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ON THE INVERSION OF THE MOMENTA RAY TRANSFORM OF SYMMETRIC TENSORS IN THE PLANE

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ABSTRACT. We present a reconstruction method which stably recovers some sufficiently smooth, real valued, symmetric tensor fields compactly supported in the Euclidean plane, from knowledge of their non/attenuated momenta ray transform. The reconstruction method extends Bukhgeim's A-analytic theory from an equation to a system.

1. INTRODUCTION

We consider the problem of recovering a real valued, symmetric m-tensor field $\mathbf{f}$ compactly supported in the plane, from knowledge of its $0, 1, ..., m$-th (non)/attenuated moment ray transforms, $m \geq 1$. When the data is limited to the 0-moment, the (non-attenuated) ray transform has a large kernel containing all the potential tensors vanishing at the boundary of the support. In response, a vast literature in tensor tomography (also on Riemannian manifolds) concerns recovery of the solenoidal part of the tensor field, see [32, 25, 24, 31, 14] and reference therein. The problem of inversion of the ray transform of 1-tensors (Doppler) is motivated by engineering practices [38, 23, 5, 34]. The ray transform of second and four order tensors arise in the linearization of the boundary rigidity problem, respectively, in the linearized inverse kinematics problem for an anisotropic elastic medium [32].

In order to recover the entire tensor two types of additional data were proposed: either the transverse data as in [10, 19, 18], or the longitudinal ray data as in [32]. This work concerns the additional longitudinal data: In [32] it is shown that the entire field is uniquely determined from the combined k^{th}-moment ray transform for $0 \leq k \leq m$; for brevity we call it the momenta ray transform. Inversion of the momenta ray transform has been the subject of recent research interests: In the Euclidean setting some inversion formulas were given in [7, 8], with reconstruction for the $m = 1$ case in [2, 18, 13], and the recent sharp stability estimates in [9]. In the non-Euclidean setting the unique determination result was shown for simple real analytic Riemannian manifolds in [1], extended to simple Riemannian surface in [17], with inversion for $m = 1$ with sources on a curve in [20], and stability estimates in [33, 16].

Different from the works above, in here we present a reconstruction method, which recovers $\mathbf{f}$ from momenta ray transform in fan-beam coordinates. Specific to the Euclidean plane, our method is based on Bukhgeim's theory of A-analyticity [6], developed in [27, 30, 29] and extended here to a system of inhomogeneous Bukhgeim-Beltrami equations. In addition, we also consider the case of the attenuated momenta ray transform. It should be mentioned that the attenuated Doppler transform in the Euclidean plane is known to determine the entire field in subdomains where the attenuation is positive [4, 15, 36].

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For $m \geq 1$ fixed integer, let $\mathbf{f} = (f_{i_1 i_2 \dots i_m})$ be a real valued symmetric m-tensor supported in a disc of known radius. Without loss of generality, we assume that $\mathbf{f}$ is supported in the unit disc $\overline{\Omega}$. Furthermore, for $s \geq 1$ an integer, we assume that $\mathbf{f}$ has components in the Sobolev space of functions with square integrable derivatives which vanish at the boundary up to order s. We denote the space of such tensors by $H^s_0(\mathbf{S}^m;\Omega) = \{\mathbf{f} = (f_{i_1\cdots i_m}) \in \mathbf{S}^m(\Omega) : f_{i_1\cdots i_m} \in H^s_0(\Omega)\}$. The symmetry refers to $f_{i_1 i_2 \dots i_m}$ being invariant under any transposition of the indexes $i_1, ..., i_m \in \{1, 2\}$. With the summation convention understood over repeated indexes, for $(x, \boldsymbol{\theta}) \in \mathbb{R}^2 \times \mathbb{S}^1$ we denote by $\langle \mathbf{f}(x), \boldsymbol{\theta}^m \rangle = f_{i_1\cdots i_m}(x)\theta^{i_1} \cdot \theta^{i_2} \cdots \theta^{i_m}$ the action of the tensor on $\underbrace{\boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \cdots \otimes \boldsymbol{\theta}}_{m}$. Both the tensor $\mathbf{f}$ and the attenuation a are extended by zero outside Ω.

As in [9] (defined for $a = 0$), in here we work with the k^{th}-moment attenuated ray transform

$$I^k_a \mathbf{f}(x, \boldsymbol{\theta}) := \int_{-\infty}^{\infty} t^k e^{-\int_t^{\infty} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} \langle \mathbf{f}(\Pi_{\boldsymbol{\theta}}(x) + t\boldsymbol{\theta}), \boldsymbol{\theta}^m \rangle dt, \quad 0 \leq k \leq m, \tag{1.1}$$

where $\Pi_{\boldsymbol{\theta}}(x) = x - (x \cdot \boldsymbol{\theta})\boldsymbol{\theta}$ is the projection of x onto $\boldsymbol{\theta}^{\perp}$.

Since $\mathbf{f}$ vanishes on the lines outside Ω, we restrict $I^k_a\mathbf{f}$ to the lines intersecting $\overline{\Omega}$. These are parametrized by points on the boundary and arbitrary directions. In other words, $I^k_a\mathbf{f}$ is a function on the torus $\varGamma \times \mathbb{S}^1$, where $\varGamma$ is the boundary of Ω. While $\varGamma$ is also the unit circle, we keep the notation to differentiate from the set of directions.

Functions u on $\Omega \times \mathbb{S}^1$ are characterized by the sequence valued map of their Fourier coefficients $u_{-n}(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z, \boldsymbol{\theta})e^{in\theta}d\theta$ (non-positive indexes are sufficient) in the angular variable,

$$\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), \cdots \rangle.$$

We work in the weighted Sobolev spaces

$$l^{2,p}(\mathbb{N}; H^q(\Omega)) := \left\{ \mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ... \rangle : \|\mathbf{u}\|^2_{p,q} := \sum_{j=0}^{\infty} (1 + j)^{2p} \|u_{-j}\|^2_{H^q(\Omega)} < \infty \right\}. \tag{1.2}$$

The first index p refers to the smoothness in the angular variable, while the second index q shows the smoothness in the spatial variable. The traces $\mathbf{g} = \mathbf{u}|_{\varGamma}$ on $\varGamma$ of maps $\mathbf{u} \in l^{2,p}(\mathbb{N}; H^q(\Omega))$ are in $l^{2,p}(\mathbb{N}; H^{q-\frac{1}{2}}(\varGamma))$ endowed with the norm

$$\|\mathbf{g}\|^2_{p,q-\frac{1}{2}} := \sum_{j=0}^{\infty} (1 + j)^{2p} \|g_{-j}\|^2_{H^{q-\frac{1}{2}}(\varGamma)}. \tag{1.3}$$

Furthermore, since Γ is the unit circle, the $H^{q-\frac{1}{2}}(\varGamma)$-norm of g_{-j} are defined in the Fourier domain by

$$\|g_{-j}\|^2_{H^{q-\frac{1}{2}}(\varGamma)} = \sum_{k=-\infty}^{\infty} (1 + |k|)^{2q-1} |g_{-j,k}|^2, \tag{1.4}$$

where $g_{-j,k} = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} g_{-j}(e^{\mathrm{i}\beta})e^{-\mathrm{i}k\beta}d\beta$, for $k \in \mathbb{Z}$, $j \geq 0$. In particular, for $\mathbf{g} \in l^{2,p}(\mathbb{N}; H^{q-\frac{1}{2}}(\varGamma))$,

$$\|\mathbf{g}\|^2_{p,q-\frac{1}{2}} = \sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} (1 + j)^{2p}(1 + |n|)^{2q-1} |g_{-j,n}|^2. \tag{1.5}$$

For brevity we adopt throughout the notation $\|\mathbf{v}\| \lesssim \|\mathbf{w}\|$, whenever $\|\mathbf{v}\| \leq C \|\mathbf{w}\|$ for some constant $C > 0$ independent of $\mathbf{v}$ and $\mathbf{w}$. We prove the following.

Theorem 1.1. *Let Ω be the unit disc, $m \geq 1$ be an integer, and $a \in C^{m+1,\mu}(\overline{\Omega})$, $\mu > 1/2$. For some unknown real valued m-tensor $\mathbf{f} \in H_0^{m+1}(\mathbf{S}^m; \Omega)$, let $\mathbb{I}_a\mathbf{f} := \langle I_a^0\mathbf{f}, I_a^1\mathbf{f}, I_a^2\mathbf{f}, \cdots, I_a^m\mathbf{f}\rangle$ be its attenuated momenta ray transform as in* (1.1)*. Then $\mathbf{f}$ is reconstructed from $\mathbb{I}_a\mathbf{f}$ with the estimate*

$$\|\mathbf{f}\|^2_{L^2(\Omega)} \lesssim \left\|I_a^0\mathbf{f}\right\|^2_{m+\frac{1}{2},\frac{1}{2}} + \sum_{j=1}^{m} \left\|I_a^j\mathbf{f}\right\|^2_{m-j+\frac{3}{2},j+\frac{1}{2}}. \tag{1.6}$$

The change in parity in the order of the tensor yields nominal changes, however these occur at every step of the method of reconstruction. We present the case of even tensors in detail, while the case of odd tensors is merely summarized.

2. A priori estimates for the inhomogenous Bukhgeim-Beltrami equation

The stability estimate in Theorem 1.1 requires a-priori estimates for higher order derivatives of solution of the inhomogenous Bukhgeim-Beltrami equation

$$\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{w}. \tag{2.1}$$

where $\overline{\partial} = \frac{1}{2}(\partial_{x_1} + \mathrm{i}\partial_{x_2})$, $\partial = \frac{1}{2}(\partial_{x_1} - \mathrm{i}\partial_{x_2})$ are the Cauchy-Riemann operators, and $L\mathbf{v} = L(v_0, v_{-1}, v_{-2}, ...) := (v_{-1}, v_{-2}, ...)$ denotes the left translation. Recall the weighted Sobolev spaces in (1.2).

Theorem 2.1 below requires an estimate on the normal derivatives of the solution of (2.1).

