Modeling large spot price deviations in electricity markets

Christian Laudagé, Florian Aichinger, Sascha Desmettre

RICAM-Report 2023-15

MODELING LARGE SPOT PRICE DEVIATIONS IN ELECTRICITY MARKETS

CHRISTIAN LAUDAGÉ*, FLORIAN AICHINGER†,‡, AND SASCHA DESMETTRE‡

Abstract. Increased insecurities on the energy markets have caused massive fluctuations of the electricity spot price within the past two years. In this work, we investigate the fit of a classical 3-factor model with a Gaussian base signal as well as one positive and one negative jump signal in this new market environment. We also study the influence of adding a second Gaussian base signal to the model. For the calibration of our model we use a Markov Chain Monte Carlo algorithm based on the so-called Gibbs sampling. The resulting 4-factor model is then compared to the 3-factor model in different time periods of particular interest and evaluated using posterior predictive checking. Additionally, we derive closed-form solutions for the price of futures contracts in our 4-factor spot price model.

We find that the 4-factor model outperforms the 3-factor model in times of non-crises. In times of crises, the second Gaussian base signal does not lead to a better the fit of the model. To the best of our knowledge, this is the first study regarding stochastic electricity spot price models in this new market environment. Hence, it serves as a solid base for future research.

Key words : Multi-factor models, Bayesian calibration, Markov Chain Monte Carlo, Ornstein-Uhlenbeck processes, Electricity spot price, Electricity Futures, Jump processes

JEL classification : C15, C11, C13, Q40, C51, Q41

1. Introduction

Due to the energy crisis, the energy market has become extremely volatile within the last two years, causing huge price peaks and large fluctuation of the electricity spot price. Figure 1 shows EEX spot price data for the last four years, where a significant increase of the volatility can be observed from the beginning of 2022 onward. In the present study, we want to investigate how classical models like the 3-Ornstein Uhlenbeck (OU) model proposed in [7], where the desaisonalized spot price is modeled as a superposition of a Gaussian OU-process and a positive and a negative jump OU-process, perform in this new market environment. We then study the impact of adding a second Gaussian component to the model, checking whether this additional degree of freedom allows a better distinction of long and short term fluctuations and hence produces a better overall fit. This extended model will be referred to as the 4-OU model. We are particularly interested in the fit of the respective models in three different time intervals, namely the pre-crisis period 2018-2021, the time period 2021-2023 when the energy crisis occurred and the total interval

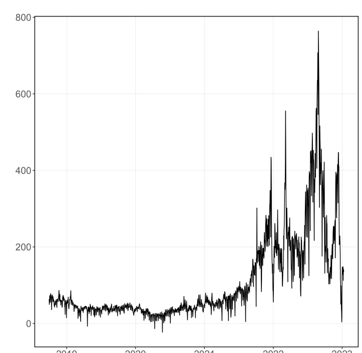


Figure 1. EEX price data.

Date: June 27, 2023.

* Rheinland-Pfälzische Technische Universität Kaiserslautern-Landau, DE-67663 Kaiserslautern, Germany.

† Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, AT-4040 Linz, Austria.

‡ Institute for Financial Mathematics and Applied Number Theory, University of Linz, AT-4040 Linz, Austria.

2018-2023. For each of these periods, we fit the 3-OU model as well as the 4-OU model to the corresponding electricity spot price data using a Markov Chain Monte Carlo algorithm. Model adequacy is then assessed through posterior predictive checking.

1.1. **Related work.** The first multi-factor electricity price models have been introduced by Lucia and Schwartz [12] and Schwartz and Smith [16]. In these models the (log-)spot price is described as a superposition of two latent stochastic processes: The long term behaviour is modeled as an arithmetic Brownian motion, the short term behavior as an OU-process. Since both components are Gaussian, the model can be calibrated using Kalman filter techniques. However, it turns out that Gaussian processes cannot appropriately describe the spikes, which frequently occur in observed spot price data. The one factor log-price model of Geman and Roncoroni [6] generates the characteristic spikes by making the jump direction and intensity level-dependent: High price levels lead to high jump intensity and downward jumps are more likely, whereas if the price is low, jumps are rare and upward-directed. In the long run, the process always fluctuates around a deterministic mean value. The calibration procedure is mainly based on likelihood estimation. The model of Benth, Kallsen and Meyer-Brandis [3] incorporates spikes and allows mean reversion to a stochastic base level by modelling the price as a sum of several OU-processes, some of which are driven by pure jump processes. A method to calibrate such a model was suggested by Tankov and Meyer-Brandis [13], who investigate a superposition of two OU-processes, one driven by a Levy jump process, the other driven by a Brownian motion. To calibrate their model, they came up with the so-called hard thresholding technique, where first the mean reversion parameters are estimated from the autocorrelation function and then maximum likelihood methods are applied to filter out the spikes path. The same approach is used by Hinderks and Wagner [9] for their two-factor model. The downside of calibrating the mean reversion rates separately is that some parameter interdependencies are being neglected. Gonzalez et al. [7] propose a superposition model with one Gaussian OU-process and several jump components, each of which having its own jump size distribution, jump frequency and mean reversion rate. This additional flexibility allows to distinguish between different jump patterns caused by different underlying physical origins and simultaneously avoids the attribution of smaller jumps to the Gaussian process. The model is then calibrated in a Bayesian framework using Markov Chain Monte Carlo (MCMC) methods, generating samples from the posterior distributions of the model parameters. In contrast to the MCMC methods based on time discretization of the involved processes developed in [17] and [8], Gonzalez et al. propose a MCMC algorithm for exact Bayesian inference and therefore do not have to take into account any approximation error. Table 1 on the next page provides a more detailed overview of the different spot price models and the corresponding calibration techniques mentioned above.

