tractability when the bound on the information complexity may depend on both d and ε^{-1}. In Section 3 we provide examples of tractability functions for both the algebraic and exponential cases. We also consider other notions of tractability such as quasi-polynomial tractability, and introduce other notions of tractability. A generalized notion of weak tractability is given in Section 4. In Section 5, we consider tractability on a restricted domain, Ω.

2 (Strong) Tractability

2.1 Notation and fundamental definitions

Let T be a function given in (4). A tractability function T is a function of a simple form that provides an upper bound on the information complexity of a problem.

Definition 1. A problem is defined as T-tractable with parameter $\boldsymbol{p}$ iff there exists a positive constant $C_{\boldsymbol{p}}$, which is independent of ε and d, such that

$$\mathrm{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p}) \qquad \forall \varepsilon > 0, \ d \in \mathbb{N}. \tag{6}$$

A problem is *strongly* T-tractable with parameter $\boldsymbol{p}$ iff the information complexity is independent of the dimension of the problem, that is, there exists a positive constant $C_{\boldsymbol{p}}$, again independent of ε and d, such that

$$\mathrm{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) \qquad \forall \varepsilon > 0, \ d \in \mathbb{N}. \tag{7}$$

Table 1: Common forms of the tractability function, T.

Tractability type	$T(\varepsilon^{-1}, d, \boldsymbol{p})$	$T(0, d, \boldsymbol{p})$
algebraic polynomial	$\max\{1, \varepsilon^{-p}\}d^q$	d^q
exponential polynomial	$[\max\{1, \log(1 + \varepsilon^{-1})\}]^p d^q$	d^q
algebraic quasi-polynomial	$\exp\{p(1 + \log(\max\{1, \varepsilon^{-1}\}))(1 + \log(d))\}$	$\exp\{p(1 + \log(d))\}$
exponential quasi-polynomial	$\exp\{p(1 + \log(\max\{1, \log(1 + \varepsilon^{-1})\}))(1 + \log(d))\}$	$\exp\{p(1 + \log(d))\}$

We allow $\boldsymbol{p}$ to be a scalar or vector as the situation dictates.

For this definition of tractability to make sense we assume that

$$T \text{ is non-decreasing in all variables,} \tag{8a}$$

which implies that the problem is expected to be no easier by decreasing ε, the tolerance, or increasing d, the dimension[2]. Furthermore, increasing $\boldsymbol{p}$ allows for a possibly looser bound on the information complexity. Since there are an infinite number of positive singular values, it also makes sense to assume that

$$\lim_{\varepsilon \to 0} T(\varepsilon^{-1}, d, \boldsymbol{p}) = \infty \qquad \forall d \in \mathbb{N}, \ \boldsymbol{p} \in [0, \infty)^s. \tag{8b}$$

Since $T(\cdot, d, \boldsymbol{p})$ is non-decreasing, we may define the following limit, which we assume will always be positive:

$$T(0, d, \boldsymbol{p}) := \lim_{\varepsilon \to \infty} T(\varepsilon^{-1}, d, \boldsymbol{p}) = \inf_{\varepsilon} T(\varepsilon^{-1}, d, \boldsymbol{p}) \geq T(0, 1, \boldsymbol{0}) > 0. \tag{8c}$$

Finally, there is a technical assumption required for our analysis in Sections 2 and 5. There exists a $K_{\boldsymbol{p}, \tau}$ depending on $\boldsymbol{p}$ and τ, but independent of ε and d, such that

$$[T(\varepsilon^{-1}, d, \boldsymbol{p})]^\tau \leq K_{\boldsymbol{p}, \tau} T(\varepsilon^{-1}, d, \tau \boldsymbol{p}) \quad \forall \varepsilon \in (0, \infty), \ d \in \mathbb{N}, \ \boldsymbol{p} \in [0, \infty)^s, \ \tau \in [1, \infty). \tag{9}$$

Certain common notions of tractability covered by the general situation described above are provided in Table 1. Here, $\boldsymbol{p} = (p, q)$ in some cases, and $\boldsymbol{p}$ is a scalar in other cases. A general reference for results on algebraic tractability is [3]. For exponential tractability we refer to [10] and [11].

If (6) holds for some $\boldsymbol{p}$, it clearly holds for larger $\boldsymbol{p}$. We are often interested in the optimal or smallest $\boldsymbol{p}$ for which (6) holds. Define the closures of the sets of parameters for which our (strong) tractability conditions hold:

$$\mathcal{P}_{\text{trct}} := \{\boldsymbol{p}^* : (6) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \boldsymbol{\infty})\}, \qquad \mathcal{P}_{\text{strct}} := \{\boldsymbol{p}^* : (7) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \boldsymbol{\infty})\}.$$

If $\boldsymbol{p}^* \in \mathcal{P}_{\text{(s)trct}}$, (strong) tractability may not hold for $\boldsymbol{p} = \boldsymbol{p}^*$, but it must hold for any $\boldsymbol{p}$ whose components are all greater than the corresponding components of $\boldsymbol{p}^*$.

Definition 2. The set of optimal parameters is defined as all of those parameters satisfying the (strong) tractability conditions that are not greater than or equal to others:

$$\mathcal{P}_{\text{opt}} := \{\boldsymbol{p}^* \in \mathcal{P}_{\text{trct}} : \boldsymbol{p}^* \notin [\boldsymbol{p}, \boldsymbol{\infty}) \ \forall \boldsymbol{p} \in \mathcal{P}_{\text{trct}} \setminus \{\boldsymbol{p}^*\}\}, \tag{10}$$

$$\mathcal{P}_{\text{sopt}} := \{\boldsymbol{p}^* \in \mathcal{P}_{\text{strct}} : \boldsymbol{p}^* \notin [\boldsymbol{p}, \boldsymbol{\infty}) \ \forall \boldsymbol{p} \in \mathcal{P}_{\text{strct}} \setminus \{\boldsymbol{p}^*\}\}. \tag{11}$$

In the sections below we prove necessary and sufficient conditions on tractability as generally defined in (6) and (7). These conditions involve the boundedness of sums defined in terms of $(T(\lambda_{i,d}^{-1}, \cdot, \cdot))_{i=1}^\infty$. In practice, it may be easier to verify whether or not these conditions hold than to verify (6) and (7) directly.

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[2]This is why we make T a function of ε^{-1} rather than of ε.

2.2 Strong tractability

We first consider the simpler case when the information complexity is essentially independent of the dimension, d. The proof also introduces the line of argument used for the case where there is d-dependence.

Theorem 1. *Let T be a tractability function as specified in (4) and satisfying (8) and (9). A problem is strongly T-tractable iff there exists $\boldsymbol{p} \in [\boldsymbol{0}, \boldsymbol{\infty})$ and an integer $L_{\boldsymbol{p}} > 0$ such that*

$$S_{\boldsymbol{p}} := \sup_{d \in \mathbb{N}} \sum_{i=L_{\boldsymbol{p}}}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p})} < \infty. \tag{12}$$

If (12) holds for some $\boldsymbol{p}$, let $\widetilde{\mathcal{P}}_{\text{strct}} := \{\boldsymbol{p}^* : (12) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \boldsymbol{\infty})\}$. Then $\mathcal{P}_{\text{strct}} = \widetilde{\mathcal{P}}_{\text{strct}}$, and the set of optimal strong tractability parameters is

$$\mathcal{P}_{\text{sopt}} = \{\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\text{strct}} : \boldsymbol{p}^* \notin [\widetilde{\boldsymbol{p}}, \boldsymbol{\infty}) \ \forall \widetilde{\boldsymbol{p}} \in \widetilde{\mathcal{P}}_{\text{strct}} \setminus \{\boldsymbol{p}^*\}\}.$$

Proof. **Sufficient condition:**
We make the first part of the argument in some generality so that it can be reused in the proof of Theorem 2 for tractability. Fix any $\varepsilon > 0$ and any $d \in \mathbb{N}$, and let $\mathscr{D} \in \mathbb{N}$ be arbitrary. Since the $\lambda_{i,d}$ are non-increasing in i, it follows that the $T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})$ are non-decreasing in i. In particular, for any $N \in \mathbb{N}$ with $n \geq N$, we have

$$\lambda_{n+1,d} \leq \lambda_{n,d} \leq \cdots \leq \lambda_{N,d} \implies \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{T(\lambda_{n,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \cdots \leq \frac{1}{T(\lambda_{N,d}^{-1}, \mathscr{D}, \boldsymbol{p})}$$

$$\implies \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{n-N+1} \sum_{i=N}^{n} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{n-N+1} \sum_{i=N}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})}.$$

Thus, we can conclude from the previous line that

$$n - N + 1 \geq T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p}) \sum_{i=N}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})}$$

$$\implies \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{n-N+1} \sum_{i=N}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p})}$$

$$\forall \mathscr{D}, n, N \in \mathbb{N}. \quad (13)$$

We note that the left side of the implication implies $n \geq N$ automatically since n is an integer and $n - N + 1$ is assumed to be positive.

Moreover, given that T is increasing in its arguments, we have the following equivalent expression for the information complexity via (3):

$$\text{comp}(\varepsilon, d) = \min\{n \in \mathbb{N}_0 : \lambda_{n+1,d} \leq \varepsilon\}$$

$$= \min\left\{n \in \mathbb{N}_0 : \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p})}\right\} \quad \forall \mathscr{D} \in \mathbb{N}$$

$$\leq \min\left\{n \in \mathbb{N}_0 : n \geq N - 1 + T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p}) \sum_{i=N}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})}\right\} \forall \mathscr{D}, N \in \mathbb{N}, \quad \text{by (13)}$$

$$\leq T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p}) \left[\frac{N}{T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p})} + \sum_{i=N}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})}\right] \quad \forall \mathscr{D}, N \in \mathbb{N}. \quad (14)$$

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Now we take this upper bound further, specializing to the case of $\mathscr{D} = 1$ and $N = L_{\boldsymbol{p}}$:

$$\operatorname{comp}(\varepsilon, d) \leq T(\varepsilon^{-1}, 1, \boldsymbol{p}) \left[\frac{L_{\boldsymbol{p}}}{T(\varepsilon^{-1}, 1, \boldsymbol{p})} + \underbrace{\sum_{i=L_{\boldsymbol{p}}}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p})}}_{\leq S_{\boldsymbol{p}} < \infty \text{ by } (12)} \right]$$

$$\leq T(\varepsilon^{-1}, 1, \boldsymbol{p}) \underbrace{\left[\frac{L_{\boldsymbol{p}}}{T(0, 1, \boldsymbol{p})} + S_{\boldsymbol{p}} \right]}_{=:C_{\boldsymbol{p}}} \qquad \text{by (8a)}$$

$$= C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}).$$

This means that we have strong T-tractability via (7), and verifies the sufficiency of (12).

