ÖAW AUSTRIAN ACADEMY OF SCIENCES

RICAM – JOHANN RADON INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

ON THE X-RAY TRANSFORM OF PLANAR SYMMETRIC TENSORS

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RICAM-Report 2023-22

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ON THE X-RAY TRANSFORM OF PLANAR SYMMETRIC TENSORS

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ABSTRACT. In this article we characterize the range of the attenuated and non-attenuated X-ray transform of compactly supported symmetric tensor fields in the Euclidean plane. The characterization is in terms of a Hilbert-transform associated with A-analytic maps in the sense of Bukhgeim.

1. INTRODUCTION

We consider here the problem of the range characterization of (non)-attenuated X-ray transform of a real valued symmetric m-tensors in a strictly convex bounded domain in the Euclidean plane. As the X-ray and Radon transform [35] for planar functions (0-tensors) differ merely by the way lines are parameterized, the $m = 0$ case is the classical Radon transform [35], for which the range characterization has been long established independently by Gelfand and Graev [12], Helgason [13], and Ludwig [18]. Models in the presence of attenuation have also been considered in the homogeneous case [17, 1], and in the non-homogeneous case in the breakthrough works [2, 28, 29], and subsequently [24, 5, 4, 14, 21]. The references here are by no means exhaustive.

The interest in the range characterization problem in the 0-tensors case stems out from their applications to data enhancement in medical imaging methods such as Single Photon Emission Computed Tomography or Positron Emission Computed Tomography [23, 11]. The X-ray transform of 1-tensors (Doppler transform [25, 44]) appears in the investigation of velocity distribution in a flow [6], in ultrasound tomography [45, 42], and also in non-invasive industrial measurements for reconstructing the velocity of a moving fluid [26, 27]. The X-ray transform of second order tensors arises as the linearization of the boundary rigidity problem [44]. The case of tensor fields of rank four describes the perturbation of travel times of compressional waves propagating in slightly anisotropic elastic media [44, Chapters 6,7]. Thus, due to the various applications the range characterization problem has been a continuing subject of research.

Unlike the scalar case, the X-ray transform of tensor fields has a non-zero kernel, and the null-space becomes larger as the order of the tensor field increases. For tensors of order $m \geq 1$, it is easy to check that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors, and it is possible to reconstruct uniquely (without additional information of moment ray transforms [44]) only the solenoidal part of a tensor field. The non-injectivity of the X-ray transform makes the range characterization problem even more interesting.

For the attenuating media in planar domains, interesting enough, the 1-tensor field can be recovered in the regions of positive absorption as shown in [15, 46, 37, 30], without using some additional data information [43, 8, 19]. It is due to a surprising fact that the two-dimensional attenuated Doppler transform with positive attenuation is injective while the non-attenuated Doppler transform is not.

Date: August 3, 2023.

2000 *Mathematics Subject Classification.* Primary 30E20; Secondary 35J56.

Key words and phrases. X-ray transform of symmetric tensors, Attenuated X-ray transform, A-analytic maps, Hilbert transform.

The systematic study of tensor tomography in non-Euclidean spaces originated in [44]. On simple Riemannian surfaces, the range characterization of the geodesic X-ray of compactly supported 0 and 1-tensors has been established in terms of the scattering relation in [34], and the results were extended in [3, 10, 16] to symmetric tensors of arbitrary order. Explicit inversion approaches in the Euclidean case have been proposed in [14, 9, 20]. In the attenuating media, tensor tomography was solved for the cases $m = 0, 1$ in [41]. Inversion for the attenuated X-ray transform for solenoidal tensors of rank two and higher can be found in [32], with a range characterization in [33, 21, 3].

The original characterization in [12, 13, 18] was extended to arbitrary symmetric m-tensors in [31]; see [9] for a partial survey on the tensor tomography in the Euclidean plane. The connection between the Euclidean version of the characterization in [34] and the characterization in [12, 13, 18] was established in [20]. Recently, in [38] the connection between the range characterization result in [36] and the original range characterization in [12, 13, 18] has been established. Moreover, the results in [38] from 0-order is extended to symmetric tensors of an arbitrary order in [39] and also establishes the connection with the generalized moment conditions in [31].

In here we build on the results in [36, 37, 40], and extends them to symmetric tensor fields of any arbitrary order. In particular, the range characterization therein are given in terms of the Bukhgeim-Hilbert transform [36] (the Hilbert-like transform associated with A-analytic maps in the sense of Bukhgeim [7]). The characterization in here can be viewed as an explicit description of the scattering relation in [32, 33] particularized to the Euclidean setting. The characterization in both the non-attenuated case (see Theorem 4.1 and Theorem 4.2 below) and in the attenuated case (see Theorem 5.1 below) are given in terms of the Bukhgeim-Hilbert transform (27). In the sufficiency part we reconstruct all possible m-tensors yielding identical X-ray data; see (44) and (62) for the non-attenuated case, and (88) and (89) for the attenuated case.

This article is organized as follows: All the details establishing notations and basic properties of symmetric tensor fields needed here are in Section 2. In Section 3 we briefly recall existing results on A-analytic maps that are used in the proofs. We provide range characterization of symmetric m-tensor field $\mathbf{f}$ in the non-attenuated case in Section 4, and in the attenuated case in Section 5. In Section 6 we gave some concluding remarks.

2. PRELIMINARIES

Given an integer $m \geq 0$, let $\mathbf{T}^m(\mathbb{R}^2)$ denote the space of all real-valued covariant tensor fields of rank m:

$$\mathbf{f}(x^1, x^2) = f_{i_1 \cdots i_m}(x^1, x^2) dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_m}, \quad i_1, \cdots, i_m \in \{1, 2\}, \tag{1}$$

where $\otimes$ is the tensor product, $f_{i_1 \cdots i_m}$ are the components of tensor field $\mathbf{f}$ in the Cartesian basis (x^1, x^2), and where by repeating superscripts and subscripts in a monomial a summation from 1 to 2 is meant.

We denote by $\mathbf{S}^m(\mathbb{R}^2)$ the space of symmetric covariant tensor fields of rank m on $\mathbb{R}^2$. Let $\sigma : \mathbf{T}^m(\mathbb{R}^2) \to \mathbf{S}^m(\mathbb{R}^2)$ be the canonical projection defined by $(\sigma \mathbf{f})_{i_1 \cdots i_m} = \dfrac{1}{m!} \displaystyle\sum_{\pi \in \Pi_m} f_{i_{\pi(1)} \cdots i_{\pi(m)}}$, where the summation is over the group Π_m of all permutations of the set $\{1, \cdots, m\}$.

A planar covariant symmetric tensor field of rank m has $m + 1$ independent component, which we denote by

$$\tilde{f}_k := f_{\underbrace{1 \cdots 1}_{m-k} \underbrace{2 \cdots 2}_{k}}, \quad (k = 0, \cdots, m), \tag{2}$$

in connection with this, a symmetric tensor $\mathbf{f} = (f_{i_1\cdots i_m},\ i_1, \cdots, i_m = 1,2)$ of rank m will be given by a pseudovector of size $m+1$: $\mathbf{f} = (\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_{m-1}, \tilde{f}_m)$.

We identify the plane $\mathbb{R}^2$ by the complex plane $\mathbb{C}$, $z^1 \equiv z = x^1 + \mathrm{i}x^2$, $z^2 \equiv \bar{z} = x^1 - \mathrm{i}x^2$. We consider the Cauchy-Riemann operators

$$\frac{\partial}{\partial z^1} \equiv \frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x^1} - \mathrm{i}\frac{\partial}{\partial x^2}\right), \quad \frac{\partial}{\partial z^2} \equiv \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x^1} + \mathrm{i}\frac{\partial}{\partial x^2}\right), \tag{3}$$

and the inverse relation by $\dfrac{\partial}{\partial x^1} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$, $\quad \dfrac{\partial}{\partial x^2} = \mathrm{i}\dfrac{\partial}{\partial z} - \mathrm{i}\dfrac{\partial}{\partial \bar{z}}$.

Let $\mathbf{f} = (f_{i_1\cdots i_m}(x^1,x^2),\ i_1, \cdots, i_m = 1,2)$ be real valued symmetric m-tensor field in Cartesian coordinates (x^1,x^2), then in complex coordinates (z^1,z^2) it will have new components $(F_{i_1\cdots i_m}(z,\bar{z}))$, which are formally expressed by the covariant tensor law:

$$\begin{aligned} F_{i_1\cdots i_m}(z,\bar{z}) &= \frac{\partial x^{s_1}}{\partial z^{i_1}} \cdots \frac{\partial x^{s_m}}{\partial z^{i_m}} f_{s_1\cdots s_m}(x^1,x^2), \quad \text{and} \\ f_{i_1\cdots i_m}(x^1,x^2) &= \frac{\partial z^{s_1}}{\partial x^{i_1}} \cdots \frac{\partial z^{s_m}}{\partial x^{i_m}} F_{s_1\cdots s_m}(z,\bar{z}), \end{aligned} \tag{4}$$

where the Jacobian matrix has the form

$$J := \begin{pmatrix} \frac{\partial x^1}{\partial z^1} & \frac{\partial x^1}{\partial z^2} \\ \frac{\partial x^2}{\partial z^1} & \frac{\partial x^2}{\partial z^2} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -\mathrm{i} & \mathrm{i} \end{pmatrix}, \quad \text{and} \quad J^{-1} = \begin{pmatrix} \frac{\partial z^1}{\partial x^1} & \frac{\partial z^1}{\partial x^2} \\ \frac{\partial z^2}{\partial x^1} & \frac{\partial z^2}{\partial x^2} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}.$$

Adopting the notation in [14], we shall write the transformations (4) as

$$\begin{aligned} \mathbf{f} = \{f_{i_1\cdots i_m}(x^1,x^2)\} &\quad \rightarrowtail \quad \mathbf{F} = \{F_{i_1\cdots i_m}(z,\bar{z})\}, \quad \text{and} \\ \mathbf{F} = \{F_{i_1\cdots i_m}(z,\bar{z})\} &\quad \rightarrowtail \quad \mathbf{f} = \{f_{i_1\cdots i_m}(x^1,x^2)\}. \end{aligned} \tag{5}$$

A symmetric tensor $\mathbf{F}$ of rank m, obtained from the real symmetric tensor $\mathbf{f}$ by passing to complex variables, we also define a pseudovector $(F_0, F_1, \cdots, F_{m-1}, F_m)$ with components

$$F_k = F_{\underbrace{1\cdots 1}_{m-k}\underbrace{2\cdots 2}_{k}}, \quad k = 0, \cdots, m, \tag{6}$$

and subject to the conditions

$$F_k = \overline{F}_{m-k}, \quad k = 0, \cdots, m. \tag{7}$$

Taking into account the tensor law (4), we obtain formulas relating the components of pseudovectors in (2) and pseudovectors in (6):

$$F_k = \frac{(-\mathrm{i})^{m-k}}{2^m} \sum_{q=0}^{m-k} \sum_{p=0}^{k} \binom{m-k}{q}\binom{k}{p} \mathrm{i}^{k-p+q} \tilde{f}_{p+q}, \quad k = 0, 1, \cdots, m, \tag{8}$$

$$\tilde{f}_k = \mathrm{i}^k \sum_{q=0}^{m-k} \sum_{p=0}^{k} \binom{m-k}{q}\binom{k}{p} (-1)^{k-p} F_{p+q}, \quad k = 0, 1, \cdots, m. \tag{9}$$

In Cartesian coordinates covariant and contravariant components are the same, and thus contravariant components of the tensor field $\mathbf{f}$ coincide with its corresponding covariant components, $f_{i_1\cdots i_m} = f^{i_1\cdots i_m}$. The dot product on $\mathbf{S}^m(\mathbb{R}^2)$ induced by the Euclidean metric is defined by

$$\langle \mathbf{f}, \mathbf{h} \rangle := f_{i_1\cdots i_m} h^{i_1\cdots i_m}. \tag{10}$$

Note that if $\mathbf{f}_1 \rightarrowtail \mathbf{F}_1$ and $\mathbf{f}_2 \rightarrowtail \mathbf{F}_2$, then the pointwise inner product of tensors is invariant:

$$\langle \mathbf{f}_1, \mathbf{f}_2 \rangle = \langle \mathbf{F}_1, \mathbf{F}_2 \rangle. \tag{11}$$

For $\boldsymbol{\theta} = (\theta^1, \theta^2) = (\cos\theta, \sin\theta) \in \mathbb{S}^1$, we denote by $\boldsymbol{\theta}^m$ the tensor product $\boldsymbol{\theta}^m := \underbrace{\boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \cdots \otimes \boldsymbol{\theta}}_{m}$ and $\boldsymbol{\theta}^m$ will be an m-contravariant tensor in Cartesian coordinates. According to the tensor law for contravariant components its representation in complex coordinates will look like

$$\boldsymbol{\theta} \rightarrowtail \Theta, \qquad \Theta^k = \frac{\partial z^k}{\partial x^s}\theta^s, \qquad \Theta = (\Theta^1, \Theta^2) = (e^{\mathrm{i}\theta}, e^{-\mathrm{i}\theta}),$$

and $\Theta^m := \underbrace{\Theta \otimes \Theta \otimes \cdots \otimes \Theta}_{m}$ be an m-contravariant tensor, and we also have $\boldsymbol{\theta}^m \rightarrowtail \Theta^m$.