1.2. **Contribution.** Based on the work of Gonzalez et al. [7], we study two superposition models, a 3-OU model consisting of one Gaussian and two jump OU-processes with different sign and a 4-OU model, where a second Gaussian component is added to the 3-OU model (Section 2). To calibrate the 4-OU model, in Section 3, we develop an extension of the MCMC algorithm presented in [7]. We point out that even in the 4-OU model, the inference performed by the MCMC algorithm is still exact at the level of distributions, since there is no discretization involved in the update of the second Gaussian component. The two models are then calibrated to EEX spot price data within different time periods and are then compared in terms of model adequacy using posterior predictive checks (Section 5). Our study is the first attempt to calibrate such models for the extremely volatile data in the period 2021-2023 and can serve as a starting point for further research in this direction. In Section 6, we also give an explicit formula for the price of a futures contract in the 4-OU model.

	Schwartz/Smith	**Meyer-Brandis/Tankov**	**Hinderks/Wagner**
Model	$\ln(P_t) = X_t$	$P_t = f_t + X_t$	$P_t = f_t + X_t$
deterministic part	none	linear trend trigonometric yearly seasonality weekend effects	linear trend trigonometric yearly seasonality weekend effects
stochastic part	$X_t = S_t + L_t$ $dL_t = \mu_l dt \sigma_L dW_t^L$ $dS_t = -\lambda S_t dt + \sigma_S dW_t^S$	$X(t) = Y_1(t) + Y_2(t)$ $dY_1(t) = -\lambda_1^{-1} Y_1(t)dt + dW(t)$ $dY_2(t) = -\lambda_2^{-1} Y_2(t)dt + dL(t)$	$X(t) = Y_1(t) + Y_2(t)$ $dY_1(t) = \sigma dW(t)$ $dY_2(t) = -\lambda Y_2(t)dt + dL(t)$
Long term modelling	Brownian motion	Gaussian OU-Process	Brownian motion
Short term modelling	Gaussian OU-Process	Jump OU-Process	Jump OU-Process
Jump specifications	no jumps	L is a Levy process possibly time varying intensity	$L(t) = \sum_{i=1}^{N(t)} B_i \cdot D_i$ B_i are iid Bernoulli distributed D_i are iid Gamma distributed
Method of calibration	Kalman filtering	Hard thresholding	Hard thresholding
Remarks	• Kalman Filtering does not work if jump components are involved	• Does not work for more than one Gaussian component	• Does not work for more than one Gaussian component

	Seifert/Uhrig-Homburg	**Gonzalez et al.**	**4-OU model**
Model	$\ln(P_t) = f_t + X_t$	$P_t = e^{f(t/260)} X_t$	$P_t = f_t + X_t$
deterministic part	linear trend trigonometric yearly seasonality weekend effects	linear trend trigonometric yearly seasonality	linear trend trigonometric yearly seasonality
stochastic part	$X_t = S_t + L_t$ $dL_t = \sigma_L dW_t^L$ $dS_t = \lambda(\mu - S_t)dt + \sigma_S dW_t^S + \xi_t dP_t$	$X_t = \sum_{i=0}^{n} w_i Y_i(t),\ w_i \in \{-1, 1\}$ $dY_0(t) = -\lambda_0^{-1}(\mu - Y_0(t))dt + \sigma dW_0(t)$ $dY_i(t) = -\lambda_i^{-1} Y_i(t)dt + d\Pi_i(t),\ i = 1, \dots, n$	$X_t = Y_1(t) + Y_2(t) + J_1(t) - J_2(t)$ $dY_i(t) = -\lambda_i^{-1} Y_i(t)dt + \sigma_i dW_i(t),\ i = 1, 2$ $dJ_i(t) = -\lambda_i^{-1} J_i(t)dt + d\Pi_i(t),\ i = 1, 2$
Long term modelling	Brownian motion	Gaussian OU-Process	Gaussian OU-process
Short term modelling	OU-Process with jumps and BM	n Jump OU-Processes	one Gaussian and two Jump OU-Processes
Jump specifications	P is a Poisson process constant intensity normally distributed jump sizes	Π_i is a compound Poisson process constant or time dependent intensity exponentially distributed jump size	Π_i is a compound Poisson process constant intensity exponentially distributed jump size
Method of calibration	MCMC methods	MCMC methods	MCMC methods
Remarks	• Brownian fluctuations and jumps both incorporated in process S • same mean reversion	• separate mean reversion for each jump component	• separate mean reversion for each component

TABLE 1. Detailed comparison of different electricity spot price models in the literature.

2. The 4-factor model

In this section, we model the electricity spot price as a superposition of a deterministic seasonality function and 4 stochastic components, two Gaussian OU-processes and two jump OU-processes. This model is an extension of the 3-OU model introduced in [7]. The idea to use two Gaussian components to model long and short-term behavior of the spot price goes back to the work of [12] and has also been used in [17].