Necessary condition:
Suppose that we have strong T-tractability as defined in (7). That is, for some $\boldsymbol{p} \geq \boldsymbol{0}$ there exists a positive constant $C_{\boldsymbol{p}}$ such that

$$\operatorname{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) \qquad \forall \varepsilon > 0, \ d \in \mathbb{N}.$$

Since the sequence of singular values $\lambda_{1,d}, \lambda_{2,d}, \dots$ is non-increasing, we have

$$\lambda_{\lfloor C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) \rfloor + 1, d} \leq \varepsilon \qquad \forall \varepsilon > 0, \ d \in \mathbb{N}. \tag{15}$$

For all positive ε, define

$$i(\varepsilon, \boldsymbol{p}) := \lfloor C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) \rfloor + 1, \quad i(\infty, \boldsymbol{p}) = \lfloor C_{\boldsymbol{p}} T(0, 1, \boldsymbol{p}) \rfloor + 1. \tag{16}$$

Thus, it follows by (15) that $\lambda_{i(\varepsilon, \boldsymbol{p}), d} \leq \varepsilon$. Note furthermore that we always have

$$i(\varepsilon, \boldsymbol{p}) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) + 1 \leq C_{\boldsymbol{p}} T(\lambda_{i(\varepsilon, \boldsymbol{p}), d}^{-1}, 1, \boldsymbol{p}) + 1 \qquad \forall \varepsilon > 0, \ d \in \mathbb{N},$$

since $T(\cdot, 1, \boldsymbol{p})$ is non-decreasing.

For ε taking on all positive values, $i(\varepsilon, \boldsymbol{p})$ takes on, by (8b), all values greater than or equal to $i(\infty, \boldsymbol{p})$, so

$$i \leq C_{\boldsymbol{p}} T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p}) + 1 \qquad \forall i \geq i(\infty, \boldsymbol{p}), \ d \in \mathbb{N}.$$

This implies via our technical assumption (9) that, for any $\tau > 1$,

$$K_{\boldsymbol{p}, \tau} T(\lambda_{i,d}^{-1}, 1, \tau \boldsymbol{p}) \geq [T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p})]^{\tau} \geq \left[\frac{(i-1)}{C_{\boldsymbol{p}}} \right]^{\tau} \qquad \forall i \geq i(\infty, \boldsymbol{p}), \ d \in \mathbb{N},$$

$$\frac{1}{T(\lambda_{i,d}^{-1}, 1, \tau \boldsymbol{p})} \leq \frac{K_{\boldsymbol{p}, \tau} C_{\boldsymbol{p}}^{\tau}}{(i-1)^{\tau}} \qquad \forall i \geq \max\{i(\infty, \boldsymbol{p}), 2\}, \ d \in \mathbb{N}.$$

Summing both sides of this last inequality over i from $i(\infty, \boldsymbol{p}) + 1 \geq 2$ to ∞ yields

$$\sup_{d \in \mathbb{N}} \sum_{i=i(\infty, \boldsymbol{p})+1}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \tau \boldsymbol{p})} \leq K_{\boldsymbol{p}, \tau} C_{\boldsymbol{p}}^{\tau} \sum_{i=2}^{\infty} \frac{1}{(i-1)^{\tau}} \leq K_{\boldsymbol{p}, \tau} C_{\boldsymbol{p}}^{\tau} \zeta(\tau) < \infty,$$

where ζ denotes the Riemann zeta function. This yields (12) with $\boldsymbol{p}$ replaced by $\boldsymbol{p}' = \tau \boldsymbol{p}$ and $L_{\boldsymbol{p}'} = i(\infty, \boldsymbol{p}) + 1$, and so we see the necessity of (12).

Optimality:
To complete the proof, we must show that $\mathcal{P}_{\text{strct}} = \widetilde{\mathcal{P}}_{\text{strct}}$. Then, the expression for $\mathcal{P}_{\text{sopt}}$ automatically follows from its definition in (11).

First we show that if $\boldsymbol{p}^* \in \mathcal{P}_{\text{strct}}$, then $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\text{strct}}$. If $\boldsymbol{p}^* \in \mathcal{P}_{\text{strct}}$, then (7) must hold for all $\boldsymbol{p}' \in (\boldsymbol{p}^*, \infty)$. For any $\boldsymbol{p}' \in (\boldsymbol{p}^*, \infty)$, we may choose $\boldsymbol{p}$ and τ such that $\boldsymbol{p}^* < \boldsymbol{p} < \boldsymbol{p}' = \tau\boldsymbol{p}$. Since $\boldsymbol{p} > \boldsymbol{p}^*$, (7) also holds for $\boldsymbol{p}$, and it follows that (12) must hold for $\boldsymbol{p}'$ by the proof of necessity above. Thus, $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\text{strct}}$.

Next we show that if $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\text{strct}}$, then $\boldsymbol{p}^* \in \mathcal{P}_{\text{strct}}$. If $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\text{strct}}$, then (12) holds for all $\boldsymbol{p} \in (\boldsymbol{p}^*, \infty)$. By the argument to prove the sufficient condition above, it follows that (7) must also hold for all $\boldsymbol{p} \in (\boldsymbol{p}^*, \infty)$. Thus, $\boldsymbol{p}^* \in \mathcal{P}_{\text{strct}}$.

This concludes the proof of Theorem 1. $\qquad\square$

$\square$

2.3 Tractability

The argument proving equivalent conditions for tractability is similar to, but somewhat more involved than the proof of Theorem 1. The lower limit on the sum is somewhat more complicated as well.

Theorem 2. *Let T be a tractability function as specified in (4) and satisfying (8) and (9). A problem is T-tractable iff there exists $\boldsymbol{p} \geq \boldsymbol{0}$ and a positive constant $L_{\boldsymbol{p}}$ such that*

$$S_{\boldsymbol{p}} := \sup_{d \in \mathbb{N}} \sum_{i=\lceil L_{\boldsymbol{p}} T(0,d,\boldsymbol{p}) \rceil}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p})} < \infty. \tag{17}$$

If (17) holds for some $\boldsymbol{p}$, let $\widetilde{\mathcal{P}}_{\text{trct}} := \{\boldsymbol{p}^* : (17) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \infty)\}$. Then $\mathcal{P}_{\text{trct}} = \widetilde{\mathcal{P}}_{\text{trct}}$, and the set of optimal tractability parameters is

$$\mathcal{P}_{\text{opt}} = \{\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\text{trct}} : \boldsymbol{p}^* \notin [\widetilde{\boldsymbol{p}}, \infty) \; \forall \widetilde{\boldsymbol{p}} \in \widetilde{\mathcal{P}}_{\text{trct}} \setminus \{\boldsymbol{p}^*\}\}.$$

Comparing the equivalent conditions for strong tractability in (12) and tractability in (17), they are seen to be nearly the same. The difference lies in the lower summation index, which is allowed to depend on d for tractability, and the second argument of T in the sum, which is 1 for strong tractability, and d in general. Both of these can allow $S_{\boldsymbol{p}}$ to be finite in Theorem 2, when it may be infinite in Theorem 1.

Proof. **Sufficient condition:**
Suppose that (17) holds for some $\boldsymbol{p} \geq \boldsymbol{0}$. By the argument in the proof of the sufficient condition for Theorem 1, it follows from (14) with $\mathscr{D} = d$ and $N = \lceil L_{\boldsymbol{p}} T(0,d,\boldsymbol{p}) \rceil$ that

$$\text{comp}(\varepsilon, d) \leq T(\varepsilon^{-1}, d, \boldsymbol{p}) \left[\frac{\lceil L_{\boldsymbol{p}} T(0,d,\boldsymbol{p}) \rceil}{T(\varepsilon^{-1}, d, \boldsymbol{p})} + \underbrace{\sum_{i=\lceil L_{\boldsymbol{p}} T(0,d,\boldsymbol{p}) \rceil}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p})}}_{\leq S_{\boldsymbol{p}} \text{ by (17)}} \right] \quad \forall \varepsilon > 0, \; d \in \mathbb{N}$$

$$\leq T(\varepsilon^{-1}, d, \boldsymbol{p}) \left[\frac{L_{\boldsymbol{p}} T(0,d,\boldsymbol{p}) + 1}{T(\varepsilon^{-1}, d, \boldsymbol{p})} + S_{\boldsymbol{p}} \right] \quad \forall \varepsilon > 0, \; d \in \mathbb{N}$$

$$\leq T(\varepsilon^{-1}, d, \boldsymbol{p}) \left[\frac{L_{\boldsymbol{p}} T(0,d,\boldsymbol{p})}{T(\varepsilon^{-1}, d, \boldsymbol{p})} + \frac{1}{T(0,1,\boldsymbol{p})} + S_{\boldsymbol{p}} \right] \quad \text{by (8a)}$$

$$\leq T(\varepsilon^{-1}, d, \boldsymbol{p}) \underbrace{\left[L_{\boldsymbol{p}} + \frac{1}{T(0,1,\boldsymbol{p})} + S_{\boldsymbol{p}} \right]}_{=:C_{\boldsymbol{p}}} \quad \text{by (8a).}$$

It follows that we have T-tractability, which shows sufficiency of (17).

Necessary condition:

Suppose that we have T-tractability. That is, for some $\boldsymbol{p} \geq \boldsymbol{0}$, there exists a positive constant $C_{\boldsymbol{p}}$ such that

$$\operatorname{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} \, T(\varepsilon^{-1}, d, \boldsymbol{p}) \qquad \forall \varepsilon > 0, \ d \in \mathbb{N}.$$

Since the sequence of singular values $\lambda_{1,d}, \lambda_{2,d}, \dots$ is non-increasing, we have

$$\lambda_{\lfloor C_{\boldsymbol{p}} \, T(\varepsilon^{-1},d,\boldsymbol{p}) \rfloor + 1, d} \leq \varepsilon \qquad \forall \varepsilon > 0, \ d \in \mathbb{N}. \tag{18}$$

Define the integers

$$i(\varepsilon, d, \boldsymbol{p}) := \lfloor C_{\boldsymbol{p}} \, T(\varepsilon^{-1}, d, \boldsymbol{p}) \rfloor + 1, \qquad i(\infty, d, \boldsymbol{p}) := \lfloor C_{\boldsymbol{p}} \, T(0, d, \boldsymbol{p}) \rfloor + 1 \geq 1.$$

Thus, it follows by (18) that $\lambda_{i(\varepsilon,d,\boldsymbol{p}),d} \leq \varepsilon$. Note furthermore that we always have

$$i(\varepsilon, d, \boldsymbol{p}) \leq C_{\boldsymbol{p}} \, T(\varepsilon^{-1}, d, \boldsymbol{p}) + 1 \leq C_{\boldsymbol{p}} T(\lambda_{i(\varepsilon,d,\boldsymbol{p}),d}^{-1}, d, \boldsymbol{p}) + 1 \qquad \forall \varepsilon > 0, \ d \in \mathbb{N},$$

since $T(\cdot, d, \boldsymbol{p})$ is non-decreasing.