Using (11), we get

$$\begin{aligned} \langle \mathbf{f}, \boldsymbol{\theta}^m \rangle = \langle \mathbf{F}, \Theta^m \rangle &= \sum_{k=0}^{m} \binom{m}{k} F_k \, e^{\mathrm{i}\theta(m-k)} e^{-\mathrm{i}\theta k} = \sum_{k=0}^{m} \binom{m}{k} F_k e^{\mathrm{i}(m-2k)\theta} \\ &= \begin{cases} f_0 + \displaystyle\sum_{k=1}^{q} \left(f_{-2k} e^{\mathrm{i}(2k)\theta} + f_{2k} e^{-\mathrm{i}(2k)\theta} \right), & (\text{if } m = 2q,\ q \geq 0), \\ \displaystyle\sum_{k=0}^{q} \left(f_{-(2k+1)} e^{\mathrm{i}(2k+1)\theta} + f_{2k+1} e^{-\mathrm{i}(2k+1)\theta} \right), & (\text{if } m = 2q+1,\ q \geq 0), \end{cases} \end{aligned} \tag{12}$$

where

$$f_{-2k} = \binom{2q}{q-k} F_{q-k}, \qquad 0 \leq k \leq q,\ q \geq 0, \quad \left(q = \frac{m}{2}, m \,\text{even} \right), \tag{13}$$

$$f_{-(2k+1)} = \binom{2q+1}{q-k} F_{q-k}, \qquad 0 \leq k \leq q,\ q \geq 0, \quad \left(q = \frac{m-1}{2}, m \,\text{odd} \right), \tag{14}$$

and $f_n = \overline{f_{-n}}$ and $F_n = \overline{F}_{m-n}$, for $0 \leq n \leq m$.

Let $\mathbf{f}$ be a real valued symmetric m-tensor, with integrable components of compact support in $\mathbb{R}^2$, and $a \in L^1(\mathbb{R}^2)$ a real valued function. The attenuated X-ray transform of $\mathbf{f}$ is given by

$$X_a \mathbf{f}(x, \boldsymbol{\theta}) := \int_{-\infty}^{\infty} \langle \mathbf{f}(x + t\boldsymbol{\theta}), \boldsymbol{\theta}^m \rangle \exp\left\{ - \int_t^{\infty} a(x + s\boldsymbol{\theta}) ds \right\} dt, \tag{15}$$

where $x \in \mathbb{R}^2$, $\boldsymbol{\theta} \in \mathbb{S}^1$, and $\langle \cdot, \cdot \rangle$ is the inner product in (10). For the non attenuated case $(a \equiv 0)$, we use the notation $X\mathbf{f}$.

In here, we consider the tensor field $\mathbf{f}$ be defined on a strongly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing boundary values on Γ; further regularity and the order of vanishing will be specified in the theorems. In the statements below we use the notations in [44]:

$$C^\mu(\mathbf{S}^m; \Omega) = \{ \mathbf{f} = (f_{i_1 \cdots i_m}) \in \mathbf{S}^m(\Omega) : f_{i_1 \cdots i_m} \in C^\mu(\Omega) \}$$

$0 < \mu < 1$, for the space of real valued, symmetric tensor fields of order m with locally Hölder continuous components. Similarly, $L^1(\mathbf{S}^m; \Omega)$ denotes the tensor fields of order m with integrable components.

For any $(x, \boldsymbol{\theta}) \in \overline{\Omega} \times \mathbb{S}^1$, let $\tau(x, \boldsymbol{\theta})$ be length of the chord passing through x in the direction of $\boldsymbol{\theta}$. Let also consider the incoming $(-)$, respectively outgoing $(+)$ submanifolds of the unit bundle restricted to the boundary

$$\Gamma_\pm := \{(x, \boldsymbol{\theta}) \in \Gamma \times \mathbb{S}^1 : \pm\boldsymbol{\theta} \cdot \nu(x) > 0\}, \tag{16}$$

and the variety

$$\Gamma_0 := \{(x, \boldsymbol{\theta}) \in \Gamma \times \mathbb{S}^1 : \boldsymbol{\theta} \cdot \nu(x) = 0\}, \tag{17}$$

where $\nu(x)$ denotes outer normal.

The a-attenuated X-ray transform of $\mathbf{f}$ is realized as a function on $\varGamma_+$ by

$$X_a\mathbf{f}(x, \boldsymbol{\theta}) = \int_{-\tau(x,\boldsymbol{\theta})}^{0} \langle \mathbf{f}(x + t\boldsymbol{\theta}), \boldsymbol{\theta}^m \rangle \, e^{-\int_t^0 a(x+s\boldsymbol{\theta})ds} \, dt, \ (x, \boldsymbol{\theta}) \in \varGamma_+. \tag{18}$$

We approach the range characterization via the well-known connection with the transport model as follows: The boundary value problem

$$\boldsymbol{\theta} \cdot \nabla u(x, \boldsymbol{\theta}) + a(x)u(x, \boldsymbol{\theta}) = \langle \mathbf{f}(x), \boldsymbol{\theta}^m \rangle, \quad (x, \boldsymbol{\theta}) \in \Omega \times \mathbb{S}^1, \tag{19a}$$

$$u|_{\varGamma_-} = 0, \tag{19b}$$

has a unique solution in $\Omega \times \mathbb{S}^1$ and

$$u|_{\varGamma_+}(x, \boldsymbol{\theta}) = X_a\mathbf{f}(x, \boldsymbol{\theta}), \quad (x, \boldsymbol{\theta}) \in \varGamma_+. \tag{20}$$

The range characterization is given in terms of the boundary value

$$g := u|_{\varGamma \times \mathbb{S}^1} = \begin{cases} X_a\mathbf{f}, & \text{on } \varGamma_+, \\ 0, & \text{on } \varGamma_- \cup \varGamma_0. \end{cases} \tag{21}$$

3. Ingredients from A-analytic theory

In this section we briefly introduce the properties of A-analytic maps needed later.
For $0 < \mu < 1$, $p = 1, 2$, we consider the Banach spaces:

$$\begin{aligned} l^{1,p}_\infty(\varGamma) &:= \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ... \rangle \ : \|\mathbf{g}\|_{l^{1,p}_\infty(\varGamma)} := \sup_{\xi \in \varGamma} \sum_{j=0}^{\infty} \langle j \rangle^p |g_{-j}(\xi)| < \infty \right\}, \\ C^\mu(\varGamma; l_1) &:= \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ... \rangle : \sup_{\xi \in \varGamma} \|\mathbf{g}(\xi)\|_{l_1} + \sup_{\substack{\xi, \eta \in \varGamma \\ \xi \neq \eta}} \frac{\|\mathbf{g}(\xi) - \mathbf{g}(\eta)\|_{l_1}}{|\xi - \eta|^\mu} < \infty \right\}, \\ Y_\mu(\varGamma) &:= \left\{ \mathbf{g} : \mathbf{g} \in l^{1,2}_\infty(\varGamma) \text{ and } \sup_{\substack{\xi, \eta \in \varGamma \\ \xi \neq \eta}} \sum_{j=0}^{\infty} \langle j \rangle \frac{|g_{-j}(\xi) - g_{-j}(\eta)|}{|\xi - \eta|^\mu} < \infty \right\}, \end{aligned} \tag{22}$$

where $l_\infty(, l_1)$ is the space of bounded (, respectively summable) sequences, and for brevity, we use the notation $\langle j \rangle = (1 + |j|^2)^{1/2}$. Similarly, we consider $C^\mu(\overline{\Omega}; l_1)$, and $C^\mu(\overline{\Omega}; l_\infty)$.

A sequence valued map $\Omega \ni z \mapsto \mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), ... \rangle$ in $C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty)$ is called L^k-*analytic* (in the sense of Bukhgeim), $k = 1, 2$, if

$$\overline{\partial}\mathbf{v}(z) + L^k \partial \mathbf{v}(z) = \mathbf{0}, \quad z \in \Omega, \tag{23}$$

where L is the left shift operator $L\langle v_0, v_{-1}, v_{-2}, \cdots \rangle = \langle v_{-1}, v_{-2}, \cdots \rangle$, and $L^2 = L \circ L$.

Bukhgeim's original theory in [7] shows that solutions of (23), satisfy a Cauchy-like integral formula,

$$\mathbf{v}(z) = \mathcal{B}[\mathbf{v}|_\varGamma](z), \quad z \in \Omega, \tag{24}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator acting on $\mathbf{v}|_{\Gamma}$. We use the formula in [11], where $\mathcal{B}$ is defined component-wise for $n \geq 0$ by

(25)
$$(\mathcal{B}\mathbf{g})_{-n}(z) := \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \right\} \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^{j}, z \in \Omega.$$

The following regularity result in [36, Proposition 4.1] is needed.

Proposition 3.1. [36, Proposition 4.1] *Let $\mu > 1/2$ and $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ... \rangle$ be the sequence valued map of non-positive Fourier modes of g.*

(i) If $g \in C^{\mu}(\Gamma; C^{1,\mu}(\mathbb{S}^1))$, then $\mathbf{g} \in l^{1,1}_{\infty}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$.

(ii) If $g \in C^{\mu}(\Gamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\Gamma; C^{2,\mu}(\mathbb{S}^1))$, then $\mathbf{g} \in Y_{\mu}(\Gamma)$.

(iii) If $\mathbf{g} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$, then $g \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$.

Similar to the analytic maps, the boundary values of L-analytic maps must satisfy some constraints, which can be expressed in terms of a corresponding Hilbert-like transform introduced in [36]. More precisely, the Bukhgeim-Hilbert transform $\mathcal{H}$ acting on g,

$$\Gamma \ni z \mapsto (\mathcal{H}\mathbf{g})(z) = \langle (\mathcal{H}\mathbf{g})_0(z), (\mathcal{H}\mathbf{g})_{-1}(z), (\mathcal{H}\mathbf{g})_{-2}(z), ... \rangle \tag{26}$$

is defined component-wise for $n \geq 0$ by

(27)
$$(\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi} \int_{\Gamma} \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi} \int_{\Gamma} \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \right\} \sum_{j=1}^{\infty} g_{-n-j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^{j}, z \in \Gamma,$$

and we refer to [36] for its mapping properties.

Note that the Bukhgeim-Cauchy integral formula in (25) above is restated in terms of L-analytic maps as opposed to L^2-analytic as in [36]. The only change is the index relabeling. In particular, the index g_{-n-j} will change to g_{-n-2j} therein to account for L^2-analytic. Moreover, the same index relabelling in the Bukhgeim-Hilbert transform formula (27) is made to account for the difference between L-analytic and L^2-analytic.

The following result recalls the necessary and sufficient conditions for a sufficiently regular map to be the boundary value of an L^k-analytic function, $k = 1, 2$.

Theorem 3.1. *Let $0 < \mu < 1$, and $k = 1, 2$. Let $\mathcal{B}$ be the Bukhgeim-Cauchy operator in* (25). *Let $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ... \rangle \in Y_{\mu}(\Gamma)$ for $\mu > 1/2$ be defined on the boundary Γ, and let $\mathcal{H}$ be the Bukhgeim-Hilbert transform acting on* $\mathbf{g}$ *as in* (27).

(i) If $\mathbf{g}$ *is the boundary value of an L^k-analytic function, then $\mathcal{H}\mathbf{g} \in C^{\mu}(\Gamma; l_1)$ and satisfies*

$$(I + \mathrm{i}\mathcal{H})\mathbf{g} = \mathbf{0}. \tag{28}$$

(ii) If $\mathbf{g}$ *satisfies* (28)*, then there exists an L^k-analytic function $\mathbf{v} := \mathcal{B}\mathbf{g} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_{\infty})$, such that*

$$\mathbf{v}|_{\Gamma} = \mathbf{g}. \tag{29}$$

For the proof of Theorem 3.1 we refer to [36, Theorem 3.2, Corollary 4.1, and Proposition 4.2] and [37, Proposition 2.3].

Another ingredient, in addition to L^2-analytic maps, consists in the one-to-one relation between solutions $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, ... \rangle$ satisfying

$$\overline{\partial} u_{-n}(z) + \partial u_{-n-2}(z) + a(z) u_{-n-1}(z) = 0, \quad z \in \Omega, \ n \geq 0, \tag{30}$$

and the L^2-analytic map $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ satisfying

$$\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z) = 0, \quad z \in \Omega,\ n \geq 0; \tag{31}$$

via a special function h, see [40, Lemma 4.2] for details. The function h is defined as

$$h(z, \boldsymbol{\theta}) := Da(z, \boldsymbol{\theta}) - \frac{1}{2}\left(I - \mathrm{i}H\right) Ra(z \cdot \boldsymbol{\theta}^{\perp}, \boldsymbol{\theta}^{\perp}), \tag{32}$$

where $\boldsymbol{\theta}^{\perp}$ is the counter-clockwise rotation of $\boldsymbol{\theta}$ by $\pi/2$, $Ra(s, \boldsymbol{\theta}^{\perp}) = \displaystyle\int_{-\infty}^{\infty} a\left(s\boldsymbol{\theta}^{\perp} + t\boldsymbol{\theta}\right) dt$ is the Radon transform in $\mathbb{R}^2$ of the attenuation a, $Da(z, \boldsymbol{\theta}) = \displaystyle\int_0^{\infty} a(z + t\boldsymbol{\theta}) dt$ is the divergent beam transform of the attenuation a, and $Hh(s) = \displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(t)}{s-t} dt$ is the classical Hilbert transform [22], taken in the first variable and evaluated at $s = z \cdot \boldsymbol{\theta}^{\perp}$. The function h appeared first in [23] and enjoys the crucial property of having vanishing negative Fourier modes yielding the expansions

$$e^{-h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \alpha_k(z) e^{\mathrm{i}k\theta}, \quad e^{h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \beta_k(z) e^{\mathrm{i}k\theta}, \quad (z, \boldsymbol{\theta}) \in \overline{\Omega} \times \mathbb{S}^1. \tag{33}$$

Using the Fourier coefficients of $e^{\pm h}$, we construct the sequence valued maps

$$\overline{\Omega} \ni z \mapsto \boldsymbol{\alpha}(z) := \langle \alpha_0(z), \alpha_1(z), ..., \rangle, \quad \overline{\Omega} \ni z \mapsto \boldsymbol{\beta}(z) := \langle \beta_0(z), \beta_1(z), ..., \rangle$$

to define the convolution operators $e^{\pm G}$ acting on some $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ via

$$e^{-G}\mathbf{u} = \sum_{k=0}^{\infty} \alpha_k L^k \mathbf{u} = \boldsymbol{\alpha} * \mathbf{u}, \quad \text{and} \quad e^{G}\mathbf{u} = \sum_{k=0}^{\infty} \beta_k L^k \mathbf{u} = \boldsymbol{\beta} * \mathbf{u}, \tag{34}$$

where L^k is the k-th composition of left translation. In particular, note that $e^{\pm G}$ commutes with L.