The model setup is as follows: We assume a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. On this probability space, we introduce two independent $\mathbb{F}$-adapted Brownian motions W^{Y_1} and W^{Y_2} and two compound Poisson processes Π_1 and Π_2 with constant jump intensity rates θ_1, θ_2 and exponentially distributed jump sizes with parameters β_1 and β_2 respectively. We model the electricity spot price

$$P_t = f(t) + X_t \tag{2.1}$$

as the sum of a deterministic seasonal function $f(t)$ and a stochastic part X_t, where the deseasonalized spot price X_t is a superposition of four stochastic processes

$$X(t) = Y_1(t) + Y_2(t) + J_1(t) - J_2(t), \tag{2.2}$$

with dynamics

$$dY_1(t) = -\lambda_{Y_1}^{-1} Y_1(t)dt + \sigma_{Y_1} dW_{Y_1}(t), \quad Y_1(0) = 0, \tag{2.3}$$

$$dY_2(t) = -\lambda_{Y_2}^{-1} Y_2(t)dt + \sigma_{Y_2} dW_{Y_2}(t), \quad Y_2(0) = 0, \tag{2.4}$$

$$dJ_1(t) = -\lambda_{J_1}^{-1} J_1(t)dt + d\Pi_1(t), \quad J_1(0) = 0, \tag{2.5}$$

$$dJ_2(t) = -\lambda_{J_2}^{-1} J_2(t)dt + d\Pi_2(t), \quad J_2(0) = 0. \tag{2.6}$$

The processes Y_1 and Y_2 admit the explicit solutions

$$Y_i(t) = \int_0^t \sigma e^{-\lambda_{Y_i}^{-1}(t-s)} dW_s, \quad i = 1, 2, \tag{2.7}$$

and they are called Gaussian OU processes since the conditional distribution Y_{t+s} given Y_t is normally distributed with

$$\mathbb{E}[Y_i(t+s)|Y_i(t) = y] = ye^{-\lambda_{Y_i}^{-1}s}, \qquad \mathbb{V}[Y_i(t+s)|Y_i(t) = y] = \lambda_{Y_i}\sigma_{Y_i}^2 \left(1 - e^{-2\lambda_{Y_i}^{-1}s}\right)/2.$$

In equation (2.3) and (2.4), the long term mean is 0, i.e., the process tends to revert to 0. The parameter $\lambda_{Y_i}^{-1}$ is responsible for the speed of the mean reversion and σ_{Y_i} governs the volatility of the respective process. The processes J_1 and J_2, driven by the compound Poisson processes Π_1 and Π_2 with interval representation $\Pi_i(t) = \sum_{j=1}^{\infty} \xi_j^i \mathbb{1}_{\{t\geq\tau_j^i\}}$, $i = 1, 2$, where τ_j^i are the arrival times of a Poisson process and ξ_j^i is the jump size at time τ_j^i are called jump OU processes and they have explicit solutions

$$J_i(t) = \sum_{j:0\leq\tau_j^i\leq t} e^{-\lambda_{J_i}^{-1}(t-\tau_j^i)}\xi_j^i.$$

The parameters $\lambda_{J_1}^{-1}$ and $\lambda_{J_2}^{-1}$ in equations (2.5) and (2.6) are the mean reversion speeds of the respective jump processes. The mean level is again 0.

3. Calibration

In this section, we explain the calibration of the superposition model introduced in the previous section. After calibrating the deterministic seasonality function applying least squares methods, we use a Bayesian approach to calibrate the parameters of the stochastic components. To this end, we apply MCMC methods to obtain samples from the posterior distributions of the model parameters. The algorithm we use is based on the algorithm described in [7, Section 3.3], extended by the update steps for the additional Gaussian component in the 4-OU model.

3.1. **Calibration of the seasonality function.** We first calibrate the deterministic seasonality function f. We assume that f is a superposition of a linear trend and trigonometric functions modeling half-yearly and quarterly seasonal variations of the spot price.

$$\begin{aligned} f(t; a_1, \ldots, a_6) &= a_1 + a_2 t + a_3 \sin(2\pi t) \\ &+ a_4 \cos(2\pi t) + a_5 \sin(4\pi t) + a_6 \cos(4\pi t). \end{aligned}$$

The parameters $a_1, \ldots, a_6$ are then obtained by applying least-squares methods. Finally f gets subtracted from the data to get the deseasonalized data. An example for a calibrated seasonality function is shown in Figure 2.

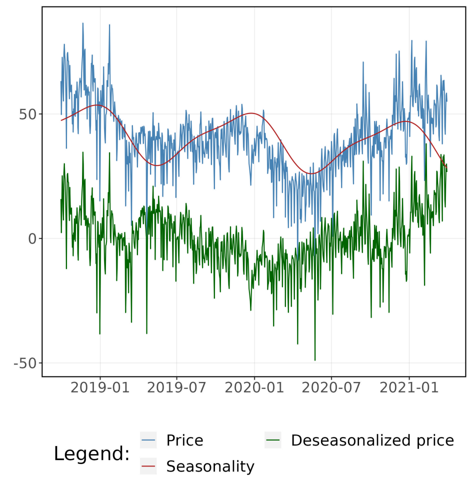


FIGURE 2. Calibrated seasonality function for EEX data from 2018-2021.

3.2. **Calibration of the stochastic part.** We calibrate the model parameters λ_{Y_1},σ_{Y_1},λ_{Y_2},σ_{Y_2}, λ_{J_1},λ_{J_2},θ_1,θ_2,β_1,β_2 simultaneously in a Bayesian framework, i.e., our aim is to obtain the joint posterior distribution

$$\pi(\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \theta_1, \theta_2, \beta_1, \beta_2 | \chi)$$

given the deseasonalized spot price data χ. To this end, we use MCMC methods, constructing a Markov chain whose stationary distribution is the posterior distribution of the model parameters. For a detailed introduction to the topic see for example [1] and [5]. For our calibration procedure, we apply Gibbs sampling, i.e., we iteratively generate samples from the distribution of each variable conditioned on the current values of the other variables. We start with samples