For ε taking on all positive values, $i(\varepsilon, d, \boldsymbol{p})$ takes on, by (8b), all values greater than or equal to $i(\infty, d, \boldsymbol{p})$, so

$$i \leq C_{\boldsymbol{p}} \, T(\varepsilon^{-1}, d, \boldsymbol{p}) + 1 \leq C_{\boldsymbol{p}} T(\lambda_{i,d}^{-1}, d, \boldsymbol{p}) + 1 \qquad \forall i \geq i(\infty, d, \boldsymbol{p}), \ d \in \mathbb{N}.$$

This implies via our technical assumption (9) that for any $\tau > 1$,

$$K_{\boldsymbol{p},\tau} \, T(\lambda_{i,d}^{-1}, d, \tau\boldsymbol{p}) \geq [T(\lambda_{i,d}^{-1}, d, \boldsymbol{p})]^{\tau} \geq \left[\frac{(i-1)}{C_{\boldsymbol{p}}} \right]^{\tau} \qquad \forall i \geq i(\infty, d, \boldsymbol{p}), \ d \in \mathbb{N},$$

$$\frac{1}{T(\lambda_{i,d}^{-1}, d, \tau\boldsymbol{p})} \leq \frac{K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau}}{(i-1)^{\tau}} \qquad \forall i \geq \max\{i(\infty, d, \boldsymbol{p}), 2\}, \ d \in \mathbb{N}.$$

Let $\boldsymbol{p}' = \tau\boldsymbol{p}$. Note that $i(\infty, d, \cdot)$ is non-decreasing. Summing both sides of the latter inequality over i from $i(\infty, d, \boldsymbol{p}') + 1 \geq i(\infty, d, \boldsymbol{p}) + 1 \geq 2$ to ∞ it follows that

$$\sup_{d \in \mathbb{N}} \sum_{i=i(\infty,d,\boldsymbol{p}')+1}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p}')} \leq K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau} \sum_{i=2}^{\infty} \frac{1}{(i-1)^{\tau}} \leq K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau} \zeta(\tau) < \infty,$$

where ζ denotes the Riemann zeta function.

Note now that, for any $d \in \mathbb{N}$,

$$i(\infty, d, \boldsymbol{p}') + 1 = \lfloor C_{\boldsymbol{p}'} \, T(0, d, \boldsymbol{p}') \rfloor + 2 \leq \left[C_{\boldsymbol{p}'} + \frac{2}{T(0, d, \boldsymbol{p}')} \right] T(0, d, \boldsymbol{p}')$$

$$\leq \underbrace{\left[C_{\boldsymbol{p}'} + \frac{2}{T(0, 1, \boldsymbol{p}')} \right]}_{=: L_{\boldsymbol{p}'}} T(0, d, \boldsymbol{p}') \leq \lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil.$$

For this choice of $L_{\boldsymbol{p}'}$ we have

$$\sup_{d \in \mathbb{N}} \sum_{i=\lceil L_{\boldsymbol{p}'} T(0,d,\boldsymbol{p}') \rceil}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p}')} \leq \sup_{d \in \mathbb{N}} \sum_{i=i(\infty,d,\boldsymbol{p}')+1}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p}')} < \infty.$$

This yields (17) with $\boldsymbol{p}$ replaced by $\boldsymbol{p}'$, so we see the necessity of (17).

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Optimality:
This proof is similar to the proof for the optimality condition for strong tractability in Theorem 1. We must show that $\mathcal{P}_{\mathrm{trct}} = \widetilde{\mathcal{P}}_{\mathrm{trct}}$. Then, the expression for $\mathcal{P}_{\mathrm{opt}}$ in this theorem automatically follows from its definition in (10).

First we show that if $\boldsymbol{p}^* \in \mathcal{P}_{\mathrm{trct}}$, then $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\mathrm{trct}}$. If $\boldsymbol{p}^* \in \mathcal{P}_{\mathrm{trct}}$, then (6) must hold for all $\boldsymbol{p}' \in (\boldsymbol{p}^*, \infty)$. For any $\boldsymbol{p}' \in (\boldsymbol{p}^*, \infty)$, we may choose $\boldsymbol{p}$ and τ such that $\boldsymbol{p}^* < \boldsymbol{p} < \boldsymbol{p}' = \tau\boldsymbol{p}$. Since $\boldsymbol{p} > \boldsymbol{p}^*$, (6) also holds for $\boldsymbol{p}$, and it follows that (17) must hold for $\boldsymbol{p}'$ by the proof of necessity above. Thus, $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\mathrm{trct}}$.

Next we show that if $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\mathrm{trct}}$, then $\boldsymbol{p}^* \in \mathcal{P}_{\mathrm{trct}}$. If $\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\mathrm{trct}}$, then (17) holds for all $\boldsymbol{p} \in (\boldsymbol{p}^*, \infty)$. By the argument to prove the sufficient condition above, it follows that (6) must also hold for all $\boldsymbol{p} \in (\boldsymbol{p}^*, \infty)$. Thus, $\boldsymbol{p}^* \in \mathcal{P}_{\mathrm{trct}}$.

This concludes the proof of Theorem 2. $\qquad\qquad\square\qquad\qquad\qquad\square$

3 Examples

In this section, we present six examples relating to Theorem 1 and Theorem 2, both of which were shown in Section 2, for the various notions of tractability that are listed in Table 1. In the subsequent sections, we will continue to present various examples.

The concept of (strong) polynomial tractability for both the algebraic case and the exponential case will be first considered as follows:

Example 1 (Algebraic polynomial tractability)**.** Let the tractability function, T, be defined by

$$T(\varepsilon^{-1}, d, \boldsymbol{p}) = \max\{1, \varepsilon^{-p}\}\, d^q \qquad \forall \varepsilon > 0,\, d \in \mathbb{N},\, \boldsymbol{p} \in [0, \infty)^2,$$

where in this case $\boldsymbol{p} = (p, q)$. Then Theorem 2 yields the equivalent condition for algebraic polynomial tractability:

$$\sup_{d \in \mathbb{N}} d^{-q} \sum_{i = \lceil L_{(p,q)} d^q \rceil}^{\infty} \min\{1, \lambda_{i,d}^p\} < \infty \text{ for some } p, q \geq 0,\, L_{(p,q)} > 0.$$

Furthermore, for optimality, let $\widetilde{\mathcal{P}}_{\mathrm{ALG\text{-}trct}} := \{(p^*, q^*) : \text{the above condition holds } \forall (p, q) \in (p^*, \infty) \times (q^*, \infty)\}$. Therefore, the set of optimal (p, q) is the set

$$\{(p^*, q^*) \in \widetilde{\mathcal{P}}_{\mathrm{ALG\text{-}trct}} : (p^*, q^*) \notin [\widetilde{p}, \infty) \times [\widetilde{q}, \infty) \,\forall (\widetilde{p}, \widetilde{q}) \in \widetilde{\mathcal{P}}_{\mathrm{ALG\text{-}trct}} \setminus \{p^*, q^*\}\}.$$

This essentially recovers the result on polynomial tractability in [3, Theorem 5.1].

Having defined the tractability function as above, we will now examine a specific case where we always set $d = 1$, i.e.,

$$T(\varepsilon^{-1}, 1, \boldsymbol{p}) = \max\{1, \varepsilon^{-\boldsymbol{p}}\} \qquad \forall \varepsilon > 0,\, d \in \mathbb{N},\, \boldsymbol{p} \in [0, \infty),$$

where in this case $\boldsymbol{p} = p$ is a scalar.

Then Theorem 1 yields the equivalent condition for algebraic strong polynomial tractability:

$$\sup_{d \in \mathbb{N}} \sum_{i = L_p}^{\infty} \min\{1, \lambda_{i,d}^p\} < \infty \text{ for some } p \geq 0,\, L_p \in \mathbb{N}.$$

Furthermore, the optimal p is the infimum of all p for which the above condition holds. This essentially recovers the result in [3, Theorem 5.1].

Next, we study the exponential case where we replace ε^{-1} by $\log(1 + \varepsilon^{-1})$ and consider the same notions of tractability as for the algebraic case.

Example 2 (Exponential polynomial tractability). Let the tractability function, T, be defined by

$$T(\varepsilon^{-1}, d, \boldsymbol{p}) = \left[\max\{1, \log(1 + \varepsilon^{-1})\}\right]^p d^q \qquad \forall \varepsilon > 0, \, d \in \mathbb{N}, \, \boldsymbol{p} \in [0, \infty)^2,$$

where $\boldsymbol{p} = (p, q)$. Then Theorem 2 yields the equivalent condition for exponential polynomial tractability:

$$\sup_{d \in \mathbb{N}} d^{-q} \sum_{i = \lceil L_{(p,q)} d^q \rceil}^{\infty} \frac{1}{[\max\{1, \log(1 + \lambda_{i,d}^{-1})\}]^p} < \infty \text{ for some } p, q \geq 0, L_{(p,q)} > 0. \tag{19}$$

Furthermore, for optimality, let $\widetilde{\mathcal{P}}_{\text{EXP-trct}} := \{(p^*, q^*) : \text{the above condition holds } \forall (p, q) \in (p^*, \infty) \times (q^*, \infty)\}$. Therefore, the set of optimal (p, q) is the set

$$\{(p^*, q^*) \in \widetilde{\mathcal{P}}_{\text{EXP-trct}} : (p^*, q^*) \notin [\widetilde{p}, \infty) \times [\widetilde{q}, \infty) \, \forall (\widetilde{p}, \widetilde{q}) \in \widetilde{\mathcal{P}}_{\text{EXP-trct}} \setminus \{p^*, q^*\}\}.$$

Again, we will look at the special case in which we always set $d = 1$. In this case, we will define T as follows.

$$T(\varepsilon^{-1}, 1, p) = [\max\{1, \log(1 + \varepsilon^{-1})\}]^{\boldsymbol{p}} \quad \forall \varepsilon > 0, \, d \in \mathbb{N}, \, p \in [0, \infty),$$

where $\boldsymbol{p} = p$ is a scalar. Then Theorem 1 yields the equivalent condition for exponential strong polynomial tractability:

$$\sup_{d \in \mathbb{N}} \sum_{i = L_p}^{\infty} \frac{1}{[\max\{1, \log(1 + \lambda_{i,d}^{-1})\}]^p} < \infty, \quad \text{for some } p \geq 0, L_p \in \mathbb{N}. \tag{20}$$

In this case, the optimal p is the infimum of all p for which the above condition holds.

In the paper [11], the authors derived conditions equivalent to exponential strong polynomial tractability and exponential polynomial tractability, respectively. These conditions read (with notation slightly adapted to our present paper) as follows:

- We have exponential strong polynomial tractability iff there exists a $\tau > 0$ and a $C \in \mathbb{N}$ such that

$$\sup_{d \in \mathbb{N}} \sum_{i = C}^{\infty} \lambda_{i,d}^{i^{-\tau}} < \infty. \tag{21}$$

- We have exponential polynomial tractability iff there exist $\tau_1, \tau_3 \geq 0$ and $\tau_2, C > 0$ such that

$$\sup_{d \in \mathbb{N}} d^{-\tau_1} \sum_{i = \lceil C d^{\tau_3} \rceil}^{\infty} \lambda_{i,d}^{i^{-\tau_2}} < \infty. \tag{22}$$

Firstly, we remark that the role of L_p in (20) corresponds to that of C in (21). Moreover, q in (19) has a similar role as τ_1 and τ_3 in (22) (i.e., (22) is slightly more precise than (19) regarding the exponents of d, but this is just a minor difference); furthermore $L_{(p,q)}$ in (19) corresponds to C in (22).

