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AN INVERSE SOURCE PROBLEM FOR LINEARLY ANISOTROPIC RADIATIVE SOURCES IN ABSORBING AND SCATTERING MEDIUM

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ABSTRACT. We consider in a two dimensional absorbing and scattering medium, an inverse source problem in the stationary radiative transport, where the source is linearly anisotropic. The medium has an anisotropic scattering property that is neither negligible nor large enough for the diffusion approximation to hold. The attenuating and scattering properties of the medium are assumed known. For scattering kernels of finite Fourier content in the angular variable, we show how to recover the anisotropic radiative sources from boundary measurements. The approach is based on the Cauchy problem for a Beltrami-like equation associated with A-analytic maps. As an application, we determine necessary and sufficient conditions for the data coming from two different sources to be mistaken for each other.

1. INTRODUCTION

In this work, we consider an inverse source problem for stationary radiative transfer (transport) [6, 7], in a two-dimensional bounded, strictly convex domain $\Omega \subset \mathbb{R}^2$, with boundary Γ. The stationary radiative transport models the linear transport of particles through a medium and includes absorption and scattering phenomena. In the steady state case, when generated solely by a linearly anisotropic source f inside Ω, the density $u(z, \boldsymbol{\theta})$ of particles at z traveling in the direction $\boldsymbol{\theta}$ solves the stationary radiative transport boundary value problem

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) + a(z, \boldsymbol{\theta})u(z, \boldsymbol{\theta}) - \int_{\mathbf{S}^1} k(z, \boldsymbol{\theta}, \boldsymbol{\theta}')u(z, \boldsymbol{\theta}')d\boldsymbol{\theta}' = f(z, \boldsymbol{\theta}), \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1, \\ &u|_{\Gamma_-} = 0. \end{aligned} \tag{1}$$

In boundary value problem (1), the function $a(z, \boldsymbol{\theta})$ is the medium capability of absorption per unit path-length at z moving in the direction $\boldsymbol{\theta}$ called the attenuation coefficient, the function $k(z, \boldsymbol{\theta}, \boldsymbol{\theta}')$ is the scattering coefficient which accounts for particles from an arbitrary direction $\boldsymbol{\theta}'$ which scatter in the direction $\boldsymbol{\theta}$ at a point z, and $\Gamma_- := \{(\zeta, \boldsymbol{\theta}) \in \Gamma \times \mathbf{S}^1 : \nu(\zeta) \cdot \boldsymbol{\theta} < 0\}$ is the incoming unit tangent sub-bundle of the boundary, with $\nu(\zeta)$ being the outer unit normal at $\zeta \in \Gamma$. The attenuation and scattering coefficients are assumed known real valued functions. The boundary condition in (1) indicates that no radiation is coming from outside the domain. Throughout, the measure $d\boldsymbol{\theta}$ on the unit sphere $\mathbf{S}^1$ is normalized to $\int_{\mathbf{S}^1} d\boldsymbol{\theta} = 1$.

The (forward) boundary value problem (1) is known to be well-posed under various assumptions, e.g in [9, 8, 2, 18], with a general result in [29] showing that, for an open and dense set of coefficients $a \in C^2(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, the boundary value problem (1) has a unique solution $u \in L^2(\Omega \times \mathbf{S}^1)$ for any $f \in L^2(\Omega \times \mathbf{S}^1)$. In [14], it is shown that for attenuation merely *once* differentiable, $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, the boundary value problem (1) has a unique

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solution $u \in L^p(\Omega \times \mathbf{S}^1)$ for any $f \in L^p(\Omega \times \mathbf{S}^1)$, $p > 1$. Moreover, uniqueness result of the forward problem (1) are also establish in weighted L^p spaces in [11], and in [27, 16] using Carleman estimates.

In our reconstruction method here, some of our arguments require solutions $u \in C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$, $\frac{1}{2} < \mu < 1$. We revisit the arguments in [29, 14] and show that such a regularity can be achieved for sources $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, $p > 4$; see Theorem A.2 (iii) below.

For a given medium, i.e., a and k both known, we consider the inverse problem of determining the linear anisotropic source $f = f_0 + \boldsymbol{\theta} \cdot \mathbf{F}$; in particular, recovering the isotropic scalar field f_0 and the vector field $\mathbf{F}$ from measurements $g_{f_0,\mathbf{F}}$ of exiting radiation on Γ,

$$u|_{\Gamma_+} = g_{f_0,\mathbf{F}}, \tag{2}$$

where $\Gamma_+ := \{(z, \boldsymbol{\theta}) \in \Gamma \times \mathbf{S}^1 : \nu(z) \cdot \boldsymbol{\theta} > 0\}$ is the outgoing unit tangent sub-bundle of the boundary.

For anisotropic sources the problem has non-uniqueness [26, 8, 4, 30]. One of our main result, Theorem 3.1 shows that from boundary measurement data $g_{f_0,\mathbf{F}}$, one can only recover the part of the linear anisotropic source $f = f_0 + \boldsymbol{\theta} \cdot \mathbf{F}$; in particular, only the solenoidal part $\mathbf{F}^s$ of the vector field $\mathbf{F}$ is recovered inside the domain. However, in Theorem 3.2, if one know apriori that the source $\mathbf{F}$ is divergence-free, then from the data $g_{f_0,\mathbf{F}}$, one can recover both isotropic field f_0 and the vector field $\mathbf{F}$ inside the domain. Moreover, instead of apriori information of the divergence-free source $\mathbf{F}$, if one has the additional data $g_{f_0,\mathbf{0}}$ information along with the data $g_{f_0,\mathbf{F}}$, then in Theorem 3.3, one can recover both sources f_0 and $\mathbf{F}$ under subcritical assumption of the medium. One of the main crux in our reconstruction method is the observation that any finite Fourier content in the angular variable of the scattering kernel splits the problem into an infinite system of non-scattering case and a boundary value problem for a finite elliptic system. The role of the finite Fourier content has been independently recognized in [13] and [19].

The inverse source problem above has applications in medical imaging: In a non-scattering ($k = 0$) and non-attenuating ($a = 0$) medium the problem is mathematically equivalent to the one occurring in classical computerized X-ray tomography (e.g., [5, 21]). In the absorbing non-scattering medium, such a problem (with only isotropic source $f = f_0$), appears in Positron/Single Photon Emission Tomography [21, 22], and $f = \boldsymbol{\theta} \cdot \mathbf{F}$ with $f_0 = 0$, appears in Doppler Tomography [22, 21, 28]. For applications in scattering media the inverse source problem formulated here is the two dimensional version of the corresponding three dimensional problem occurring in imaging techniques such as Bioluminescence tomography and Optical Molecular Imaging, see [31, 15, 17] and references therein.

In this work, except for the results in the appendix, the attenuation coefficient are assumed isotropic $a = a(z)$, and that the scattering kernel $k(z, \boldsymbol{\theta}, \boldsymbol{\theta}') = k(z, \boldsymbol{\theta} \cdot \boldsymbol{\theta}')$ depends polynomially on the angle between the directions. Moreover, the functions a, k and the source f are assumed real valued.

In Section 2, we recall some basic properties of A-analytic theory, and in Section 3 we provide the reconstruction method for the full (part) of the linearly anisotropic source. Our approach is based on the Cauchy problem for a Beltrami-like equation associated with A-analytic maps in the sense of Bukhgeim [5]. The A-analytic approach developed in [5] treats the non-attenuating case, and the absorbing but non-scattering case is treated in [3]. The original idea of Bukhgeim from the absorbing non-scattering media [5, 3] to the absorbing and scattering media has been extended in [13, 14]. In here we extend the results in [13, 14] to linear anisotropic sources.

In Section 4, the method used will explain when the data coming from two different linear anisotropic field sources can be mistaken for each other.

In the appendix, we revisit the arguments in [29, 14] and remark on the existence and regularity of the forward boundary value problem. The results in the appendix consider both attenuation coefficient and scattering kernel in general setting.

2. Ingredients from A-analytic theory

In this section we briefly introduce the properties of A-analytic maps needed later, and introduce notation. We recall some of the existing results and concepts used in our reconstruction method.