We refer [40, Lemma 4.1] for the properties of h, and we restate the following result [36, Proposition 5.2] to incorporate the operators $e^{\pm G}$ notation used in here.

Proposition 3.2. [36, Proposition 5.2] *Let $a \in C^{1,\mu}(\overline{\Omega})$, $\mu > 1/2$. Then $\boldsymbol{\alpha}, \partial\boldsymbol{\alpha}, \boldsymbol{\beta}, \partial\boldsymbol{\beta} \in l^{1,1}_{\infty}(\overline{\Omega})$, and the operators*

(i) $e^{\pm G} : C^{\mu}(\overline{\Omega}; l_{\infty}) \to C^{\mu}(\overline{\Omega}; l_{\infty})$; *(ii)* $e^{\pm G} : C^{\mu}(\overline{\Omega}; l_1) \to C^{\mu}(\overline{\Omega}; l_1)$; *(iii)* $e^{\pm G} : Y_{\mu}(\varGamma) \to Y_{\mu}(\varGamma)$.

Lemma 3.1. [37, Lemma 4.2] *Let $a \in C^{1,\mu}(\overline{\Omega})$, $\mu > 1/2$, and $e^{\pm G}$ be operators as defined in* (34).

(i) If $\mathbf{u} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = \mathbf{0}$, then $\mathbf{v} = e^{-G}\mathbf{u} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$.

(ii) Conversely, if $\mathbf{v} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$, then $\mathbf{u} = e^{G}\mathbf{v} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = \mathbf{0}$.

4. m-TENSOR - NON-ATTENUATED CASE

In the non-attenuated $a \equiv 0$ case, using (12) the transport equation (19a) becomes

$$\boldsymbol{\theta} \cdot \nabla u = \begin{cases} f_0 + \displaystyle\sum_{k=1}^{\frac{m}{2}} f_{-2k} e^{\mathrm{i}(2k)\theta} + f_{2k} e^{-\mathrm{i}(2k)\theta}, & \text{(if } m \text{ is even)}, \\ \displaystyle\sum_{k=0}^{\frac{m-1}{2}} f_{-(2k+1)} e^{\mathrm{i}(2k+1)\theta} + f_{2k+1} e^{-\mathrm{i}(2k+1)\theta}, & \text{(if } m \text{ is odd)}, \end{cases} \tag{35}$$

where $f_n's$ are defined as in (13) and (14), and $f_n = \overline{f_{-n}}$, for $0 \leq n \leq m$. Note that f_0 is real-valued while other modes are complex conjugates.

For $z = x_1 + \mathrm{i}x_2 \in \Omega$, the advection operator $\boldsymbol{\theta} \cdot \nabla$ in complex notation becomes $e^{-\mathrm{i}\theta}\overline{\partial} + e^{\mathrm{i}\theta}\partial$, where $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$, and $\overline{\partial}, \partial$ are the Cauchy-Riemann operators in (3).

If $\sum_{n\in\mathbb{Z}} u_n(z)e^{\mathrm{i}n\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution u of (35), then by identifying the Fourier coefficients of the same order, (35) reduces to the system for even order m-tensor:

$$\overline{\partial}u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = f_{2n}(z), \qquad 0 \leq n \leq \frac{m}{2},\ m \text{ even}, \tag{36}$$

$$\overline{\partial}u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = 0, \qquad n \geq \frac{m}{2} + 1, \tag{37}$$

$$\overline{\partial}u_{-2n}(z) + \partial u_{-(2n+2)}(z) = 0, \qquad n \geq 0, \tag{38}$$

and for odd order m-tensor we have:

$$\overline{\partial}u_{-2n}(z) + \partial u_{-(2n+2)}(z) = f_{2n+1}(z), \qquad 0 \leq n \leq \frac{m-1}{2},\ m \text{ odd}, \tag{39}$$

$$\overline{\partial}u_{-2n}(z) + \partial u_{-(2n+2)}(z) = 0, \qquad n \geq \frac{m+1}{2}, \tag{40}$$

$$\overline{\partial}u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = 0, \qquad n \geq 0. \tag{41}$$

Recall that the boundary value $u|_{\varGamma\times\mathbb{S}^1} := g$ as in (21), with $g = X\mathbf{f}$ on $\varGamma_+$ and $g = 0$ on $\varGamma_- \cup \varGamma_0$. The range characterization is given in terms of the Fourier modes of g in the angular variables: $g(\zeta, \boldsymbol{\theta}) = \sum_{n=-\infty}^{\infty} g_n(\zeta)e^{\mathrm{i}n\theta}$, for $\zeta \in \varGamma$. Since the data g is also real valued, its modes satisfies

$$g_{-n}(\zeta) = \overline{g_n(\zeta)}, \qquad \text{for } n \geq 0,\ \zeta \in \varGamma. \tag{42}$$

From the non-positive Fourier modes, we built the sequences

$$\mathbf{g}^{\text{even}} := \langle g_0, g_{-2}, g_{-4}, ...\rangle, \qquad \text{and} \qquad \mathbf{g}^{\text{odd}} := \langle g_{-1}, g_{-3}, g_{-5}, ...\rangle. \tag{43}$$

4.1. **Even order m-tensor.** In this subsection, we establish necessary and sufficient conditions for a sufficiently smooth function on $\varGamma \times \mathbb{S}^1$ to be the non-attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $\mathbf{f}$ of even order $m = 2q,\ q \geq 0$.

We characterize next the non-attenuated X-ray data g in terms of the Bukhgeim-Hilbert Transform $\mathcal{H}$ in (27). We will construct the solution u of the transport equation (35) in the m-even case, whose boundary value matches the boundary data g, and also construct the right hand side of the (35). The construction of solution u is in terms of its Fourier modes in the angular variable. We first construct the non-positive Fourier modes and then the positive Fourier modes are constructed by conjugation. For even $m = 2q$, $q \geq 1$, apart from q many Fourier modes $u_{-1}, u_{-3}, \cdots u_{-(2q-1)}$, all non-positive Fourier modes are defined by Bukhgeim-Cauchy integral formula (25) using boundary data. Other than having the boundary value $u_{-(2j-1)}\big|_{\varGamma} = g_{-(2j-1)},\ 1 \leq j \leq q,\ q \geq 1$, the q many Fourier modes $u_{-(2j-1)},\ 1 \leq j \leq q,\ q \geq 1$, are unconstrained. They are chosen arbitrarily from the class Ψ_g^{even} of functions of cardinality $q = \frac{m}{2}$ with prescribed restriction on the boundary $\varGamma$ defined:

$$\Psi_g^{\text{even}} := \Big\{ \big(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\big) \in \left(C^{1,\mu}(\overline{\Omega}; \mathbb{C})\right)^q, 2\mu > 1 : \psi_{-(2j-1)}\big|_{\varGamma} = g_{-(2j-1)},\ 1 \leq j \leq q,\ q \geq 1 \Big\}. \tag{44}$$

Remark 4.1. *Any arbitrary $C^{1,\mu}(\overline{\Omega})$ functions such that its restriction on the boundary matches with the respective Fourier mode of data will suffice in* (44)*. Note for the 0-tensor case, there is no class, and the characterization of the X-ray data g is in terms of the Fourier modes* $\mathbf{g}$.

Theorem 4.1 (Range characterization for even order tensors)**.** *(i) Let $\mathbf{f} \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$, $\mu > 1/2$, be a real-valued symmetric tensor field of even order $m = 2q$, $q \geq 0$, and*

$$g = X\mathbf{f} \text{ on } \varGamma_+ \text{ and } g = 0 \text{ on } \varGamma_- \cup \varGamma_0.$$

Then $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in l^{1,1}_{\infty}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$ satisfy

$$[I + \mathrm{i}\mathcal{H}]\mathbf{g}^{\text{even}} = \mathbf{0}, \quad \text{and} \quad [I + \mathrm{i}\mathcal{H}]L^{\frac{m}{2}}\mathbf{g}^{\text{odd}} = \mathbf{0}, \tag{45}$$

where $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}}$ are sequences in (43)*, and $\mathcal{H}$ is the Bukhgeim-Hilbert operator in* (27)*.*

(ii) Let $g \in C^{\mu}\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. For $q = 0$, if the corresponding sequences $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\varGamma)$ satisfies (45)*, then there is a unique real valued symmetric 0-tensor $\mathbf{f}$ such that $g|_{\varGamma_+} = X\mathbf{f}$. Moreover, for $q \geq 1$, if $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\varGamma)$ satisfies* (45)*, and for each element $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_g^{\text{even}}$, then there is a unique real valued symmetric m-tensors $\mathbf{f}_{\Psi} \in C^{\mu}(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+} = X\mathbf{f}_{\Psi}$.*

Proof. (i) **Necessity:** Let $\mathbf{f} = (f_{i_1\cdots i_m}) \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$. Since all components $f_{i_1\cdots i_m} \in C_0^{1,\mu}(\Omega)$ are compactly supported inside Ω, then for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\varGamma_0$ which yields $g \in C^{1,\mu}(\varGamma \times \mathbb{S}^1)$. Moreover, g is the boundary value on $\varGamma \times \mathbb{S}^1$ of a solution $u \in C^{1,\mu}(\overline{\Omega} \times \mathbb{S}^1)$ of the transport equation (35) in the m-even case. By Proposition 3.1 (i), $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in l^{1,1}_{\infty}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$.

If u solves (35), then its Fourier modes satisfy (36) - (38). Since the negative even Fourier modes u_{2n} for $n \leq 0$, satisfies the system (38), then the sequence $\mathbf{u}^{\text{even}} := \langle u_0, u_{-2}, u_{-4}, \cdots \rangle$ is L-analytic in Ω and the necessity part in Theorem 3.1 yields the first condition in (45).

The equation (37) for negative odd Fourier modes starting from negative $2q+1$ mode, yield that the sequence $\langle u_{-(2q+1)}, u_{-(2q+3)}, ... \rangle$ is L-analytic in Ω and the necessity part in Theorem 3.1 gives the last condition in (45).

(ii) **Sufficiency:** Let $g \in C^{\mu}\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. Since g is real valued, its Fourier modes in the angular variable occurs in conjugates (42). Let the corresponding sequences $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}}$ satisfy (45). By Proposition (3.1), $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\varGamma)$.

Let $m = 2q$, $q \geq 0$, be an even integer. To prove the sufficiency we will construct a real valued symmetric m-tensor $\mathbf{f}$ in Ω and a real valued function $u \in C^1(\Omega \times \mathbb{S}^1) \cap C(\overline{\Omega} \times \mathbb{S}^1)$ such that $u|_{\varGamma \times \mathbb{S}^1} = g$ and u solves (35) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of even modes u_{2n} for $n \in \mathbb{Z}$.

Apply the Bukhgeim-Cauchy Integral operator (25) to construct the negative even Fourier modes:

$$\langle u_0(z), u_{-2}(z), u_{-4}(z), ... \rangle := \mathcal{B}\mathbf{g}^{\text{even}}(z), \quad z \in \Omega. \tag{46}$$

By Theorem 3.1, the sequence $\langle u_0, u_{-2}, ... \rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$ is L-analytic in Ω, thus

$$\overline{\partial} u_{-2n} + \partial u_{-2n-2} = 0, \tag{47}$$

are satisfied for all $n \geq 0$. Moreover, the hypothesis (45) and the sufficiency part of Theorem 3.1 yields that they extend continuously to $\varGamma$ and $u_{-2n}|_{\varGamma} = g_{-2n}$, for all $n \geq 0$.

Construct the positive even Fourier modes by conjugation: $u_{2n} := \overline{u_{-2n}}$, for all $n \geq 1$.