Secondly, it should be noted that our condition (20) is easier to check in practice than condition (21). Since in (21) the summation indices i show up in the exponents of the summands, we need the exact ordering of the singular values $\lambda_{i,d}$, whereas in (20) we only have a summation

of the values of $1/T(\lambda_{i,d}^{-1}, d, \boldsymbol{p})$, and the exact order of the singular values can be neglected. A similar observation holds for (19) and (22).

The following proposition states that the conditions (20) and (21), as well as the conditions (19) and (22), are indeed equivalent. The proof of this result is deferred to the appendix, where we will prove this statement for the latter two conditions, i.e., for the conditions on exponential polynomial tractability. The case of exponential strong polynomial tractability is not explicitly included, as it is actually simpler and can be treated analogously.

Proposition 1. *The conditions* (20) *and* (21) *are equivalent. Furthermore, the conditions* (19) *and* (22) *are equivalent.*

In the following example, we provide a brief summary of the above examples of tractability. We would like to emphasize that this example was previously known (see, e.g., [6]) where in this case the roles of ε^{-1} and d are separated. In order to accomplish this, we will first define two functions ϕ and ψ with positive values that are increasing in both of their arguments. The equivalent conditions for this example take the same form as those in Theorems 1 and 2.

Example 3 ((Strong) Separable tractability)**.** Suppose that the tractability function is defined by
$$T(\varepsilon^{-1}, d, \boldsymbol{p}) = \phi(\varepsilon^{-1}, p)\, \psi(d, q) \qquad \forall \varepsilon > 0,\, d \in \mathbb{N},\, \boldsymbol{p} \in [0, \infty)^2,$$
with non-decreasing functions $\phi : (0, \infty) \times (0, \infty) \to (0, \infty)$ and $\psi : \mathbb{N} \times (0, \infty) \to (0, \infty)$ and where $\boldsymbol{p} = (p, q)$, hence T is non-decreasing in all variables.

We say that a problem is T-separably tractable with parameter $\boldsymbol{p}$ iff there exists a positive constant $C_{\boldsymbol{p}}$ which does not depend on ε and d such that
$$\mathrm{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}}\, \phi(\varepsilon^{-1}, p)\, \psi(d, q) \qquad \forall \varepsilon > 0,\, d \in \mathbb{N}.$$

A problem is strongly T-separably tractable with parameter $\boldsymbol{p}$ iff the information complexity is independent of d, that is, there exists a positive constant $C_{\boldsymbol{p}}$ which again does not depend on ε and d such that
$$\mathrm{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}}\, \phi(\varepsilon^{-1}, p) \qquad \forall \varepsilon > 0,\, d \in \mathbb{N},$$
where in this case $\boldsymbol{p} = p$ is a scalar. Furthermore, we assume the following, which corresponds to the conditions in (8b), (8c), and (9). We assume that
$$\lim_{\varepsilon \to 0} \phi(\varepsilon^{-1}, p) = \infty \qquad \forall d \in \mathbb{N}, p > 0,$$
since there are an infinite number of positive singular values. Furthermore, we have that $\phi(\cdot, p)$ is non-decreasing, hence we define the following limit
$$\lim_{\varepsilon \to \infty} \phi(\varepsilon^{-1}, p) > 0.$$

Additionally, we have the following technical assumptions. Suppose that, for any real $\tau_1, \tau_2 > 0$ and any $p, q > 0$, we have that
$$[\phi(\varepsilon^{-1}, p)]^{\tau_1} \leq \hat{K}_{p,\tau_1} \phi(\varepsilon^{-1}, \tau_1 p) \quad \text{and} \quad [\psi(d, q)]^{\tau_2} \leq \hat{K}_{q,\tau_2} \psi(d, \tau_2 q) \qquad \forall\, \varepsilon > 0,\, d \in \mathbb{N},$$
where $\hat{K}_{p,\tau_1} > 0$ is a constant that may depend on p and τ_1, but is independent of ε, and where $\hat{K}_{q,\tau_2} > 0$ is a constant that may depend on q and τ_2, but is independent of d.

Theorem 1 yields the following equivalent condition for strong T-separable tractability:
$$\sup_{d \in \mathbb{N}} \sum_{i=L_p}^{\infty} \frac{1}{\phi(\lambda_{i,d}^{-1}, p)} < \infty,$$

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for some $p > 0$ and $L_p \in \mathbb{N}$. Furthermore, the optimal p is also the infimum of all p satisfying the latter condition. Also, Theorem 2 yields the following equivalent condition for T-separable tractability:

$$\sup_{d \in \mathbb{N}} \frac{1}{\psi(d,q)} \sum_{i=\lceil L_{(p,q)} \psi(d,q) \rceil}^{\infty} \frac{1}{\phi(\lambda_{i,d}^{-1}, p)} < \infty,$$

for some $p, q > 0$ and a positive constant $L_{(p,q)}$.

Next, we introduce another notion of tractability called *non-separable tractability*. See [7], which is probably where the term "non-separable tractability" was first introduced. In the following, we will define two functions $\hat{\phi}$ and $\hat{\psi}$ that take the roles of ε^{-1} and d. As will be seen below, this kind of tractability function includes the special case of the so-called *quasi-polynomial tractability*.

Example 4 (Non-separable tractability)**.** Suppose we define the tractability function T by

$$T(\varepsilon^{-1}, d, p) = \exp(p\,\hat{\phi}(\varepsilon^{-1})\,\hat{\psi}(d)) = [\exp(\hat{\phi}(\varepsilon^{-1}))]^{p\,\hat{\psi}(d)} \qquad \forall \varepsilon > 0,\, d \in \mathbb{N},\, p \in [0,\infty),$$

with non-decreasing functions $\hat{\phi} : (0,\infty) \to (0,\infty)$ and $\hat{\psi} : \mathbb{N} \to (0,\infty)$. We say that a problem is non-separably tractable with scalar parameter p iff there exists a positive C_p such that

$$\operatorname{comp}(\varepsilon, d) \leq C_p \exp(p(\hat{\phi}(\varepsilon^{-1})\hat{\psi}(d))).$$

Hence we can apply Theorem 2 as follows. We have the following equivalent condition for non-separable tractability:

$$\sup_{d \in \mathbb{N}} \sum_{i=\lceil L_p \hat{\phi}(0)^{p\,\hat{\psi}(d)} \rceil} [\exp(\hat{\phi}(\lambda_{i,d}^{-1}))]^{-p\,\hat{\psi}(d)} \quad \text{for some } p \geq 0, L_p > 0.$$

For $\hat{\phi}$ and $\hat{\psi}$ of special form, the condition above can be simplified further as in the following two special cases of non-separable tractability.

Example 5 (Algebraic quasi-polynomial tractability)**.** Let the tractability function, T, be defined by

$$T(\varepsilon^{-1}, d, p) = \exp\{p(1 + \log(\max\{1, \varepsilon^{-1}\}))(1 + \log(d))\} \qquad \forall \varepsilon > 0,\, d \in \mathbb{N},\, p \in [0,\infty).$$

We note that the choice of T is equivalent to the definition in [9]. This can be rewritten as

$$T(\varepsilon^{-1}, d, p) = e^p (\max\{1, \varepsilon^{-1}\})^{p(1+\log(d))} d^p.$$

Then Theorem 2 yields the following equivalent condition for quasi-polynomial tractability in the algebraic case:

$$\sup_{d \in \mathbb{N}} d^{-p} \sum_{i=\lceil L_p d^p \rceil} \frac{1}{(\max\{1, \lambda_{i,d}\})^{p(1+\log(d))}} < \infty \qquad \text{for some } p \geq 0, L_p > 0.$$

This essentially recovers [5, Theorem 23.1].

Next, we will consider the exponential case as follows.

Example 6 (Exponential quasi-polynomial tractability)**.** Suppose that the tractability function, T, is defined by

$$T(\varepsilon^{-1}, d, p) = \exp\{p(1 + \log(\max\{1, \log(1 + \varepsilon^{-1})\}))(1 + \log(d))\} \qquad \forall \varepsilon > 0,\, d \in \mathbb{N},\, p \in [0,\infty).$$

This can be rewritten as

$$T(\varepsilon^{-1}, d, p) = e^p(\max\{1, \log(1 + \varepsilon^{-1})\})^{p(1+\log(d))} d^p.$$

Hence we can apply Theorem 2 as follows. We have the following equivalent condition for quasi-polynomial tractability in the exponential case:

$$\sup_{d \in \mathbb{N}} d^{-p} \sum_{i=\lceil L_p d^p \rceil}^{} \frac{1}{\left(\max\{1, \log(1 + \lambda_{i,d}^{-1})\}\right)^{p(1+\log(d))}} < \infty \quad \text{for some } p \geq 0, L_p > 0.$$

This essentially recovers [11, Theorem 2].

4   Sub-h Tractability

In this section we generalize the concept of weak tractability (see again [3]–[5]). Let $h : [0, \infty) \to [1, \infty)$ satisfy

$$h \text{ is strictly increasing}, \qquad h(0) = 1, \qquad h(x + y) \geq h(x) \cdot h(y) \quad \forall x, y \geq 0. \tag{23}$$

This means that $\log(h)$ is a superadditive function. An example of the function h is the exponential function, from which we can recover weak tractability.

Furthermore, let us denote the inverse function of h by h^{-1}. Then, for arbitrary $z, w \in [1, \infty)$ we choose $x = h^{-1}(z)$ and $y = h^{-1}(w)$, to obtain from (23)

$$zw \leq h(h^{-1}(z) + h^{-1}(w)),$$

which yields by the monotonicity of h^{-1} that

$$h^{-1}(zw) \leq h^{-1}(z) + h^{-1}(w). \tag{24}$$

Now we define a generalized notion of weak tractability.

Definition 3. A problem is sub-h-T tractable for parameter $\boldsymbol{p} > \boldsymbol{0}$ if

$$\lim_{\varepsilon^{-1}+d \to \infty} \frac{h^{-1}(\max(1, \text{comp}(\varepsilon, d)))}{T(\varepsilon^{-1}, d, \boldsymbol{p})} = 0.$$

Note that this definition of sub-h-T tractability implies that

$$\lim_{\varepsilon^{-1}+d \to \infty} \frac{\text{comp}(\varepsilon, d)}{h(cT(\varepsilon^{-1}, d, \boldsymbol{p}))} = 0 \qquad \forall c > 0, \tag{25}$$

which means that for $\varepsilon \to 0$ or $d \to \infty$ $\text{comp}(\varepsilon, d)$ must increase slower than $h(cT(\varepsilon^{-1}, d, \boldsymbol{p}))$, no matter how small c is.

Theorem 3. *Let T be a tractability function as specified in (4) and satisfying (8). Moreover, assume that $\lim_{\varepsilon^{-1}+d \to \infty} T(\varepsilon^{-1}, d, \boldsymbol{p}) = \infty$ for all $\boldsymbol{p} > \boldsymbol{0}$. The problem is sub-$h$-$T$ tractable if and only if*

$$S_c = S_{c,\boldsymbol{p}} := \sup_{d \in \mathbb{N}} \sum_{i=1}^{\infty} \frac{1}{h(c\,T(\lambda_{i,d}^{-1}, d, \boldsymbol{p}))} < \infty \qquad \forall c > 0.$$

Proof. The idea of this proof follows [12]. We suppress the $\boldsymbol{p}$ dependence in the proof, because it is insignificant.