For $0 < \mu < 1$, $p = 1, 2$, we consider the Banach spaces:

$$C^{\mu}(\varGamma; l_1) := \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ...\rangle : \sup_{\xi\in\varGamma} \|\mathbf{g}(\xi)\|_{l_1} + \sup_{\substack{\xi,\eta\in\varGamma\\ \xi\neq\eta}} \frac{\|\mathbf{g}(\xi)-\mathbf{g}(\eta)\|_{l_1}}{|\xi-\eta|^{\mu}} < \infty \right\},$$

(3)

$$Y_{\mu}(\varGamma) := \left\{ \mathbf{g} : \sup_{\xi\in\varGamma} \sum_{j=0}^{\infty} \langle j\rangle^2 |g_{-j}(\xi)| < \infty, \text{ and } \sup_{\substack{\xi,\eta\in\varGamma\\ \xi\neq\eta}} \sum_{j=0}^{\infty} \langle j\rangle \frac{|g_{-j}(\xi)-g_{-j}(\eta)|}{|\xi-\eta|^{\mu}} < \infty \right\},$$

where $\langle j\rangle = (1+|j|^2)^{1/2}$. Similarly, we consider $C^{\mu}(\overline{\Omega}; l_1)$, and $C^{\mu}(\overline{\Omega}; l_{\infty})$.

For $z = x_1 + \mathrm{i}x_2$, we consider the Cauchy-Riemann operators $\overline{\partial} = (\partial_{x_1} + \mathrm{i}\partial_{x_2})/2$, $\partial = (\partial_{x_1} - \mathrm{i}\partial_{x_2})/2$. A sequence valued map $\Omega \ni z \mapsto \mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), ...\rangle$ in $C(\overline{\Omega}; l_{\infty}) \cap C^1(\Omega; l_{\infty})$ is called L^2-*analytic* (in the sense of Bukhgeim [5]), if

$$\overline{\partial}\mathbf{v}(z) + L^2\partial\mathbf{v}(z) = 0, \quad z \in \Omega, \tag{4}$$

where L is the left shift operator $L\langle v_0, v_{-1}, v_{-2}, \cdots\rangle = \langle v_{-1}, v_{-2}, \cdots\rangle$, and $L^2 = L \circ L$.

Bukhgeim's original theory [5] shows that solutions of (4), satisfy a Cauchy-like integral formula,

$$\mathbf{v}(z) = \mathcal{B}[\mathbf{v}|_{\varGamma}](z), \quad z \in \Omega, \tag{5}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator acting on $\mathbf{v}|_{\varGamma}$. We use the formula in [12], where $\mathcal{B}$ is defined component-wise for $n \geqslant 0$ by

$$(\mathcal{B}\mathbf{v})_{-n}(z) := \frac{1}{2\pi\mathrm{i}} \int_{\varGamma} \frac{v_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\mathrm{i}} \int_{\varGamma} \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \right\} \sum_{j=1}^{\infty} v_{-n-2j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^j, z \in \Omega. \tag{6}$$

Similar to the analytic maps, the traces of L^2-analytic maps on the boundary must satisfy some constraints, which can be expressed in terms of a corresponding Hilbert-like transform introduced in [23]. More precisely, the Bukhgeim-Hilbert transform $\mathcal{H}$ is defined component-wise for $n \geqslant 0$ by

$$(\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi} \int_{\varGamma} \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi} \int_{\varGamma} \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \right\} \sum_{j=1}^{\infty} g_{-n-2j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^j, \ z \in \varGamma, \tag{7}$$

and we refer to [23] for its mapping properties.

Another ingredient, in addition to L^2-analytic maps, consists in the one-to-one relation between solutions $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, ...\rangle$ satisfying

$$\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = \mathbf{0}, \tag{8}$$

and the L^2-analytic map $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ satisfying (4), via a special function h, see [25, Lemma 4.2] for details. The function h is defined as $h(z, \boldsymbol{\theta}) := \displaystyle\int_0^{\infty} a(z + t\boldsymbol{\theta})dt - \frac{1}{2}(I - \mathrm{i}H)\, Ra(z \cdot \boldsymbol{\theta}^{\perp}, \boldsymbol{\theta}^{\perp})$, where $Ra(s, \boldsymbol{\theta}^{\perp}) = \displaystyle\int_{-\infty}^{\infty} a\left(s\boldsymbol{\theta}^{\perp} + t\boldsymbol{\theta}\right) dt$ is the Radon transform of the attenuation a, and $Hh(s) = \displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{h(t)}{s-t} dt$ is the classical Hilbert transform [20]. The function h has vanishing negative Fourier modes yielding the expansions $e^{-h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \alpha_k(z) e^{\mathrm{i}k\theta},\ e^{h(z,\boldsymbol{\theta})} := \sum_{k=0}^{\infty} \beta_k(z) e^{\mathrm{i}k\theta}$, for $(z,\boldsymbol{\theta}) \in \overline{\Omega} \times \mathbf{S}^1$. Using the Fourier coefficients of $e^{\pm h}$, define the sequence valued maps

$$\overline{\Omega} \ni z \mapsto \boldsymbol{\beta}(z) := \langle \beta_0(z), \beta_1(z), ..., \rangle \quad \overline{\Omega} \ni z \mapsto \boldsymbol{\alpha}(z) := \langle \alpha_0(z), \alpha_1(z), ..., \rangle,$$

and define the operators $e^{\pm G}$ component-wise for each $n \leqslant 0$, by

$$(e^{-G}\mathbf{u})_n = (\boldsymbol{\alpha} * \mathbf{u})_n = \sum_{k=0}^{\infty} \alpha_k u_{n-k}, \quad \text{and} \quad (e^{G}\mathbf{u})_n = (\boldsymbol{\beta} * \mathbf{u})_n = \sum_{k=0}^{\infty} \beta_k u_{n-k}. \tag{9}$$

Note the commutating property $[e^{\pm G}, L] = \mathbf{0}$.

Lemma 2.1. [24, Lemma 4.2] *Let $a \in C^{1,\mu}(\overline{\Omega})$, $\mu > 1/2$, and $e^{\pm G}$ be operators as defined in* (9).

(i) If $\mathbf{u} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = \mathbf{0}$, then $\mathbf{v} = e^{-G}\mathbf{u} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$.

(ii) Conversely, if $\mathbf{v} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$, then $\mathbf{u} = e^{G}\mathbf{v} \in C^1(\Omega, l_1)$ solves $\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + aL\mathbf{u} = \mathbf{0}$.

3. Reconstruction of a sufficiently smooth linearly anisotropic source

For an isotropic real valued vector field $\mathbf{F}$ and real map f_0, recall the boundary value problem (1):

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla u(z,\boldsymbol{\theta}) + a(z)u(z,\boldsymbol{\theta}) - \int_{\mathbf{S}^1} k(z, \boldsymbol{\theta}\cdot\boldsymbol{\theta}') u(z,\boldsymbol{\theta}') d\boldsymbol{\theta}' = \underbrace{f_0(z) + \boldsymbol{\theta}\cdot\mathbf{F}(z)}_{f(z,\boldsymbol{\theta})}, \ (z,\boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1, \\ &u|_{\Gamma_-} = 0, \end{aligned} \tag{10}$$

with an isotropic attenuation $a = a(z)$, and with the scattering kernel $k(z,\boldsymbol{\theta},\boldsymbol{\theta}') = k(z,\boldsymbol{\theta}\cdot\boldsymbol{\theta}')$ depending polynomially on the angle between the directions,

$$k(z, \cos\theta) = k_0(z) + 2\sum_{n=1}^{M} k_{-n}(z)\cos(n\theta), \tag{11}$$

for some fixed integer $M \geqslant 1$. Note that, since $k(z,\cos\theta)$ is both real valued and even in θ, the coefficient k_{-n} is the $(-n)^{th}$ Fourier coefficient of $k(z,\cos(\cdot))$. Moreover k_{-n} is real valued, and $k_n(z) = k_{-n}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(z,\cos\theta)e^{in\theta}d\theta$.

For the real vector field $\mathbf{F} = \langle F_1, F_2 \rangle$, let

$$f_1 := (F_1 + \mathrm{i}F_2)/2, \tag{12}$$

and for $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in \mathbf{S}^1$, a calculation shows that the linear anisotropic source

$$f(z,\boldsymbol{\theta}) = f_0(z) + \boldsymbol{\theta}\cdot\mathbf{F}(z) = f_0(z) + \overline{f_1(z)}e^{\mathrm{i}\theta} + f_1(z)e^{-\mathrm{i}\theta}. \tag{13}$$

We assume that the coefficients $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ are such that the forward problem (10) has a unique solution $u \in L^p(\Omega \times \mathbf{S}^1)$ for any $f \in L^p(\Omega \times \mathbf{S}^1)$, $p > 1$, see Theorem A.1. Moreover, we assume also an *unknown* source of a priori regularity $f \in W^{3,p}(\overline{\Omega};\mathbb{R})$, $p > 4$, and by Theorem A.2 part (iii), the solution $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$, $p > 4$. Furthermore, the functions a, k and source f are assumed real valued, so that the solution u is also real valued.