By conjugating (47) we note that the positive even Fourier modes also satisfy

$$\overline{\partial} u_{2n+2} + \partial u_{2n} = 0, \quad n \geq 0.$$

Moreover, by reality of g in (42) they extend continuously to $\varGamma$ and

$$u_{2n}|_{\varGamma} = \overline{u_{-2n}}|_{\varGamma} = \overline{g_{-2n}} = g_{2n}, \quad n \geq 1.$$

Thus, as a summary from above equations, we have shown that the even modes u_{2n} satisfy

(48) $$\overline{\partial} u_{2n} + \partial u_{2n-2} = 0, \quad \text{and} \quad u_{2n}\Big|_{\varGamma} = g_{2n}, \quad \text{for all } n \in \mathbb{Z}.$$

Step 2: The construction of odd modes u_{2n-1} for $|n| \geq q$, $q \geq 0$.
Apply the Bukhgeim-Cauchy Integral operator (25) to construct the other odd negative modes:

(49) $$\langle u_{-(2q+1)}(z), u_{-(2q+3)}(z), \cdots \rangle := \mathcal{B} L^q \mathbf{g}^{\text{odd}}(z), \quad z \in \Omega.$$

By Theorem 3.1, the sequence $\langle u_{-(2q+1)}, u_{-(2q+3)}, ..., \rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$, is L-analytic in Ω, thus the equations

(50) $$\overline{\partial} u_{-(2n+1)} + \partial u_{-(2n+3)} = 0,$$

are satisfied for all $n \geq q$, $q \geq 0$. Moreover, the hypothesis (45) : $[I + \mathrm{i}\mathcal{H}] L^{\frac{m}{2}} \mathbf{g}^{\text{odd}} = \mathbf{0}$, and the sufficiency part of Theorem 3.1 yields that they extend continuously to $\varGamma$ and

(51) $$u_{-(2n+1)}|_{\varGamma} = g_{-(2n+1)}, \quad \forall n \geq q, \ q \geq 0.$$

Construct the positive odd Fourier modes by conjugation: $u_{2n+1} := \overline{u}_{-(2n+1)}$, for all $n \geq q$, $q \geq 0$.
By conjugating (50) we note that the positive odd Fourier modes also satisfy

(52) $$\overline{\partial} u_{2n+3} + \partial u_{2n+1} = 0, \quad \forall n \geq q, \ q \geq 0.$$

Moreover, by (42) they extend continuously to $\varGamma$ and

(53) $$u_{2n+1}|_{\varGamma} = \overline{u}_{-(2n+1)}|_{\varGamma} = \overline{g}_{-(2n+1)} = g_{2n+1}, \quad n \geq q, \ q \geq 0.$$

Step 3: The construction of the 0- tensor field $\mathbf{f}$. In the case of the 0-tensor, $\mathbf{f} = f_0$ is uniquely determined from the odd mode u_{-1} in (49), by

(54) $$f_0 := 2\,\mathbb{R}\mathrm{e}\, \partial u_{-1}, \quad (\text{for } q = 0 \text{ case}).$$

We consider next the case $q \geq 1$ of tensors of order 2 or higher. In this case the construction of the tensor field $\mathbf{f}_\Psi$ is in terms of the Fourier mode $u_{-(2q+1)}$ in (49) and the class Ψ_g^{even} in (44).
Step 4: The construction of odd modes $u_{\pm(2n-1)}$, for $1 \leq n \leq q$, $q \geq 1$.
For $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_g^{\text{even}}$ arbitrary, define the modes $u_{\pm 1}, u_{\pm 3}, ..., u_{\pm(2q-1)}$ in Ω by

(55) $$u_{-(2n-1)} := \psi_{-(2n-1)} \text{ and } u_{2n-1} := \overline{\psi}_{-(2n-1)}, \quad 1 \leq n \leq q, \ q \geq 1.$$

By the definition of the class Ψ_g^{even} in (44), and the reality of g in (42), we have

(56) $$u_{-(2n-1)}|_{\varGamma} = g_{-(2n-1)}, \quad \text{and} \quad u_{2n-1}|_{\varGamma} = \overline{g}_{-(2n-1)} = g_{2n-1}, \quad 1 \leq n \leq q, \ q \geq 1.$$

Step 5: The construction of the tensor field $\mathbf{f}_\Psi$ whose X-ray data is g.
For $q \geq 1$, we define f_{2q} by using $\psi_{-(2q-1)}$ from the non-uniqueness class (44), and Fourier mode $u_{-(2q+1)}$ from the Bukhgeim-Cauchy formula (49). Then, define $\{f_{2n} : \ 0 \leq n \leq q-1\}$ solely from

the information in the non-uniqueness class. Finally, define $\{f_{-2n} :\ 1 \leq n \leq q\}$ by conjugation.

$$
\begin{aligned}
&f_{2q} := \overline{\partial}\psi_{-(2q-1)} + \partial u_{-(2q+1)}, && q \geq 1, \\
&f_{2n} := \overline{\partial}\psi_{-(2n-1)} + \partial \psi_{-(2n+1)}, && 1 \leq n \leq q-1,\ q \geq 2, \\
&f_0 := 2\,\mathbb{R}\mathrm{e}\,\partial\psi_{-1}, && q \geq 1, \quad \text{and} \\
&f_{-2n} := \overline{f_{2n}}, && 1 \leq n \leq q,\ q \geq 1,
\end{aligned} \tag{57}
$$

By construction, $f_{2n} \in C^{\mu}(\Omega)$, for $-q \leq n \leq q$, as $\psi_{-1}, \cdots, \psi_{-2q+1} \in C^{1,\mu}(\Omega)$. We use these Fourier modes $f_0, f_{\pm 2}, f_{\pm 4}, \cdots, f_{\pm 2q}$ for $q \geq 1$, and equations (13), (7) and (9) to construct the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle$, and thus the m-tensor field $\mathbf{f}_{\Psi} \in C^{\mu}(\mathbf{S}^m; \Omega)$.

In order to show $g|_{\Gamma_+} = X\mathbf{f}_{\Psi}$ for $q \geq 1$, with $\mathbf{f}_{\Psi}$ being constructed as in (57), we define the real valued function u via its Fourier modes for $q \geq 1$,

$$
u(z, \boldsymbol{\theta}) = \sum_{n=-\infty}^{\infty} u_{2n} e^{\mathrm{i}2n\theta} + \sum_{|n| \geq q} u_{2n+1} e^{\mathrm{i}(2n+1)\theta} + \sum_{n=1}^{q} \psi_{-(2n-1)} e^{-\mathrm{i}(2n-1)\theta} + \sum_{n=1}^{q} \overline{\psi}_{-(2n-1)} e^{\mathrm{i}(2n-1)\theta}. \tag{58}
$$

Since $g \in C^{\mu}\left(\Gamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C\left(\Gamma; C^{2,\mu}(\mathbb{S}^1)\right)$, we use Proposition 3.1 (ii)-(iii), to conclude that u defined in (58) belongs to $C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$. Using (48), (51), (53), (56), and definition of $\left(\psi_{-1}, \cdots, \psi_{-(2q-1)}\right) \in \Psi_g^{\text{even}}$ for $q \geq 1$, the $u(\cdot, \boldsymbol{\theta})$ in (58) extends to the boundary,

$$
u(\cdot, \boldsymbol{\theta})|_{\Gamma} = g(\cdot, \boldsymbol{\theta}).
$$

Since $u \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$, then the term by term differentiation in (58) is now justified, and u satisfy (35):

$$
\begin{aligned}
\boldsymbol{\theta} \cdot \nabla u &= \overline{\partial}\, \overline{\psi_{-1}} + \partial \psi_{-1} + \sum_{n=1}^{q-1} (\overline{\partial} \psi_{-(2n-1)} + \partial \psi_{-(2n+1)}) e^{-\mathrm{i}(2n)\theta} + \sum_{n=1}^{q-1} (\overline{\partial}\, \overline{\psi}_{-(2n+1)} + \partial \overline{\psi}_{-(2n-1)}) e^{\mathrm{i}(2n)\theta} \\
&\quad + e^{-\mathrm{i}(2q)\theta} \big(\overline{\partial} \psi_{-(2q-1)} + \partial u_{-(2q+1)}\big) + e^{\mathrm{i}(2q)\theta} \big(\partial \overline{\psi}_{-(2q-1)} + \overline{\partial}\, \overline{u}_{-(2q+1)}\big) \\
&= \sum_{n=-q}^{q} f_{2n}(z) e^{-\mathrm{i}(2n)\theta} = \langle \mathbf{f}, \boldsymbol{\theta}^{2q} \rangle,
\end{aligned}
$$

where we use (48), (50), (52), (55), and the second equality uses the definition of f_{2k}'s in (57). □

4.2. **Odd order m-tensor.** In this subsection we establish necessary and sufficient conditions for a sufficiently smooth function on $\Gamma \times \mathbb{S}^1$ to be the non-attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $\mathbf{f}$ of odd order $m = 2r+1,\ r \geq 0$.

In the m-tensor case, the even and odd Fourier modes of u plays a different role, unlike the even m-tensor case in the previous section. To emphasize this difference we separate the non-positive even modes $\mathbf{u}^{\text{even}} := \langle u_0, u_{-2}, u_{-4} \ldots \rangle$, and negative odd modes $\mathbf{u}^{\text{odd}} := \langle u_{-1}, u_{-3}, \ldots \rangle$, and note that if $\langle u_0(z), u_{-1}(z), u_{-2}(z), \ldots \rangle$ is L^2-analytic, then $\mathbf{u}^{\text{even}}, \mathbf{u}^{\text{odd}}$ are L-analytic.

Let us consider the sequence $\{\mathbf{u}^{2k-1}\}_{k \geq 1} \subset C(\overline{\Omega}; l_{\infty}) \cap C^1(\Omega; l_{\infty})$ given by

$$
\mathbf{u}^{2k-1} := \langle u_{2k-1}, u_{2k-3}, \ldots., u_1, u_{-1}, u_{-3}, u_{-5}, \ldots \rangle, \quad k \geq 1, \tag{59}
$$

obtained by augmenting the sequence of negative odd indices $\langle u_{-1}, u_{-3}, u_{-5}, \ldots \rangle$ by k many terms in the order $u_{2k-1}, u_{2k-3}, \ldots., u_1$.

One of the ingredients in our characterization of the odd m-tensor is the following simple property of L-analytic maps, shown in [36, Lemma 2.6].

Lemma 4.1. [36, Lemma 2.6] *Let $\{\mathbf{u}^{2k-1}\}_{k\geq 1}$ be the sequence of L-analytic maps defined in* (59)*. Assume that $u_{2k-1}|_{\varGamma}= \overline{u}_{-(2k-1)}|_{\varGamma}$, for all $k\geq 1$. Then, for each $k\geq 1$,*

$$u_{2k-1}(z) = \overline{u}_{-(2k-1)}(z), \quad z\in\Omega. \tag{60}$$

Similar to the even m-tensor case, the range characterization of data g will be given in terms of its Fourier modes.From the non-positive even modes, we build the sequences $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}}$ as in (43).

For each $k\geq 1$, we use the odd modes $\{g_{-1}, g_{-3}, g_{-5}, ...\}$ to build the sequence

$$\mathbf{g}^{2k-1} := \langle g_{2k-1}, g_{2k-3}, ..., g_1, g_{-1}, g_{-3}, g_{-5}, ...\rangle \tag{61}$$

by augmenting the negative odd indices by k-many terms in the order $g_{2k-1}, g_{2k-3}, ..., g_1$.

Similar to the even m-tensor case before, we will construct the solution u of the transport equation (35) in the m-odd case, whose boundary value matches the boundary data g, and also construct the right hand side of the (35). The construction of solution u is in terms of its Fourier modes in the angular variable. Except for non-positive modes $u_0, u_{-2}, \cdots, u_{-2r}$, all non-positive modes are defined by Bukhgeim-Cauchy integral formula in (25) using boundary data. Other than having the restrictions $u_{-2j}\big|_{\varGamma} = g_{-2j},\ 0\leq j\leq r,\ r\geq 0$, on the boundary, the $r+1$ many Fourier modes $u_{-2j},\ 0\leq j\leq r,\ r\geq 0$, are unconstrained. They are chosen arbitrarily from the class of functions

$$\begin{aligned}\Psi_g^{\text{odd}} := \Big\{ &(\psi_0, \psi_{-2}, \cdots, \psi_{-2r}) \in C^{1,\mu}(\overline{\Omega};\mathbb{R}) \times \left(C^{1,\mu}(\overline{\Omega};\mathbb{C})\right)^r : 2\mu > 1 : \\ &\psi_{-2j}\big|_{\varGamma} = g_{-2j},\ 0\leq j\leq r,\ r\geq 0 \Big\}.\end{aligned} \tag{62}$$

Remark 4.2. *Any arbitrary $C^{1,\mu}(\overline{\Omega})$ functions such that its restriction on the boundary matches with the respective Fourier mode of data will suffice in* (62)*. In the 1-tensor case ($m=1$), only Fourier mode u_0 be an* arbitrary *function in $C^1(\Omega)\cap C(\overline{\Omega})$ with $u_0|_{\varGamma} = g_0$. The arbitrariness of u_0 characterizes the non-uniqueness (up to the gradient field of a function which vanishes at the boundary) in the reconstruction of a vector field from its Doppler data.*

Theorem 4.2 (Range characterization for odd tensors.)**.** *Let $\mathbf{f}\in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$, $\mu>1/2$, be a real-valued symmetric tensor field of odd order $m=2r+1$, $r\geq 0$, and*

$$g = X\mathbf{f} \text{ on } \varGamma_+ \text{ and } g = 0 \text{ on } \varGamma_-\cup\varGamma_0.$$

Then $\mathbf{g}^{\text{even}}, \mathbf{g}^{2k-1}\in l^{1,1}_{\infty}(\Gamma)\cap C^{\mu}(\Gamma;l_1)$ for $k\geq 1$, and satisfy

$$[I+\mathrm{i}\mathcal{H}]L^{\frac{m+1}{2}}\mathbf{g}^{\text{even}} = \mathbf{0}, \quad \text{and} \quad [I+\mathrm{i}\mathcal{H}]\mathbf{g}^{2k-1} = \mathbf{0}, \quad \forall k\geq 1, \tag{63}$$

where $\mathbf{g}^{\text{even}}$ is the sequence in (43)*, $\mathbf{g}^{2k-1}$ for $k\geq 1$ is the sequence in* (61)*, and $\mathcal{H}$ is the Bukhgeim-Hilbert operator in* (27)*.*

(ii) Let $g\in C^{\mu}\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right)\cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_-\cup\varGamma_0}=0$. If the corresponding sequence $\mathbf{g}^{\text{even}}, \mathbf{g}^{2k-1}\in Y_{\mu}(\varGamma)$ for $k\geq 1$, satisfies (63)*, and for each element $(\psi_0,\cdots,\psi_{-2r})\in\Psi_g^{\text{odd}}$, then there is a unique real valued symmetric m-tensor $\mathbf{f}_{\Psi}\in C^{\mu}(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+}= X\mathbf{f}_{\psi}$.*

Proof. (i) **Necessity:** Let $\mathbf{f} = (f_{i_1\cdots i_m})\in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$. Since all components $f_{i_1\cdots i_m}\in C_0^{1,\mu}(\Omega)$, $X\mathbf{f}\in C^{1,\mu}(\varGamma_+)$, and, thus, the solution u to the transport equation (35) is in $C^{1,\mu}(\overline{\Omega}\times\mathbb{S}^1)$. Moreover, its boundary value $g = u|_{\varGamma\times\mathbb{S}^1}\in C^{1,\mu}(\varGamma\times\mathbb{S}^1)$.