Lemma 2.1. *Let $\mathbf{w} \in l^{2,p+\frac{1}{2}}(\mathbb{N}; H^q(\Omega))$, for some fixed $p \geq 1$ and $q \geq 1$. If $\mathbf{v} \in l^{2,p+\frac{1}{2}}(\mathbb{N}; H^{q+1}(\Omega))$ solves* (2.1)*, then*

$$\|\partial_\nu \mathbf{v}|_\Gamma\|^2_{p-\frac{1}{2},q-\frac{1}{2}} \lesssim \|\mathbf{v}|_\Gamma\|^2_{p+\frac{1}{2},q+\frac{1}{2}} + \|\mathbf{w}|_\Gamma\|^2_{p+\frac{1}{2},q-\frac{1}{2}}, \tag{2.2}$$

where ∂_ν is the normal derivative.

Proof. A change of coordinates at the boundary point $e^{\mathrm{i}\eta} \in \Gamma$ rewrites $\partial, \overline{\partial}$ in terms of the tangential and normal derivatives $\partial_\tau, \partial_\nu$ as $\partial = e^{-\mathrm{i}\eta}(\partial_\nu - \mathrm{i}\partial_\tau)/2$ respectively $\overline{\partial} = e^{\mathrm{i}\eta}(\partial_\nu + \mathrm{i}\partial_\tau)/2$.

We reason by induction in p. The case $p = 1$:
If $\mathbf{v} \in l^{2,1}(\mathbb{N}; H^{q+1}(\Omega))$ for some fixed $q \geq 1$, then $\partial_\nu \mathbf{v}|_\Gamma \in l^{2,1}(\mathbb{N}; H^{q-\frac{1}{2}}(\Gamma))$ and the restriction of (2.1) to the boundary,

$$[e^{2\mathrm{i}\eta} + L^2](\partial_\nu \mathbf{v}|_\Gamma) = -\mathrm{i}[e^{2\mathrm{i}\eta} - L^2](\partial_\tau \mathbf{v}|_\Gamma) + 2e^{\mathrm{i}\eta}\mathbf{w}|_\Gamma$$

holds in $H^{q-\frac{1}{2}}(\Gamma)$. With $\mu = \mathrm{i}e^{\mathrm{i}\eta}$ we get

$$[\mu^2 - L^2](\partial_\nu \mathbf{v}|_\Gamma) = -\mathrm{i}[\mu^2 + L^2](\partial_\tau \mathbf{v}|_\Gamma) + 2\mathrm{i}\mu\mathbf{w}|_\Gamma. \tag{2.3}$$

While the unit circle is in the spectrum of the left translation $L : l^2 \mapsto l^2$, the resolvent $(\lambda - L^2)^{-1}$ extends continuously from $|\lambda| > 1$ to $|\lambda| = 1$ as a bounded operator from $l^{2,1}$ to l^2; see Lemma A.2 in the appendix.

An application of Lemma A.2 to (2.3) estimates the normal derivative of solutions of (2.1) in terms of their tangential derivative,

$$\|\partial_\nu \mathbf{v}|_\Gamma\|_{H^{q-\frac{1}{2}}(\Gamma;l^2)} \lesssim \|\partial_\tau \mathbf{v}|_\Gamma\|_{H^{q-\frac{1}{2}}(\Gamma;l^{2,1})} + \|\mathbf{w}|_\Gamma\|_{H^{q-\frac{1}{2}}(\Gamma;l^{2,1})}. \tag{2.4}$$

In terms of the Sobolev norms on the unit circle Γ in (1.5), the above estimate rewrites

$$\|\partial_\nu \mathbf{v}|_\Gamma\|^2_{0,q-\frac{1}{2}} \lesssim \|\partial_\tau \mathbf{v}|_\Gamma\|^2_{1,q-\frac{1}{2}} + \|\mathbf{w}|_\Gamma\|^2_{1,q-\frac{1}{2}} \lesssim \|\mathbf{v}|_\Gamma\|^2_{1,q+\frac{1}{2}} + \|\mathbf{w}|_\Gamma\|^2_{1,q-\frac{1}{2}}. \tag{2.5}$$

Since $\mathbf{v}$ solves (2.1), for each $n \geq 0$, the left shifted sequence $L^n\mathbf{v}$ solves the shifted inhomogeneous Bukhgeim-Beltrami equation

$$\overline{\partial} L^n\mathbf{v} + L^2\partial L^n\mathbf{v} = L^n\mathbf{w}. \tag{2.6}$$

Thus, it satisfies the estimate (2.5) with $\mathbf{v}$ replaced by $L^n\mathbf{v}$, and $\mathbf{w}$ replaced by $L^n\mathbf{w}$. A summation in n yields

$$\sum_{n=0}^{\infty} \|\partial_\nu L^n\mathbf{v}|_\Gamma\|^2_{0,q-\frac{1}{2}} \lesssim \sum_{n=0}^{\infty} \|L^n\mathbf{v}|_\Gamma\|^2_{1,q+\frac{1}{2}} + \sum_{n=0}^{\infty} \|L^n\mathbf{w}|_\Gamma\|^2_{1,q-\frac{1}{2}} \tag{2.7}$$

provided the right-hand-side is finite.

By applying the Lemma A.1 (with $B = l^2(\mathbb{N}; H^q(\Omega))$) in (2.7) yields

$$\|\partial_\nu \mathbf{v}|_\Gamma\|^2_{\frac{1}{2},q-\frac{1}{2}} \lesssim \|\mathbf{v}|_\Gamma\|^2_{\frac{3}{2},q+\frac{1}{2}} + \|\mathbf{w}|_\Gamma\|^2_{\frac{3}{2},q-\frac{1}{2}}. \tag{2.8}$$

Since $\mathbf{v} \in l^{2,\frac{3}{2}}(\mathbb{N}; H^{q+1}(\Omega))$ for $p = 1$, the right hand of (2.8) is finite.

Assume next that (2.2) holds for p:

$$\|\partial_\nu \mathbf{v}|_\Gamma\|^2_{p-\frac{1}{2},q-\frac{1}{2}} \lesssim \|\mathbf{v}|_\Gamma\|^2_{p+\frac{1}{2},q+\frac{1}{2}} + \|\mathbf{w}|_\Gamma\|^2_{p+\frac{1}{2},q-\frac{1}{2}}. \tag{2.9}$$

By using the estimate (2.9), for the sequence $L^m\mathbf{v}$ replacing $\mathbf{v}$ and $L^m\mathbf{w}$ replacing $\mathbf{w}$, a summation over m, and another application of Lemma A.1 yields

$$\|\partial_\nu \mathbf{v}|_\Gamma\|^2_{p,q-\frac{1}{2}} \lesssim \|\mathbf{v}|_\Gamma\|^2_{p+1,q+\frac{1}{2}} + \|\mathbf{w}|_\Gamma\|^2_{p+1,q-\frac{1}{2}}. \tag{2.10}$$

Now repeat the process for the estimate (2.10) with $\mathbf{v}$ replaced by $L^m\mathbf{v}$, $\mathbf{w}$ replaced by $L^m\mathbf{w}$, sum over m, and apply Lemma A.1 to obtain

$$\|\partial_\nu \mathbf{v}|_\Gamma\|^2_{p+\frac{1}{2},q-\frac{1}{2}} \lesssim \|\mathbf{v}|_\Gamma\|^2_{p+\frac{3}{2},q+\frac{1}{2}} + \|\mathbf{w}|_\Gamma\|^2_{p+\frac{3}{2},q-\frac{1}{2}}. \tag{2.11}$$

Note that, by hypothesis, the right-hand-side is finite.

□

The following results establish the bootstrapping in regularity of the solutions of (2.1).

Theorem 2.1. *(a) Let* $\mathbf{w} \in l^{2,p+1}(\mathbb{N}; L^2(\Omega))$*, for some fixed* $p \geq 0$*. If* $\mathbf{v} \in l^{2,p+\frac{1}{2}}(\mathbb{N}; H^1(\Omega))$ *solves* (2.1)*, then*

$$\|\mathbf{v}\|^2_{p,1} \lesssim \|\mathbf{w}\|^2_{p+1,0} + \|\mathbf{v}|_\Gamma\|^2_{p+\frac{1}{2},\frac{1}{2}}. \tag{2.12}$$

(b) Moreover, if $\mathbf{w} \in l^{2,p+\frac{1}{2}}(\mathbb{N}; H^q(\Omega)) \cap l^{2,p+1}(\mathbb{N}; L^2(\Omega))$*, for some fixed* $p, q \geq 1$ *and* $\mathbf{v} \in l^{2,p+\frac{1}{2}}(\mathbb{N}; H^{q+1}(\Omega))$ *solves* (2.1)*, then*

$$\|\mathbf{v}\|^2_{p-1,q+1} \lesssim \|\mathbf{w}\|^2_{p,q} + \|\mathbf{w}|_\Gamma\|^2_{p+\frac{1}{2},q-\frac{1}{2}} + \|\mathbf{v}|_\Gamma\|^2_{p+\frac{1}{2},q+\frac{1}{2}}. \tag{2.13}$$

Proof of part (a). We reason by induction in p. The case $p = 0$,

$$\|\mathbf{v}\|^2_{0,1} \lesssim \|\mathbf{w}\|^2_{1,0} + \|\mathbf{v}|_\Gamma\|^2_{\frac{1}{2},\frac{1}{2}}, \tag{2.14}$$

is established in [12, Corollary 4.1]. Assume next that (2.12) holds for p:

$$\|\mathbf{v}\|^2_{p,1} \lesssim \|\mathbf{w}\|^2_{p+1,0} + \|\mathbf{v}|_\Gamma\|^2_{p+\frac{1}{2},\frac{1}{2}}. \tag{2.15}$$

Since $L^n\mathbf{v}$ solves (2.6), via the estimate (2.15), and a summation over n, we get

$$\sum_{n=0}^{\infty} \|L^n\mathbf{v}\|^2_{p,1} \lesssim \sum_{n=0}^{\infty} \|L^n\mathbf{w}\|^2_{p+1,0} + \sum_{n=0}^{\infty} \|L^n\mathbf{v}|_\Gamma\|^2_{p+\frac{1}{2},\frac{1}{2}}, \tag{2.16}$$

provided the right-hand-side is finite.