$$\left(\lambda_{Y_1}^{(0)}, \lambda_{Y_2}^{(0)}, \sigma_{Y_1}^{(0)}, \sigma_{Y_2}^{(0)}, \lambda_{J_1}^{(0)}, \lambda_{J_2}^{(0)}, \theta_1^{(0)}, \theta_2^{(0)}, \beta_1^{(0)}, \beta_2^{(0)}\right)$$

from the a-priori distribution of the parameters. Based on a sample

$$\left(\lambda_{Y_1}^{(k)}, \lambda_{Y_2}^{(k)}, \sigma_{Y_1}^{(k)}, \sigma_{Y_2}^{(k)}, \lambda_{J_1}^{(k)}, \lambda_{J_2}^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}\right),$$

we then obtain

$$\left(\lambda_{Y_1}^{(k+1)}, \lambda_{Y_2}^{(k+1)}, \sigma_{Y_1}^{(k+1)}, \sigma_{Y_2}^{(k+1)}, \lambda_{J_1}^{(k+1)}, \lambda_{J_2}^{(k+1)}, \theta_1^{(k+1)}, \theta_2^{(k+1)}, \beta_1^{(k+1)}, \beta_2^{(k+1)}\right)$$

via the following algorithm:

Algorithm 3.1 (Gibbs Sampling)**.** *Starting from a given initial state of the chain, the MCMC algorithm cycles through the following steps:*

(1) $\sigma_{Y_1}^{2\ (k+1)} \sim \pi\left(\sigma_{Y_1}^{2\ (k+1)} | \sigma_{Y_2}^{(k)}, \lambda_{Y_1}^{(k)}, \lambda_{Y_2}^{(k)}, \lambda_{J_1}^{(k)}, \lambda_{J_2}^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \mathcal{X}\right)$

(2) $\sigma_{Y_2}^{2\ (k+1)} \sim \pi\left(\sigma_{Y_2}^{2\ (k+1)} | \sigma_{Y_1}^{(k+1)}, \lambda_{Y_1}^{(k)}, \lambda_{Y_2}^{(k)}, \lambda_{J_1}^{(k)}, \lambda_{J_2}^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \mathcal{X}\right)$

(3) $\lambda_{Y_1}^{(k+1)} \sim \pi\left(\lambda_{Y_1}^{(k+1)} | \sigma_{Y_1}^{(k+1)}, \sigma_{Y_2}^{(k+1)}, \lambda_{Y_2}^{(k)}, \lambda_{J_1}^{(k)}, \lambda_{J_2}^{(k)}, \theta_1^{(k)}, \theta_2^{(k)}, \beta_1^{(k)}, \beta_2^{(k)}, \mathcal{X}\right)$

(4) $\lambda_{Y_2}^{(k+1)} \sim \pi\left(\lambda_{Y_2}^{(k+1)} | \dots\right)$

(5) $\lambda_{J_1}^{(k+1)} \sim \pi\left(\lambda_{J_1}^{(k+1)} | \dots\right)$

(6) $\lambda_{J_2}^{(k+1)} \sim \pi\left(\lambda_{J_2}^{(k+1)} | \dots\right)$

(7) $\theta_1^{(k+1)} \sim \pi\left(\theta_1^{(k+1)} | \dots\right)$

(8) $\theta_2^{(k+1)} \sim \pi\left(\theta_2^{(k+1)} | \dots\right)$

(9) $\beta_1^{(k+1)} \sim \pi\left(\beta_1^{(k+1)} | \sigma_{Y_1}^{(k+1)}, \sigma_{Y_2}^{(k+1)}, \lambda_{Y_1}^{(k+1)}, \lambda_{Y_2}^{(k+1)}, \lambda_{J_1}^{(k+1)}, \lambda_{J_2}^{(k+1)}, \theta_1^{(k+1)}, \theta_2^{(k+1)}, \beta_2^{(k)}, \mathcal{X}\right)$

(10) $\beta_2^{(k+1)} \sim \pi\left(\beta_2^{(k+1)} | \sigma_{Y_1}^{(k+1)}, \sigma_{Y_2}^{(k+1)}, \lambda_{Y_1}^{(k+1)}, \lambda_{Y_2}^{(k+1)}, \lambda_{J_1}^{(k+1)}, \lambda_{J_2}^{(k+1)}, \theta_1^{(k+1)}, \theta_2^{(k+1)}, \beta_1^{(k+1)}, \mathcal{X}\right)$

3.3. **Data augmentation.** The Gibbs sampler would yield samples of the posterior distribution of the parameters, however, in order to calculate densities involved in the Gibbs algorithm, we need the likelihood $l(\chi|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \theta_1, \theta_2, \beta_1, \beta_2)$ of the data given the parameters, which for our superposition model cannot be calculated analytically or numerically. Thus, to overcome this hurdle, we use so-called data augmentation methods (cf. [2], [14]). We point out that the augmentation method we use is an extension of the procedure described in [7] when adding a second Gaussian component.
Let $\chi = \{x_0, \dots, x_N\}$ be the vector of observations of the deseasonalized spot price at times $0 = t_0, \dots, t_N = T$ and denote by $\Delta_i = t_i - t_{i-1}$ the time increment between two observations. In order to get an explicit expression for the likelihood $l(\chi|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \theta_1, \theta_2, \beta_1, \beta_2)$, we augment the state space with observations $\mathcal{Y}_2 = \{y_{2,0}, \dots, y_{2,N}\}$ of the short-term process and observations $\mathcal{J}_1 = \{j_{1,0}, \dots, j_{1,N}\}$ and $\mathcal{J}_2 = \{j_{2,0}, \dots, j_{2,N}\}$ of the jump processes. The likelihood of the data $\mathcal{X}$ given $\mathcal{Y}_2$, $\mathcal{J}_1$ and $\mathcal{J}_2$ becomes independent of $\lambda_{Y_2}, \sigma_{Y_2}, \beta_1, \beta_2, \theta_1, \theta_2$ and can be calculated explicitly as