By Proposition 3.1 (i), $\mathbf{g}^{\text{even}}, \mathbf{g}^{2k-1}\in l^{1,1}_{\infty}(\Gamma)\cap C^{\mu}(\Gamma;l_1)$ for all $k\geq 1$.

If u solves (35) then its Fourier modes satisfy (39) - (41). Since the negative even Fourier modes u_{-2n} for $n\geq \frac{m+1}{2}$, satisfies the system (40), then the sequence $\langle u_{-(m+1)}, u_{-(m+3)}, \cdots\rangle$ is L-analytic in Ω and the necessity part in Theorem 3.1 yields the first condition in (63).

The system (41) yield that the sequence $\mathbf{u}^1 := \langle u_1, u_{-1}, u_{-3} \cdots \rangle$ is L-analytic in Ω with the boundary value satisfying $u_{2k-1}|_{\Gamma}= g_{2k-1}$, for all $k \leq 1$. By Theorem 3.1 necessity part, the sequence $\mathbf{g}^1 = \langle g_1, g_{-1}, g_{-3}, ...\rangle$ must satisfy $[I + \mathrm{i}\mathcal{H}]\mathbf{g}^1 = \mathbf{0}$.

Recall that u is real valued so that its Fourier modes occur in conjugates $u_n = \overline{u_{-n}}$ for all $n \geq 0$. Consider now the equation (41) for $n = 1$ and take its conjugate to yield

$$\overline{\partial} u_3 + \partial u_1 = 0. \tag{64}$$

The above equation (64) together with (41) yield that the sequence $\mathbf{u}^3 := \langle u_3, u_1, u_{-1}, u_{-3} \cdots \rangle$ is L-analytic in Ω with the boundary value satisfying $u_{2k-1}|_{\Gamma}= g_{2k-1}$ for all $k \leq 2$. By the necessity part in Theorem 3.1, it must be that $\mathbf{g}^3 = \langle g_3, g_1, g_{-1}, g_{-3}, ...\rangle$ satisfies $[I + \mathrm{i}\mathcal{H}]\mathbf{g}^3 = \mathbf{0}$.

Inductively, the argument above holds for any odd index $2k - 1$ to yield that the sequence

$$\Omega \ni z \mapsto \mathbf{u}^{2k-1}(z) := \langle u_{2k-1}(z), u_{2k-3}(z), ..., u_1(z), u_{-1}(z), u_{-3}(z) \cdots \rangle$$

is L-analytic in Ω. Then, again by the necessity part in Theorem 3.1, its boundary value $\mathbf{u}^{2k-1}|_{\Gamma}= \mathbf{g}^{2k-1}$ must satisfy the last condition in (63).

(ii) **Sufficiency:** Let $g \in C^{\mu}(\Gamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\Gamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0}= 0$. Since g is real valued, its Fourier modes in the angular variable occurs in conjugates (42). Let the corresponding sequences $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}}$ satisfies (45). By Proposition (3.1), $\mathbf{g}^{\text{even}}, \mathbf{g}^{\text{odd}} \in Y_{\mu}(\Gamma)$.

Let $m = 2r + 1,\ r \geq 0$, be an odd integer. To prove the sufficiency we will construct a real valued symmetric m-tensor $\mathbf{f}$ in Ω and a real valued function $u \in C^1(\Omega \times \mathbb{S}^1) \cap C(\overline{\Omega} \times \mathbb{S}^1)$ such that $u|_{\Gamma \times \mathbb{S}^1}= g$ and u solves (35) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of even modes u_{2n} for $|n| \geq 2r + 1,\ r \geq 0$.

Apply the Bukhgeim-Cauchy integral formula (25) to construct the negative even Fourier modes:

$$\langle u_{-2(r+1)}, u_{-2(r+2)}, ...\rangle := \mathcal{B}L^{r+1}\mathbf{g}^{\text{even}}. \tag{65}$$

By Theorem 3.1, the sequence $\langle u_{-2(r+1)}, u_{-2(r+2)}, ...\rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$is L-analytic in Ω:

$$\overline{\partial} u_{-2n} + \partial u_{-(2n+2)} = 0, \tag{66}$$

are satisfied for all $n \geq r + 1$, $r \geq 0$. Moreover, the hypothesis (63) and the sufficiency part of Theorem 3.1 yields that they extend continuously to Γ and

$$u_{-2n}|_{\Gamma} = g_{-2n}, \quad n \geq r + 1,\ r \geq 0. \tag{67}$$

Construct the positive even Fourier modes by conjugation: $u_{2n} := \overline{u_{-2n}}$, for all $n \geq r+1$, $r \geq 0$.

By conjugating (66) we note that the positive even Fourier modes also satisfy

$$\overline{\partial} u_{2n+2} + \partial u_{2n} = 0, \quad n \geq r + 1,\ r \geq 0. \tag{68}$$

Moreover, by reality of g in (42), they extend continuously to Γ and

$$u_{2n}|_{\Gamma} = \overline{u_{-2n}}|_{\Gamma} = \overline{g_{-2n}} = g_{2n}, \quad n \geq r + 1,\ r \geq 0. \tag{69}$$

Step 2: The construction of even modes u_{2n}, for $|n| \leq 2r,\ r \geq 0$.

For $(\psi_0, \psi_{-2}, \cdots, \psi_{-2r}) \in \Psi_g^{\text{odd}}$ arbitrary, define the modes $u_0, u_{\pm 2}, u_{\pm 4}, ..., u_{\pm 2r}$ in Ω by

$$u_{-2n} := \psi_{-2n}, \quad \text{and} \quad u_{2n} := \overline{\psi_{-2n}}, \quad 0 \leq n \leq r. \tag{70}$$

By the definition of the class (62), and reality of g in (42), we have

$$u_{2n}|_{\Gamma}= \overline{g_{-2n}} = g_{2n}, \quad 0 \leq n \leq r. \tag{71}$$

Step 3: The construction of negative modes u_{2n-1} for $n \in \mathbb{Z}$.

Use the Bukhgeim-Cauchy Integral formula (25) to construct the negative odd Fourier modes:

$$\langle u_{-1}(z), u_{-3}(z), ...\rangle := \mathcal{B}\mathbf{g}^{\text{odd}}(z), \quad z \in \Omega. \tag{72}$$

By Theorem 3.1, the sequence $\langle u_{-1}, u_{-3}...\rangle \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$ is L-analytic in Ω, thus

$$\overline{\partial} u_{-2n-1} + \partial u_{-2n-3} = 0, \tag{73}$$

are satisfied for all $n \geq 0$.

Note that $L\mathbf{g}^1 = \mathbf{g}^{\text{odd}}$. By hypothesis (63), $[I + \mathrm{i}\mathcal{H}]\mathbf{g}^1 = \mathbf{0}$. Since $\mathcal{H}$ commutes with the left translation L, then

$$\mathbf{0} = L[I + \mathrm{i}\mathcal{H}]\mathbf{g}^1 = [I + \mathrm{i}\mathcal{H}]L\mathbf{g}^1 = [I + \mathrm{i}\mathcal{H}]\mathbf{g}^{\text{odd}}.$$

By applying Theorem 3.1 sufficiency part, we have that each u_{2n-1} extends continuously to Γ:

$$u_{-2n-1}|_{\varGamma} = g_{-2n-1}, \quad n \geq 1.$$

If we were to define the positive odd index modes by conjugating the negative ones (as we did for the non-attenuated even tensor case) it would not be clear why the equation (41) for $n = 0$:

$$\overline{\partial} u_1 + \partial u_{-1} = 0,$$

should hold. To solve this problem we will define the positive odd modes by using the Bukhgeim-Cauchy integral formula (25) inductively.

Let $\mathbf{u}^1 = \langle u_1, u^1_{-1}, u^1_{-3}, \cdots \rangle$ be the L-analytic map defined by

$$\mathbf{u}^1 := \mathcal{B}\mathbf{g}^1. \tag{74}$$

The hypothesis (63) for $k = 1$: $[I + \mathrm{i}\mathcal{H}]\mathbf{g}^1 = \mathbf{0}$, allows us to apply the sufficiency part of Theorem 3.1 to yield that $\mathbf{u}^1$ extends continuously to $\varGamma$ and has boundary value $\mathbf{g}^1$ on $\varGamma$. However, $L\mathbf{u}^1 = \mathbf{u}^{\text{odd}}$ is also L-analytic with the same boundary value $\mathbf{g}^{\text{odd}}$ as $\mathbf{u}^{\text{odd}}$. By the uniqueness of L-analytic maps with the given boundary value we must have the equality

$$\langle u^1_{-1}, u^1_{-3}, \cdots \rangle = \langle u_{-1}, u_{-3}, \cdots \rangle.$$

In other words the formula (74) constructs only one new function u_1 and recovers the previously defined negative odd functions $u_{-1}, u_{-3}, ...$ In particular $\mathbf{u}^1 = \langle u_1, u_{-1}, u_{-3}, \cdots \rangle$ is L-analytic, and the equation $\overline{\partial} u_1 + \partial u_{-1} = 0$ holds in Ω. We stress here that, at this stage, we do not know that u_1 is the complex conjugate of u_{-1}.

Inductively, for $k \geq 1$, the formula

$$\mathbf{u}^{2k-1} = \langle u_{2k-1}, u^{2k-1}_{2k-3}, ..., u^{2k-1}_1, u^{2k-1}_{-1}, \cdots \rangle := \mathcal{B}\mathbf{g}^{2k-1} \tag{75}$$

defines a sequence $\{\mathbf{u}^{2k-1}\}_{k\geq 1}$ of L-analytic maps with $\mathbf{u}^{2k-1}|_{\varGamma} = \mathbf{g}^{2k-1}$. By the uniqueness of L-analytic maps with the given boundary value, a similar reasoning as above shows

$$L\mathbf{u}^{2k-1} = \mathbf{u}^{2k-3}, \quad \forall k \geq 2.$$

In particular, for all $k \geq 1$, the sequence $\mathbf{u}^{2k-1} = \langle u_{2k-1}, u_{2k-3}, ..., u_1, u_{-1}, \cdots \rangle$ is L-analytic. Note that the sequence $\{\mathbf{u}^{2k-1}\}_{k\geq 1}$ constructed above satisfies the hypotheses of the Lemma 4.1, and therefore for each $k \geq 1$,

$$u_{2k-1}(z) = \overline{u_{-(2k-1)}}(z), \quad z \in \Omega. \tag{76}$$

We stress here that the identities (76) need the hypothesis (63) for all $k \geq 1$, cannot be inferred directly from the Bukhgeim-Cauchy integral formula (25) for finitely many k's.

We have shown that

$$\overline{\partial} u_{2n-1} + \partial u_{2n-3} = 0, \quad \text{and} \quad u_{2n-1}|_{\Gamma} = g_{2n-1}, \quad \forall n \in \mathbb{Z}. \tag{77}$$

Step 4: The construction of the tensor field $\mathbf{f}_\psi$ whose X-ray data is g.

For $r \geq 0$, we define first f_{2r+1} by using ψ_{-2r} from the non-uniqueness class in (62), and mode $u_{-(2r+2)}$ from (65). Then, next define $\{f_{2n+1} : \ 0 \leq n \leq r-1\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-(2n+1)} : \ 0 \leq n \leq r\}$ by conjugation:

$$\begin{aligned} f_{2r+1} &:= \overline{\partial}\psi_{-2r} + \partial u_{-(r+2)}, && r \geq 0, \\ f_{2n+1} &:= \overline{\partial}\psi_{-2n} + \partial \psi_{-(2n+2)}, && 0 \leq n \leq r-1, \ r \geq 1, \quad \text{and} \\ f_{-(2n+1)} &:= \overline{f_{2n+1}}, && 0 \leq n \leq r, \ r \geq 0. \end{aligned} \tag{78}$$

By construction, $f_{\pm(2n+1)} \in C^{\mu}(\Omega)$, for $0 \leq n \leq r$, as $\psi_0, \psi_{-2}, \cdots, \psi_{-2r} \in C^{1,\mu}(\Omega)$. We use these Fourier modes $f_{\pm 1}, f_{\pm 3}, \cdots, f_{\pm m}$ for $m = 2r+1, \ r \geq 0$, and equations (14), (7) and (9) to construct the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle$, and thus the m-tensor field $\mathbf{f}_\Psi \in C^{\mu}(\mathbf{S}^m; \Omega)$.