By applying Lemma A.1 with $B = l^2(\mathbb{N}; H^q(\Omega))$ for $q \in \{0, \frac{1}{2}, 1\}$ to (2.16), we obtain

$$\|\mathbf{v}\|^2_{p+\frac{1}{2},1} \lesssim \|\mathbf{w}\|^2_{p+\frac{3}{2},0} + \|\mathbf{v}|_\Gamma\|^2_{p+1,\frac{1}{2}}. \tag{2.17}$$

Now repeat the process for the estimate (2.17) with $\mathbf{v}$ replaced by $L^m\mathbf{v}$, $\mathbf{w}$ replaced by $L^m\mathbf{w}$, sum over m, and apply Lemma A.1 to conclude

$$\|\mathbf{v}\|^2_{p+1,1} \lesssim \|\mathbf{w}\|^2_{p+2,0} + \|\mathbf{v}|_\Gamma\|^2_{p+\frac{3}{2},\frac{1}{2}}. \tag{2.18}$$

Note that, by hypothesis, the right-hand-side is finite.

Proof of part (b): To prove the estimate (2.13), we differentiate (2.1) in the spatial variables:

$$\overline{\partial}(\nabla^q \mathbf{v}) + L^2\partial(\nabla^q\mathbf{v}) = \nabla^q\mathbf{w}, \quad q \geq 1, \tag{2.19}$$

where ∇ stands for either $\overline{\partial}$ or ∂. We apply the estimate (2.12) for $\nabla^q\mathbf{v}$ and p replaced by $p-1$:

$$\|\mathbf{v}\|^2_{p-1,q+1} \lesssim \|\mathbf{w}\|^2_{p,q} + \|\mathbf{v}|_\Gamma\|^2_{p-\frac{1}{2},q+\frac{1}{2}} + \|\partial_\nu\mathbf{v}|_\Gamma\|^2_{p-\frac{1}{2},q-\frac{1}{2}}. \tag{2.20}$$

An application of Lemma 2.1 to the last term in (2.20) establishes (2.13). □

3. Proof of Theorem 1.1 in the non-attenuated case for even m-tensors

The inversion of the momenta ray transform can be reduced to solving an inverse boundary value problem for a system of transport equations as follows. Let $\Gamma_\pm := \{(x, \boldsymbol{\theta}) \in \partial\Omega \times \mathbb{S}^1 : \pm\nu(x)\cdot\boldsymbol{\theta} > 0\}$ be the incoming (-), respectively outgoing (+), unit tangent sub-bundles of the boundary; where $\nu(x)$ is the outer unit normal at $x \in \partial\Omega$.

The following result holds for both m even and odd, and where the attenuation a is not necessarily zero.

Proposition 3.1. *Let* $\mathbf{f} \in H^s_0(\mathbf{S}^m; \Omega)$, $s, m \geq 1$, *and* $a \in C^{s,\mu}(\overline{\Omega})$, $\mu > 1/2$. *The system*

$$\boldsymbol{\theta}\cdot\nabla u^0(z,\boldsymbol{\theta}) + a(z)u^0(z,\boldsymbol{\theta}) = \langle \mathbf{f}(z), \boldsymbol{\theta}^m\rangle, \quad \textit{for } (z,\boldsymbol{\theta}) \in \overline{\Omega}\times\mathbb{S}^1, \tag{3.1a}$$

$$\boldsymbol{\theta}\cdot\nabla u^k(z,\boldsymbol{\theta}) + a(z)u^k(z,\boldsymbol{\theta}) = u^{k-1}(z,\boldsymbol{\theta}), \quad \textit{for } 1 \leq k \leq m, \tag{3.1b}$$

subject to the boundary conditions

$$u^k|_{\Gamma_-} = 0, \quad 0 \leq k \leq m, \tag{3.1c}$$

has a unique solution $u^k \in H^s(\Omega\times\mathbb{S}^1)$, $s \geq 1$. *In particular* $u^k|_{\Gamma\times\mathbb{S}^1} \in H^s(\mathbb{S}^1; H^{s-\frac{1}{2}}(\Gamma))$.

Moreover, $\langle u^0|_{\Gamma_+}, u^1|_{\Gamma_+}, \cdots, u^m|_{\Gamma_+}\rangle$ *are in a one-to-one correspondence with the attenuated momenta ray transform* $\langle I^0_a\mathbf{f}, I^1_a\mathbf{f}, \cdots, I^m_a\mathbf{f}\rangle$ *in* (1.1) *via the relations*

$$\begin{aligned} &u^0|_{\Gamma_+}(x,\boldsymbol{\theta}) = I^0_a\mathbf{f}(x,\boldsymbol{\theta}),\\ &u^k|_{\Gamma_+}(x,\boldsymbol{\theta}) = \sum_{n=1}^{k}(-1)^{n-1}\frac{(x\cdot\boldsymbol{\theta})^n}{n!}u^{k-n}|_{\Gamma_+}(x,\boldsymbol{\theta}) + \frac{(-1)^k}{k!}I^k_a\mathbf{f}(x,\boldsymbol{\theta}), \quad \textit{for } 1\leq k\leq m. \end{aligned} \tag{3.2}$$

The elementary proof of the Proposition above is in the Appendix B.

For specificity, we refer to the solution $u^k(z,\boldsymbol{\theta})$ of (3.1) as the *k-level flux*.

For the remaining of this section we assume m even, and the attenuation $a \equiv 0$.

It is easy to see (e.g., in [29, Lemma A.1]) that

$$\langle \mathbf{f}, \boldsymbol{\theta}^m\rangle = f_0 + \sum_{k=1}^{m/2}\left(f_{2k}e^{-\mathrm{i}(2k)\theta} + f_{-2k}e^{\mathrm{i}(2k)\theta}\right), \tag{3.3}$$

for some functions $\{f_{2k}:\ -m/2 \leq k \leq m/2\}$ in an explicit one-to-one correspondence (linear combination) with $\{f_{\underbrace{1\cdots 1}_{m-k}\underbrace{2\cdots 2}_{k}}:\ 0 \leq k \leq m\}$. By symmetry, the latter coincide with $f_{i_1\cdots i_m}$ for all multi-indexes $(i_1, \cdots, i_m) \in \{1,2\}^m$ in which 2 occurs exactly k times.

To stress the non-attenuated case, we denote by v^k the k-level flux solution of the boundary value problem (3.1) where $a \equiv 0$. The solution v^k for $0 \leq k \leq m$ is given by Proposition 3.1, and let g^k be its trace on $\varGamma \times \mathbb{S}^1$,

$$g^k = \begin{cases} v^k|_{\varGamma_+} & \text{on } \varGamma_+, \\ 0 & \text{on } \varGamma_-. \end{cases} \tag{3.4}$$

We use the Fourier approach to the transport problem and work with the sequence of the (non-positive) Fourier coefficients of the k-level flux $v^k(z,\cdot)$,

$$v_n^k(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v_n^k(z,\boldsymbol{\theta})e^{-\mathrm{i}n\theta}d\theta.$$

The upper index k denotes the level of the flux, while the lower index n is the Fourier coefficient in the angular variable.

For $\theta = \arg\boldsymbol{\theta} \in (-\pi,\pi]$, the advection operator in polar coordinates becomes $\boldsymbol{\theta}\cdot\nabla = e^{-\mathrm{i}\theta}\overline{\partial} + e^{\mathrm{i}\theta}\partial$. By identifying the Fourier coefficients in (3.1) and by using (3.3), the modes solve

$$\overline{\partial} v^0_{-(2n-1)}(z) + \partial v^0_{-(2n+1)}(z) = f_{2n}(z), \qquad 0 \leq n \leq m/2, \tag{3.5}$$

$$\overline{\partial} v^0_{-(2n-1)}(z) + \partial v^0_{-(2n+1)}(z) = 0, \qquad n \geq m/2+1, \tag{3.6}$$

$$\overline{\partial} v^0_{-2n}(z) + \partial v^0_{-(2n+2)}(z) = 0, \qquad n \geq 0, \tag{3.7}$$

$$\overline{\partial} v^k_{-n}(z) + \partial v^k_{-n-2}(z) = v^{k-1}_{-n-1}(z), \qquad n \in \mathbb{Z},\ 1 \leq k \leq m, \tag{3.8}$$

and

$$v^k_{-n}|_{\varGamma} = g^k_{-n}, \qquad 0 \leq k \leq m. \tag{3.9}$$

The existence of the solution to the boundary value problem (3.5) - (3.9) is postulated by the forward problem. Moreover, since $\mathbf{f} \in H_0^{m+1}(\mathbf{S}^m;\Omega)$, the k-level solution $v^k \in H^{m+1}(\mathbb{S}^1; H^{m+1}(\Omega))$.

For $0 \leq k \leq m$, let $\mathbf{v}^k$ be the sequence valued map of the Fourier coefficients of the solution v^k and $\mathbf{g}^k$ be its corresponding trace on the boundary:

$$\mathbf{v}^k(z) = \langle v_0^k(z), v_{-1}^k(z), v_{-2}^k(z), v_{-3}^k(z), \cdots\rangle, \quad z \in \Omega, \tag{3.10}$$

$$\mathbf{g}^k = \langle g_0^k, g_{-1}^k, g_{-2}^k, g_{-3}^k, \cdots\rangle := \mathbf{v}^k|_{\varGamma}. \tag{3.11}$$

Since $v^k \in H^{m+1}(\mathbb{S}^1; H^{m+1}(\Omega))$, $\mathbf{v}^k \in l^{2,m+1}(\mathbb{N}; H^{m+1}(\Omega))$ and $\mathbf{g}^k \in l^{2,m+1}(\mathbb{N}; H^{m+\frac{1}{2}}(\varGamma))$.

In the sequence valued map notation the boundary value problem (3.5) - (3.9), becomes

$$\overline{\partial}\overline{v^0_{-1}} + \partial v^0_{-1} = f_0, \tag{3.12a}$$

$$\overline{\partial}\mathbf{v}^0 + L^2\partial\mathbf{v}^0 = L\mathbf{F}, \tag{3.12b}$$

$$\overline{\partial}\mathbf{v}^k + L^2\partial\mathbf{v}^k = L\mathbf{v}^{k-1}, \quad 1 \leq k \leq m, \tag{3.12c}$$

subject to

$$\mathbf{v}^k|_{\varGamma} = \mathbf{g}^k, \quad \text{for } 0 \leq k \leq m, \tag{3.12d}$$

where

$$\mathbf{F} := \langle f_0, 0, f_2, 0, f_4, 0, \cdots, f_{m-2}, 0, f_m, 0, 0, \cdots \rangle \tag{3.13}$$

is the sequence valued map of the Fourier modes f_{2n} in (3.3).