$$
\begin{aligned}
&l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \mathcal{Y}_2, \mathcal{J}_1, \mathcal{J}_2) \\
&\qquad = \frac{1}{\sqrt{2\pi}\prod_{i=1}^N \sqrt{\lambda_{Y_1}\sigma_{Y_1}^2\left(1 - e^{2\lambda_{Y_1}^{-1}\Delta t}\right)/2}} \exp\left\{-\frac{1}{2}\sum_{i=1}^N \frac{\left(y_{1,i} - y_{1,i-1}e^{-\lambda_{Y_1}^{-1}\Delta t}\right)^2}{\lambda_{Y_1}\sigma_{Y_1}^2\left(1 - e^{-2\lambda_{Y_1}^{-1}\Delta t}\right)/2}\right\},
\end{aligned}
$$

with $y_{1,i} = x_i - y_{2,i} - j_{1,i} + j_{2,i}$.

Since for N iid random variables $\epsilon_i \sim \mathcal{N}(0,1)$, $i = 1, \dots, N$, the transition densities of the Gaussian OU-process Y_2 are given by

$$
Y_2(j\Delta t) = Y_2((j-1)\Delta t)e^{-\lambda_{Y_2}^{-1}\Delta t} + \left(\frac{\sigma_{Y_2}^2\lambda_{Y_2}}{2}\left(1 - e^{-2\lambda_{Y_2}^{-1}\Delta t}\right)\right)^{1/2}\epsilon_{2,i}, \quad i = 1, \dots, N,
$$

in order to make the parameters independent of the latent variables, instead of treating the process $Y_2(t)$ itself as missing data, the random vector $\mathcal{E} = \{\epsilon_{2,1}, \dots, \epsilon_{2,N}\}$ is considered as the hidden variable. Similarly the set of pairs $\Phi_1 = \{(\tau_{1,j}, \xi_{1,j})\}_{1 \le j \le N_{T_1}}$ and $\Phi_2 = \{(\tau_{2,j}, \xi_{2,j})\}_{1 \le j \le N_{T_2}}$ of jump times and corresponding jump sizes are treated as the missing data instead of $J_1(t)$ and $J_2(t)$. The sets Φ_i can be interpreted as realisations of marked Poisson processes taking values in $[0,T]\times(0,\infty)$ and their likelihood $l(\Phi_i|\theta_i,\beta_i)$ can be calculated with respect to a dominating measure (for details see Appendix B). Thus with

$$y_{2,0}=0;\quad y_{2,i}=y_{2,i-1}e^{-\lambda_{Y_2}^{-1}\Delta t}+\left(\frac{\sigma_{Y_2}^2\lambda_{Y_2}}{2}\left(1-e^{-2\lambda_{Y_2}^{-1}\Delta t}\right)\right)^{1/2}\epsilon_i,\quad i=1,\dots,N,$$

$$j_{1,i}=\sum_{n=1}^{N_T^1}e^{-\lambda_{J1}^{-1}(i\cdot\Delta t-\tau_{1,n})}\xi_{1,n},\mathbb{1}_{\{\tau_{1,n}\le i\cdot\Delta t\}},\quad i=0,\dots,N,$$

$$j_{2,i}=\sum_{n=1}^{N_T^2}e^{-\lambda_{J1}^{-1}(i\cdot\Delta t-\tau_{2,n})}\xi_{2,n}\mathbb{1}_{\{\tau_{2,n}\le i\cdot\Delta t\}},\quad i=0,\dots,N,$$

we obtain

$$\begin{aligned}&l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_1,\Phi_2,\mathcal{E})\\&\quad=\frac{1}{\sqrt{2\pi}\prod_{i=1}^N\sqrt{\lambda_{Y_1}\sigma_{Y_1}^2\left(1-e^{2\lambda_{Y_1}^{-1}\Delta t}\right)/2}}\exp\left\{-\frac{1}{2}\sum_{i=1}^N\frac{\left(y_{1,i}-y_{1,i-1}e^{-\lambda_{Y_1}^{-1}\Delta t}\right)^2}{\lambda_{Y_1}\sigma_{Y_1}^2\left(1-e^{-2\lambda_{Y_1}^{-1}\Delta t}\right)/2}\right\},\end{aligned}$$

with $y_{1,i}=x_i-y_{2,i}-j_{1,i}+j_{2,i}$.

To see, why the explicit likelihood of the data is particularly important for the Gibbs sampling, consider the following factorization which is frequently used within the algorithm:

$$\begin{aligned}&\pi(\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\theta_1,\theta_2,\beta_1,\beta_2,\Phi_1,\Phi_2,\mathcal{E}|\mathcal{X})\propto l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_1,\Phi_2,\mathcal{E})\\&\cdot l(\Phi_1|\theta_1,\beta_1)\cdot l(\Phi_2|\theta_2,\beta_2)\cdot\pi(\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\theta_1,\theta_2,\beta_1,\beta_2,\mathcal{E}).\end{aligned}\tag{3.1}$$

3.4. **Classes of prior distributions.** We specify the classes of prior distributions for the model parameters. As in [7], conjugate priors are chosen whenever it is possible for the sake of computational efficiency. The starting values for the first MCMC iteration are the starting values of [7] scaled to the yearly time framework. Here $\mathrm{IG}(a,b)$ denotes the inverse Gamma distribution with shape parameter a and scale parameter b and $\mathrm{Ga}(a,b)$ denotes the Gamma distribution with shape parameter a and scale parameter b.