Let $u(z,\boldsymbol{\theta}) = \sum_{-\infty}^{\infty} u_n(z)e^{in\theta}$ be the formal Fourier series representation of the solution of (10) in the angular variable $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$. Since u is real valued, the Fourier modes $\{u_n\}$ occurs in complex-conjugate pairs $u_{-n} = \overline{u_n}$, and the angular dependence is completely determined by the sequence of its nonpositive Fourier modes

$$\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), ...\rangle. \tag{14}$$

For the derivatives $\partial, \overline{\partial}$ in the spatial variable, the advection operator $\boldsymbol{\theta} \cdot \nabla = e^{-\mathrm{i}\theta}\overline{\partial} + e^{\mathrm{i}\theta}\partial$. By identifying the Fourier coefficients of the same order, the equation (10) reduces to the system:

$$\overline{\partial}u_1(z) + \partial u_{-1}(z) + [a(z) - k_0(z)]u_0(z) = f_0(z), \tag{15}$$

$$\overline{\partial}u_0(z) + \partial u_{-2}(z) + [a(z) - k_{-1}(z)]u_{-1}(z) = f_1(z), \tag{16}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-n-2}(z) + [a(z) - k_{-n-1}(z)]u_{-n-1}(z) = 0, \quad 1 \leqslant n \leqslant M-1, \tag{17}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-n-2}(z) + a(z)u_{-n-1}(z) = 0, \qquad n \geqslant M, \tag{18}$$

where f_1 as in (12).

By Hodge decomposition [26], any vector field $\mathbf{F} = \langle F_1, F_2\rangle \in H^1(\Omega;\mathbb{R}^2)$ decomposes into a gradient field and a divergence-free (solenoidal) field :

$$\mathbf{F} = \nabla\varphi + \mathbf{F}^s, \quad \varphi|_{\partial\Omega} = 0, \, \mathrm{div}\,\mathbf{F}^s = 0, \tag{19}$$

where $\varphi \in H_0^2(\Omega;\mathbb{R})$ and $\mathbf{F}^s = \langle F_1^s, F_2^s\rangle \in H^1_{\mathrm{div}}(\Omega;\mathbb{R}^2) := \{\mathbf{F}^s \in H^1(\Omega;\mathbb{R}^2) : \mathrm{div}\,\mathbf{F}^s = 0\}$.

Note that for f_1 in (12), we have

$$4\partial f_1 = \mathrm{div}\,\mathbf{F} + \mathrm{i}\,\mathrm{curl}\,\mathbf{F}. \tag{20}$$

Using $4\overline{\partial}\partial f_1 = \Delta f_1$, we have $\Delta F_1 = \partial_{x_1}\mathrm{div}\,\mathbf{F} - \partial_{x_2}\mathrm{curl}\,\mathbf{F}$, and $\Delta F_2 = \partial_{x_2}\mathrm{div}\,\mathbf{F} + \partial_{x_1}\mathrm{curl}\,\mathbf{F}$. Moreover, for $f_1^s = (F_1^s + \mathrm{i}F_2^s)/2$, the Hodge decomposition (19) can be rewritten as

$$f_1 = \overline{\partial}\varphi + f_1^s, \quad \varphi|_{\partial\Omega} = 0, \quad \mathbb{R}e(\partial f_1^s) = 0. \tag{21}$$

The following result show that from the knowledge of boundary data, one can only recover the part of the linear anisotropic source f; in particular, only the solenoidal part $\mathbf{F}^s$ of the vector field source $\mathbf{F}$ can be recovered inside Ω.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex bounded domain, and Γ be its boundary. Consider the boundary value problem* (10) *for some known real valued $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ such that* (10) *is well-posed. If scalar and vector field sources f_0 and $\mathbf{F}$ are real valued, $W^{3,p}(\Omega;\mathbb{R})$ and $W^{3,p}(\Omega;\mathbb{R}^2)$-regular, respectively, with $p > 4$, then the data $g_{f_0,\mathbf{F}}$ defined in* (2)*, uniquely determine the solenoidal part $\mathbf{F}^s$ in Ω. Moreover, $u - u_0$ is also uniquely determined in Ω, where u is the solution of* (10) *and u_0 is the zeroth Fourier mode of u in the angular variable.*

Proof. Let u be the solution of the boundary value problem (10) and let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ be the sequence valued map of its non-positive Fourier modes. Since the scalar field $f_0 \in W^{3,p}(\Omega;\mathbb{R})$, $p > 4$, and isotropic vector field $\mathbf{F} \in W^{3,p}(\Omega;\mathbb{R}^2)$, $p > 4$, then the anisotropic source $f = f_0 + \boldsymbol{\theta} \cdot \mathbf{F}$ belong to $W^{3,p}(\Omega \times \mathbf{S}^1)$ for $p > 4$. By applying Theorem A.2 (iii), we have $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$, $p > 4$. Moreover, by the Sobolev embedding [1], $W^{3,p}(\Omega \times \mathbf{S}^1) \subset C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$ with $\mu = 1 - \frac{2}{p} > \frac{1}{2}$, we have $u \in C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$, and thus, by [23, Proposition 4.1 (ii)], the sequence valued map $\mathbf{u} \in Y_\mu(\Gamma)$.

Since $\mathbf{F} \in W^{3,p}(\Omega;\mathbb{R}^2), p > 4$, then by compact imbedding of Sobolev spaces [1], $\mathbf{F} \in H^1(\Omega;\mathbb{R}^2)$. By Hodge decomposition (19), field $\mathbf{F} = \nabla\varphi + \mathbf{F}^s$, with $\varphi|_\Gamma = 0$, and $\mathrm{div}\,\mathbf{F}^s = 0$.

We note from (18) that the shifted sequence valued map $L^M\mathbf{u} = \langle u_{-M}, u_{-M-1}, u_{-M-2}, ...\rangle$ solves

$$\overline{\partial}L^M\mathbf{u}(z) + L^2\partial L^M\mathbf{u}(z) + a(z)L^{M+1}\mathbf{u}(z) = \mathbf{0}, \quad z \in \Omega. \tag{22}$$

Let $\mathbf{v} := e^{-G}L^M\mathbf{u}$. By Lemma 2.1, the system (22) becomes $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$, i.e $\mathbf{v}$ is L^2 analytic.

By (2), the data $u|_{\Gamma_+} = g$ determines the sequence valued map $L^M\mathbf{u}$ on Γ. By Proposition 9 (iii), and the convolution formula (9), traces $L^M\mathbf{u}|_\Gamma$ determines the traces $\mathbf{v} \in Y_\mu(\varGamma)$ on Γ.

Since $\mathbf{v}|_\Gamma$ is the boundary value of an L^2-analytic function in Ω, then [23, Theorem 3.2 (i)] yields

$$[I + \mathrm{i}\mathcal{H}]\mathbf{v}|_\Gamma = \mathbf{0}, \tag{23}$$

where $\mathcal{H}$ is the Bukhgeim-Hilbert transform in (7).

From $\mathbf{v}$ on $\varGamma$, we use the Bukhgeim-Cauchy Integral formula (6) to construct the sequence valued map $\mathbf{v}$ inside Ω:

$$\mathbf{v}(z) = \mathcal{B}[\mathbf{v}|_\varGamma](z), \quad z \in \Omega, \tag{24}$$

By [24, Proposition 2.3] and [23, Theorem 3.2 (ii)], the constructed sequence valued map $\mathbf{v} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty)$ is L^2-analytic in Ω.

From the convolution formula (9), we construct the sequence valued map

$$L^M\mathbf{u} := e^G\mathbf{v}. \tag{25}$$

Thus, determining u_{-n} inside Ω for $n \geqslant M$. In particular, we recover modes $u_{-M-1}, u_{-M} \in C^2(\Omega)$.

Recall that the modes $u_{-1}, u_{-2}, \cdots, u_{-M}, u_{-M-1}$ satisfy

$$\overline{\partial}u_{-M+j} = -\partial u_{-M+j-2} - \left[(a - k_{-M+j-1})u_{-M+j-1}\right], \quad 1 \leqslant j \leqslant M-1, \tag{26a}$$

$$u_{-M+j}|_\Gamma = g_{-M+j}. \tag{26b}$$

By applying 4∂ to (26a), the mode u_{-M+1} (for $j = 1$) is then the solution to the Dirichlet problem for the Poisson equation

$$\Delta u_{-M+1} = -4\partial^2 u_{-M-1} - 4\partial\left[(a - k_{-M})u_{-M}\right], \tag{27a}$$

$$u_{-M+1}|_\varGamma = g_{-M+1}, \tag{27b}$$

where the right hand side of (27) is known.