It is crucial to note that $L^{m+1}\mathbf{F} = \mathbf{0} = \langle 0, 0, ... \rangle$, so that for $0 \leq k \leq m$, $L^{m-k}\mathbf{v}^k$ solve the following boundary value problem for the elliptic system:

$$\overline{\partial}[L^m\mathbf{v}^0] + L^2\partial[L^m\mathbf{v}^0] = \mathbf{0}, \tag{3.14a}$$

$$\overline{\partial}[L^{m-k}\mathbf{v}^k] + L^2\partial[L^{m-k}\mathbf{v}^k] = L^{m+1}\mathbf{v}^{k-1}, \quad 1 \leq k \leq m, \tag{3.14b}$$

subject to

$$L^{m-k}\mathbf{v}^k|_{\Gamma} = L^{m-k}\mathbf{g}^k, \quad 0 \leq k \leq m. \tag{3.14c}$$

Note that (3.14) does not involve the source $\mathbf{F}$ (encoding the tensor).

The solution $\mathbf{v}^k$ of (3.12c) and (3.12d) is given by an explicit Pompeiu-like formula for the $(\overline{\partial} + L^2\partial)$ operator (see equation (C.10) and its derivation in the Appendix C):

$$\mathbf{v}^k(z) = (\mathcal{B}\mathbf{g}^k)(z) + (\mathcal{T}L\mathbf{v}^{k-1})(z),\ z \in \Omega, \tag{3.15}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator in (C.3), and $\mathcal{T}$ is the Pompeiu-like operator in (C.11). Moreover, by Theorem 2.1 (a),

$$\left\|\mathbf{v}^k\right\|_{0,1}^2 \lesssim \left\|L\mathbf{v}^{k-1}\right\|_{1,0}^2 + \left\|\mathbf{v}^k|_{\Gamma}\right\|_{\frac{1}{2},\frac{1}{2}}^2 \quad \text{for } 1 \leq k \leq m. \tag{3.16}$$

Proposition 3.2. *Let* $(\mathbf{g}^0, \mathbf{g}^1, \cdots, \mathbf{g}^m)$ *be the data as in* (3.11) *obtained for some unknown even order* m*-tensor* $\mathbf{f} \in H_0^{m+1}(\mathbf{S}^m; \Omega)$*, and* $\mathbf{F}$ *be its corresponding unknown sequence as in* (3.13)*. Then the boundary value problem* (3.12) *has a unique solution* $\mathbf{v}^k$ *satisfying*

$$L^{m-k}\mathbf{v}^k(z) = \sum_{j=0}^{k} \mathcal{T}^j L^{m-k+j}[\mathcal{B}\mathbf{g}^{k-j}](z), \quad z \in \Omega,\ 0 \leq k \leq m. \tag{3.17}$$

Moreover,

$$\begin{aligned} &\left\|L^m\mathbf{v}^0\right\|_{m,1}^2 \lesssim \left\|L^m\mathbf{g}^0\right\|_{m+\frac{1}{2},\frac{1}{2}}^2, \\ &\left\|L^{m-k}\mathbf{v}^k\right\|_{m-k,k+1}^2 \lesssim \left\|L^m\mathbf{g}^0\right\|_{m+\frac{1}{2},\frac{1}{2}}^2 + \sum_{j=1}^{k} \left\|L^{m-j}\mathbf{g}^j\right\|_{m-j+\frac{3}{2},j+\frac{1}{2}}^2, \quad 1 \leq k \leq m. \end{aligned} \tag{3.18}$$

Proof. Since $v^k,\ 0 \leq k \leq m$ solves the forward problem (3.1) (with $a \equiv 0$) for some $\mathbf{f}$, existence of solution of the boundary value problem (3.12) is postulated. More precisely, $\mathbf{v}^k$ is the solution of (3.12) with $\mathbf{F}$ in (3.13).

We show the formula (3.17) with estimate (3.18) by induction in k, for $0 \leq k \leq m$.

By applying L^m to (3.12b):

$$\overline{\partial}[L^m\mathbf{v}^0] + L^2\partial[L^m\mathbf{v}^0] = \mathbf{0}, \tag{3.19a}$$

subject to

$$L^m\mathbf{v}^0|_{\Gamma} = L^m\mathbf{g}^0. \tag{3.19b}$$

Since $L^m\mathbf{v}^0$ is L^2-analytic, the Bukhgeim-Cauchy Integral formula (C.3) determines the sequence $L^m\mathbf{v}^0$ inside Ω from its boundary values:

$$L^m\mathbf{v}^0(z) := \mathcal{B}L^m\mathbf{g}^0(z), \quad z \in \Omega. \tag{3.20}$$

Applying Theorem 2.1 (a) to the boundary value problem (3.19) and using estimate (2.12) yields

$$\left\|L^m \mathbf{v}^0\right\|_{m,1}^2 \lesssim \left\|L^m \mathbf{g}^0\right\|_{m+\frac{1}{2},\frac{1}{2}}^2 . \tag{3.21}$$

Thus showing the $k=0$ case.

Next, we assume (3.17) and (3.18) holds for k:

$$L^{m-k}\mathbf{v}^k(z) = \sum_{j=0}^{k} \mathcal{T}^j L^{m-k+j}[\mathcal{B}\mathbf{g}^{k-j}](z), \quad z\in\Omega, \tag{3.22}$$

satisfying

$$\left\|L^{m-k}\mathbf{v}^k\right\|_{m-k,k+1}^2 \lesssim \left\|L^m \mathbf{g}^0\right\|_{m+\frac{1}{2},\frac{1}{2}}^2 + \sum_{j=1}^{k}\left\|L^{m-j}\mathbf{g}^j\right\|_{m-j+\frac{3}{2},j+\frac{1}{2}}^2 , \tag{3.23}$$

and prove it for $k+1$.

The equation (3.12c) for $k+1$ yields

$$\overline{\partial}[L^{m-(k+1)}\mathbf{v}^{k+1}] + L^2\partial[L^{m-(k+1)}\mathbf{v}^{k+1}] = L^{m-k}\mathbf{v}^k, \tag{3.24a}$$

subject to

$$L^{m-(k+1)}\mathbf{v}^{k+1}|_{\varGamma} = L^{m-(k+1)}\mathbf{g}^{k+1}. \tag{3.24b}$$

The solution $L^{m-(k+1)}\mathbf{v}^{k+1}$ of (3.24) is given by the Bukhgeim-Pompeiu formula (3.15):

$$L^{m-(k+1)}\mathbf{v}^{k+1} = \mathcal{B}L^{m-(k+1)}\mathbf{g}^{k+1} + \mathcal{T}(L^{m-k}\mathbf{v}^k).$$

Following directly from their definitions, the operators L and $\mathcal{B}$ commute, and the operators L and $\mathcal{T}$ commute. In particular,

$$(\mathcal{T}L)^j = \mathcal{T}^j L^j, \quad \text{and} \quad L^j\mathcal{B} = \mathcal{B}L^j, \quad j\geq 0. \tag{3.25}$$

Using the commutating properties (3.25) and the induction hypothesis (3.22) yields

$$\begin{aligned} L^{m-(k+1)}\mathbf{v}^{k+1} &= L^{m-(k+1)}\left[\mathcal{B}\mathbf{g}^{k+1}\right] + \mathcal{T}(L^{m-k}\mathbf{v}^k) \\ &= L^{m-(k+1)}\left[\mathcal{B}\mathbf{g}^{k+1}\right] + \sum_{j=0}^{k}\mathcal{T}^{j+1}L^{m-k+j}[\mathcal{B}\mathbf{g}^{k-j}] \\ &= \sum_{j=0}^{k+1}\mathcal{T}^j L^{m-(k+1)+j}[\mathcal{B}\mathbf{g}^{k+1-j}]. \end{aligned}$$

Moreover, by applying Theorem 2.1 (b) to the boundary value problem (3.24) and using the estimate (2.13) yields

$$\begin{aligned} \left\|L^{m-1-k}\mathbf{v}^{k+1}\right\|_{m-1-k,k+2}^2 &\lesssim \left\|L^{m-k}\mathbf{v}^k\right\|_{m-k,k+1}^2 + \left\|L^{m-k}\mathbf{g}^k\right\|_{m-k+\frac{1}{2},k+\frac{1}{2}}^2 \\ &\quad + \left\|L^{m-(k+1)}\mathbf{g}^{k+1}\right\|_{m-(k+1)+\frac{3}{2},(k+1)+\frac{3}{2}}^2 \\ &\lesssim \left\|L^m\mathbf{g}^0\right\|_{m+\frac{1}{2},\frac{1}{2}}^2 + \sum_{j=1}^{k+1}\left\|L^{m-j}\mathbf{g}^j\right\|_{m-j+\frac{3}{2},j+\frac{1}{2}}^2 , \end{aligned}$$

where the last inequality uses the induction hypothesis (3.23).

$\square$

3.1. **The reconstruction method.** We recover $\mathbf{v}^k, 0 \leq k \leq m$ in two steps, see Figure 1.

- Step I (Sweep down):

  Level by level, starting from $k = 0$ to $k = m$, we recover $L^{m-k}\mathbf{v}^k$ by solving the boundary value problem

$$\overline{\partial}\left(L^{m-k}\mathbf{v}^k\right) + L^2\partial\left(L^{m-k}\mathbf{v}^k\right) = L^{m+1-k}\mathbf{v}^{k-1}, \quad 1 \leq k \leq m,$$

  subject to

$$L^{m-k}\mathbf{v}^k|_{\Gamma} = L^{m-k}\mathbf{g}^k, \quad \text{for } 1 \leq k \leq m.$$

  Proposition 3.2 ensures that the unique solution $L^{m-k}\mathbf{v}^k$ given by (3.17) satisfies (3.18). In particular, when $k = m$, the sequence $\mathbf{v}^m$ is recovered from the data $\mathbf{g}^j$, $0 \leq j \leq m$,

$$\mathbf{v}^m(z) = \sum_{j=0}^{m} [\mathcal{T}L]^j \left(\mathcal{B}\mathbf{g}^{m-j}\right)(z), \quad z \in \Omega, \tag{3.26}$$

  with the estimate

$$\|\mathbf{v}^m\|^2_{0,m+1} \lesssim \left\|L^m\mathbf{g}^0\right\|^2_{m+\frac{1}{2},\frac{1}{2}} + \sum_{j=1}^{m}\left\|L^{m-j}\mathbf{g}^j\right\|^2_{m-j+\frac{3}{2},j+\frac{1}{2}}. \tag{3.27}$$