In order to show $g|_{\Gamma_+} = X\mathbf{f}_\Psi$ with $\mathbf{f}_\Psi$ being constructed from pseudovectors via Fourier modes as in (78) from class Ψ_g^{odd}, we define the real valued function u via its Fourier modes

$$u(z, \boldsymbol{\theta}) := \sum_{n=-\infty}^{\infty} u_{2n-1}(z) e^{\mathrm{i}(2n-1)\theta} + \sum_{|n| \geq r+1} u_{2n}(z) e^{\mathrm{i}2n\theta} + \sum_{n=0}^{r} \psi_{-2n}(z) e^{-\mathrm{i}2n\theta} + \sum_{n=0}^{r} \overline{\psi}_{-2n}(z) e^{\mathrm{i}2n\theta}. \tag{79}$$

Since $g \in C^{\mu}(\Gamma; C^{1,\mu}(\mathbb{S}^1)) \cap C(\Gamma; C^{2,\mu}(\mathbb{S}^1))$, we use Proposition 3.1 (ii) and (iii), to conclude that u defined in (79) belongs to $C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$.

Using (67), (69), (71), (77), and element $(\psi_0, \psi_{-2}, \cdots, \psi_{-2r}) \in \Psi_g^{\text{odd}}$, the $u(\cdot, \boldsymbol{\theta})$ in (79) extends to the boundary $u(\cdot, \boldsymbol{\theta})|_{\Gamma} = g(\cdot, \boldsymbol{\theta})$.

Since $u \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$, then the term by term differentiation in (79) is now justified, satisfying the transport equation (35):

$$\begin{aligned} \boldsymbol{\theta} \cdot \nabla u &= 2\,\mathbb{R}\mathrm{e}\left\{ (\overline{\partial}\psi_{-2r} + \partial u_{-(2r+2)}) e^{\mathrm{i}(2r+1)\theta} \right\} + 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{n=0}^{r-1} (\overline{\partial}\psi_{-2n} + \partial \psi_{-(2n+2)}) e^{\mathrm{i}(2n+1)\theta} \right\} \\ &= \sum_{n=0}^{r} \left(f_{2n+1} e^{-\mathrm{i}(2n+1)\theta} + f_{-(2n+1)} e^{\mathrm{i}(2n+1)\theta} \right) = \langle \mathbf{f}, \boldsymbol{\theta}^{2r+1} \rangle, \end{aligned}$$

where we use (66), (68), (77), and the second equality uses the definition of f_{2k+1}'s in (78).

□

5. m-TENSOR - ATTENUATED CASE

Let $a \in C^{2,\mu}(\overline{\Omega})$, $\mu > 1/2$, with $\min_{\overline{\Omega}} a > 0$. In this case, the transport equation (19a) becomes

$$\boldsymbol{\theta} \cdot \nabla u + au = \begin{cases} f_0 + \displaystyle\sum_{k=1}^{\frac{m}{2}} f_{-2k} e^{\mathrm{i}(2k)\theta} + f_{2k} e^{-\mathrm{i}(2k)\theta}, & \text{(if } m \text{ is even)}, \\ \displaystyle\sum_{k=0}^{\frac{m-1}{2}} f_{-(2k+1)} e^{\mathrm{i}(2k+1)\theta} + f_{2k+1} e^{-\mathrm{i}(2k+1)\theta}, & \text{(if } m \text{ is odd)}, \end{cases} \tag{80}$$

where $f_n' s$ are defined as in (13) and (14), and $f_n = \overline{f_{-n}}, \ $ for $0 \leq n \leq m$.

If $\sum_{n\in\mathbb{Z}} u_n(z)e^{\mathrm{i}n\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution u of (80), then by identifying the Fourier coefficients of the same order, (80) reduces to the system for even order m-tensor:

$$\overline{\partial}u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) + au_{-2n}(z) = f_{2n}(z), \qquad 0 \le n \le \frac{m}{2},\ m \text{ even}, \tag{81}$$

$$\overline{\partial}u_{-2n}(z) + \partial u_{-(2n+2)}(z) + au_{-2n-1}(z) = 0, \qquad 0 \le n \le \frac{m}{2} - 1, \tag{82}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-(n+2)}(z) + au_{-(n+1)}(z) = 0, \qquad n \ge m, \tag{83}$$

and for odd order m-tensor we have:

$$\overline{\partial}u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) + au_{-2n}(z) = 0, \qquad 0 \le n \le \frac{m-1}{2},\ m \text{ odd}, \tag{84}$$

$$\overline{\partial}u_{-2n}(z) + \partial u_{-(2n+2)}(z) + au_{-(2n+1)}(z) = f_{2n+1}(z), \qquad 0 \le n \le \frac{m-1}{2}, \tag{85}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-(n+2)}(z) + au_{-(n+1)}(z) = 0, \qquad n \ge m. \tag{86}$$

Given the data g for attenuated X-ray transform for even or odd order tensor field, we expand the data g in terms of its Fourier modes in the angular variables: $g(\zeta,\boldsymbol{\theta}) = \sum_{n=-\infty}^{\infty} g_n(\zeta)e^{\mathrm{i}n\theta}$, for $\zeta \in \varGamma$. Since the data g is also real valued, its Fourier modes will satisfy $g_{-n} = \overline{g_n}$, for $n \ge 0$. From the negative modes, we built the sequence $\mathbf{g} := \langle g_0, g_{-1}, g_{-2}, ...\rangle$. From the special function h defined in (32) and the data g, we built the sequence

$$\mathbf{g}_h := e^{-G}\mathbf{g} := \langle \gamma_0, \gamma_{-1}, \gamma_{-2}, ...\rangle, \tag{87}$$

where e^{-G} as defined in (34).

Next we characterize the attenuated X-ray data g in terms of its Fourier modes $g_0, g_{-1}, \cdots g_{-(m-1)}$, and the Fourier modes

$$L^m\mathbf{g}_h := L^m e^{-G}\mathbf{g} := \langle \gamma_{-m}, \gamma_{-(m+1)}, \gamma_{-(m+2)}, ...\rangle.$$

Similar to the non-attenuated case as before, we construct simultaneously the right hand side of the transport equation (80) together with the solution u via its Fourier modes. In both cases, apart from modes $u_0, u_{-1}, u_{-2}, \cdots u_{-(m-1)}$, all Fourier modes are constructed uniquely from the data $L^m\mathbf{g}_h$. For even $m \ge 2$, the modes $u_0, u_{-2}, u_{-4}, \cdots u_{-(m-2)}$ will be chosen arbitrarily from the class $\Psi_{a,g}^{\text{even}}$ of cardinality $\frac{m}{2}$ with prescribed boundary value and gradient on $\varGamma$ defined as

$$\begin{aligned}\Psi_{a,g}^{\text{even}} := \Big\{ &\big(\psi_0, \psi_{-2}, \cdots, \psi_{-(m-2)}\big) \in C^2(\overline{\Omega};\mathbb{R}) \times \big(C^2(\overline{\Omega};\mathbb{C})\big)^{\frac{m}{2}-1} : \\ &\psi_{-2j}\big|_{\varGamma} = g_{-2j}, \qquad 0 \le j \le \frac{m}{2} - 1,\ m \ge 2, \\ &\overline{\partial}\psi_{-(m-2)}\big|_{\varGamma} = -\partial(e^{G}\mathcal{B}e^{-G}\mathbf{g})_{-m}\big|_{\varGamma} - a\big|_{\varGamma}\, g_{-(m-1)}, \quad m \ge 2, \\ &\overline{\partial}\psi_{-2j}\big|_{\varGamma} = -\partial\psi_{-(2j+2)}\big|_{\varGamma} - a\big|_{\varGamma}\, g_{-(2j+1)}, \quad 0 \le j \le \frac{m}{2} - 2,\ m \ge 4 \Big\},\end{aligned} \tag{88}$$

and for odd $m \geq 1$, the modes $u_{-1}, u_{-3}, \cdots, u_{-(m-2)}$ will be chosen arbitrarily from the class $\Psi^{\text{odd}}_{a,g}$ of cardinality $\frac{m-1}{2}$ with prescribed boundary value and gradient on $\varGamma$ defined as

$$\begin{aligned}\Psi^{\text{odd}}_{a,g} := \Big\{ \big(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(m-2)}\big) \in \big(C^2(\overline{\Omega};\mathbb{C})\big)^{\frac{m-1}{2}} :\\
&\psi_{-(2j-1)}\big|_{\varGamma} = g_{-(2j-1)}, \qquad 1 \leq j \leq \frac{m-1}{2},\ m \geq 3,\\
&\overline{\partial}\psi_{-(m-2)}\big|_{\varGamma} = -\partial(e^{G}\mathcal{B}e^{-G}\mathbf{g})_{-m}\big|_{\varGamma} - a\big|_{\varGamma}\, g_{-(m-1)}, \quad m \geq 3,\\
&\overline{\partial}\psi_{-(2j-1)}\big|_{\varGamma} = -\partial\psi_{-(2j+1)}\big|_{\varGamma} - a\big|_{\varGamma}\, g_{-2j}, \quad 1 \leq j \leq \frac{m-3}{2}, \quad m \geq 5,\\
&2\left(\mathbb{R}\mathrm{e}\,\partial\psi_{-1}\big|_{\varGamma}\right) = -a|_{\varGamma}\, g_0 \Big\},\end{aligned} \tag{89}$$

where $\mathcal{B}$ be the Bukhgeim-Cauchy operator in (25), and the operators $e^{\pm G}$ as defined in (34).

Theorem 5.1 (Range characterization)**.** *Let $a \in C^{2,\mu}(\overline{\Omega})$, $\mu > 1/2$ with $\min_{\overline{\Omega}} a > 0$, and $\mathbb{Z} \ni m \geq 0$.*

(i) Let $\mathbf{f} \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$ be a real-valued m-order symmetric tensor field, and

$$g = X_a\mathbf{f} \text{ on } \varGamma_+ \text{ and } g = 0 \text{ on } \varGamma_- \cup \varGamma_0.$$

Then $\mathbf{g}_h \in l^{1,1}_{\infty}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$ satisfy

$$[I + \mathrm{i}\mathcal{H}]L^m\mathbf{g}_h = \mathbf{0}, \tag{90}$$

where $\mathbf{g}_h$ is sequence in (87) *and $\mathcal{H}$ is the Bukhgeim-Hilbert operator in* (27)*. Additionally, in $m = 1$ case, for each $\zeta \in \varGamma$, the zero-th Fourier mode g_0 of g satisfy*

$$g_0(\zeta) = \lim_{\Omega \ni z \to \zeta \in \varGamma} \frac{-2\,\mathbb{R}\mathrm{e}\,\partial(e^{G}\mathcal{B}\mathbf{g}_h)_{-1}(z)}{a(z)}, \quad \text{for} \quad m = 1, \tag{91}$$

where $\mathcal{B}$ be the Bukhgeim-Cauchy operator in (25)*, and the operators $e^{\pm G}$ as defined in* (34)*.*

(ii) Let $g \in C^{\mu}\left(\varGamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\varGamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\varGamma_- \cup \varGamma_0} = 0$. For $m = 0$, if the corresponding sequences $\mathbf{g}_h \in Y_{\mu}(\varGamma)$ satisfies (90)*, then there is a unique real valued symmetric 0-tensor $\mathbf{f}$ such that $g|_{\varGamma_+} = X_a\mathbf{f}$. Moreover, for $m = 1$, if the corresponding sequences $\mathbf{g}_h \in Y_{\mu}(\varGamma)$ satisfies* (90)*, and g_0 satisfies* (91)*, then there exists a unique real valued vector field (1-tensor) $\mathbf{f} \in C(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+} = X_a\mathbf{f}$. Furthermore, for $m \geq 2$, if $\mathbf{g}_h \in Y_{\mu}(\varGamma)$ satisfies* (90)*, and for each element $\big(\psi_0, \psi_{-2}, \cdots, \psi_{-(m-2)}\big) \in \Psi^{\text{even}}_{a,g}$ for even m-tensor, and $\big(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(m-2)}\big) \in \Psi^{\text{odd}}_{a,g}$ for m-odd tensor, then there is a unique real valued symmetric m-tensor $\mathbf{f}_{\Psi} \in C(\mathbf{S}^m;\Omega)$ such that $g|_{\varGamma_+} = X_a\mathbf{f}_{\Psi}$.*

Proof. (i) **Necessity:** Let $\mathbf{f} = (f_{i_1\cdots i_m}) \in C_0^{1,\mu}(\mathbf{S}^m;\Omega)$. Since all components $f_{i_1\cdots i_m} \in C_0^{1,\mu}(\Omega)$ are compactly supported inside Ω, then for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\varGamma_0$ which yields $g \in C^{1,\mu}(\varGamma \times \mathbb{S}^1)$. Moreover, g is the boundary value on $\varGamma \times \mathbb{S}^1$ of a solution $u \in C^{1,\mu}(\overline{\Omega} \times \mathbb{S}^1)$ of the transport equation (80). By Proposition 3.1(i) and Proposition 3.2, $\mathbf{g}_h = e^{-G}\mathbf{g} \in l^{1,1}_{\infty}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$.

If u solves (80) then its Fourier modes satisfies (81) - (83) for even m-tensors, and (85)- (86) for odd m-tensors.