We solve repeatedly (27) for $j = 2, ..., M-1$ in (26), to recover the modes

$$u_{-M+1}, u_{-M+2}, \cdots, u_{-1}, \quad \text{in } \Omega. \tag{28}$$

From determined $L^M\mathbf{u} = \langle u_{-M}, u_{-M-1}, u_{-M-2}, ...\rangle$ in (25) and modes $u_{-M+1}, u_{-M+2}, \cdots, u_{-1}$ in (28), the sequence $L\mathbf{u} = \langle u_{-1}, u_{-2}, ...\rangle$ is determined in Ω. Thus $u - u_0$ is determined in Ω.

Since $u_0, u_{-1}, u_{-2} \in C^2(\Omega)$, we can take 4∂ on both sides of the equation (16) to get

$$\Delta u_0 + 4\partial^2 u_{-2} + 4\partial([a - k_{-1}]u_{-1}) = 4\partial f_1 = \operatorname{div}\mathbf{F} + \mathrm{i}\operatorname{curl}\mathbf{F}, \tag{29}$$

where in the last equality we use (20).

Moreover, since u_0 is real valued and $\operatorname{div}\mathbf{F} = \Delta\varphi$, by equating the real part in (29) yields the boundary value problem:

$$\Delta(u_0 - \varphi) = -4\,\mathbb{R}\mathrm{e}\left[\partial^2 u_{-2} + \partial([a - k_{-1}]u_{-1})\right], \tag{30a}$$

$$(u_0 - \varphi)|_\varGamma = g_0, \tag{30b}$$

where the right hand side of (30) is known.

Thus, real valued function $(u_0 - \varphi)$ is recovered in Ω, by solving the Dirichlet problem for the above Poisson equation (30).

Even though u_0 is not determined, the function $(u_0 - \varphi)$ is uniquely determined in Ω. Moreover, modes u_{-1} and u_{-2} are also uniquely determined in Ω. Furthermore, using expression of f_1 from (16) and $f_1^s = f_1 - \overline{\partial}\varphi$ from (21), we define

$$f_1^s := \overline{\partial}(u_0(z) - \varphi(z)) + \partial u_{-2}(z) + [a(z) - k_{-1}(z)]u_{-1}(z), \quad z \in \Omega, \tag{31}$$

with f_1^s satisfying $\mathbb{R}e(\partial f_1^s) = 0$.

Thus, the solenoidal part $\mathbf{F}^s = \langle 2\,\mathrm{Re}\, f_1^s, 2\,\mathrm{Im}\, f_1^s\rangle$, of the vector field $\mathbf{F}$ is recovered in Ω. □

If we know apriori that the vector field $\mathbf{F}$ is incompressible (i.e divergenceless), then we can reconstruct both scalar field source f_0 and vector field source $\mathbf{F}$ in Ω.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex bounded domain, and Γ be its boundary. Consider the boundary value problem* (10) *for some known real valued $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ such that* (10) *is well-posed. If the unknown scalar field source f_0 and divergenceless vector field sources $\mathbf{F}$ are real valued, $W^{3,p}(\Omega;\mathbb{R})$ and $W^{3,p}(\Omega;\mathbb{R}^2)$-regular, respectively, with $p > 4$, then the data $g_{f_0,\mathbf{F}}$ defined in* (2) *uniquely determine both f_0 and $\mathbf{F}$ in Ω.*

Proof. Let u be the solution of the boundary value problem (10) and let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ be the sequence valued map of its non-positive Fourier modes, Since the isotropic scalar and vector field $f_0 \in W^{3,p}(\Omega;\mathbb{R})$, and $\mathbf{F} \in W^{3,p}(\Omega;\mathbb{R}^2)$ respectively for $p > 4$, then the anistropic source $f = f_0 + \boldsymbol{\theta}\cdot\mathbf{F} \in W^{3,p}(\Omega \times \mathbf{S}^1)$ and by applying Theorem A.2 (iii), we have $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$. By the Sobolev embedding [1], $W^{3,p}(\Omega \times \mathbf{S}^1) \subset C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$ with $\mu = 1 - \frac{2}{p} > \frac{1}{2}$, we have $u \in C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$, and thus, by [23, Proposition 4.1 (ii)], $\mathbf{u} \in Y_\mu(\varGamma)$.

Since $\mathbf{F} \in W^{3,p}(\Omega;\mathbb{R}^2), p > 4$, then by compact imbedding of Sobolev spaces [1], $\mathbf{F} \in H^1(\Omega;\mathbb{R}^2)$. By Hodge decomposition (19), field $\mathbf{F} = \nabla\varphi + \mathbf{F}^s$, with $\varphi|_\varGamma = 0$, and $\mathrm{div}\,\mathbf{F}^s = 0$.

If we know apriori that the vector field $\mathbf{F}$ is incompressible (i.e divergenceless $\nabla\cdot\mathbf{F} = 0$). Then $\triangle\varphi = \mathrm{div}\,\mathbf{F} = 0$ and $\varphi|_{\partial\Omega} = 0$ implies $\varphi \equiv 0$ inside Ω. Thus, vector field $\mathbf{F} = \mathbf{F}^s$ inside Ω.

By Theorem 3.1, the data $u|_{\Gamma_+} = g_{f_0,\mathbf{F}}$ uniquely determine the solenoidal field $\mathbf{F}^s = \mathbf{F}$ in Ω by equation (31) with $\varphi \equiv 0$, and the sequence valued map $L\mathbf{u} = \langle u_{-1}, u_{-2}, ...\rangle$ in Ω. Moreover, the real valued mode u_0 is also then recovered (with $\varphi \equiv 0$) in Ω, by solving the Dirichlet problem for the Poisson equation (30).

Thus, from modes u_{-1} and u_0, the scalar field f_0 is also recovered in Ω by

$$f_0 := 2\,\mathrm{Re}[\partial u_{-1}] + [a - k_0]u_0. \tag{32}$$

□

In the radiative transport literature, the attenuation coefficient $a = \sigma_a + \sigma_s$, where σ_a represents pure loss due to absorption and $\sigma_s(z) = \frac{1}{2\pi}\int_0^{2\pi} k(z,\theta)d\theta = k_0(z)$ is the isotropic part of scattering kernel. We consider the subcritical region:

$$\sigma_a := a - k_0 \geqslant \delta > 0, \quad \text{for some positive constant } \delta. \tag{33}$$

Remark 3.1. *In addition to the hypothesis to Theorem 3.1, if we assume that coefficients a, k_0 satisfies* (33)*, then in the region $\{z \in \Omega : \ f_0(z) = 0\}$, one can recover explicitly the entire vector field $\mathbf{F} = \langle 2\,\mathrm{Re}\, f_1, 2\,\mathrm{Im}\, f_1\rangle$. Indeed, the equation (15) gives $u_0 = -2\,\mathrm{Re}(\partial u_{-1})/\sigma_a$ and, following (16), the vector field $\mathbf{F}$ can be recovered by the formula*

$$f_1 = \partial u_{-2} + [a - k_{-1}]u_{-1} - 2\overline{\partial}\left(\frac{\mathrm{Re}(\partial u_{-1})}{\sigma_a}\right). \tag{34}$$

Next, we show that one can also determine both scalar field f_0 and vector field $\mathbf{F}$, if one has the additional data $g_{f_0,\mathbf{0}}$ (or $g_{0,\mathbf{F}}$) information, instead of $\mathbf{F}$ being incompressible as in Theorem 3.2.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex bounded domain, and Γ be its boundary. Consider the boundary value problem* (10) *for some known real valued $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ such that* (10) *is well-posed. If the unknown scalar field source f_0 and vector field source $\mathbf{F}$ are real valued, $W^{3,p}(\Omega;\mathbb{R})$*

and $W^{3,p}(\Omega;\mathbb{R}^2)$-regular, respectively, with $p > 4$, and coefficients a, k_0 satisfying (33)*, then the data $g_{f_0,\mathbf{F}}$ and $g_{f_0,0}$ defined in* (2) *uniquely determine both f_0 and $\mathbf{F}$ in Ω.*

Proof. Let u be the solution of the boundary value problem (10) and let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ be the sequence valued map of its non-positive Fourier modes. Since the scalar field $f_0 \in W^{3,p}(\Omega;\mathbb{R})$, $p > 4$, and isotropic vector field $\mathbf{F} \in W^{3,p}(\Omega;\mathbb{R}^2)$, $p > 4$, then the anisotropic source $f = f_0 + \boldsymbol{\theta} \cdot \mathbf{F}$ belong to $W^{3,p}(\Omega \times \mathbf{S}^1)$ for $p > 4$. By applying Theorem A.2 (iii), we have $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$, $p > 4$. Moreover, by the Sobolev embedding, $W^{3,p}(\Omega \times \mathbf{S}^1) \subset C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$ with $\mu = 1 - \frac{2}{p} > \frac{1}{2}$, we have $u \in C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$, and thus, by [23, Proposition 4.1 (ii)], the sequence valued map $\mathbf{u} \in Y_\mu(\Gamma)$.