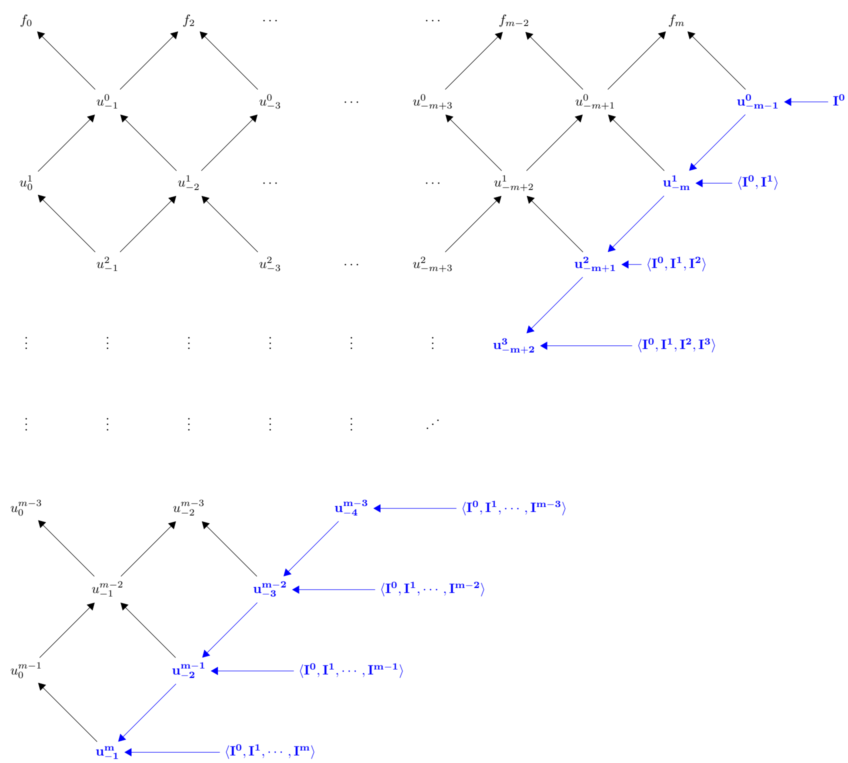

FIGURE 1. Flow of the reconstruction of even order tensors. In the sweep down, all the modes colored in blue are determined layer by layer from the momenta ray data and the previous layer. In the sweep up, also layer by layer starting from the bottom, the remaining coefficients are recovered. The arrows indicate which modes determine what.

- Step II (Sweep up):
  Level by level, starting from $k = m$ to $k = 1$, use (3.12c) in its component-wise form

$$v^{k-1}_{-n-1} := \overline{\partial} v^k_{-n} + \partial v^k_{-n-2}, \quad n \geq 0, \tag{3.28}$$

to recover $L\mathbf{v}^{k-1}$ from the knowledge of $\mathbf{v}^k$.
Moreover, repeated differentiation of (3.28) yields

$$\nabla^q \left(L\mathbf{v}^{k-1}\right) = \overline{\partial}[\nabla^q \mathbf{v}^k] + L^2\partial[\nabla^q \mathbf{v}^k], \quad q \geq 1,$$

with the estimate

$$\left\|L\mathbf{v}^{k-1}\right\|^2_{0,q} \lesssim \left\|\mathbf{v}^k\right\|^2_{0,q+1}, \tag{3.29}$$

where ∇ stands for either $\overline{\partial}$ or ∂.
We use (3.28) recursively to recover the entire sequences $\mathbf{v}^{m-1}, \cdots, \mathbf{v}^1, \mathbf{v}^0$ with estimate

$$\left\|L\mathbf{v}^0\right\|^2_{0,1} \lesssim \|\mathbf{v}^m\|^2_{0,m+1}. \tag{3.30}$$

With $\mathbf{v}^0$ known, we recover $\mathbf{F}$ via (3.12b) and (3.12a),

$$f_0 := 2\,\mathbb{R}\mathrm{e}\left[\partial v^0_{-1}\right], \quad \text{and} \quad L\mathbf{F} := \overline{\partial}\mathbf{v}^0 + L^2\partial\mathbf{v}^0. \tag{3.31}$$

Moreover, using (3.30) and (3.27), we have the estimate

$$\|\mathbf{F}\|^2_{0,0} \lesssim \left\|\mathbf{v}^0\right\|^2_{0,1} \lesssim \|\mathbf{v}^m\|^2_{0,m+1} \lesssim \left\|L^m\mathbf{g}^0\right\|^2_{m+\frac{1}{2},\frac{1}{2}} + \sum_{j=1}^{m} \left\|L^{m-j}\mathbf{g}^j\right\|^2_{m-j+\frac{3}{2},j+\frac{1}{2}}. \tag{3.32}$$

Finally, the components $f_{i_1\cdots i_m} = f_{\underbrace{1\cdots 1}_{m-k}\underbrace{2\cdots 2}_{k}}$ are defined via the explicit one-to-one correspondence (linear combination) with $\{f_{2k} : \ -m/2 \leq k \leq m/2\}$.

Since $\|\mathbf{f}\|^2_{0,0} \lesssim \|\mathbf{F}\|^2_{0,0} \lesssim \|\mathbf{f}\|^2_{0,0}$, Theorem 1.1 is, thus, proven for m even and $a \equiv 0$.

4. The odd m-tensors in the non-attenuated case

The proof of Theorem 1.1 for odd m-order tensors follows similarly to the even case. We highlight below the nominal changes due to the change in parity.

For m odd, the action of the tensors becomes

$$\langle \mathbf{f}, \boldsymbol{\theta}^m \rangle = \sum_{k=0}^{(m-1)/2} \left(f_{2k+1} e^{-\mathrm{i}(2k+1)\theta} + f_{-(2k+1)} e^{\mathrm{i}(2k+1)\theta} \right), \tag{4.1}$$

where $\{f_{2k+1} : \ -(m-1)/2 \leq k \leq (m-1)/2\}$ are in an explicit one-to-one correspondence (linear combination) with $\{f_{\underbrace{1\cdots 1}_{m-k}\underbrace{2\cdots 2}_{k}} : \ 0 \leq k \leq m\}$, see [29, Lemma A.1]. Let

$$\mathbf{F} := \langle 0, f_1, 0, f_3, 0, f_4, 0, \cdots, f_{m-2}, 0, f_m, 0, 0, \cdots \rangle \tag{4.2}$$

be the sequence valued map of the positive Fourier modes f_{2n+1} in (4.1). Note the change in the definition of $\mathbf{F}$ above from the one in the even tensor case in (3.13).

Given by Proposition 3.1, let v^k be the k-level flux solution of the boundary value problem (3.1) (with $a \equiv 0$) for some unknown tensor $\mathbf{f}$, and let g^k be its trace on $\varGamma \times \mathbb{S}^1$. As before, let $\mathbf{v}^k = \langle v_0^k, v_{-1}^k, v_{-2}^k, v_{-3}^k, ...\rangle$ be the sequence valued map of the Fourier coefficients of v^k, and $\mathbf{g}^k = \langle g_0^k, g_{-1}^k, g_{-2}^k, g_{-3}^k, ...\rangle := \mathbf{v}^k|_\Gamma$ be its trace. Then $\mathbf{v}^k$ solves

$$\overline{\partial}\mathbf{v}^0 + L^2\partial\mathbf{v}^0 = \mathbf{F}, \tag{4.3a}$$

$$\overline{\partial}\mathbf{v}^k + L^2\partial\mathbf{v}^k = L\mathbf{v}^{k-1}, \quad 1 \le k \le m, \tag{4.3b}$$

subject to

$$\mathbf{v}^k|_\Gamma = \mathbf{g}^k, \quad \text{for } 0 \le k \le m \tag{4.3c}$$

where $\mathbf{F}$ is now given by (4.2). Proposition 3.2 holds verbatim for odd m.

Following the two step reconstruction method in Section 3.1, $\mathbf{f}$ is similarly recovered with the estimate (1.6).

5. The attenuated case

We detail the reduction for even order tensors while the odd order case follows similarly.

As in [27] we treat the attenuated case by the reduction to the non-attenuated case via the special integrating factor function introduced in [11]:

$$h(z,\boldsymbol{\theta}) := \int_0^\infty a(z+t\boldsymbol{\theta})dt - \frac{1}{2}\left(I - \mathrm{i}H\right) Ra(z\cdot\boldsymbol{\theta}^\perp, \boldsymbol{\theta}^\perp),$$

where $H\psi(s,\boldsymbol{\theta}) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\psi(t,\boldsymbol{\theta})}{s-t}dt$ is the Hilbert transform taken in the linear variable, and $Ra(s,\boldsymbol{\theta}^\perp) = \int_{-\infty}^{\infty} a\left(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}\right) dt$ is the Radon transform of a.

It is known that all the negative Fourier modes of h vanish [11, 22], yielding

$$e^{-h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \alpha_k(z) e^{\mathrm{i}k\theta}, \quad e^{h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \beta_k(z) e^{\mathrm{i}k\theta}, \quad (z,\boldsymbol{\theta}) \in \overline{\Omega}\times\mathbb{S}^1. \tag{5.1}$$

As in [12], we use the sequence valued maps

$$\overline{\Omega} \ni z \mapsto \boldsymbol{\alpha}(z) := \langle \alpha_0(z), \alpha_1(z), ..., \rangle, \quad \overline{\Omega} \ni z \mapsto \boldsymbol{\beta}(z) := \langle \beta_0(z), \beta_1(z), ..., \rangle \tag{5.2}$$

to define the convolution operators $e^{\pm G}$ acting on some $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ via

$$e^{-G}\mathbf{u} = \sum_{k=0}^{\infty} \alpha_k L^k \mathbf{u} = \boldsymbol{\alpha} * \mathbf{u} \quad \text{and} \quad e^{G}\mathbf{u} = \sum_{k=0}^{\infty} \beta_k L^k \mathbf{u} = \boldsymbol{\beta} * \mathbf{u}. \tag{5.3}$$

In particular, note that $e^{\pm G}$ commutes with L and has the following mapping property.