In either of the even m-tensor case or the odd m-tensor case, (from (83) or (86)), the sequence $L^m\mathbf{u} := \langle u_{-m}, u_{-m-1}, u_{-m-2}, \cdots \rangle$ satisfies

$$\overline{\partial}L^m\mathbf{u} + L^2\partial L^m\mathbf{u} + aL^{m+1}\mathbf{u} = \mathbf{0}.$$

Let $\mathbf{v} := e^{-G}L^m\mathbf{u}$, then by Lemma 3.1, and the fact that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = \mathbf{0}$, the sequence $\mathbf{v} = L^m e^{-G}\mathbf{u}$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$, i.e $\mathbf{v}$ is L^2 analytic, with boundary value $L^m\mathbf{g}_h$. The necessity part in Theorem 3.1 yields (90).

Additionally, in the $m = 1$ case, the Fourier modes u_0, u_{-1}, u_1 of u solve (84) for $n = 0$. Since $a > 0$ in Ω, we have

$$u_0(z) = \frac{-2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}(z)}{a(z)}, \quad z \in \Omega. \tag{92}$$

Since the left hand side of (92) is continuous all the way to the boundary, so is the right hand side. Moreover, the limit below exists and in the $m = 1$ case, we have

$$g_0(z_0) = \lim_{\Omega \ni z \to z_0 \in \Gamma} u_0(z) = \lim_{\Omega \ni z \to z_0 \in \Gamma} \frac{-2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}(z)}{a(z)},$$

thus (91) holds. This proves part (i) of the theorem.

(ii) **Sufficiency:** Let $g \in C^{\mu}\left(\Gamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\Gamma; C^{2,\mu}(\mathbb{S}^1))$ be real valued with $g|_{\Gamma_- \cup \Gamma_0} = 0$. Let the corresponding sequence $\mathbf{g}_h$ as in (87) satisfying (90). By Proposition 3.1(ii) and Proposition 3.2(iii), we have $\mathbf{g}_h \in Y_\mu(\Gamma)$.

To prove the sufficiency we will construct a real valued symmetric m-tensor $\mathbf{f}$ in Ω and a real valued function $u \in C^1(\Omega \times \mathbb{S}^1) \cap C(\overline{\Omega} \times \mathbb{S}^1)$ such that $u|_{\Gamma \times \mathbb{S}^1} = g$ and u solves (80) in Ω. The construction of such u is in terms of its Fourier modes in the angular variable and it is done in several steps. We first construct modes u_{-n} for $|n| \geq m$ from data $\mathbf{g}_h$ in either of the m-even case or the m-odd case.

Step 1: The construction of modes u_{-n} for $|n| \geq m$.

Use the Bukhgeim-Cauchy Integral formula (25) to define the L^2-analytic maps

$$\mathbf{v}(z) = \langle v_0(z), v_{-1}(z), v_{-2}(z), ...\rangle := \mathcal{B}L^m\mathbf{g}_h(z), \quad z \in \Omega.$$

By Theorem 3.1 (ii),

$$\mathbf{v} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty). \tag{93}$$

Moreover, since $\mathbf{g}_h$ satisfy the hypothesis (90), by Theorem 3.1 sufficiency part, we have

$$\mathbf{v}|_{\Gamma} = L^m\mathbf{g}_h = L^m e^{-G}\mathbf{g}. \tag{94}$$

Define the sequence valued map

$$\Omega \ni z \mapsto L^m\mathbf{u}(z) = \langle u_{-m}(z), u_{-m-1}(z), \cdots\rangle := e^G\mathbf{v}(z), \tag{95}$$

where the operator e^G as defined in (34). Since convolution preserves l_1, by Proposition 3.2, $L^m\mathbf{u} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1)$. Moreover, since $\mathbf{v} \in C^2(\Omega; l_\infty)$ as in (93), we also conclude from convolution that $L^m\mathbf{u} \in C^2(\Omega; l_\infty)$. Thus,

$$L^m\mathbf{u} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty). \tag{96}$$

As $\mathbf{v}$ is L^2 analytic, by Lemma 3.1, $L^m\mathbf{u}$ satisfies $\overline{\partial}L^m\mathbf{u} + L^2\partial L^m\mathbf{u} + aL^{m+1}\mathbf{u} = \mathbf{0}$, which in component form:

$$\overline{\partial}u_{-n} + \partial u_{-n-2} + au_{-n-1} = 0, \quad n \geq m. \tag{97}$$

Moreover, the restriction to the boundary satisfy

$$L^m\mathbf{u}|_{\Gamma} = e^G\mathbf{v}|_{\Gamma} = e^G L^m e^{-G}\mathbf{g} = L^m\mathbf{g}, \tag{98}$$

where the second equality follows from (94) and in the last equality we use the fact that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = \mathbf{0}$.

Construct the positive Fourier modes by conjugation: $u_n := \overline{u_{-n}}$, for all $n \geq m$. Moreover using (98), the boundary value $u_n|_\Gamma$ for each $n \geq m$, satisfy

$$u_n|_\Gamma = \overline{u_{-n}}|_\Gamma = \overline{g_{-n}} = g_n, \quad n \geq m. \tag{99}$$

By conjugating (97) we note that the positive Fourier modes also satisfy

$$\overline{\partial} u_{n+2} + \partial u_n + a u_{n+1} = 0, \quad n \geq m. \tag{100}$$

Step 2: The construction of the 0-tensor field $\mathbf{f}$.

In the case of the 0-tensor, all the modes in (95) is constructed from the data. Using the modes $u_0, u_{-1} \in C^2(\Omega)$ from (95), the real valued 0-tensor $\mathbf{f} = f_0 \in C(\Omega; \mathbb{R})$ is uniquely determined by

$$\mathbf{f} := 2\,\mathbb{R}\mathrm{e}\, \partial u_{-1} + a u_0. \tag{101}$$

Step 3: The construction of the 1-tensor field $\mathbf{f}$.

In the 1-tensor case, all the modes except u_0 are constructed in (95). Using (84) for $n = 0$, and $a > 0$ in Ω, we can define u_0 via the mode u_{-1} from (95) by

$$u_0(z) := -\frac{2\,\mathbb{R}\mathrm{e}\, \partial u_{-1}(z)}{a(z)}, \quad z \in \Omega. \tag{102}$$

From (91), u_0 defined above extends continuously to the boundary Γ and $u_0\big|_\Gamma = g_0$. Moreover, since $u_{-1} \in C^2(\Omega)$ from (96) and $a \in C^2(\Omega)$ we get $u_0 \in C^1(\Omega)$.

Using modes u_{-1}, u_{-2} from (95) and u_0 from (102), the real valued 1-tensor (vector field) $\mathbf{f} \in C(\Omega; \mathbb{R}^2)$ is uniquely determined by

$$\mathbf{f} = \langle 2\,\mathbb{R}\mathrm{e}\, f_1, 2\,\mathbb{I}\mathrm{m}\, f_1 \rangle, \quad \text{where} \quad f_1 := \overline{\partial} u_0 + \partial u_{-2} + a u_{-1}. \tag{103}$$

Remark 5.1. *In the attenuated case, both the 0-tensor and the 1-tensor are uniquely recovered, and there is no class.*

For m-tensor with $m \geq 2$, we next consider separately the m-even and m-odd cases. Using the first step, where modes u_{-n} for $|n| \geq m$ are already constructed from the data $\mathbf{g}_h$ in either of the m-even case or the m-odd case, we construct the remaining modes u_n for $|n| \leq m-1$ separately first in the m-even case ($m = 2q$, $q \geq 1$) and then in the m-odd case ($m = 2r+1$, $r \geq 1$).

Step 4: In the m even case, the construction of modes u_n for $|n| \leq m-1$.

Given $\left(\psi_0, \psi_{-2}, \cdots, \psi_{-2(q-1)}\right) \in \Psi_{a,g}^{\text{even}}$ arbitrary, define the modes $u_0, u_{\pm 2}, \dots, u_{\pm(2(q-1))}$ in Ω by

$$u_{-2j} := \psi_{-2j}, \quad \text{and} \quad u_{2j} := \overline{\psi_{-2j}}, \quad 0 \leq j \leq q-1,\ q \geq 1. \tag{104}$$

Using the mode u_{-2q} from (95) and $\psi_{-2(q-1)}$, define the modes $u_{\pm(2q-1)}$ by

$$u_{-(2q-1)} := -\frac{\overline{\partial} \psi_{-2(q-1)} + \partial u_{-2q}}{a}, \quad \text{and} \quad u_{2q-1} := \overline{u}_{-(2q-1)}, \quad \text{for all } q \geq 1. \tag{105}$$

As $\psi_0 \in C^2(\overline{\Omega}; \mathbb{R})$ and $\psi_{-(2j+2)} \in C^2(\overline{\Omega}; \mathbb{C})$, for $0 \leq j \leq q-2,\ q \geq 2$, define modes

$$u_{-(2j+1)} := -\frac{\overline{\partial} \psi_{-2j} + \partial \psi_{-(2j+2)}}{a}, \text{ and } u_{2j+1} := \overline{u}_{-(2j+1)}, \quad \text{for all } 0 \leq j \leq q-2,\ q \geq 2.$$

By the construction in the above equations, we have

$$\begin{aligned} &u_{-2j} \in C^2(\Omega; l_\infty), && \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1, \\ &u_{-(2j+1)} \in C^1(\Omega; l_\infty), && \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1, \quad \text{and} \\ &\overline{\partial} u_{-2j} + \partial u_{-(2j+2)} + a u_{-(2j+1)} = 0, && \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1, \end{aligned} \tag{106}$$

are satisfied. Moreover, by conjugating the last equation in (106) yields

$$\partial u_{2j} + \overline{\partial} u_{(2j+2)} + a u_{(2j+1)} = 0, \quad \text{for} \quad 0 \leq j \leq q-1,\ q \geq 1. \tag{107}$$

By the definition of the class (88), and reality of g, we have

$$u_{-2j}|_{\varGamma} = g_{-2j}, \quad \text{and} \quad u_{2j}|_{\varGamma} = \overline{g_{-2j}} = g_{2j}, \quad 0 \leq j \leq q-1,\ q \geq 1. \tag{108}$$

We check next that the boundary value of $u_{-(2j+1)}$ is $g_{-(2j+1)}$ for $0 \leq j \leq q-2,\ q \geq 2$:

$$u_{-(2j+1)}\big|_{\varGamma} = -\frac{\overline{\partial}\psi_{-2j} + \partial\psi_{-(2j+2)}}{a}\Bigg|_{\varGamma} = g_{-(2j+1)}, \tag{109}$$

where the last equality uses the condition in class (88). Similar calculation to (109) for mode $u_{-(2q-1)}$ yields $u_{-(2q-1)}\big|_{\varGamma} = g_{-(2q-1)}$. Thus, from the above equations, we have

$$u_n\big|_{\varGamma} = g_n, \quad \forall\, |n| \leq m-1. \tag{110}$$

Step 5: The construction of even m-tensor $\mathbf{f}_\Psi$ whose attenuated X-ray data is g.

We define first f_{2q} by using $\psi_{-(m-2)}$ from the non-uniqueness class, and modes $u_{-m}, u_{-m-1} \in C^2(\Omega; l_\infty)$ from (95). Then, next define f_{2q-2} by using $\psi_{-2(q-1)}, \psi_{-2(q-2)}$ from the non-uniqueness class $\Psi^{\text{even}}_{a,g}$, and Fourier mode u_{-2q} from (95). Then, define $\{f_{2n} :\ 0 \leq n \leq q-2\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-2n} :\ 1 \leq n \leq q\}$ by conjugation.

(111)

$$
\begin{aligned}
&f_{2q} := -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}\right) + \partial u_{-(2q+1)} + a u_{-2q}, \quad q \geq 1,\\
&f_{2q-2} := -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-2(q-2)} + \partial\psi_{-2(q-1)}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}\right) + a\psi_{-2(q-1)}, \quad q \geq 2,\\
&f_{2n} := -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-2(n-1)} + \partial\psi_{-2n}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-2n} + \partial\psi_{-2(n+1)}}{a}\right) + a\psi_{-2n}, \quad 1 \leq n \leq q-2, \quad q \geq 3,\\
&f_0 := \begin{cases} -2\,\mathbb{R}\mathrm{e}\,\partial\left(\dfrac{\overline{\partial}\psi_0 + \partial u_{-2}}{a}\right) + a\psi_0, & q = 1,\\ -2\,\mathbb{R}\mathrm{e}\,\partial\left(\dfrac{\overline{\partial}\psi_0 + \partial\psi_{-2}}{a}\right) + a\psi_0, & q \geq 2, \end{cases}\\
&f_{-2n} := \overline{f_{2n}}, \quad 1 \leq n \leq q,\ q \geq 1,
\end{aligned}
$$

By construction, $f_{2n} \in C(\Omega)$, for $0 \leq n \leq q,\ q \geq 1$, as $\psi_{-2n} \in C^2(\Omega; l_\infty)$, for $0 \leq n \leq q-1$, from (88). Note that f_{2n} satisfy (81). We use these Fourier modes $\langle f_0, f_{\pm 2}, \cdots, f_{\pm m}\rangle$ and equations (13), (7) and (9) to construct $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m\rangle$, and thus even m-tensor field $\mathbf{f}_\Psi \in C(\mathbf{S}^m; \Omega)$.