We consider the boundary value problems

$$\boldsymbol{\theta} \cdot \nabla v + av - Kv = f_0, \qquad \text{subject to} \quad v|_{\Gamma_-} = 0, \quad v|_{\Gamma_+} = g_{f_0,0}, \quad \text{and} \tag{35}$$

$$\boldsymbol{\theta} \cdot \nabla w + aw - Kw = \boldsymbol{\theta} \cdot \mathbf{F}, \quad \text{subject to} \quad w|_{\Gamma_-} = 0, \quad w|_{\Gamma_+} = \widetilde{g} := g_{f_0,\mathbf{F}} - g_{f_0,\mathbf{0}}. \tag{36}$$

Then $u = v + w$ satisfy the boundary value problem (10).

We consider first the boundary value problem (35), and reconstruct the scalar field f_0 from the given boundary data $g_{f_0,0}$ as follows.

If $\sum_{n\in\mathbb{Z}} v_n(z)e^{in\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution v of boundary value problem (35), then, by identifying the Fourier modes of the same order, (35) reduces to the system:

$$\overline{\partial}\overline{v_{-1}}(z) + \partial v_{-1}(z) + [a(z) - k_0(z)]v_0(z) = f_0(z), \tag{37}$$

$$\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z) + [a(z) - k_{-n-1}(z)]v_{-n-1}(z) = 0, \quad 0 \leqslant n \leqslant M-1, \tag{38}$$

$$\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z) + a(z)v_{-n-1}(z) = 0, \qquad n \geqslant M. \tag{39}$$

Let $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ be the sequence valued map of its non-positive Fourier modes. By Theorem 3.1, the data $g_{f_0,0}$, uniquely determine the sequence $L\mathbf{v} = \langle v_{-1}, v_{-2}, ...\rangle$ in Ω. Moreover, as (38) holds also for $n = 0$ ($f_1 = 0$ in this case), the mode v_0 is also determined in Ω by solving the Dirichlet problem for the Poisson equation

$$\Delta v_0 = -4\partial^2 v_{-2} - 4\partial\left[(a - k_{-1})v_{-1}\right], \tag{40a}$$

$$v_0|_\Gamma = g_0, \tag{40b}$$

where the right hand side of (40) is known.

Thus, using modes v_0 and v_{-1}, the isotropic scalar source f_0 is recovered in Ω by

$$f_0(z) := 2\,\mathbb{R}\mathrm{e}\left(\partial v_{-1}(z)\right) + \left(a(z) - k_0(z)\right)v_0(z), \quad z \in \Omega. \tag{41}$$

Next, we consider the boundary value problem (36), and reconstruct the vector field $\mathbf{F}$ from the given boundary data $\widetilde{g} = g_{f_0,\mathbf{F}} - g_{f_0,0}$ as follows.

If $\sum_{n\in\mathbb{Z}} w_n(z)e^{in\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution w of the boundary value problem (36), then (36) reduces to the system:

$$\overline{\partial}\overline{w_{-1}}(z) + \partial w_{-1}(z) + [a(z) - k_0(z)]w_0(z) = 0, \tag{42}$$

$$\overline{\partial} w_0(z) + \partial w_{-2}(z) + [a(z) - k_{-1}(z)]w_{-1}(z) = \left(F_1(z) + \mathrm{i}F_2(z)\right)/2, \tag{43}$$

$$\overline{\partial} w_{-n}(z) + \partial w_{-n-2}(z) + [a(z) - k_{-n-1}(z)]w_{-n-1}(z) = 0, \quad 1 \leqslant n \leqslant M-1, \tag{44}$$

$$\overline{\partial} w_{-n}(z) + \partial w_{-n-2}(z) + a(z)w_{-n-1}(z) = 0, \qquad n \geqslant M. \tag{45}$$

Let $\mathbf{w} = \langle w_0, w_{-1}, w_{-2}, ...\rangle$ be the sequence valued map of its non-positive Fourier modes. By Theorem 3.1, the data $\widetilde{g} = g_{f_0,\mathbf{F}} - g_{f_0,\mathbf{0}}$, uniquely determine the sequence $L\mathbf{w} = \langle w_{-1}, w_{-2}, ...\rangle$ in Ω.

Using the subcriticality condition (33): $\sigma_a = a - k_0 > 0$, we define via (42):

$$w_0(z) := -\frac{2\,\mathbb{R}\mathrm{e}\,\partial w_{-1}(z)}{a(z) - k_0(z)} = -\frac{2\,\mathbb{R}\mathrm{e}\,\partial w_{-1}(z)}{\sigma_a(z)}, \quad z \in \Omega. \tag{46}$$

Using determined modes w_{-1}, w_{-2} in $L\mathbf{w}$ and mode w_0 from (42), the real valued vector field $\mathbf{F} = \langle 2\,\mathbb{R}\mathrm{e}\, f_1, 2\,\mathbb{I}\mathrm{m}\, f_1\rangle$ is recovered in Ω by

$$f_1(z) := \overline{\partial} w_0(z) + \partial w_{-2}(z) + [a(z) - k_{-1}(z)]\, w_{-1}(z), \quad z \in \Omega. \tag{47}$$

□

4. When can the data coming from two sources be mistaken for each other ?

In this section we show when the data coming from two different linear anisotropic field sources can be mistaken for each other.

In the Theorem 4.1 below the data are assuming the same attenuation a and scattering coefficient k.

Theorem 4.1. *(i) Let $a \in C^3(\overline{\Omega})$, $k \in C^3(\overline{\Omega} \times \mathbf{S}^1)$ be real valued, with $\sigma_a = a - k_0 > 0$, and $f_0, \widetilde{f}_0 \in W^{3,p}(\Omega)$, $p > 4$ be real valued with $(f_0 - \widetilde{f}_0)/\sigma_a \in C_0(\overline{\Omega})$. Then $\mathbf{F} := \widetilde{\mathbf{F}} + \nabla\left(\frac{f_0 - \widetilde{f}_0}{\sigma_a}\right)$ is a real valued vector field such that the data $g_{f_0,\mathbf{F}}$ coming from the linear anisotropic source $f_0 + \boldsymbol{\theta}\cdot\mathbf{F}$, is the same as data $g_{\widetilde{f}_0,\widetilde{\mathbf{F}}}$ coming from a different linear anisotropic source $\widetilde{f}_0 + \boldsymbol{\theta}\cdot\widetilde{\mathbf{F}}$:*

$$g_{f_0,\widetilde{\mathbf{F}}+\nabla\left(\frac{f_0-\widetilde{f}_0}{\sigma_a}\right)} = g_{\widetilde{f}_0,\widetilde{\mathbf{F}}}.$$

(ii) Let $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ be real valued with $\sigma_a = a - k_0 > 0$. Assume that there are real valued linear anisotropic sources $f = f_0 + \boldsymbol{\theta}\cdot\mathbf{F}$ and $\widetilde{f} = \widetilde{f}_0 + \boldsymbol{\theta}\cdot\widetilde{\mathbf{F}}$, with isotropic fields $f_0, \widetilde{f}_0 \in W^{3,p}(\Omega)$, $p > 4$, and vector fields $\mathbf{F}, \widetilde{\mathbf{F}} \in W^{3,p}(\Omega;\mathbb{R}^2)$, $p > 4$. If the data $g_{f_0,\mathbf{F}}$ of the linear anisotropic source f equals the data $g_{\widetilde{f}_0,\widetilde{F}}$ of the linear anisotropic source $\widetilde{f}$. Then

$$\mathbf{F} = \widetilde{\mathbf{F}} + \nabla\left(\frac{f_0 - \widetilde{f}_0}{\sigma_a}\right).$$

Proof. (i) Assume $g_{\widetilde{f}_0,\widetilde{\mathbf{F}}}$ is the data of some real valued anisotropic source $\widetilde{f} = \widetilde{f}_0 + \boldsymbol{\theta}\cdot\widetilde{\mathbf{F}}$, i.e., it is the trace on $\varGamma \times \mathbf{S}^1$ of solution w to the stationary transport boundary value problem:

$$\begin{aligned} \boldsymbol{\theta}\cdot\nabla w + aw - Kw &= \widetilde{f}, \\ w|_{\varGamma\times\mathbf{S}^1} &= g_{\widetilde{f}_0,\widetilde{\mathbf{F}}}, \end{aligned} \tag{48}$$

where the operator $[Kw](z,\boldsymbol{\theta}) := \int_{\mathbf{S}^1} k(z, \boldsymbol{\theta}\cdot\boldsymbol{\theta}')w(z,\boldsymbol{\theta}')d\boldsymbol{\theta}'$, for $z \in \Omega$ and $\boldsymbol{\theta} \in \mathbf{S}^1$.