Proposition 5.1. [12, Proposition 2.1] *Let $a \in C^{1,\mu}(\overline{\Omega})$, $\mu > 1/2$. Then*

$$e^{\pm G} : l^{2,p}(\mathbb{N}; H^q(\Omega)) \to l^{2,p}(\mathbb{N}; H^q(\Omega)) \tag{5.4}$$

are bounded. In particular,

$$\left\| e^{-G}\mathbf{u} \right\|_{p,0} \le \|\boldsymbol{\alpha}\|_{l^{1,1}_\infty(\overline{\Omega})} \|\mathbf{u}\|_{p,0}\,, \tag{5.5}$$

$$\left\| e^{-G}\mathbf{u} \right\|_{p,1} \lesssim \left(\|\boldsymbol{\alpha}\|_{l^{1,1}_\infty(\overline{\Omega})} + \|\partial\boldsymbol{\alpha}\|_{l^{1,1}_\infty(\overline{\Omega})} \right) \|\mathbf{u}\|_{p,1}\,, \tag{5.6}$$

where $\|\boldsymbol{\alpha}\|_{l^{1,1}_\infty(\overline{\Omega})} := \sup_{z\in\overline{\Omega}} \sum_{j=0}^{\infty} (1+j)|\alpha_j(z)| < \infty.$

The same estimates work for $e^{G}\mathbf{u}$ with $\boldsymbol{\alpha}$ replaced by $\boldsymbol{\beta}$.

The following result is a slight generalization of [12, Lemma 2.1]; see the appendix for its proof.

Lemma 5.1. *Let $a \in C^{1,\mu}(\overline{\Omega})$, $\mu > 1/2$, and $e^{\pm G}$ be the operators in* (5.3).

(i) If $\mathbf{u} \in l^2(\mathbb{N}; H^1(\Omega))$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = \mathbf{w}$, then $\mathbf{v} = e^{-G}\mathbf{u} \in l^2(\mathbb{N}; H^1(\Omega))$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = e^{-G}\mathbf{w}$.

(ii) Conversely, if $\mathbf{v} \in l^2(\mathbb{N}; H^1(\Omega))$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = e^{-G}\mathbf{w}$, then $\mathbf{u} = e^{G}\mathbf{v} \in l^2(\mathbb{N}; H^1(\Omega))$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = \mathbf{w}$.

Given by Proposition 3.1 with m even, let u^k be the k-level flux solution of the boundary value problem (3.1), and let g^k be its trace on $\varGamma \times \mathbb{S}^1$. By identifying the same order modes, the Fourier coefficients $u_n^k(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n^k(z,\boldsymbol{\theta})e^{-in\theta}d\theta$, solve

$$\overline{\partial}u^0_{-(2n-1)}(z) + \partial u^0_{-(2n+1)}(z) + a(z)u^0_{-2n}(z) = f_{2n}(z), \qquad 0 \leq n \leq m/2, \tag{5.7}$$

$$\overline{\partial}u^0_{-(2n-1)}(z) + \partial u^0_{-(2n+1)}(z) + a(z)u^0_{-2n}(z) = 0, \qquad n \geq m/2+1, \tag{5.8}$$

$$\overline{\partial}u^0_{-2n}(z) + \partial u^0_{-(2n+2)}(z) + a(z)u^0_{-2n-1}(z) = 0, \qquad n \geq 0, \tag{5.9}$$

$$\overline{\partial}u^k_{-n}(z) + \partial u^k_{-n-2}(z) + a(z)u^k_{-n-1}(z) = u^{k-1}_{-n-1}(z), \qquad n \in \mathbb{Z},\ 1 \leq k \leq m, \tag{5.10}$$

and

$$u^k_{-n}|_{\varGamma} = g^k_{-n}, \qquad 0 \leq k \leq m. \tag{5.11}$$

The existence of the solution to the boundary value problem (5.7) - (5.11) is postulated by the forward problem. Moreover, since $\mathbf{f} \in H_0^{m+1}(\mathbf{S}^m;\Omega)$ and $a \in C^{m+1,\mu}(\overline{\Omega}), \mu > 1/2$, the k-level solution $u^k \in H^{m+1}(\mathbb{S}^1; H^{m+1}(\Omega))$.

For $0 \leq k \leq m$, let $\mathbf{u}^k$ be the sequence valued map of the Fourier coefficients of the solution u^k and $\mathbf{g}^k$ be its corresponding trace on the boundary:

$$\mathbf{u}^k(z) = \langle u_0^k(z), u_{-1}^k(z), u_{-2}^k(z), u_{-3}^k(z), \cdots \rangle, \quad z \in \Omega, \tag{5.12}$$

$$\mathbf{g}^k = \langle g_0^k, g_{-1}^k, g_{-2}^k, g_{-3}^k, \cdots \rangle := \mathbf{u}^k|_{\varGamma}. \tag{5.13}$$

Since $u^k \in H^{m+1}(\mathbb{S}^1; H^{m+1}(\Omega))$, $\mathbf{u}^k \in l^{2,m+1}(\mathbb{N}; H^{m+1}(\Omega))$ and $\mathbf{g}^k \in l^{2,m+1}(\mathbb{N}; H^{m+\frac{1}{2}}(\varGamma))$.

In the sequence valued map notation the boundary value problem (5.7) - (5.11), becomes

$$\overline{\partial \overline{u^0_{-1}}} + \partial u^0_{-1} + au^0_0 = f_0, \tag{5.14a}$$

$$\overline{\partial}\mathbf{u}^0 + L^2\partial\mathbf{u}^0 + aL\mathbf{u}^0 = L\mathbf{F}, \tag{5.14b}$$

$$\overline{\partial}\mathbf{u}^k + L^2\partial\mathbf{u}^k + aL\mathbf{u}^k = L\mathbf{u}^{k-1}, \quad 1 \leq k \leq m, \tag{5.14c}$$

subject to

$$\mathbf{g}^k = \mathbf{u}^k|_{\varGamma}, \quad \text{for } 0 \leq k \leq m, \tag{5.14d}$$

where $\mathbf{F}$ is as defined in (3.13).

For e^{-G} as in (5.3), let

$$\mathbf{v}^k := e^{-G}\mathbf{u}^k, \quad \text{for } 0 \leq k \leq m. \tag{5.15}$$

By Proposition 5.1, $\mathbf{v}^k \in l^{2,m+1}(\mathbb{N}; H^{m+1}(\Omega))$, $\mathbf{v}^k|_{\varGamma} \in l^{2,m+1}(\mathbb{N}; H^{m+1/2}(\varGamma))$, and by Lemma 5.1, $\mathbf{v}^k$ solves

$$\overline{\partial}\overline{v^0_{-1}} + \partial v^0_{-1} = \left(e^{-G}\mathbf{F}\right)_0, \tag{5.16a}$$

$$\overline{\partial}\mathbf{v}^0 + L^2\partial\mathbf{v}^0 = L[e^{-G}\mathbf{F}], \tag{5.16b}$$

$$\overline{\partial}\mathbf{v}^k + L^2\partial\mathbf{v}^k = L\mathbf{v}^{k-1}, \quad 1 \le k \le m, \tag{5.16c}$$

subject to

$$\mathbf{v}^k|_{\varGamma} = e^{-G}\mathbf{g}^k, \quad \text{for } 0 \le k \le m. \tag{5.16d}$$

Note that $L^{m+1}\mathbf{F} = \mathbf{0} = \langle 0, 0, \ldots\rangle$. Moreover, by the commutating property $[e^{\pm G}, L] = \mathbf{0}$, we also have $L^{m+1}[e^{-G}\mathbf{F}] = e^{-G}L^{m+1}\mathbf{F} = e^{-G}\mathbf{0} = \mathbf{0}$. Thus, for $0 \le k \le m$, $L^{m-k}\mathbf{v}^k$ solve the following boundary value problem for the elliptic system:

$$\begin{aligned}\overline{\partial}[L^m\mathbf{v}^0] + L^2\partial[L^m\mathbf{v}^0] &= \mathbf{0},\\ \overline{\partial}[L^{m-k}\mathbf{v}^k] + L^2\partial[L^{m-k}\mathbf{v}^k] &= L^{m+1}\mathbf{v}^{k-1}, \quad 1 \le k \le m,\end{aligned}$$

subject to

$$L^{m-k}\mathbf{v}^k|_{\varGamma} = e^{-G}L^{m-k}\mathbf{g}^k, \quad 0 \le k \le m.$$

Proposition 3.2 with $\mathbf{g}$ replaced by $e^{-G}\mathbf{g}$ therein yields:

Proposition 5.2. *Let* $(\mathbf{g}^0, \mathbf{g}^1, \cdots, \mathbf{g}^m)$ *be the data as in* (3.11) *obtained for some unknown even order* m*-tensor* $\mathbf{f} \in H_0^{m+1}(\mathbf{S}^m; \Omega)$*, and* $\mathbf{F}$ *be its corresponding unknown sequence as in* (3.13)*. Then the boundary value problem* (5.16) *has a unique solution* $\mathbf{v}^k$ *satisfying*

$$L^{m-k}\mathbf{v}^k(z) = \sum_{j=0}^{k} \mathcal{T}^j L^{m-k+j}[\mathcal{B}e^{-G}\mathbf{g}^{k-j}](z), \quad z \in \Omega,\ 0 \le k \le m, \tag{5.17}$$

where $\mathcal{B}$ *is the Bukhgeim-Cauchy operator in* (C.3)*, and* $\mathcal{T}$ *is the operator in* (C.11)*. Moreover, it has the estimates*

$$\begin{aligned}&\left\|L^m\mathbf{v}^0\right\|^2_{m,1} \lesssim \left\|e^{-G}L^m\mathbf{g}^0\right\|^2_{m+\frac{1}{2},\frac{1}{2}},\\ &\left\|L^{m-k}\mathbf{v}^k\right\|^2_{m-k,k+1} \lesssim \left\|e^{-G}L^m\mathbf{g}^0\right\|^2_{m+\frac{1}{2},\frac{1}{2}} + \sum_{j=1}^{k}\left\|e^{-G}L^{m-j}\mathbf{g}^j\right\|^2_{m-j+\frac{3}{2},j+\frac{1}{2}}, \quad 1 \le k \le m.\end{aligned} \tag{5.18}$$

The reconstruction method of the non-attenuated case recovers $e^{-G}\mathbf{F}$ via

$$\left(e^{-G}\mathbf{F}\right)_0 := 2\,\mathbb{R}\mathrm{e}\left[\partial v^0_{-1}\right] \quad \text{and} \quad L[e^{-G}\mathbf{F}] := \overline{\partial}\mathbf{v}^0 + L^2\partial\mathbf{v}^0,$$

with the estimate $\left\|e^{-G}\mathbf{F}\right\|^2_{0,0} \lesssim \left\|e^{-G}L^m\mathbf{g}^0\right\|^2_{m+\frac{1}{2},\frac{1}{2}} + \sum_{j=1}^{m}\left\|e^{-G}L^{m-j}\mathbf{g}^j\right\|^2_{m-j+\frac{3}{2},j+\frac{1}{2}}$.