In order to show $g|_{\varGamma_+} = X_a \mathbf{f}_\Psi$ with $\mathbf{f}_\Psi$ being constructed from pseudovectors via Fourier modes as in (111) from class $\Psi^{\text{even}}_{a,g}$, we define the real valued function u via its Fourier modes

(112)

$$
\begin{aligned}
u(z, \boldsymbol{\theta}) := &\sum_{|n| \geq 2q} u_n(z) e^{\mathrm{i}n\theta} + 2\,\mathbb{R}\mathrm{e}\left(-\frac{\overline{\partial}\psi_{-2(q-1)} + \partial u_{-2q}}{a}\right) e^{-\mathrm{i}(2q-1)\theta}\\
&+ 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=0}^{q-1} \psi_{-2n}(z) e^{-\mathrm{i}(2n)\theta}\right\} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=0}^{q-2}\left(-\frac{\overline{\partial}\psi_{-2j} + \partial\psi_{-(2j+2)}}{a}\right) e^{-\mathrm{i}(2n+1)\theta}\right\}
\end{aligned}
$$

and check that it has the boundary value g on Γ and satisfies the transport equation (80).

Since $g \in C^{\mu}\left(\Gamma; C^{1,\mu}(\mathbb{S}^1)\right) \cap C(\Gamma; C^{2,\mu}(\mathbb{S}^1))$, we use Proposition 3.1 (ii) and (iii), to conclude that u defined in (112) belongs to $C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$. In particular $u(\cdot, \boldsymbol{\theta})$ for $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$ extends to the boundary and it satisfies

$$u(\cdot,\boldsymbol{\theta})|_{\Gamma} = \sum_{|n|\geq 2q} u_n\big|_{\Gamma} e^{\mathrm{i}n\theta} + \sum_{|n|\leq 2q-1} u_n\big|_{\Gamma} e^{\mathrm{i}n\theta} = \sum_{|n|\geq 2q} g_n e^{\mathrm{i}n\theta} + \sum_{|n|\leq 2q-1} g_n e^{\mathrm{i}n\theta} = g(\cdot,\boldsymbol{\theta}),$$

where in the second equality above we use (94), (99) and (110).

Since $u \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$, then using (97), (100), (105), (106), (107), and the definition of f_{2n} for $-q \leq n \leq q,\ q \geq 1$ in (111), the real valued u defined in (112) satisfies the transport equation (80) in the m-even case.

We consider next the m-odd tensor case of order $m = 2r+1$, $r \geq 1$, Using the first step, where modes u_{-n} for $|n| \geq m$ are already constructed from the data $\mathbf{g}_h$, we construct the remaining modes u_n for $|n| \leq m-1$ in the m-odd case.

Step 6: In the m odd case, the construction of modes u_n for $|n| \leq m-1$.

Given $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(m-2)}\right) \in \Psi^{\mathrm{odd}}_{a,g}$ arbitrary, firstly define the odd modes

$$u_{-(2n-1)} := \psi_{-(2n-1)}, \quad \text{and} \quad u_{2n-1} := \overline{\psi}_{-(2n-1)}, \quad 1 \leq n \leq r,\ r \geq 1. \tag{113}$$

Secondly, by using $\psi_{-1}, \psi_{-(m-2)}$ and the mode u_{-m} from (95), we define the modes

$$u_0 := -\frac{2\,\mathbb{R}\mathrm{e}\,\partial\psi_{-1}}{a}, \quad u_{-2r} := -\frac{\overline{\partial}\psi_{-(2r-1)} + \partial u_{-(2r+1)}}{a}, \text{ and } \quad u_{2r} := \overline{u_{-2r}} \quad \text{for} \quad r \geq 1. \tag{114}$$

Lastly, by using $\psi_{-(2n-1)} \in C^2(\overline{\Omega}; \mathbb{C})$, for $1 \leq n \leq r-1,\ r \geq 2$, we define the even modes $u_{-2n} := -\dfrac{\overline{\partial}\psi_{-(2n-1)} + \partial\psi_{-(2n+1)}}{a}$, and $u_{2n} := \overline{u_{-2n}}$, for $1 \leq n \leq r-1,\ r \geq 2$. By the construction in the above equations, we have

$$\begin{aligned} &u_{-(2n-1)} \in C^2(\Omega; l_\infty), \quad \text{for} \quad 1 \leq n \leq r,\ r \geq 1,\\ &u_{-2n} \in C^1(\Omega; l_\infty), \quad \text{for} \quad 0 \leq n \leq r,\ r \geq 1, \quad \text{and}\\ &\overline{\partial} u_{-(2n-1)} + \partial u_{-(2n+1)} + a u_{-2n} = 0, \quad \text{for} \quad 0 \leq n \leq r,\ r \geq 1, \end{aligned} \tag{115}$$

is satisfied. Moreover, by conjugating the last equation in (115), we have the Fourier modes satisfy

$$\overline{\partial} u_{-(2n-1)} + \partial u_{-(2n+1)} + a u_{-2n} = 0, \quad \text{for} \quad |n| \leq r,\ r \geq 1. \tag{116}$$

By the class (89), and reality of g, we have the boundary value of $u_{-(2n-1)}$ in (113) satisfy

$$u_{-(2n-1)}|_{\Gamma} = g_{-(2n-1)}, \quad \text{and} \quad u_{2n-1}|_{\Gamma} = \overline{g}_{-(2n-1)} = g_{2n-1}, \quad 1 \leq n \leq r,\ r \geq 1. \tag{117}$$

We check next that the boundary value of u_{-2n} is g_{-2n} for $1 \leq n \leq r-1,\ r \geq 2$:

$$u_{-2n}\big|_{\Gamma} = -\frac{\overline{\partial}\psi_{-(2n-1)} + \partial\psi_{-(2n+1)}}{a}\Bigg|_{\Gamma} = g_{-2n}, \tag{118}$$

where the last equality uses the condition in class (89). Similar calculation to (118) for modes u_0 and u_{-2r} in (114), yields $u_0\big|_{\Gamma} = g_0$, and $u_{-2r}\big|_{\Gamma} = g_{-2r}$, for $r \geq 1$. Thus, we have

$$u_n\big|_{\Gamma} = g_n, \quad \forall\, |n| \leq m-1. \tag{119}$$

Step 7: The construction of odd m-tensor $\mathbf{f}_\Psi$ whose attenuated X-ray data is g.

We first define f_{2r+1} by using $\psi_{-(2r-1)}$ from the non-uniqueness class, and the Fourier modes $u_{-m}, u_{-(m+1)}$ in (95). Next, define f_{2r-1} by using $\psi_{-(m-2)}, \psi_{-(m-4)}$ from the non-uniqueness class, and Fourier mode u_{-m} in (95). Then, define $\{f_{2n+1} : \ 0 \leq n \leq r-2\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-(2n+1)} : \ 0 \leq n \leq r\}$ by conjugation.

(120)

$$
\begin{aligned}
f_{2r+1} &:= -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-(2r-1)} + \partial u_{-(2r+1)}}{a}\right) + \partial u_{-(2r+2)} + a u_{-(2r+1)}, \quad r \geq 1,\\
f_{2r-1} &:= -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-(2r-3)} + \partial \psi_{-(2r-1)}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-(2r-1)} + \partial u_{-(2r+1)}}{a}\right) + a\psi_{-(2r-1)}, \quad r \geq 2,\\
f_{2n+1} &:= -\overline{\partial}\left(\frac{\overline{\partial}\psi_{-(2n-1)} + \partial \psi_{-(2n+1)}}{a}\right) - \partial\left(\frac{\overline{\partial}\psi_{-(2n+1)} + \partial \psi_{-(2n+3)}}{a}\right) + a\psi_{-(2n+1)}, \ 1 \leq n \leq r-2,\\
f_1 &:= \begin{cases} -2\overline{\partial}\left(\dfrac{\mathbb{R}\mathrm{e}\, \partial\psi_{-1}}{a}\right) - \partial\left(\dfrac{\overline{\partial}\psi_{-1} + \partial u_{-3}}{a}\right) + a\psi_{-1}, \quad r = 1,\\ -2\overline{\partial}\left(\dfrac{\mathbb{R}\mathrm{e}\, \partial\psi_{-1}}{a}\right) - \partial\left(\dfrac{\overline{\partial}\psi_{-1} + \partial \psi_{-3}}{a}\right) + a\psi_{-1}, \quad r \geq 2,\end{cases}\\
f_{-(2n+1)} &:= \overline{f_{2n+1}}, \quad 0 \leq n \leq r, \ r \geq 1.
\end{aligned}
$$

By construction, $f_{2n+1} \in C(\Omega)$ for $0 \leq n \leq r, \ r \geq 1$, as $u_{-(2r+1)} \in C^2(\Omega; l_\infty)$ from (96), and $\psi_{-(2n-1)} \in C^2(\Omega; l_\infty)$, for $1 \leq n \leq r-1, \ r \geq 1$, from (89). We use these $m+1$ Fourier modes $\langle f_{\pm 1}, f_{\pm 3}, \cdots, f_{\pm m}\rangle$, and equations (14), (7) and (9) to construct the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m\rangle$, and thus the odd m-tensor field $\mathbf{f}_\Psi \in C(\mathbf{S}^m; \Omega)$.

Define the real valued function u via its Fourier modes

$$
\begin{aligned}
u(z, \boldsymbol{\theta}) &:= \sum_{|n| \geq 2r+1} u_n(z) e^{\mathrm{i}n\theta} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=1}^{r} \psi_{-(2n-1)}(z) e^{-\mathrm{i}(2n-1)\theta}\right\} + \frac{-2\,\mathbb{R}\mathrm{e}\,\partial\psi_{-1}(z)}{a}\\
&\quad + 2\,\mathbb{R}\mathrm{e}\left(-\frac{\overline{\partial}\psi_{-(2r-1)}(z) + \partial u_{-(2r+1)}(z)}{a}\right) e^{-\mathrm{i}(2r)\theta} + 2\,\mathbb{R}\mathrm{e}\left\{\sum_{n=1}^{r-1} u_{-2n} e^{-\mathrm{i}(2n\theta)}\right\}.
\end{aligned} \tag{121}
$$

Using (116) and (119), and definition of $\left(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2r-1)}\right) \in \Psi^{\text{odd}}_{a,g}$ for $r \geq 1$, then $u(\cdot, \boldsymbol{\theta})$ in (121) extends to the boundary, and its boundary value satisfy $u(\cdot, \boldsymbol{\theta})|_\varGamma = g(\cdot, \boldsymbol{\theta})$.

Moreover, by using (115), (116) and the definition of f_{2n-1} for $|n| \leq r, \ r \geq 1$ in (120), the real valued u defined in (121) satisfies the transport equation (80) in the m-odd case.

□

6. Conclusion

In conclusion, we characterize the X-ray data g in both the non-attenuated case (Theorem 4.1 and Theorem 4.2) and the attenuated case (Theorem 5.1) for arbitrary m-tensor in terms of its Fourier modes $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \dots\rangle$ in the spatial variable and the Bukhgeim-Hilbert transform (27). In the necessity, while the non-attenuated case separates the even modes $\mathbf{g}^{\text{even}} = \langle g_0, g_{-2}, \dots\rangle$ and the odd modes $\mathbf{g}^{\text{odd}} = \langle g_{-1}, g_{-3}, \dots\rangle$ (see Table 1), the attenuated case mixes all the even and odd Fourier modes: $\mathbf{g}_h = e^{-G}\mathbf{g}$ (see (90) and Table 2). In the sufficiency part, in both cases we showed that reconstruction of the m-tensor field for $m \geq 2$ is possible upto a non-uniqueness class of functions that extends continuously to the boundary with prescribed boundary values. Moreover, the 1-tensor is uniquely recovered in the attenuated case, and in the non-attenuated case, the 1-tensor is recovered upto an arbitrary function (see Remark 4.2). Furthermore, the 0-tensor field is uniquely recovered (see also Remark 4.1 and Remark 5.1) in both cases.

TABLE 1. Range Characterization in the non-attenuated case

Data $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, g_{-3}, \dots \rangle$	Theorem 4.1, m-even	Theorem 4.2, m-odd
$\mathbf{g}^{\text{even}} = \langle g_0, g_{-2}, g_{-4}, \dots \rangle$	$[I + \mathrm{i}\mathcal{H}]\mathbf{g}^{\text{even}} = \mathbf{0}$	$[I + \mathrm{i}\mathcal{H}]L^{\frac{m+1}{2}}\mathbf{g}^{\text{even}} = \mathbf{0}$
$\mathbf{g}^{\text{odd}} = \langle g_{-1}, g_{-3}, g_{-5}, \dots \rangle$	$[I + \mathrm{i}\mathcal{H}]L^{\frac{m}{2}}\mathbf{g}^{\text{odd}} = \mathbf{0}$	
$\mathbf{g}^{2k-1} = \langle g_{2k-1}, \dots, g_{-1}, g_{-3}, \dots \rangle, k \geq 1$		$[I + \mathrm{i}\mathcal{H}]\mathbf{g}^{2k-1} = \mathbf{0}$

TABLE 2. Range Characterization in the attenuated case

Data $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, g_{-3}, \dots \rangle$	Theorem 5.1, m-tensor
$\mathbf{g}_h = e^{-G}\mathbf{g} = \langle \gamma_0, \gamma_{-1}, \gamma_{-2}, ... \rangle$	$[I + \mathrm{i}\mathcal{H}]L^m\mathbf{g}_h = \mathbf{0}$

ACKNOWLEDGMENT

The work of D. Omogbhe and K. Sadiq were supported by the Austrian Science Fund (FWF), Project P31053-N32 and by the FWF Project F6801N36 within the Special Research Program SFB F68 Tomography Across the Scales.

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