Using the subcriticality condition (33): $\sigma_a = a - k_0$ with $\sigma_a > 0$, and isotropic real valued functions ψ and σ_a, we note:

$$\begin{aligned} \left[K\frac{\psi}{\sigma_a}\right](z,\boldsymbol{\theta}) &= \int_{\mathbf{S}^1} k(z,\boldsymbol{\theta}\cdot\boldsymbol{\theta}')\left[\frac{\psi}{\sigma_a}\right](z,\boldsymbol{\theta}')d\boldsymbol{\theta}' \\ &= \frac{\psi(z)}{\sigma_a(z)}\int_{\mathbf{S}^1} k(z,\boldsymbol{\theta}\cdot\boldsymbol{\theta}')d\boldsymbol{\theta}' = \frac{\psi(z)}{\sigma_a(z)}k_0(z), \end{aligned} \tag{49}$$

where second equality use the fact that both ψ and σ_a are angularly independent functions.

Let $u := w + (f_0 - \widetilde{f}_0)/\sigma_a$ and $\mathbf{F} := \widetilde{\mathbf{F}} + \nabla \left(\dfrac{f_0 - \widetilde{f}_0}{\sigma_a}\right)$. Then

$$\begin{aligned}
\boldsymbol{\theta}\cdot\nabla u + au - Ku &= \boldsymbol{\theta}\cdot\nabla\left(w + \frac{f_0-\widetilde{f}_0}{\sigma_a}\right) + a\left(w + \frac{f_0-\widetilde{f}_0}{\sigma_a}\right) - K\left(w + \frac{f_0-\widetilde{f}_0}{\sigma_a}\right)\\
&= \boldsymbol{\theta}\cdot\nabla w + aw - Kw - \left(\frac{a}{\sigma_a}\right)\widetilde{f}_0 + \left(\frac{k_0}{\sigma_a}\right)\widetilde{f}_0 + \left(\frac{a}{\sigma_a}\right)f_0 - \left(\frac{k_0}{\sigma_a}\right)f_0 + \boldsymbol{\theta}\cdot\nabla\left(\frac{f_0-\widetilde{f}_0}{\sigma_a}\right)\\
&= \left(1 - \frac{a}{\sigma_a} + \frac{k_0}{\sigma_a}\right)\widetilde{f}_0 + \left(\frac{a-k_0}{\sigma_a}\right)f_0 + \boldsymbol{\theta}\cdot\left(\widetilde{\mathbf{F}} + \nabla\left(\frac{f_0-\widetilde{f}_0}{\sigma_a}\right)\right) = f_0 + \boldsymbol{\theta}\cdot\mathbf{F} = f,
\end{aligned}$$

where the second equality uses the linearty of K and (49), the third equality uses (48), and the last equality uses the definition of $\mathbf{F}$. Moreover, since $f_0 - \widetilde{f}_0/\sigma_a$ vanishes on $\varGamma$, we get

$$g_{f_0,\mathbf{F}} = u|_{\varGamma\times\mathbf{S}^1} = w|_{\varGamma\times\mathbf{S}^1} + \left.\frac{f_0-\widetilde{f}_0}{\sigma_a}\right|_{\varGamma} = w|_{\varGamma\times\mathbf{S}^1} = g_{\widetilde{f}_0,\widetilde{\mathbf{F}}}.$$

(ii) For isotropic scalar fields $f_0, \widetilde{f}_0 \in W^{3,p}(\Omega)$, $p > 4$, and vector fields $\mathbf{F} = \langle F_1, F_2\rangle, \widetilde{\mathbf{F}} = \langle \widetilde{F_1}, \widetilde{F_2}\rangle \in W^{3,p}(\Omega;\mathbb{R}^2)$, $p > 4$, the real valued linear anisotropic sources $f = f_0 + \boldsymbol{\theta}\cdot\mathbf{F} \in W^{3,p}(\Omega\times\mathbf{S}^1), p > 4$, and $\widetilde{f} = \widetilde{f}_0 + \boldsymbol{\theta}\cdot\widetilde{\mathbf{F}} \in W^{3,p}(\Omega\times\mathbf{S}^1)$, $p > 4$.

Let the data $g_{f_0,\mathbf{F}}$ equals data $g_{\widetilde{f},\widetilde{\mathbf{F}}}$ i.e.

$$g_{\widetilde{f},\widetilde{\mathbf{F}}} = g = g_{f_0,\mathbf{F}}.$$

Consider the corresponding boundary value problems

$$\boldsymbol{\theta}\cdot\nabla u + au - Ku = f \tag{50a}$$

$$\boldsymbol{\theta}\cdot\nabla w + aw - Kw = \widetilde{f}, \tag{50b}$$

respectively, subject to

$$u|_{\varGamma\times\mathbf{S}^1} = g = w|_{\varGamma\times\mathbf{S}^1}. \tag{50c}$$

Since $f, \widetilde{f} \in W^{3,p}(\Omega\times\mathbf{S}^1), p > 4$, Theorem A.2 (iii), yields solutions $u, w \in W^{3,p}(\Omega\times\mathbf{S}^1), p > 4$. Moreover, by the Sobolev embedding, $u, w \in C^{2,\mu}(\overline{\Omega}\times\mathbf{S}^1)$ with $\mu = 1 - \frac{2}{p} > \frac{1}{2}$.

The corresponding sequences of non-positive Fourier modes $\{u_{-n}\}_{n\geqslant 0}$ of u satisfy

$$\overline{\partial}\overline{u_{-1}}(z) + \partial u_{-1}(z) + [a(z) - k_0(z)]u_0(z) = f_0(z), \tag{51}$$

$$\overline{\partial}u_0(z) + \partial u_{-2}(z) + [a(z) - k_{-1}(z)]u_{-1}(z) = (F_1(z) + iF_2(z))/2, \tag{52}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-n-2}(z) + [a(z) - k_{-n-1}(z)]u_{-n-1}(z) = 0, \quad 1 \leqslant n \leqslant M-1, \tag{53}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-n-2}(z) + a(z)u_{-n-1}(z) = 0, \qquad n \geqslant M, \tag{54}$$

whereas the non-positive Fourier modes $\{w_{-n}\}_{n\geqslant 0}$ of w satisfy

$$\overline{\partial}\overline{w_{-1}}(z) + \partial w_{-1}(z) + [a(z) - k_0(z)]w_0(z) = \widetilde{f}_0(z), \tag{55}$$

$$\overline{\partial} w_0(z) + \partial w_{-2}(z) + [a(z) - k_{-1}(z)]w_{-1}(z) = \left(\widetilde{F_1}(z) + i\widetilde{F_2}(z)\right)/2, \tag{56}$$

$$\overline{\partial} w_{-n}(z) + \partial w_{-n-2}(z) + [a(z) - k_{-n-1}(z)]w_{-n-1}(z) = 0, \quad 1 \leqslant n \leqslant M-1, \tag{57}$$

$$\overline{\partial} w_{-n}(z) + \partial w_{-n-2}(z) + a(z)w_{-n-1}(z) = 0, \qquad n \geqslant M. \tag{58}$$

Since the boundary data g is the same $u|_{\varGamma\times\mathbf{S}^1} = w|_{\varGamma\times\mathbf{S}^1}$, we also have the sequence valued map

$$\mathbf{g} := \mathbf{w}|_{\varGamma} = \mathbf{u}|_{\varGamma}. \tag{59}$$

Moreover, by [23, Proposition 4.1 (ii)], the sequence $\mathbf{g} \in Y_\mu(\varGamma)$ with $\mu > \frac{1}{2}$.

Claim 4.1. *The above systems subject to boundary condition* (59) *yields*

$$u_{-n} = w_{-n}, \qquad \textit{for all } \; n \geqslant 1, \tag{60}$$

inside Ω.

Proof of Claim 4.1. We first show that the systems (54) and (58) subject to (59) yields

$$u_{-n}(z) = w_{-n}(z), \quad z \in \Omega, \quad \text{for all } \; n \geqslant M. \tag{61}$$

From (54) and (58), the shifted sequence valued maps $L^M\mathbf{u} = \langle u_{-M}, u_{-M-1}, ...\rangle$ and $L^M\mathbf{w} = \langle w_{-M}, w_{-M-1}, ...\rangle$, respectively, solves

$$\overline{\partial} L^M\mathbf{u} + L^2\partial L^M\mathbf{u} + aL^{M+1}\mathbf{u} = \mathbf{0}, \text{ and } \overline{\partial} L^M\mathbf{w} + L^2\partial L^M\mathbf{w} + aL^{M+1}\mathbf{w} = \mathbf{0}. \tag{62}$$

From $\mathbf{g}$ in (59), we use the Bukhgeim-Cauchy Integral formula (6) to construct the sequence valued map $\mathbf{v}$ and $\mathbf{w}$ inside Ω:

$$\mathbf{v} := \mathcal{B}\left(L^M e^{-G}\mathbf{g}\right), \quad \boldsymbol{\rho} := \mathcal{B}\left(L^M e^{-G}\mathbf{g}\right), \tag{63}$$

where e^{-G} is the operator in (9).