Since $\mathbf{F} = e^G\left[e^{-G}\mathbf{F}\right]$, an application of Proposition 5.1 yields

$$\|\mathbf{F}\|^2_{0,0} \lesssim \|\boldsymbol{\beta}\|^2_{l^{1,1}_\infty(\overline{\Omega})}\|\boldsymbol{\alpha}\|^2_{l^{1,1}_\infty(\overline{\Omega})}\left(\left\|L^m\mathbf{g}^0\right\|^2_{m+\frac{1}{2},\frac{1}{2}} + \sum_{j=1}^{m}\left\|L^{m-j}\mathbf{g}^j\right\|^2_{m-j+\frac{3}{2},j+\frac{1}{2}}\right).$$

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APPENDIX A. A HIERACHY OF NORMS INDUCED BY THE LEFT TRANSLATION

The results in this section are from [13, Appendix A]. We repeat it here for reader's convenience.

For sequence valued maps with elements in a Banach space $(B, \|\cdot\|)$, we introduce here a hierarchy of norms compatible with the left translation operator.

Recall the notation $\|\mathbf{v}\| \lesssim \|\mathbf{w}\|$, whenever $\|\mathbf{v}\| \leq C\,\|\mathbf{w}\|$ for some constant $C > 0$ independent of $\mathbf{v}$ and $\mathbf{w}$. We also denote $\|\mathbf{v}\| \approx \|\mathbf{w}\|$ if $\|\mathbf{v}\| \lesssim \|\mathbf{w}\| \lesssim \|\mathbf{v}\|$.

We define inductively the spaces $l^{2,\frac{p}{2}}(\mathbb{N};B)$, for $p \geq 0$ integer as follows:
$l^{2,0}(\mathbb{N};B)$ is the space of sequences $\mathbf{u}$ with

$$\|\mathbf{u}\|_0 := \left(\sum_{j=0}^{\infty} \|u_j\|^2 \right)^{\frac{1}{2}} < \infty, \tag{A.1}$$

while for $p \geq 1$, we define

$$l^{2,\frac{p}{2}}(\mathbb{N};B) := \left\{ \mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ... \rangle : \|\mathbf{u}\|_{\frac{p}{2}} < \infty \right\}, \tag{A.2}$$

where

$$\|\mathbf{u}\|_{\frac{p}{2}} := \left(\sum_{n=0}^{\infty} \|L^n \mathbf{u}\|^2_{\frac{p-1}{2}} \right)^{\frac{1}{2}}. \tag{A.3}$$

The following result shows the equivalence of the norm in (A.3) with the weighted l^2-norms.

Lemma A.1. *Let $(B, \|\cdot\|)$ be a Banach space, and $\mathbf{u} \in l^{2,\frac{p}{2}}(\mathbb{N};B)$ as in (A.2), for some $p \geq 0$ integer. Then*

$$\|\mathbf{u}\|^2_{\frac{p}{2}} = \sum_{j=0}^{\infty} C_p^{j+p} \;\|u_j\|^2 \approx \sum_{j=0}^{\infty} (1+j)^p \;\|u_j\|^2, \tag{A.4}$$

where $C_p^{j+p} = \frac{(j+p)!}{j!p!}$.

Proof. We first show the equality in (A.4) by induction in p.

The case $p = 0$ holds by definition (A.1).

Assume next that the equality in (A.4) holds for some fixed p:

$$\|\mathbf{u}\|^2_{\frac{p}{2}} = \sum_{j=0}^{\infty} C_p^{j+p} \;\|u_j\|^2. \tag{A.5}$$

By definition (A.3),

$$\|\mathbf{u}\|^2_{\frac{p+1}{2}} = \sum_{n=0}^{\infty} \|L^n \mathbf{u}\|^2_{\frac{p}{2}} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_p^{m+p} \;\|u_{m+n}\|^2. \tag{A.6}$$

By changing the index $j = m+n$, for $m \geq 0$, ($j - n \geq 0$, and $n \leq j$) we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} C_p^{m+p} \ \|u_{m+n}\|^2 = \sum_{j=0}^{\infty}\sum_{n=0}^{j} C_p^{j-n+p} \ \|u_j\|^2 = \sum_{j=0}^{\infty}\|u_j\|^2 \sum_{n=0}^{j} C_p^{j-n+p}. \tag{A.7}$$

By using Pascal's recurrence and the telescopic cancellations, we have

$$\sum_{n=0}^{j} C_p^{j-n+p} = C_{p+1}^{j+p+1}.$$

Thus, using (A.7), $\|\mathbf{u}\|^2_{\frac{p+1}{2}} = \sum_{j=0}^{\infty} C_{p+1}^{j+p+1}\|u_j\|^2$.

The equivalence of the norm in (A.4) follows from the inequalities

$$\frac{1}{p!}(1+j)^p \leq C_p^{j+p} \leq (1+j)^p.$$

□

The following result recalls the extension of the resolvent of the left translation operator L from outside the unit disc to the unit circle.

Lemma A.2. *Let $\mathbf{a} \in l^2$, $\mathbf{c} \in l^{2,1}$ be sequences, L be the left translation operator, and $\lambda \in \mathbb{C}$ with $|\lambda| \geq 1$. If $\mathbf{a}, \mathbf{c}$ satisfy $(\lambda - L)\mathbf{a} = \mathbf{c}$, then there exists an $M > 0$ independent of λ such that*

$$\|\mathbf{a}\|_{l^2} \leq M\|\mathbf{c}\|_{l^{2,1}}.$$

We refer to [35, Lemma 3.1.1, Step 1] for its proof.

Appendix B. Elementary results

To improve the readability, we moved the proof of the more elementary claims to this section. The presentation follows the order of their occurrence.

Proof of Proposition 3.1. From (3.1a) and (3.1b), we note that for $(x,\boldsymbol{\theta}) \in \Omega\times\mathbb{S}^1$ and $1 \leq k \leq m$,

$$\frac{d}{dt}\left[e^{-\int_t^\infty a(x+s\boldsymbol{\theta})ds}u^0(x+t\boldsymbol{\theta},\boldsymbol{\theta})\right] = e^{-\int_t^\infty a(x+s\boldsymbol{\theta})ds}\langle \mathbf{f}(x+t\boldsymbol{\theta}),\boldsymbol{\theta}^m\rangle, \tag{B.1}$$

$$\frac{d}{dt}\left[e^{-\int_t^\infty a(x+s\boldsymbol{\theta})ds}u^k(x+t\boldsymbol{\theta},\boldsymbol{\theta})\right] = e^{-\int_t^\infty a(x+s\boldsymbol{\theta})ds}u^{k-1}(x+t\boldsymbol{\theta},\boldsymbol{\theta}). \tag{B.2}$$

For $(x,\boldsymbol{\theta}) \in \Omega\times\mathbb{S}^1$ an integration along the line through x in the direction of $\boldsymbol{\theta}$ in (3.1a) together with the zero incoming condition (3.1c) yield

$$\begin{aligned} e^{-\int_{x\cdot\boldsymbol{\theta}}^\infty a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds}u^0(x,\boldsymbol{\theta}) &= \int_{-\infty}^{x\cdot\boldsymbol{\theta}} \frac{d}{dt}\left[e^{-\int_t^\infty a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds}u^0(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta},\boldsymbol{\theta})\right]dt \\ &= \int_{-\infty}^{x\cdot\boldsymbol{\theta}} e^{-\int_t^\infty a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds}\langle \mathbf{f}(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta}),\boldsymbol{\theta}^m\rangle dt. \end{aligned} \tag{B.3}$$

Note that $\int_{x\cdot\boldsymbol{\theta}}^\infty a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds = \int_0^\infty a(x+s\boldsymbol{\theta})ds.$

Similarly, for each $1 \leq k \leq m$ a recursive integration by parts in (3.1b) together with (3.1c) yield

$$
\begin{aligned}
e^{-\int_{x\cdot\boldsymbol{\theta}}^{\infty} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} u^k(x,\boldsymbol{\theta}) &= \int_{-\infty}^{x\cdot\boldsymbol{\theta}} \frac{d}{dt}\left[e^{-\int_t^{\infty} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} u^k(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta},\boldsymbol{\theta})\right] dt \\
&= \int_{-\infty}^{x\cdot\boldsymbol{\theta}} e^{-\int_t^{\infty} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} u^{k-1}(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta},\boldsymbol{\theta}) dt \\
&= e^{-\int_{x\cdot\boldsymbol{\theta}}^{\infty} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} \sum_{n=1}^{k} (-1)^{n-1}\frac{(x\cdot\boldsymbol{\theta})^n}{n!} u^{k-n}(x,\boldsymbol{\theta}) \\
&\quad + (-1)^k \int_{-\infty}^{x\cdot\boldsymbol{\theta}} \frac{t^k}{k!} e^{-\int_t^{\infty} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} \langle \mathbf{f}(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta}), \boldsymbol{\theta}^m\rangle dt,
\end{aligned}
$$

where in the last equality we use (B.3). Thus, multiplying both sides of the above equation with $e^{\int_{x\cdot\boldsymbol{\theta}}^{\infty} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds}$ yields

(B.4)

$$
u^k(x,\boldsymbol{\theta}) = \sum_{n=1}^{k} (-1)^{n-1}\frac{(x\cdot\boldsymbol{\theta})^n}{n!} u^{k-n}(x,\boldsymbol{\theta}) + (-1)^k \int_{-\infty}^{x\cdot\boldsymbol{\theta}} \frac{t^k}{k!} e^{-\int_t^{x\cdot\boldsymbol{\theta}} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} \langle \mathbf{f}(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta}), \boldsymbol{\theta}^m\rangle dt,
$$

Since $\mathbf{f}(x+(t-x\cdot\boldsymbol{\theta})\boldsymbol{\theta}) = \mathbf{0}$ for every $(x,\boldsymbol{\theta}) \in \varGamma_+$ and $t > x\cdot\boldsymbol{\theta}$,

$$
\begin{aligned}
\int_{-\infty}^{x\cdot\boldsymbol{\theta}} t^k e^{-\int_t^{x\cdot\boldsymbol{\theta}} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} \langle \mathbf{f}(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta}), \boldsymbol{\theta}^m\rangle dt &= \int_{-\infty}^{\infty} t^k e^{-\int_t^{x\cdot\boldsymbol{\theta}} a(\Pi_{\boldsymbol{\theta}}(x)+s\boldsymbol{\theta})ds} \langle \mathbf{f}(\Pi_{\boldsymbol{\theta}}(x)+t\boldsymbol{\theta}), \boldsymbol{\theta}^m\rangle dt \\
&= I_a^k \mathbf{f}(x,\boldsymbol{\theta}).
\end{aligned} \tag{B.5}
$$

The relations (3.2) now follow from (B.4), (B.3), and (B.5).