Sufficient condition:
Suppose that $S_c < \infty$ for all $c > 0$. Since the terms of the series in the definition of S_c are non-increasing, we have

$$\frac{n}{h(cT(\lambda_{n,d}^{-1}, d))} \le S_c \qquad \forall n, d \in \mathbb{N},$$

or equivalently

$$n \le S_c \, h(cT(\lambda_{n,d}^{-1}, d)) \qquad \forall n, d \in \mathbb{N}. \tag{26}$$

This implies that for the special choice

$$n = \lceil S_c \, h(cT(\varepsilon^{-1}, d)) \rceil \ge 1$$

we have

$$T(\lambda_{n,d}^{-1}, d) \ge T(\varepsilon^{-1}, d) \qquad \forall \varepsilon > 0, \, d \in \mathbb{N}.$$

Since $T(\cdot, d)$ is non-decreasing, it follows that $\lambda_{n,d} \le \varepsilon$, and thus

$$\text{comp}(\varepsilon, d) \le \lceil S_c \, h(cT(\varepsilon^{-1}, d)) \rceil \le \lceil S_c \, h(cT(\lambda_{1,1}^{-1}, 1)) \, h(cT(\varepsilon^{-1}, d)) \rceil \qquad \forall c, \varepsilon > 0, \, d \in \mathbb{N}$$

because h is no smaller than 1.

As $S_c \ge 1/h(cT(\lambda_{1,1}^{-1}, 1))$ by (26), it follows that $S_c \, h(cT(0,1)) \, h(cT(\varepsilon^{-1}, d)) \ge 1$. Define $\widetilde{S}_c := 2S_c \, h(cT(\lambda_{1,1}^{-1}, 1))$, and note that $\widetilde{S}_c$ must also be finite for all $c > 0$. It follows that

$$\max(1, \text{comp}(\varepsilon, d)) \le \lceil (\widetilde{S}_c/2) \, h(cT(\varepsilon^{-1}, d)) \rceil \le \widetilde{S}_c \, h(cT(\varepsilon^{-1}, d)) \qquad \forall c, \varepsilon > 0, \, d \in \mathbb{N}.$$

Since h is strictly increasing, so is h^{-1}, and thus by (24),

$$h^{-1}(\max(1, \text{comp}(\varepsilon, d))) \le h^{-1}\left(\widetilde{S}_c \, h(cT(\varepsilon^{-1}, d))\right)$$
$$\le h^{-1}(\widetilde{S}_c) + cT(\varepsilon^{-1}, d) \quad \forall c, \varepsilon > 0, \, d \in \mathbb{N}.$$

By the hypothesis of this theorem, $\lim_{\varepsilon^{-1} + d \to \infty} T(\varepsilon^{-1}, d) = \infty$. Therefore

$$\lim_{\varepsilon^{-1} + d \to \infty} \frac{h^{-1}(\max(1, \text{comp}(\varepsilon, d)))}{T(\varepsilon^{-1}, d)} \le \lim_{\varepsilon^{-1} + d \to \infty} \frac{h^{-1}(2S_c)}{T(\varepsilon^{-1}, d)} + c = c \qquad \forall c > 0.$$

Since this limit is bounded above by all positive c, it must be zero, and the problem is sub-h-T tractable.

Necessary condition:
Suppose that the problem is sub-h-T tractable. Then by (25), for any $c > 0$ there exists a positive integer V_c such that

$$\text{comp}(\varepsilon, d) \le \lfloor h(cT(\varepsilon^{-1}, d)) \rfloor \qquad \forall \varepsilon^{-1} + d \ge V_c.$$

By the definition of the information complexity in (3), it follows that

$$\lambda_{n(\varepsilon,d,c),d} \le \varepsilon \quad \forall \varepsilon^{-1} + d \ge V_c, \qquad \text{where } n(\varepsilon, d, c) := \lfloor h(cT(\varepsilon^{-1}, d)) \rfloor + 1.$$

Since $h(cT(\cdot, d))$ is non-decreasing, it follows that

$$n(\varepsilon, d, c) \le h(cT(\varepsilon^{-1}, d)) + 1 \le h(cT(\lambda_{n(\varepsilon,d,c),d}^{-1}, d)) + 1 \qquad \forall \varepsilon^{-1} + d \ge V_c. \tag{27}$$

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For a fixed c and d, define

$$\varepsilon_{\max}^{-1}(d,c) := \max(V_c - d, \lambda_{1,d}^{-1}), \qquad n^*(d,c) := n(\varepsilon_{\max}^{-1}(d,c),d,c) \le h(cT(\varepsilon_{\max}^{-1}(d,c),d))+1. \quad (28)$$

By varying ε^{-1} in the interval $[\varepsilon_{\max}^{-1}(d,c),\infty)$, the integer $n(\varepsilon,d,c)$ attains, by Assumption (8b), all values greater than or equal to $n^*(d,c)$. Thus, by (27),

$$n \le h(cT(\lambda_{n,d}^{-1},d)) \qquad \forall n \ge n^*(d,c), \quad (29)$$

$$\frac{1}{h(2cT(\lambda_{n,d}^{-1},d))} \le \frac{1}{[h(cT(\lambda_{n,d}^{-1},d))]^2} \le \frac{1}{n^2} \qquad \forall n \ge n^*(d,c) \qquad \text{by (23).}$$

Combining this inequality with the upper bound on $n^*(d,c)$ yields

$$\begin{aligned}
S_{2c} &= \sup_{d\in\mathbb{N}} \sum_{i=1}^{\infty} \frac{1}{h(2cT(\lambda_{i,d}^{-1},d))} \\
&= \sup_{d\in\mathbb{N}} \left[\sum_{i=1}^{n^*(d,c)-1} \frac{1}{h(2cT(\lambda_{i,d}^{-1},d))} + \sum_{i=n^*(d,c)}^{\infty} \frac{1}{h(2cT(\lambda_{i,d}^{-1},d))} \right] \\
&\le \sup_{d\in\mathbb{N}} \left[\frac{n^*(d,c)-1}{h(2cT(\lambda_{1,d}^{-1},d))} + \sum_{i=1}^{\infty} \frac{1}{i^2} \right] \qquad \text{by (29)} \\
&\le \sup_{d\in\mathbb{N}} \left[\frac{h(cT(\varepsilon_{\max}^{-1}(d,c),d))}{h(2cT(\lambda_{1,d}^{-1},d))} \right] + \frac{\pi^2}{6} \qquad \text{by (28).} \quad (30)
\end{aligned}$$

Note that

$$\varepsilon_{\max}^{-1}(d,c) = \max(V_c - d, \lambda_{1,d}^{-1}) = \begin{cases} V_c - d \le V_c, & d \in \{1,\dots,d_c^* := \lfloor V_c - \lambda_{1,d}^{-1} \rfloor\}, \\ \lambda_{1,d}^{-1}, & d \in \{d_c^* + 1,\dots\}, \end{cases}$$

and furthermore, it follows from above that we always have $d_c^* \le \lfloor V_c \rfloor$. Recalling that T is non-decreasing in its arguments, this implies that

$$\begin{aligned}
&\sup_{d\in\mathbb{N}} \frac{h(cT(\varepsilon_{\max}^{-1}(d,c),d))}{h(2cT(\lambda_{1,d}^{-1},d))} \\
&\le \max\left\{ \sup_{d\in\{1,\dots,d_c^*\}} \frac{h(cT(\varepsilon_{\max}^{-1}(d,c),d))}{h(2cT(\lambda_{1,d}^{-1},d))}, \sup_{d\in\{d_c^*+1,\dots\}} \frac{h(cT(\varepsilon_{\max}^{-1}(d,c),d))}{h(2cT(\lambda_{1,d}^{-1},d))} \right\} \\
&\le \max\left\{ \sup_{d\in\{1,\dots,d_c^*\}} \frac{h(cT(V_c,d))}{h(2cT(\lambda_{1,d}^{-1},d))}, \sup_{d\in\{d_c^*+1,\dots\}} \frac{h(cT(\lambda_{1,d}^{-1},d))}{h(2cT(\lambda_{1,d}^{-1},d))} \right\} \\
&\le \max\left\{ \frac{h(cT(V_c,d_c^*))}{1}, \sup_{d\in\{d_c^*+1,\dots\}} \frac{h(cT(\lambda_{1,d}^{-1},d))}{[h(cT(\lambda_{1,d}^{-1},d))]^2} \right\} \\
&\le \max\left\{ \frac{h(cT(V_c,\lfloor d_c^* \rfloor))}{1}, 1 \right\} \\
&= h(cT(V_c,\lfloor V_c \rfloor)) \\
&< \infty,
\end{aligned}$$

where we used that $h(\cdot)$ is always at least 1. Combining this bound with the upper bound on S_c in (30) establishes the finiteness of S_{2c} and completes the proof of the necessary condition. This concludes the proof of Theorem 3. $\qquad\square\qquad\qquad\qquad\qquad\square$

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Example 7 (Weak Tractability). Suppose that

$$T(\varepsilon^{-1}, d, s, t) = \max(1, \varepsilon^{-1})^s + d^t \quad \forall \varepsilon, s, t > 0, \ d \in \mathbb{N}, \qquad h(x) = \exp(x) \quad \forall x \geq 0.$$

Then Theorem 3 implies that

$$\lim_{\varepsilon^{-1}+d \to \infty} \frac{\log(\text{comp}(\varepsilon, d))}{\varepsilon^{-s} + d^t} = 0 \iff \sup_{d \in \mathbb{N}} \sum_{i=1}^{\infty} \exp(-c\,(\min\{1, \lambda_{i,d}\}^s + d^t)) < \infty \qquad \forall c > 0.$$

The expression on the left is the notion of (s,t)-weak tractability introduced in [13], and the condition on the right was derived in [12, Theorem 3.1]. Weak tractability corresponds to $(s,t) = (1,1)$, and uniform weak tractability corresponds to the case when a problem is weakly tractable for all positive s and t.

5 (Strong) Tractability on a Restricted Domain

Suppose that the domain of interest (ε^{-1}, d) is not all of $(0, \infty) \times \mathbb{N}$ but some subset

$$\Omega := \{(\varepsilon^{-1}, d) : \varepsilon \in (\mathscr{E}(d), \infty), \ d \in \mathbb{N}\}, \qquad \text{where } \mathscr{E} : \mathbb{N} \to [0, \infty). \tag{31}$$

We then expect that the equivalent conditions for (strong) tractability will be similar in form, but weaker than the conditions in Theorems 1 and 2.