By [24, Proposition 2.3] and [23, Theorem 3.2 (ii)], $\mathbf{v}, \boldsymbol{\rho} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty)$ are L^2-analytic in Ω:

$$\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}, \text{ and } \overline{\partial}\boldsymbol{\rho} + L^2\partial\boldsymbol{\rho} = \mathbf{0}, \tag{64}$$

and also coincide at the boundary Γ. By uniqueness of L^2-analytic functions with a given trace, they coincide inside:

$$\mathbf{v}(z) = \boldsymbol{\rho}(z), \quad \text{for } z \in \Omega. \tag{65}$$

Using the operator e^G in (9), we construct the sequence valued map

$$L^M\mathbf{u}(z) := e^G\mathbf{v}(z), \text{ and } L^M\mathbf{w}(z) := e^G\mathbf{v}(z) = e^G\boldsymbol{\rho}(z) \quad z \in \Omega, \tag{66}$$

and conclude that (61) holds.

Moreover, by Lemma 2.1, the sequences $L^M\mathbf{u}$ and$L^M\mathbf{w}$ in (66) satisfies (62).

Next, we show that the systems (53) and (57) subject to boundary condition (59) yield

$$u_{-n} = w_{-n}, \quad \text{for all } \; 1 \leqslant n \leqslant M-1, \tag{67}$$

inside Ω.

Define the function

$$\psi_{-j}(z) := u_{-j}(z) - w_{-j}(z), \quad \text{for } z \in \Omega, \quad \text{and } j \geqslant 1. \tag{68}$$

Since the boundary data g is the same, equation (50c) yields

$$\psi_{-j}|_{\Gamma} = 0, \quad \text{for } j \geqslant 1. \tag{69}$$

From (61), we note that

$$\psi_{-j} \equiv 0, \quad \text{in } \Omega, \quad \text{for } j \geqslant M. \tag{70}$$

By subtracting system (57) from (53), and using (68) and (69), yields the boundary value problem

$$\overline{\partial}\psi_{-M+j} = -\partial\psi_{-M+j-2} - \left[(a - k_{-M+j-1})\psi_{-M+j-1}\right], \quad 1 \leqslant j \leqslant M-1, \tag{71a}$$

$$\psi_{-M+j}|_{\Gamma} = 0. \tag{71b}$$

Note that for $j = 1$ in (71), the right hand side of (71a) contains modes ψ_{-M-1} and ψ_{-M} which are both zero inside Ω by (70). Thus, for $j = 1$, the mode ψ_{-M+1} satisfy the Cauchy problem for the $\overline{\partial}$-equation,

$$\overline{\partial}\Psi = 0, \quad \text{in } \Omega, \tag{72a}$$

$$\Psi = 0, \quad \text{on } \Gamma. \tag{72b}$$

The unique solution of the above Cauchy problem is $\Psi \equiv 0$. Therefore, resulting $\psi_{-M+1} \equiv 0$.

We then solve repeatedly (71) starting for $j = 2, ..., M-1$, where the right hand side of (71a) in each step is zero, yielding the Cauchy problem (72) for each subsequent modes, and thus, resulting in the recovering of the modes $\psi_{-M+1} = \psi_{-M+2} = \cdots \psi_{-2} = \psi_{-1} \equiv 0$ in Ω. Hence, establishing (67).

From (61) and (67), we establish (60), and thus Claim 4.1. □

By subtracting (55) from (51), and using (60) yields $(a - k_0)(u_0 - w_0) = f_0 - \widetilde{f}_0$. Using $\sigma_a = a - k_0$ with $\sigma_a > 0$, yields

$$u_0 - w_0 = \frac{f_0 - \widetilde{f}_0}{\sigma_a}. \tag{73}$$

Moreover, by subtracting (56) from (52), and using (60) yields

$$2\overline{\partial}(u_0 - w_0) = (F_1 - \widetilde{F_1}) + \mathrm{i}(F_2 - \widetilde{F_2}).$$

Since both u_0 and w_0 are real valued we have from (73):

$$\mathbf{F} - \widetilde{\mathbf{F}} = \nabla(u_0 - w_0) = \nabla\left(\frac{f_0 - \widetilde{f}_0}{\sigma_a}\right).$$

□

Remark 4.1. *Note that in Theorem 4.1(i), the assumption on scattering kernels of finite Fourier content in the angular variable is not assumed, and the result holds for a general scattering kernels which depends polynomially on the angle between the directions.*

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APPENDIX A. SOME REMARKS ON THE REGULARITY OF THE FORWARD PROBLEM

In this section, we revisit the arguments in [29, 14], and remark on the well posedness in $L^p(\Omega \times \mathbf{S}^1)$ of the boundary value problem (1).

The results in appendix consider both attenuation coefficient and scattering kernel in general setting. Adopting the notation in [29, 14], we consider the operators

$$[T_1^{-1}g](x,\boldsymbol{\theta}) = \int_{-\infty}^{0} e^{-\int_s^0 a(x+t\boldsymbol{\theta},\boldsymbol{\theta})dt} g(x+s\boldsymbol{\theta},\boldsymbol{\theta})ds, \text{ and } [Kg](x,\boldsymbol{\theta}) = \int_{\mathbf{S}^1} k(x,\boldsymbol{\theta},\boldsymbol{\theta}')g(x,\boldsymbol{\theta}')d\boldsymbol{\theta}',$$

where the intervening functions are extended by 0 outside Ω.

Using the above operators, the boundary value problem (1) can be rewritten as

$$(I - T_1^{-1}K)u = T_1^{-1}f, \qquad u|_{\Gamma_-} = 0. \tag{74}$$

If the operator $I - T_1^{-1}K$ is invertible, then the problem (74) is uniquely solvable, and has the form $u = (I - T_1^{-1}K)^{-1}T_1^{-1}f$. By using the formal expansion

$$u = T_1^{-1}f + T_1^{-1}KT_1^{-1}f + T_1^{-1}(KT_1^{-1}K)[I - T_1^{-1}K]^{-1}T_1^{-1}f. \tag{75}$$

We recall some results in [14].

Proposition A.1. [14, Proposition 2.1] *Let $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$. Then the operator*

$$KT_1^{-1}K : L^p(\Omega \times \mathbf{S}^1) \longrightarrow W^{1,p}(\Omega \times \mathbf{S}^1) \textit{ is bounded}, \ 1 < p < \infty. \tag{76}$$

Theorem A.1. [14, Theorem 2.1] *Let $p > 1$, $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$, and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$. At least one of the following statements is true.*
(i) $I - T_1^{-1}K$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$.
(ii) there exists $\epsilon > 0$ such that $I - T_1^{-1}(\lambda K)$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$, for any $0 < |\lambda - 1| < \epsilon$.

For our main Theorems, we require $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$, $p > 4$ and such a regularity can be achieved for sources $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, $p > 4$; see Theorem A.2 (iii) below. We refer to [14, Theorem 2.2] for part (i) and (ii) of Theorem A.2, and we include the proof here.

The regularity of the solution u of (1) increases with the regularity of f as follows.

Theorem A.2. *Consider the boundary value problem* (1) *with $a \in C^3(\overline{\Omega} \times \mathbf{S}^1)$. For $p > 1$, let $k \in C^3(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$ be such that $I - T_1^{-1}K$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$, and let $u \in L^p(\Omega \times \mathbf{S}^1)$ in* (75) *be the solution of* (1).

(i) If $f \in W^{1,p}(\Omega \times \mathbf{S}^1)$, then $u \in W^{1,p}(\Omega \times \mathbf{S}^1)$.
(ii) If $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, then $u \in W^{2,p}(\Omega \times \mathbf{S}^1)$.
(iii) If $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, then $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$.

Proof. (i) We consider the regularity of the solution u of (1) term by term as in (75). It is easy to see that the operator T_1^{-1} preserve the space $W^{1,p}(\Omega \times \mathbf{S}^1)$, and also the operator K preserve the space $W^{1,p}(\Omega \times \mathbf{S}^1)$, so that the first two terms, $T_1^{-1}f$ and $T_1^{-1}KT_1^{-1}f$, both belong to $W^{1,p}(\Omega \times \mathbf{S}^1)$. Moreover, $(I - T_1^{-1}K)^{-1}T_1^{-1}f \in L^p(\Omega \times \mathbf{S}^1)$, and now, by using Proposition A.1, the last term is also belong in $W^{1,p}(\Omega \times \mathbf{S}^1)$.