Since $\mathbf{f} \in H_0^s(\mathbf{S}^m;\Omega)$, $s \geq 1$ and $a \in C^{s,\mu}(\overline{\Omega})$, $\mu > 1/2$, the solution u^0 given by (B.3) preserves the regularity and $u^0 \in H^s(\Omega\times\mathbb{S}^1)$. Moreover, by (3.1b) and (B.4), $u^k \in H^s(\Omega\times\mathbb{S}^1)$, $s \geq 1$ for $1 \leq k \leq m$. □

Proof of Lemma 5.1. (i) Let $\mathbf{v} = e^{-G}\mathbf{u} = \sum_{k=0}^{\infty} \alpha_k L^k \mathbf{u}$. Since $\mathbf{u} \in l^2(\mathbb{N}; H^1(\Omega))$, then from Proposition 5.1, $\mathbf{v} \in l^2(\mathbb{N}; H^1(\Omega))$. Then $\mathbf{v}$ solves

$$
\begin{aligned}
\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} &= \overline{\partial}\alpha_0\mathbf{u} + \overline{\partial}\alpha_1 L\mathbf{u} + \sum_{k=0}^{\infty}\left(\overline{\partial}\alpha_{k+2} + \partial\alpha_k\right)L^{k+2}\mathbf{u} + \sum_{k=0}^{\infty}\alpha_k L^k\left(\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u}\right) \\
&= \overline{\partial}\alpha_0\mathbf{u} + \left(\overline{\partial}\alpha_1 - a\alpha_0\right)L\mathbf{u} + \sum_{k=0}^{\infty}\left(\overline{\partial}\alpha_{k+2} + \partial\alpha_k - a\alpha_{k+1}\right)L^{k+2}\mathbf{u} + \sum_{k=0}^{\infty}\alpha_k L^k\mathbf{w} \\
&= \sum_{k=0}^{\infty}\alpha_k L^k\mathbf{w} = e^{-G}\mathbf{w}.
\end{aligned}
$$

where in the last equality we have used [28, Lemma 4.2].

An analogue calculation using the properties in [28, Lemma 4.2 (iv)] shows the converse. □

APPENDIX C. AN EXPLICIT POMPEIU FORMULA FOR L^2-ANALYTIC MAPS

The results in this section are from [13, Appendix B]. We repeat it here for reader's convenience. In here the Pompeiu type formula corresponding to A-analytic maps is derived.

Bukhgeim's original theory in [6] shows that solutions (called L^2-analytic) of the homogenous Beltrami-like equation

$$\overline{\partial}\mathbf{v}(z) + L^2\partial\mathbf{v}(z) = \mathbf{0}, \quad z \in \Omega, \tag{C.1}$$

satisfy a Cauchy-like integral formula,

$$\mathbf{v}(z) = \mathcal{B}[\mathbf{v}|_{\Gamma}](z), \quad z \in \Omega, \tag{C.2}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator acting on $\mathbf{v}|_{\Gamma}$ defined component-wise [11] for $n \geq 0$ by

$$(\mathcal{B}\mathbf{v})_{-n}(z) := \frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\frac{v_{-n}(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi \mathrm{i}}\int_{\Gamma}\left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}\sum_{j=1}^{\infty} v_{-n-2j}(\zeta)\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^{j}, \ z \in \Omega. \tag{C.3}$$

We next give the solutions of the inhomogenous Bukhgeim-Beltrami equation (2.1), which leads to Bukhgeim-Pompeiu formula.

Solutions of the inhomogenous Beltrami like equation (2.1)*, in component-wise form :*

$$\overline{\partial}v_{-n}(z) + \partial v_{-n-2}(z) = w_{-n}(z), \quad n \in \mathbb{Z}. \tag{C.4}$$

Assume that Ω is bounded convex domain with C^1 boundary, and for $n \geq 0$, let $\sigma_{-n}(z,\varphi) = \displaystyle\sum_{j=0}^{\infty} v_{-n-2j}(z)e^{-\mathrm{i}(n+2j)\varphi}$, and $\sigma_{-n} \in C^1(\Omega)\cap C(\overline{\Omega})$. Let $z \in \Omega$ and $\zeta \in \overline{\Omega}$, we write the parametrization $\zeta(\varphi) = z + l(\varphi)e^{\mathrm{i}\varphi}$. For $n \geq 0$, we have

$$\begin{aligned}
\sigma_{-n}(\zeta,\varphi) - \sigma_{-n}(z,\varphi) &= \int_0^l \frac{\partial \sigma_{-n}}{\partial t}(z + te^{\mathrm{i}\varphi},\varphi)dt = \int_0^l \Big(\frac{\partial\sigma_{-n}}{\partial \overline{z}}e^{-\mathrm{i}\varphi} + \frac{\partial \sigma_{-n}}{\partial z}e^{\mathrm{i}\varphi}\Big)dt\\
&= \int_0^l \frac{\partial v_{-n}}{\partial z}e^{-\mathrm{i}(n-1)\varphi}dt + \int_0^l\sum_{j=0}^{\infty}\Big(\overline{\partial}v_{-n-2j} + \partial v_{-n-2j-2}\Big)e^{-\mathrm{i}(n+2j+1)\varphi}dt. \qquad \text{(C.5)}
\end{aligned}$$

We obtain the Fourier coefficients v_{-n} in Ω for $n \geq 0$ as follows:

$$\begin{aligned}
v_{-n}(z) &= \frac{1}{2\pi}\int_0^{2\pi}\sigma_{-n}(z,\varphi)e^{\mathrm{i}n\varphi}d\varphi\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sigma_{-n}(\zeta,\varphi)e^{\mathrm{i}n\varphi}d\varphi - \frac{1}{2\pi}\int_0^{2\pi}\int_0^{l(\varphi)}\frac{\partial v_{-n}}{\partial z}\frac{1}{te^{-\mathrm{i}\varphi}}tdtd\varphi\\
&\qquad - \frac{1}{2\pi}\int_0^{2\pi}\int_0^{l(\varphi)}\sum_{j=0}^{\infty}\Big(\overline{\partial}v_{-n-2j} + \partial v_{-n-2j-2}\Big)e^{-2\mathrm{i}j\varphi}\frac{1}{te^{\mathrm{i}\varphi}}tdtd\varphi\\
&= \frac{1}{2\pi}\int_0^{2\pi}\sum_{j=0}^{\infty}v_{-n-2j}(\zeta)e^{-2\mathrm{i}j\varphi}d\varphi - \frac{1}{2\pi}\int_{\Omega}\frac{\partial v_{-n}}{\partial z}\frac{1}{\overline{\zeta} - \overline{z}}dA \qquad \text{(C.6)}\\
&\qquad - \frac{1}{2\pi}\int_{\Omega}\sum_{j=0}^{\infty}\Big(\overline{\partial}v_{-n-2j} + \partial v_{-n-2j-2}\Big)e^{-2\mathrm{i}j\varphi}\frac{1}{\zeta - z}dA
\end{aligned}$$

where dA is the element of the area, and in the second equality we use (C.5).

From $\zeta = z + l(\varphi)e^{\mathrm{i}\varphi}$ we get

$$e^{-2\mathrm{i}\varphi} = \frac{\overline{\zeta} - \overline{z}}{\zeta - z}, \qquad d\varphi = \frac{1}{2\mathrm{i}}\left(\frac{1}{\zeta - z}d\zeta - \frac{1}{\overline{\zeta} - \overline{z}}d\overline{\zeta}\right), \tag{C.7}$$

and by the conjugate form of the Cauchy- Pompeiu formula (e.g. see [37]), we have

$$\frac{1}{2\pi}\int_{\Omega} \frac{\partial c_{-n}}{\partial \overline{z}} \frac{1}{\overline{\zeta - z}} dA = \frac{1}{2} v_{-n}(z) + \frac{1}{4\pi i}\int_{\partial\Omega} v_{-n}(\zeta)\frac{1}{\overline{\zeta} - \overline{z}}. \tag{C.8}$$

Substituting (C.7), (C.8) and (C.4) into (C.6) yields an explicit form of the Bukhgeim-Pompeiu formula [6], defined component-wise for $n \geq 0$ by

$$\begin{aligned} v_{-n}(z) = {} & \frac{1}{2\pi\mathrm{i}}\int_{\varGamma} \frac{v_{-n}(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi\mathrm{i}}\int_{\varGamma}\left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}\sum_{j=1}^{\infty} v_{-n-2j}(\zeta)\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^{j} \\ & - \frac{1}{\pi}\sum_{j=0}^{\infty}\int_{\Omega} w_{-n-2j}(\zeta)\frac{1}{\zeta - z}\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^{j} d\xi d\eta, \quad \zeta = \xi + i\eta. \end{aligned} \tag{C.9}$$

The following (C.10) is the Bukhgeim-Pompeiu formula, given in [6], and in an explicit form, defined component-wise for $n \geq 0$ by

$$\mathbf{v}_{-n}(z) = (\mathcal{B}\mathbf{v}|_{\varGamma})_{-n}(z) + (\mathcal{T}\mathbf{w})_{-n}(z),\ z \in \Omega, \tag{C.10}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator in (C.3), and $\mathcal{T}$ is a Pompeiu like operator defined component-wise for $n \geq 0$ by

$$(\mathcal{T}\mathbf{w})_{-n}(z) := -\frac{1}{\pi}\sum_{j=0}^{\infty}\int_{\Omega} w_{-n-2j}(\zeta)\frac{1}{\zeta - z}\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^{j} d\xi d\eta, \quad \zeta = \xi + i\eta, \quad z \in \Omega. \tag{C.11}$$

□

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