Definition 4. A problem is strongly T-tractable with parameter $\boldsymbol{p}$ *on the restricted domain* Ω iff the information complexity is independent of the dimension of the problem, that is, there exists a positive constant $C_{\boldsymbol{p}}$, again independent of ε and d, such that

$$\text{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}}\, T(\varepsilon^{-1}, 1, \boldsymbol{p}) \qquad \forall (\varepsilon^{-1}, d) \in \Omega. \tag{32}$$

A problem is T-tractable with parameter $\boldsymbol{p}$ *on the restricted domain* Ω iff there exists a positive constant $C_{\boldsymbol{p}}$, again independent of ε and d, such that

$$\text{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}}\, T(\varepsilon^{-1}, d, \boldsymbol{p}) \qquad \forall (\varepsilon^{-1}, d) \in \Omega. \tag{33}$$

Define the closures of the sets of parameters for which our (strong) tractability conditions hold:

$$\mathcal{P}_{\text{rtrct}} := \{\boldsymbol{p}^* : (33) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \infty)\}, \qquad \mathcal{P}_{\text{rstrct}} := \{\boldsymbol{p}^* : (32) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \infty)\}.$$

The set of optimal parameters is defined as all of those parameters satisfying the (strong) tractability conditions that are not greater than or equal to others:

$$\mathcal{P}_{\text{ropt}} := \{\boldsymbol{p}^* \in \mathcal{P}_{\text{rtrct}} : \boldsymbol{p}^* \notin [\boldsymbol{p}, \infty) \ \forall \boldsymbol{p} \in \mathcal{P}_{\text{rtrct}} \setminus \{\boldsymbol{p}^*\}\}, \tag{34}$$

$$\mathcal{P}_{\text{rsopt}} := \{\boldsymbol{p}^* \in \mathcal{P}_{\text{rstrct}} : \boldsymbol{p}^* \notin [\boldsymbol{p}, \infty) \ \forall \boldsymbol{p} \in \mathcal{P}_{\text{rstrct}} \setminus \{\boldsymbol{p}^*\}\}. \tag{35}$$

Define $\mathscr{U} : \mathbb{N} \to \mathbb{N}_0 \cup \{\infty\}$

$$\mathscr{U}(d) := \begin{cases} \min\{n \in \mathbb{N}_0 : \lambda_{n+1,d} \leq \mathscr{E}(d)\}, & \mathscr{E}(d) > 0, \\ \infty, & \mathscr{E}(d) = 0. \end{cases} \tag{36}$$

Also, for each $d \in \mathbb{N}$ define the set

$$\mathcal{I}_d := \begin{cases} \emptyset, & \mathscr{U}(d) = 0, \\ \{1, \ldots, \mathscr{U}(d)\}, & 0 < \mathscr{U}(d) < \infty, \\ \mathbb{N}, & \mathscr{U}(d) = \infty. \end{cases} \tag{37}$$

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From the definition of $\mathscr{U}(d)$ it follows that $(\lambda_{i,d}^{-1}, d) \in \Omega \iff i \in \mathcal{I}_d$, and

$$\text{comp}(\varepsilon, d) \begin{cases} = 0, & \mathscr{U}(d) = 0, \\ \in \mathcal{I}_d, & \mathscr{U}(d) > 0, \end{cases} \quad \forall (\varepsilon^{-1}, d) \in \Omega. \tag{38}$$

The equivalent conditions for (strong) T-tractability on a restricted domain in Theorems 4 and 5 mimic the conditions given in Theorems 1 and 2, except that the upper limits on the sums now correspond to $\mathscr{U}(d)$ rather than ∞. The proofs are a bit more delicate, but similar arguments are used.

We note that it may be possible to define Ω such that $\lambda_{1,d} \leq \mathscr{E}(d)$ for all $d \in \mathbb{N}$. In this case, $\mathscr{U}(d) = 0$ for all $d \in \mathbb{N}$, the equivalent conditions for (strong) tractability are trivially satisfied, and the zero algorithm satisfies the error tolerance.

5.1 Strong Tractability on a Restricted Domain

Theorem 4. *Let T be a tractability function as specified in (4) and satisfying (8) and (9). A problem is strongly T-tractable on the restricted domain Ω iff there exists $\boldsymbol{p} \in [\boldsymbol{0}, \infty)$ and an integer $L_{\boldsymbol{p}} > 0$ such that*

$$S_{\boldsymbol{p}} := \sup_{d \in \mathbb{N}} \sum_{i=L_{\boldsymbol{p}}}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p})} < \infty. \tag{39}$$

By convention, if $\mathscr{U}(d) < L_{\boldsymbol{p}}$, the sum is zero.

If (39) holds for some $\boldsymbol{p}$, let $\widetilde{\mathcal{P}}_{\text{rstrct}} := \{\boldsymbol{p}^* : (39) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \infty)\}$. Then $\mathcal{P}_{\text{rstrct}} = \widetilde{\mathcal{P}}_{\text{rstrct}}$, and the set of optimal strong tractability parameters on the restricted domain is

$$\mathcal{P}_{\text{rsopt}} = \{\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\text{rstrct}} : \boldsymbol{p}^* \notin [\widetilde{\boldsymbol{p}}, \infty) \; \forall \widetilde{\boldsymbol{p}} \in \widetilde{\mathcal{P}}_{\text{rstrct}} \setminus \{\boldsymbol{p}^*\}\}.$$

Proof. **Sufficient condition:**
Fix d. If $\mathscr{U}(d) = 0$, then $\text{comp}(\varepsilon, d) = 0 \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p})$ automatically for all $\varepsilon \in [\mathscr{E}(d), \infty)$.

For the case $\mathscr{U}(d) > 0$, following the argument for the sufficient condition in Theorem 1 it follows that for all $\mathscr{D} \in \mathbb{N}$ and integers $n, N \in \mathcal{I}_d$ with $N \leq n$, we have

$$\lambda_{n+1,d} \leq \lambda_{n,d} \leq \cdots \leq \lambda_{N,d} \implies \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{T(\lambda_{n,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \cdots \leq \frac{1}{T(\lambda_{N,d}^{-1}, \mathscr{D}, \boldsymbol{p})}$$

$$\implies \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{n-N+1} \sum_{i=N}^{n} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{n-N+1} \sum_{i=N}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})}.$$

Thus, we can conclude from the previous line that

$$n - N + 1 \geq T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p}) \sum_{i=N}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})}$$

$$\implies \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{n-N+1} \sum_{i=N}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p})}$$

$$\forall \mathscr{D} \in \mathbb{N}, \; n, N \in \mathcal{I}_d, \; N \leq n. \tag{40}$$

Moreover, given that T is increasing in its arguments, we have the following equivalent expression for the information complexity via (3):

$$\text{comp}(\varepsilon, d) = \min\{n \in \mathcal{I}_d : \lambda_{n+1,d} \leq \varepsilon\} \quad \text{by (38)}$$

$$= \min\left\{ n \in \mathcal{I}_d : \frac{1}{T(\lambda_{n+1,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \leq \frac{1}{T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p})} \right\} \qquad \forall \mathscr{D} \in \mathbb{N}$$

$$\leq \min\left\{ n \in \mathcal{I}_d : n \geq N - 1 + T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p}) \sum_{i=N}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \right\} \forall \mathscr{D}, \ 1 \leq N \leq \mathscr{U}(d)$$

by (40).

This last statement holds for all $N \leq \mathscr{U}(d)$ because $n \geq N$ automatically in the expression above. Thus,

$$\text{comp}(\varepsilon, d) \leq T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p}) \left[\frac{N}{T(\varepsilon^{-1}, \mathscr{D}, \boldsymbol{p})} + \sum_{i=N}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, \mathscr{D}, \boldsymbol{p})} \right] \quad \forall \mathscr{D}, \ 1 \leq N \leq \mathscr{U}(d). \tag{41}$$

Now we take this upper bound further, specializing to the case of $\mathscr{D} = 1$ and $N = \min\{L_{\boldsymbol{p}}, \mathscr{U}(d)\}$:

$$\text{comp}(\varepsilon, d) \leq T(\varepsilon^{-1}, 1, \boldsymbol{p}) \left[\frac{\min\{L_{\boldsymbol{p}}, \mathscr{U}(d)\}}{T(\varepsilon^{-1}, 1, \boldsymbol{p})} + \underbrace{\sum_{i=\min\{L_{\boldsymbol{p}}, \mathscr{U}(d)\}}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p})}}_{\leq S_{\boldsymbol{p}} + 1/T(\lambda_{\mathscr{U}(d),d}^{-1}, 1, \boldsymbol{p}) \text{ by (39)}} \right]$$

$$\leq T(\varepsilon^{-1}, 1, \boldsymbol{p}) \underbrace{\left[\frac{L_{\boldsymbol{p}} + 1}{T(0, 1, \boldsymbol{p})} + S_{\boldsymbol{p}} \right]}_{=: C_{\boldsymbol{p}}} \qquad \text{by (8a)}$$

$$= C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}).$$

This means that we have strong T-tractability on the restricted domain Ω via (32), and verifies the sufficiency of (39).

Necessary condition:
Suppose that we have strong T-tractability on the restricted domain Ω as defined in (32). That is, for some $\boldsymbol{p} \geq \boldsymbol{0}$ there exists a positive constant $C_{\boldsymbol{p}}$ such that

$$\text{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) \qquad \forall (\varepsilon^{-1}, d) \in \Omega.$$

Since the sequence of singular values $\lambda_{1,d}, \lambda_{2,d}, \dots$ is non-increasing, we have

$$\lambda_{\lfloor C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) \rfloor + 1, d} \leq \varepsilon \qquad \forall (\varepsilon^{-1}, d) \in \Omega. \tag{42}$$

For all $\varepsilon > 0$, define the positive integers

$$i(\varepsilon, \boldsymbol{p}) := \lfloor C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) \rfloor + 1, \quad i(\infty, \boldsymbol{p}) = \lfloor C_{\boldsymbol{p}} T(0, 1, \boldsymbol{p}) \rfloor + 1 \geq 1,$$

just as in (16). Thus, it follows by (42) that

$$\lambda_{i(\varepsilon, \boldsymbol{p}), d} \leq \varepsilon \qquad \forall (\varepsilon^{-1}, d) \in \Omega. \tag{43}$$

Note furthermore that we always have

$$i(\varepsilon, \boldsymbol{p}) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, 1, \boldsymbol{p}) + 1 \leq C_{\boldsymbol{p}} T(\lambda_{i(\varepsilon, \boldsymbol{p}), d}^{-1}, 1, \boldsymbol{p}) + 1 \qquad \forall (\varepsilon^{-1}, d) \in \Omega,$$

since $T(\cdot, 1, \boldsymbol{p})$ is non-decreasing.