(ii) We define the following operators

$$
\begin{aligned}
&T_0^{-1}u(x,\boldsymbol{\theta}) := \int_{-\infty}^{0} u(x+t\boldsymbol{\theta},\boldsymbol{\theta})dt, \qquad K_{\xi_j}u(x,\boldsymbol{\theta}) := \int_{\mathbf{S}^1} \frac{\partial k}{\partial \xi_j}(x,\boldsymbol{\theta},\boldsymbol{\theta}')u(x,\boldsymbol{\theta}')d\boldsymbol{\theta}',\\
&\widetilde{T}_{0,j}^{-1}u(x,\boldsymbol{\theta}) := \int_{-\infty}^{0} u(x+t\boldsymbol{\theta},\boldsymbol{\theta})t^j dt, \quad K_{\eta_i\xi_j}u(x,\boldsymbol{\theta}) := \int_{\mathbf{S}^1} \frac{\partial^2 k}{\partial \eta_i \partial \xi_j}(x,\boldsymbol{\theta},\boldsymbol{\theta}')u(x,\boldsymbol{\theta}')d\boldsymbol{\theta}',
\end{aligned} \tag{77}
$$

where $\eta_i = \{x_i, \theta_i\}$ and $\xi_j = \{x_j, \theta_j\}$ for $i, j = 1, 2$.
It is easy to see that $T_0^{-1}, \widetilde{T}_{0,j}^{-1}, K_{\xi_j}$ and $K_{\eta_i\xi_j}$ preserve $W^{1,p}(\Omega \times \mathbf{S}^1)$.

By evaluating the radiative transport equation in (1) at $x + t\boldsymbol{\theta}$ and integrating in t from $-\infty$ to 0, the boundary value problem (1) with zero incoming fluxes is equivalent to the integral equation:

$$
u + T_0^{-1}(au) - T_0^{-1}Ku = T_0^{-1}f. \tag{78}
$$

According to part (i), for $f \in W^{1,p}(\Omega\times\mathbf{S}^1)$, the solution $u \in W^{1,p}(\Omega\times\mathbf{S}^1)$, and so $u_{x_j} \in L^p(\Omega\times\mathbf{S}^1)$. In particular u_{x_j} solves the integral equation:

$$
u_{x_j} + T_0^{-1}(au_{x_j}) - T_0^{-1}Ku_{x_j} = -T_0^{-1}(a_{x_j}u) + T_0^{-1}K_{x_j}u + T_0^{-1}f_{x_j}. \tag{79}
$$

Moreover, since $a \in C^2(\overline{\Omega} \times \mathbf{S}^1)$, $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, the right-hand-side of (79) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. By applying part (i) above, we get that the unique solution to (79)

$$
u_{x_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1, 2. \tag{80}
$$

For $f \in W^{1,p}(\Omega \times \mathbf{S}^1)$, also according to part (i), $u_{\theta_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular u_{θ_j} is the unique solution of the integral equation

$$
u_{\theta_j} + T_0^{-1}(au_{\theta_j}) = -\widetilde{T}_{0,1}^{-1}(au_{x_j}) - T_0^{-1}(a_{\theta_j}u) - \widetilde{T}_{0,1}^{-1}(a_{x_j}u) + T_0^{-1}K_{\theta_j}u + \widetilde{T}_{0,1}^{-1}K_{x_j}u + T_0^{-1}f_{\theta_j}, \tag{81}
$$

which is of the type (78) with $K = 0$. Moreover, since $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, and, according to (80), $u_{x_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1, 2$, the right-hand-side of (81) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. Again, by applying part (i), we get

$$
u_{\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1, 2.
$$

Thus, $u \in W^{2,p}(\Omega \times \mathbf{S}^1)$.

(iii) For $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, according to part (ii), $u_{x_j}, u_{\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1)$, and $u_{x_ix_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular $u_{x_ix_j}$ is the unique solution of the integral equation

$$
\begin{aligned}
u_{x_ix_j} + T_0^{-1}(au_{x_ix_j}) - T_0^{-1}(Ku_{x_ix_j}) = {} & T_0^{-1}f_{x_ix_j} - T_0^{-1}(a_{x_j}u_{x_i}) - T_0^{-1}(a_{x_ix_j}u) + T_0^{-1}(K_{x_j}u_{x_i})\\
& + T_0^{-1}(K_{x_ix_j}u) - T_0^{-1}(a_{x_i}u_{x_j}) - T_0^{-1}(K_{x_i}u_{x_j}).
\end{aligned} \tag{82}
$$

Moreover, since $a \in C^3(\overline{\Omega} \times \mathbf{S}^1)$, $k \in C^3(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, the right-hand-side of (82) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. By applying part (i) above, we get that the unique solution to (82)

$$
u_{x_ix_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ i, j = 1, 2. \tag{83}
$$

For $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, also according to part (ii), $u_{x_j}, u_{\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1)$, and $u_{\theta_i\theta_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular $u_{\theta_i\theta_j}$ is the unique solution of the integral equation

$$\begin{aligned} u_{\theta_i\theta_j} + T_0^{-1}(a u_{\theta_i\theta_j}) = {} & T_0^{-1}(f_{\theta_i\theta_j}) - \widetilde{T}_{0,2}^{-1}(a_{x_i}u_{x_j}) - \widetilde{T}_{0,1}^{-1}(a_{x_i}u_{\theta_j}) - \widetilde{T}_{0,1}^{-1}(a_{\theta_i}u_{x_j}) \\ & - T_0^{-1}(a_{\theta_i}u_{\theta_j}) - \widetilde{T}_{0,2}^{-1}(a_{x_j}u_{x_i}) - \widetilde{T}_{0,1}^{-1}(a_{x_j}u_{\theta_i}) - \widetilde{T}_{0,2}^{-1}(a_{x_ix_j}u) \\ & - \widetilde{T}_{0,1}^{-1}(a_{x_j\theta_i}u) - \widetilde{T}_{0,1}^{-1}(a_{\theta_j}u_{x_i}) - T_0^{-1}(a_{\theta_j}u_{\theta_i}) - \widetilde{T}_{0,1}^{-1}(a_{\theta_i\theta_j}u) \\ & - \widetilde{T}_{0,1}^{-1}(K_{\theta_j}u_{x_i}) - T_0^{-1}(K_{\theta_i\theta_j}u) - \widetilde{T}_{0,2}^{-1}(Ku_{x_ix_j}) - \widetilde{T}_{0,1}^{-1}(K_{\theta_i}u_{x_j}), \end{aligned} \tag{84}$$

which is of the type (78) with $K = 0$.

Moreover, since $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, and, according to (83), $u_{x_ix_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1, 2$, the right-hand-side of (84) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. Again, by applying part (i), we get

$$u_{\theta_i\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ i, j = 1, 2. \tag{85}$$

For $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, also according to part (ii), $u_{x_i}u_{\theta_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular $u_{x_i\theta_j}$ is the unique solution of the integral equation

$$\begin{aligned} u_{x_i\theta_j} + T_0^{-1}(a u_{x_i\theta_j}) - T_0^{-1}(K u_{x_j\theta_i}) = {} & T_0^{-1}(f_{x_j\theta_i}) - \widetilde{T}_{0,1}^{-1}(a_{x_i}u_{x_j}) - T_0^{-1}(a_{\theta_i}u_{x_j}) \\ & - T_0^{-1}(a_{x_j\theta_i}u) - \widetilde{T}_{0,1}^{-1}(a_{x_j}u_{x_i}) - T_0^{-1}(u_{\theta_i}a_{x_j}) \\ & + \widetilde{T}_{0,1}^{-1}(K_{x_j}u_{x_i}) + T_0^{-1}(K_{\theta_i}u_{x_j}) + T_0^{-1}(K_{x_j\theta_i}u), \end{aligned} \tag{86}$$

which is of the type (78). Moreover, since $a \in C^3(\overline{\Omega} \times \mathbf{S}^1)$, $k \in C^3(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, the right-hand-side of (86) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. Again, by applying part (i), we get

$$u_{x_i\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ i, j = 1, 2. \tag{87}$$

From (83), (85), and (87), we get $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$.

□

We remark that for Theorem A.2 part (i) we only need $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and we only require $a \in C^2(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$ for Theorem A.2 part (ii). Moreover, in a similar fashion, one can show that under sufficiently increased regularity of a and k, the solution u of (1) belong to $u \in W^{m,p}(\Omega \times \mathbf{S}^1)$ for $\mathbb{Z} \ni m \geqslant 1$, provided $f \in W^{m,p}(\Omega \times \mathbf{S}^1)$.

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