For ε taking on all values in $(\mathscr{E}(d), \infty)$, $i(\varepsilon, \boldsymbol{p})$ takes on (at least) all values in

$$\mathscr{I}_{d,\boldsymbol{p}} := \begin{cases} \emptyset, & i(\mathscr{E}(d), \boldsymbol{p}) = i(\infty, \boldsymbol{p}), \\ \{i(\infty, \boldsymbol{p}), \ldots, i(\mathscr{E}(d), \boldsymbol{p}) - 1\}, & \mathscr{E}(d) > 0, \\ \{i(\infty, \boldsymbol{p}), i(\infty, \boldsymbol{p}) + 1, \ldots\}, & \mathscr{E}(d) = 0. \end{cases}$$

So,

$$i \leq C_{\boldsymbol{p}} T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p}) + 1 \qquad \forall i \in \mathscr{I}_{d,\boldsymbol{p}}, \ d \in \mathbb{N}.$$

This implies via our technical assumption (9) that

$$K_{\boldsymbol{p},\tau} \, T(\lambda_{i,d}^{-1}, 1, \tau\boldsymbol{p}) \geq [T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p})]^{\tau} \geq \left[\frac{(i-1)}{C_{\boldsymbol{p}}}\right]^{\tau} \qquad \forall i \in \mathscr{I}_{d,\boldsymbol{p}}, \ d \in \mathbb{N},$$

$$\frac{1}{T(\lambda_{i,d}^{-1}, 1, \tau\boldsymbol{p})} \leq \frac{K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau}}{(i-1)^{\tau}} \qquad \forall i \in \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}, \ d \in \mathbb{N}. \tag{44}$$

To complete the proof, we will sum both sides of (44) over the range of i appearing in (39), to establish that $S_{\tau\boldsymbol{p}}$ is finite. To do this we need to show that (44) holds for that range. Choose $L_{\tau\boldsymbol{p}} = i(\infty, \boldsymbol{p}) + 1 \geq 2$. There are three cases.

i) If $\mathscr{U}(d) < L_{\tau\boldsymbol{p}}$, then

$$\sum_{i=L_{\tau\boldsymbol{p}}}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \tau\boldsymbol{p})} = 0,$$

which does not affect the finiteness of $S_{\tau\boldsymbol{p}}$.

ii) If $\mathscr{U}(d) \geq L_{\tau\boldsymbol{p}}$ and $\mathscr{E}(d) = 0$, then $\{L_{\tau\boldsymbol{p}}, L_{\tau\boldsymbol{p}} + 1, \ldots\} \subseteq \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}$.

iii) If $\mathscr{U}(d) \geq L_{\tau\boldsymbol{p}}$ and $\mathscr{E}(d) > 0$, then (36) implies that $\lambda_{\mathscr{U}(d)+1,d} \leq \mathscr{E}(d) < \lambda_{\mathscr{U}(d),d}$. Moreover, there must be some ε^* satisfying $\mathscr{E}(d) < \varepsilon^* < \lambda_{\mathscr{U}(d),d}$, for which $\lambda_{i(\varepsilon^*,\boldsymbol{p})} \leq \varepsilon^* < \lambda_{\mathscr{U}(d),d}$ by (43). Because of the ordering of the singular values, this implies that $i(\varepsilon^*, \boldsymbol{p}) \geq \mathscr{U}(d) + 1$. Since $i(\cdot, \boldsymbol{p})$ is nonincreasing, $i(\mathscr{E}(d), \boldsymbol{p}) \geq \mathscr{U}(d) + 1$. This means that $\mathscr{U}(d) \leq i(\mathscr{E}(d), \boldsymbol{p}) - 1$. So, $\{L_{\tau\boldsymbol{p}}, \ldots, \mathscr{U}(d)\} \subseteq \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}$.

In both the second and third cases,

$$S_{\tau\boldsymbol{p}} = \sup_{d \in \mathbb{N}} \sum_{i=L_{\tau\boldsymbol{p}}}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \tau\boldsymbol{p})} \leq \sup_{d \in \mathbb{N}} \sum_{i \in \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \tau\boldsymbol{p})}$$

$$\leq K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau} \sum_{i=2}^{\infty} \frac{1}{(i-1)^{\tau}} \leq K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau} \zeta(\tau) < \infty \qquad \text{by (44)},$$

where ζ denotes the Riemann zeta function.

Thus, in all three cases, the assumption of strong tractability on the restricted domain implies (39) with $\boldsymbol{p}$ replaced by $\boldsymbol{p}' = \tau\boldsymbol{p}$, and so we see the necessity of (39).

Optimality:
The proof of optimality is analogous to that for Theorem 1 and is omitted.

This concludes the proof of Theorem 4. $\qquad \square \qquad\qquad\qquad\qquad \square$

5.2 Tractability on a Restricted Domain

The equivalent condition for tractability on a restricted domain is analogous to the conditions for tractability on an unrestricted domain and for strong tractability on a restricted domain. The proof is analogous as well.

Theorem 5. *Let T be a tractability function as specified in (4) and satisfying (8) and (9). A problem is T-tractable on the restricted domain Ω iff there exists $\boldsymbol{p} \geq \boldsymbol{0}$ and a positive constant $L_{\boldsymbol{p}}$ such that*

$$S_{\boldsymbol{p}} := \sup_{d \in \mathbb{N}} \sum_{i=\lceil L_{\boldsymbol{p}} T(0,d,\boldsymbol{p}) \rceil}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p})} < \infty. \tag{45}$$

If (45) holds for some $\boldsymbol{p}$, let $\widetilde{\mathcal{P}}_{\mathrm{rtrct}} := \{\boldsymbol{p}^* : (45) \text{ holds } \forall \boldsymbol{p} \in (\boldsymbol{p}^*, \infty)\}$. Then $\mathcal{P}_{\mathrm{rtrct}} = \widetilde{\mathcal{P}}_{\mathrm{rtrct}}$, and the set of optimal tractability parameters is

$$\mathcal{P}_{\mathrm{ropt}} = \{\boldsymbol{p}^* \in \widetilde{\mathcal{P}}_{\mathrm{rtrct}} : \boldsymbol{p}^* \notin [\widetilde{\boldsymbol{p}}, \infty) \; \forall \widetilde{\boldsymbol{p}} \in \widetilde{\mathcal{P}}_{\mathrm{rtrct}} \setminus \{\boldsymbol{p}^*\}\}.$$

Proof. **Sufficient condition:**
Fix d. If $\mathscr{U}(d) = 0$, then $\mathrm{comp}(\varepsilon, d) = 0 \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p})$ automatically for all $\varepsilon \in [\mathscr{E}(d), \infty)$.

For the case $\mathscr{U}(d) > 0$, we utilize the upper bound (41), specializing to the case of $\mathscr{D} = d$ and $N = \min\{\lceil L_{\boldsymbol{p}} T(0, d, \boldsymbol{p}) \rceil, \mathscr{U}(d)\}$:

$$\mathrm{comp}(\varepsilon, d) \leq T(\varepsilon^{-1}, d, \boldsymbol{p}) \left[\frac{\min\{\lceil L_{\boldsymbol{p}} T(0, d, \boldsymbol{p}) \rceil, \mathscr{U}(d)\}}{T(\varepsilon^{-1}, d, \boldsymbol{p})} + \underbrace{\sum_{i=\min\{\lceil L_{\boldsymbol{p}} T(0,d,\boldsymbol{p}) \rceil, \mathscr{U}(d)\}}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p})}}_{\leq S_{\boldsymbol{p}} + 1/T(\lambda_{\mathscr{U}(d),d}^{-1}, d, \boldsymbol{p}) \text{ by (39)}} \right]$$

$$\leq T(\varepsilon^{-1}, d, \boldsymbol{p}) \left[\frac{L_{\boldsymbol{p}} T(0, d, \boldsymbol{p}) + 1}{T(0, d, \boldsymbol{p})} + S_{\boldsymbol{p}} + \frac{1}{T(\lambda_{\mathscr{U}(d),d}^{-1}, d, \boldsymbol{p})} \right] \quad \text{by (8a)}$$

$$\leq T(\varepsilon^{-1}, d, \boldsymbol{p}) \underbrace{\left[L_{\boldsymbol{p}} + \frac{2}{T(0, 1, \boldsymbol{p})} + S_{\boldsymbol{p}} \right]}_{=:C_{\boldsymbol{p}}} \quad \text{by (8a)}$$

$$= C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p}).$$

This means that we have strong T-tractability on the restricted domain Ω via (33), and verifies the sufficiency of (45).

Necessary condition:
Suppose that we have T-tractability on the restricted domain. That is, for some $\boldsymbol{p} \geq \boldsymbol{0}$, there exists a positive constant $C_{\boldsymbol{p}}$ such that

$$\mathrm{comp}(\varepsilon, d) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p}) \qquad \forall (\varepsilon^{-1}, d) \in \Omega.$$

Since the sequence of singular values $\lambda_{1,d}, \lambda_{2,d}, \ldots$ is non-increasing, we have

$$\lambda_{\lfloor C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p}) \rfloor + 1, d} \leq \varepsilon \qquad \forall (\varepsilon^{-1}, d) \in \Omega. \tag{46}$$

For $(\varepsilon^{-1}, d) \in \Omega$, define the positive integers

$$i(\varepsilon, d, \boldsymbol{p}) := \lfloor C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p}) \rfloor + 1, \qquad i(\infty, d, \boldsymbol{p}) := \lfloor C_{\boldsymbol{p}} T(0, d, \boldsymbol{p}) \rfloor + 1 \geq 1.$$

It follows by (46) that $\lambda_{i(\varepsilon,d,\boldsymbol{p}),d} \leq \varepsilon$ for any $(\varepsilon^{-1}, d) \in \Omega$. Note furthermore that since $T(\cdot, d, \boldsymbol{p})$ is non-decreasing, we have

$$i(\varepsilon, d, \boldsymbol{p}) \leq C_{\boldsymbol{p}} T(\varepsilon^{-1}, d, \boldsymbol{p}) + 1 \leq C_{\boldsymbol{p}} T(\lambda_{i(\varepsilon,d,\boldsymbol{p}),d}^{-1}, d, \boldsymbol{p}) + 1 \qquad \forall (\varepsilon^{-1}, d) \in \Omega.$$

For ε taking on all values in $(\mathscr{E}(d), \infty)$, $i(\varepsilon, d, \boldsymbol{p})$ takes on (at least) all values in

$$\mathscr{I}_{d,\boldsymbol{p}} := \begin{cases} \emptyset, & i(\mathscr{E}(d), d, \boldsymbol{p}) = i(\infty, d, \boldsymbol{p}), \\ \{i(\infty, d, \boldsymbol{p}), \ldots, i(\mathscr{E}(d), d, \boldsymbol{p}) - 1\}, & \mathscr{E}(d) > 0, \\ \{i(\infty, d, \boldsymbol{p}), i(\infty, d, \boldsymbol{p}) + 1, \ldots\}, & \mathscr{E}(d) = 0. \end{cases}$$

So,

$$i \leq C_{\boldsymbol{p}} T(\lambda_{i,d}^{-1}, d, \boldsymbol{p}) + 1, \qquad \forall i \in \mathscr{I}_d, \ d \in \mathbb{N}.$$

This implies via our technical assumption (9) that

$$K_{\boldsymbol{p},\tau} T(\lambda_{i,d}^{-1}, d, \tau\boldsymbol{p}) \geq [T(\lambda_{i,d}^{-1}, d, \boldsymbol{p})]^{\tau} \geq \left[\frac{(i-1)}{C_{\boldsymbol{p}}}\right]^{\tau} \qquad \forall i \in \mathscr{I}_{d,\boldsymbol{p}}, \ d \in \mathbb{N},$$

$$\frac{1}{T(\lambda_{i,d}^{-1}, d, \tau\boldsymbol{p})} \leq \frac{K_{\boldsymbol{p},\tau} C_{\boldsymbol{p}}^{\tau}}{(i-1)^{\tau}} \qquad \forall i \in \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}, \ d \in \mathbb{N}. \tag{47}$$

To complete the proof, we will sum both sides of (47) over the range of i appearing in (45), to establish that $S_{\boldsymbol{p}'}$ is finite, where $\boldsymbol{p}' = \tau\boldsymbol{p}$. To do this we need to show that (47) holds for that range.

For any $d \in \mathbb{N}$, let

$$L_{\boldsymbol{p}'} := C_{\boldsymbol{p}'} + \frac{2}{T(0, 1, \boldsymbol{p}')},$$

and note that

$$\max\{i(\infty, d, \boldsymbol{p}'), 2\} \leq i(\infty, d, \boldsymbol{p}') + 1 = \lfloor C_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rfloor + 2 \leq \left[C_{\boldsymbol{p}'} + \frac{2}{T(0, d, \boldsymbol{p}')}\right] T(0, d, \boldsymbol{p}')$$

$$= L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \leq \lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil.$$

This is the inequality needed for the lower limit of the sum.

There are three cases.

i) If $\mathscr{U}(d) < \lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil$, then

$$\sum_{i=\lceil L_{\boldsymbol{p}'} T(0,d,\boldsymbol{p}') \rceil}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, d, \boldsymbol{p}')} = 0,$$

which does not affect the finiteness of $S_{\boldsymbol{p}'}$.

ii) If $\mathscr{U}(d) \geq \lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil$ and $\mathscr{E}(d) = 0$, then $\{\lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil, \lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil + 1, \ldots\} \subseteq \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}$.

iii) If $\mathscr{U}(d) \geq \lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil$ and $\mathscr{E}(d) > 0$, then (36) and (46) imply that

$$\lambda_{i(\mathscr{E}(d),d,\boldsymbol{p}),d} \leq \lambda_{\mathscr{U}(d)+1,d} \leq \mathscr{E}(d) < \lambda_{\mathscr{U}(d),d}.$$

By the same argument as in the proof of strong tractability, $\mathscr{U}(d) \leq i(\mathscr{E}(d), d, \boldsymbol{p}) - 1$. So, $\{\lceil L_{\boldsymbol{p}'} T(0, d, \boldsymbol{p}') \rceil, \ldots, \mathscr{U}(d)\} \subseteq \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}$.

In both the second and third cases,

$$S_{\boldsymbol{p}'} = \sup_{d \in \mathbb{N}} \sum_{i = \lceil L_{\boldsymbol{p}'} T(0,d,\boldsymbol{p}') \rceil}^{\mathscr{U}(d)} \frac{1}{T(\lambda_{i,d}^{-1}, d, \tau \boldsymbol{p})} \leq \sup_{d \in \mathbb{N}} \sum_{i \in \mathscr{I}_{d,\boldsymbol{p}} \setminus \{1\}} \frac{1}{T(\lambda_{i,d}^{-1}, d, \tau \boldsymbol{p})}$$

$$\leq K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau} \sum_{i=2}^{\infty} \frac{1}{(i-1)^{\tau}} \leq K_{\boldsymbol{p},\tau} \, C_{\boldsymbol{p}}^{\tau} \zeta(\tau) < \infty \qquad \text{by (47)},$$

where ζ denotes the Riemann zeta function.

Thus, in all three cases, the assumption of strong tractability on the restricted domain implies (45) with $\boldsymbol{p}$ replaced by $\boldsymbol{p}' = \tau \boldsymbol{p}$, and so we see the necessity of (45).

Optimality:
The proof of optimality is analogous to that for Theorem 2 and is omitted.

This concludes the proof of Theorem 5. $\qquad \square \qquad\qquad\qquad \square$

6 Discussion and Further Work

We have shown that many proofs of equivalent conditions for (strong) tractability of various types can be unified into a handful of proofs. Even the arguments underlying these handful are quite similar and could be consolidated even further. This unification simplifies our understanding of these equivalent conditions and spotlights the key ideas about what makes a class of problems hard.

However, some questions remain:

- What interesting cases of T do not correspond to known tractability measures?

- What interesting cases are there where there is no (strong) tractability on an unrestricted domain, but there is (strong) tractability on a nontrivial restricted domain, such as $(0, \infty) \times \{1, \ldots, d_{\max}\} \cup (\varepsilon_{\min}^{-1}, \infty) \times \mathbb{N}$?

- For Hilbert spaces and solution operators, $\{\mathrm{SOL}_d : \mathcal{F}_d \to \mathcal{G}_d\}_{d \in \mathbb{N}}$, of tensor product form, the singular values take the form $\lambda_{i,d} = \lambda_{i,d,1} \cdots \lambda_{i,d,d}$. What more can be said about the equivalent conditions for (strong) tractability in this case?

- For information complexity defined over cones of functions rather than balls (see e.g., [14, 15]), what can be said about equivalent conditions for (strong) tractability?

7 Acknowledgments

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Appendix

We give the proof of Proposition 1 here.

Proof. We will give the proof of equivalence of the conditions (19) and (22). The proof of the equivalence of (20) and (21) is analogous and simpler (actually, the latter proof can be seen as a special case of the former, by assuming all exponents of d to be zero).

Let us first assume that (22) holds, i.e.,

$$M := \sup_{d \in \mathbb{N}} d^{-\tau_1} \sum_{i = \lceil Cd^{\tau_3} \rceil}^{\infty} \lambda_{i,d}^{i^{-\tau_2}} < \infty$$

for some $\tau_1, \tau_3 \geq 0$ and $\tau_2, C > 0$. Here, M and other constants above may depend on τ_1, τ_2, and τ_3. Define B_d as the set of indices that are contained in the above sum and for which the terms in the sum are relatively large:

$$B_d := \left\{ i \in \mathbb{N} \colon i \geq \lceil Cd^{\tau_3} \rceil \quad \text{and} \quad \lambda_{i,d}^{i^{-\tau_2}} > \frac{1}{e} \right\}.$$

Due to our assumption, we see that $\text{card}(B_d) < e\, M\, d^{\tau_1}$ and $\text{card}(B_d) \leq \lfloor e\, M\, d^{\tau_1} \rfloor$. Suppose now that $i \geq \lceil Cd^{\tau_3} \rceil$ but $i \notin B_d$, which means that the term in the above sum is relatively small:

$$\lambda_{i,d}^{i^{-\tau_2}} \leq \frac{1}{e} \iff \log(\lambda_{i,d}^{-1}) \geq i^{\tau_2}. \tag{48}$$

Now, choose $p > 1/\tau_2$, $q = \max\{\tau_1, \tau_3\}$, and $L_{(p,q)} = \max\{C, e\, M\}$, which implies that $\lceil L_{(p,q)} d^q \rceil \geq e\, M\, d^{\tau_1}$. Then, for any $d \in \mathbb{N}$, the desired sum in (19) can be bounded by splitting it into the sum of the larger terms and the sum of the smaller terms:

$$d^{-q} \sum_{i = \lceil L_{(p,q)} d^q \rceil}^{\infty} \frac{1}{[\max\{1, \log(1 + \lambda_{i,d}^{-1})\}]^p}$$

$$= d^{-q} \sum_{\substack{i = \lceil L_{(p,q)} d^q \rceil \\ i \in B_d}}^{\infty} \frac{1}{[\max\{1, \log(1 + \lambda_{i,d}^{-1})\}]^p} + d^{-q} \sum_{\substack{i = \lceil L_{(p,q)} d^q \rceil \\ i \notin B_d}}^{\infty} \frac{1}{[\max\{1, \log(1 + \lambda_{i,d}^{-1})\}]^p}$$

$$\leq d^{-q} e\, M\, d^{\tau_1} + d^{-q} \sum_{\substack{i = \lceil L_{(p,q)} d^q \rceil \\ i \notin B_d}}^{\infty} \frac{1}{[\log(\lambda_{i,d}^{-1})]^p}$$

$$\leq e\, M + d^{-q} \sum_{\substack{i = \lceil L_{(p,q)} d^q \rceil \\ i \notin B_d}}^{\infty} \frac{1}{i^{\tau_2 p}} \qquad \text{by (48)}$$

$$\leq e\, M + \zeta(\tau_2 p).$$

Since p was chosen to be strictly larger than $1/\tau_2$, it follows that the latter expression is finite and its value is independent of d. Therefore, Condition (19) holds.

Conversely, let us now assume that Condition (19) holds for some $p > 0$ and $q \geq 0$ that are assumed to be fixed. All constants in this part of the proof may depend on p and q. We will show next that this implies Condition (22). Indeed, due to (19), there exist constants $M > 1$ and $L_{(p,q)} > 0$ such that for all $d \in \mathbb{N}$,

$$d^{-q} \sum_{i = \lceil L_{(p,q)} d^q \rceil}^{\infty} \frac{1}{[\max\{1, \log(1 + \lambda_{i,d}^{-1})\}]^p} \leq M,$$

which is equivalent to

$$\sum_{i=\lceil L_{(p,q)}d^q \rceil}^{\infty} \frac{1}{[\max\{1, \log(1 + \lambda_{i,d}^{-1})\}]^p} \le d^q M.$$

Similarly to the proof of the sufficient condition in Theorem 1, we see that for any $n > \lceil L_{(p,q)}d^q \rceil$,

$$\frac{1}{T(\lambda_{n+1,d}^{-1}, 1, \boldsymbol{p})} \le \frac{1}{n - \lceil L_{(p,q)}d^q \rceil + 1} \sum_{i=\lceil L_{(p,q)}d^q \rceil}^{\infty} \frac{1}{T(\lambda_{i,d}^{-1}, 1, \boldsymbol{p})} \le \frac{d^q M}{n - \lceil L_{(p,q)}d^q \rceil + 1}.$$

This inequality implies for our specific choice of T that

$$\frac{n}{d^q M} - \frac{L_{(p,q)}}{M} \le \frac{n - \lceil L_{(p,q)}d^q \rceil + 1}{d^q M} \le [\max\{1, \log(1 + \lambda_{n+1,d}^{-1})\}]^p \le [1 + \log(1 + \lambda_{n+1,d}^{-1})]^p.$$

Since both $L_{(p,q)}$ and M in the latter chain of inequalities are independent of d and n, there exists a positive constant K such that

$$\frac{n}{d^q K} \le [1 + \log(1 + \lambda_{n+1,d}^{-1})]^p \quad \forall n > \lceil L_{(p,q)}d^q \rceil,$$

which implies

$$\frac{n^{1/p}}{d^{q/p}K^{1/p}} - 1 \le \log(1 + \lambda_{n+1,d}^{-1}) \quad \forall n > \lceil L_{(p,q)}d^q \rceil.$$

This, in turn, implies the existence of another constant $\widetilde{K} > 0$ such that

$$\lambda_{n,d} \le \exp\left(-\frac{n^{1/p}}{\widetilde{K}^{1/p}d^{q/p}}\right) \quad \forall n > \lceil L_{(p,q)}d^q \rceil - 1,$$

which is essentially the situation described in [11, p. 118]. We can then choose $\tau_2 \in (0, 1/p)$, and proceed as in [11] to obtain

$$\sum_{i=\lceil L_{(p,q)}d^q \rceil}^{\infty} \lambda_{i,d}^{i^{-\tau_2}} = \mathcal{O}(d^{q/(1-\tau_2 p)}),$$

with the implied factor in the $\mathcal{O}$-notation independent of d. Then, we can choose $\tau_1 = q/(1 - \tau_2 p)$ and $\tau_3 = q$ and see that (22) holds.

This concludes the proof of the equivalence of (19) and (22). $\qquad\square$

$\square$