	Parameter	Prior distribution	Starting value (3OU)	Starting value (4OU)
Volatility of 1. Gaussian OU process	σ_{Y_1}	$\mathrm{IG}(1.5,0.005\cdot 365)$	$0.2\sqrt{365}$	$0.1\sqrt{365}$
Volatility of 2. Gaussian OU process	σ_{Y_2}	-	-	10
Mean reversion speed of 1. Gaussian OU process	λ_{Y_1}	$\mathrm{IG}(1,1)$	$\frac{5}{365}$	0.001
Mean reversion speed of 2. Gaussian OU process	λ_{Y_2}	$\mathrm{IG}(1,1)$	1	1
Mean reversion speed of pos. jump process	λ_{J_1}	$\mathrm{IG}(1,1)$	$\frac{5}{365}$	$\frac{1}{365}$
Mean reversion speed of pos. jump process	λ_{J_2}	$\mathrm{IG}(1,1)$	$\frac{1}{365}$	$\frac{1}{365}$
Positive jump intensity	θ_1	$\mathrm{Ga}(1,\frac{10}{365})$	$0.001\cdot 365$	$0.001\cdot 365$
Negative jump intensity	θ_2	$\mathrm{Ga}(1,\frac{10}{365})$	$0.001\cdot 365$	$0.001\cdot 365$
Positive jump size	β_1	$\mathrm{IG}(1,1)$	0.5	0.5
Negative jump size	β_2	$\mathrm{IG}(1,1)$	0.5	0.5

TABLE 2. Prior distributions and starting values for the model parameters. For the prior distribution of σ_{Y_2} we refer to Section 3.5.

3.5. **MCMC algorithm for the 4-factor model.** We use Gibbs sampling to iteratively generate samples from the distribution of each variable conditioned on the current values of the other variables. Compared to the algorithm in [7, Section 3.3], we have additional update steps for the parameters σ and λ of the second Gaussian OU process and its path (the additional hidden variable $\mathcal{E}$). The update steps for the jump paths (the hidden variables Φ_i) have been developed in [14] and [4] in the context of calibrating volatility models and have first been used for electricity spot price models in [7]. To ensure that the mixing is of the same order for small and large values of λ, a transformation of λ to $\rho := e^{-\lambda^{-1}}$ is applied in the inference procedure.

Algorithm 3.2 (Augmented Gibbs Sampling)**.** *Starting from a given initial state of the chain, the MCMC algorithm cycles through the following steps:*

(1) $\sigma^2_{Y_1} \sim \pi(\sigma^2_{Y_1}|\lambda_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E}, \mathcal{X})$
(2) $\sigma^2_{Y_2} \sim \pi(\sigma^2_{Y_2}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E}, \mathcal{X})$
(3) $\rho_{Y_1} \sim \pi(\rho_{Y_1}|\sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E}, \mathcal{X})$
(4) $\rho_{Y_2} \sim \pi(\rho_{Y_2}|\lambda_{Y_1}, \sigma_{Y_1}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E}, \mathcal{X})$
(5) $\rho_{J_1} \sim \pi(\rho_{J_1}|\sigma_{Y_1}, \lambda_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E}, \mathcal{X})$
(6) $\rho_{J_2} \sim \pi(\rho_{J_2}, \sigma_{Y_1}, \lambda_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \Phi_1, \Phi_2, \mathcal{E}, \mathcal{X})$
(7) $\theta_1 \sim \pi(\theta_1|\Phi_1)$
(8) $\theta_2 \sim \pi(\theta_2|\Phi_2)$
(9) $\beta_1 \sim \pi(\beta_1|\Phi_1)$
(10) $\beta_2 \sim \pi(\beta_2|\Phi_2)$
(11) $\mathcal{E} \sim \pi(\mathcal{E}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{X})$
(12) $\Phi_1 \sim \pi(\Phi_1|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \theta_1, \beta_1, \Phi_2, \mathcal{E}, \mathcal{X})$
(13) $\Phi_2 \sim \pi(\Phi_2|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \theta_2, \beta_2, \Phi_1, \mathcal{E}, \mathcal{X})$

3.5.1. *Detailed update steps.* Here we give a detailed instruction for each update step. As described in Algorithm 3.1, if a parameter has already been updated within the cycle (1)-(13), then in the following steps, we condition on its updated value. Whenever it is possible, we directly sample from the conditional distribution, otherwise, we apply a random walk Metropolis step using the factorization (3.1) within the Gibbs procedure.

Update $\sigma_{\mathbf{Y_1}}$:

With the choice of an $\mathrm{IG}(a_{\sigma_{Y_1}}, b_{\sigma_{Y_1}})$ distributed prior for σ_{Y_1}, for the posterior we get

$$\sigma^2_{Y_1}|\lambda_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \mathcal{X}, \mathcal{E}, \Phi_1, \Phi_2 \sim \mathrm{IG}\left(a_{\sigma_{Y_1}} + \frac{N}{2}, b_{\sigma_{Y_1}} + \frac{1}{\lambda_{Y_1}} \sum_{i=1}^{N} \frac{\left(l_i - l_{i-1} e^{-\lambda_{Y_1}^{-1}\Delta_i}\right)^2}{1 - e^{-2\lambda_{Y_1}^{-1}\Delta_i}}\right).$$

Update $\sigma_{\mathbf{Y_2}}$:

Generate candidate σ'_{Y_2} from $\sigma'_{Y_2}|\sigma_{Y_2} \sim \mathcal{N}(\sigma_{Y_2}, \sigma, 0, \infty)$. Perform Metropolis-step with acceptance ratio

$$\alpha(\sigma_S{}', \sigma_S) = \min\left\{1, \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma'_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\sigma'_{Y_2})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\sigma_{Y_2})} \cdot \frac{1 - \Phi(-\frac{\sigma_{Y_2}}{\sigma})}{1 - \Phi(-\frac{\sigma'_{Y_2}}{\sigma})}\right\}.$$

Since we do not know the magnitude of σ_{Y_2} in relation to σ_{Y_1}, we decided to approximate the quotient of the a-priori distributions $\frac{\pi(\sigma'_{Y_2})}{\pi(\sigma_{Y_2})}$ with 1. This can, e.g., be justified by the choice of an inverse gamma distribution $\mathrm{IG}(1, \beta)$ for the a-priori distribution π together with the limiting result $\lim_{\beta\to\infty} \frac{\pi(\sigma'_{Y_2})}{\pi(\sigma_{Y_2})} = 1$. The same argument holds by using a uniform distribution and considering the limiting case in which the interval length goes to infinity.

Update $\rho_{\mathbf{Y_1}} = \mathbf{e}^{-\lambda_{\mathbf{Y_1}}^{-1}}$:

Generate candidate ρ'_{Y_1} from $\rho'_{Y_1}|\rho_{Y_1} \sim \mathcal{N}(\rho_{Y_1}, \sigma, 0, 1)$. Perform Metropolis-step with acceptance ratio

$$\alpha(\rho_{Y_1}{}', \rho_{Y_1}) = \min \left\{ 1, \frac{l(\mathcal{X}|\rho'_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho'_{Y_1})}{l(\mathcal{X}|\rho_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho_{Y_1})} \cdot \frac{\Phi(\frac{1-\rho_{Y_1}}{\sigma}) - \Phi(-\frac{\rho_{Y_1}}{\sigma})}{\Phi(\frac{1-\rho'_{Y_1}}{\sigma}) - \Phi(-\frac{\rho'_{Y_1}}{\sigma})} \right\}.$$

Since λ_{Y_1} has prior distribution $\mathrm{IG}(a_{\lambda_{Y_1}}, b_{\lambda_{Y_1}})$, $Y = e^{-\lambda_{Y_1}^{-1}}$ has a-priori density

$$f_Y(y) = \frac{1}{y} \frac{b_{\lambda_{Y_1}}}{\Gamma(a_{\lambda_{Y_1}})} (-\ln(y))^{a_{\lambda_{Y_1}}-1} e^{b_{\lambda_{Y_1}} \ln(y)}.$$

Update $\rho_{\mathbf{Y_2}} = \mathbf{e}^{-\lambda_{\mathbf{Y_2}}^{-1}}$:

Generate candidate ρ'_{Y_2} from $\rho'_{Y_2}|\rho_{Y_2} \sim \mathcal{N}(\rho_{Y_2}, \sigma, 0, 1)$. Perform Metropolis-step with acceptance ratio

$$\alpha(\rho_{Y_2}{}', \rho_{Y_2}) = \min \left\{ 1, \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \rho'_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho'_S)}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \rho_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho_{Y_2})} \cdot \frac{\Phi(\frac{1-\rho_{Y_2}}{\sigma}) - \Phi(-\frac{\rho_{Y_2}}{\sigma})}{\Phi(\frac{1-\rho'_{Y_2}}{\sigma}) - \Phi(-\frac{\rho'_{Y_2}}{\sigma})} \right\}.$$

Since λ_{Y_2} is $\mathrm{IG}(a_{\lambda_{Y_2}}, b_{\lambda_{Y_2}})$ distributed, $Y = e^{-\lambda_{Y_2}^{-1}}$ has density

$$f_Y(y) = \frac{1}{y} \frac{b_{\lambda_{Y_2}}}{\Gamma(a_{\lambda_{Y_2}})} (-\ln(y))^{a_{\lambda_{Y_2}}-1} e^{b_{\lambda_{Y_2}} \ln(y)}.$$

Update $\rho_{\mathbf{J_1}} = \mathbf{e}^{-\lambda_{\mathbf{J1}}^{-1}}$:

Generate candidate ρ'_{J_1} from $\rho'_{J_1}|\rho_{J_1} \sim \mathcal{N}(\rho_{J_1}, \sigma, 0, 1)$. Perform Metropolis-step with acceptance ratio

$$\alpha(\rho_{J_1}{}', \rho_{J_1}) = \min \left\{ 1, \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \rho'_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho'_{J_1})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \rho_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho_{J_1})} \cdot \frac{\Phi(\frac{1-\rho_{J_1}}{\sigma}) - \Phi(-\frac{\rho_{J_1}}{\sigma})}{\Phi(\frac{1-\rho'_{J_1}}{\sigma}) - \Phi(-\frac{\rho'_{J_1}}{\sigma})} \right\}.$$

Since λ_{J_1} is $\mathrm{IG}(a_{\lambda_{J_1}}, b_{\lambda_{J_1}})$ distributed, $Y = e^{-\lambda_{J_1}^{-1}}$ has density

$$f_Y(y) = \frac{1}{y} \frac{b_{\lambda_{J_1}}}{\Gamma(a_{\lambda_{J_1}})} (-\ln(y))^{a_{\lambda_{J_1}}-1} e^{b_{\lambda_{J_1}} \ln(y)}.$$

Update $\rho_{\mathbf{J_2}} = \mathbf{e}^{-\lambda_{\mathbf{J_2}}^{-1}}$:

Generate candidate ρ'_{J_2} from $\rho'_{J_2}|\rho_{J_2} \sim \mathcal{N}(\rho_{J_2}, \sigma, 0, 1)$. Perform Metropolis-step with acceptance ratio

$$\alpha(\rho_{J_2}{}', \rho_{J_2}) = \min \left\{ 1, \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \rho'_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho'_{J_2})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \rho_{J_2}, \Phi_1, \Phi_2, \mathcal{E})\pi(\rho_{J_2})} \cdot \frac{\Phi(\frac{1-\rho_{J_2}}{\sigma}) - \Phi(-\frac{\rho_{J_2}}{\sigma})}{\Phi(\frac{1-\rho'_{J_2}}{\sigma}) - \Phi(-\frac{\rho'_{J_2}}{\sigma})} \right\}.$$

Since λ_{J_2} is $\mathrm{IG}(a_{\lambda_{J_2}}, b_{\lambda_{J_2}})$ distributed, $Y = e^{-\lambda_{J_2}^{-1}}$ has density

$$f_Y(y) = \frac{1}{y}\frac{b_{\lambda_{J_2}}}{\Gamma(a_{\lambda_{J_2}})}(-\ln(y))^{a_{\lambda_{J_2}}-1}e^{b_{\lambda_{J_2}}\ln(y)}$$

Update θ_1:

With the choice of an $\mathrm{Ga}(a_{\theta 1}, b_{\theta 1})$ distributed prior for θ_1, for the posterior we get

$$\theta_1|\Phi_1 \sim \mathrm{Ga}\left(a_{\theta_1} + N_T^1, b_{\theta_1} + T\right).$$

Update θ_2:

With the choice of an $\mathrm{Ga}\left(a_{\theta_2}, b_{\theta_2}\right)$ distributed prior for θ_2, for the posterior we get

$$\theta_2|\Phi_2 \sim \mathrm{Ga}\left(a_{\theta_2} + N_T^2, b_{\theta_2} + T\right).$$

Update β_1

With the choice of an $\mathrm{IG}(a_{\beta_1}, b_{\beta_1})$ distributed prior for β_1, for the posterior we get

$$\beta_1|\Phi_1 \sim \mathrm{IG}\left(a_{\beta_1} + N_T^1, b_{\beta_1} + \sum_{i=1}^{N_T^1}\xi_{1i}\right).$$

Update β_2:

With the choice of an $\mathrm{IG}(a_{\beta_2}, b_{\beta_2})$ distributed prior for β_2, for the posterior we get

$$\beta_2|\Phi_2 \sim \mathrm{IG}\left(a_{\beta_2} + N_T^2, b_{\beta_2} + \sum_{i=1}^{N_T^2}\xi_{2i}\right).$$

Update $\mathcal{E}$:

Generating values step: For $n \in \{1, \dots, N\}$ randomly choose indices $i_1, \dots, i_n \in \{1, \dots, N\}$ and define $I := \{i_1, \dots, i_n\}$, $\hat{I} := \{1, \dots, N\} \setminus I$. Generate candidate $\mathcal{E}'$ by setting $\epsilon_i' = \epsilon_i$ for $i \in \hat{I}$ and drawing $\epsilon_i' \sim \mathcal{N}(0,1)$ for $i \in I$. Since

$$q(\mathcal{E}'|\mathcal{E}) = \frac{1}{N}\frac{1}{\binom{N}{n}}\prod_{i\in I}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{(\epsilon_i')^2}{2}\Big)$$

we have

$$\frac{\pi(\mathcal{E}')}{\pi(\mathcal{E})}\cdot\frac{q(\mathcal{E}|\mathcal{E}')}{q(\mathcal{E}|\mathcal{E}')} = 1.$$

Thus for the Metropolis-step, the acceptance ratio is given by

$$\alpha(\mathcal{E}', \mathcal{E}) = \min\left\{1, \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E}')}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})}\right\}.$$

Permutation step: For $n \in \{1, \dots, N\}$ randomly choose indices $i_1, \dots, i_n \in \{1, \dots, N\}$ and define $I := \{i_1, \dots, i_n\}$, $\hat{I} := \{1, \dots, N\}\setminus I$. Let $\pi : I \to I$ be a permutation. Generate candidate $\mathcal{E}'$ by setting $\epsilon_i' = \epsilon_i$ for $i \in \hat{I}$ and $\epsilon_i' = \epsilon_{\pi(i)}$ for $i \in I$. Here $q(\mathcal{E}'|\mathcal{E}) = \frac{1}{N}\frac{1}{\binom{N}{n}}\frac{1}{n!}$ and again we have

$$\frac{\pi(\mathcal{E}')}{\pi(\mathcal{E})}\cdot\frac{q(\mathcal{E}|\mathcal{E}')}{q(\mathcal{E}|\mathcal{E}')} = 1.$$

Thus for the Metropolis-step, the acceptance ratio is given by

$$\alpha(\mathcal{E}', \mathcal{E}) = \min\left\{1, \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E}')}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_1, \Phi_2, \mathcal{E})}\right\}.$$

Multiplicative update: For $n \in \{1,\dots,N\}$ randomly choose indices $i_1,\dots,i_n \in \{1,\dots,N\}$ and define $I := \{i_1,\dots,i_n\}$, $\hat{I} := \{1,\dots,N\} \setminus I$. Generate candidate $\mathcal{E}'$ by setting $\epsilon_i' = \epsilon_i$ for $i \in \hat{I}$ and $\epsilon_i' = \xi_i \epsilon_i$ with $\xi_i \sim \mathcal{N}(1, c^2)$ for $i \in I$. We have

$$q(\mathcal{E}'|\mathcal{E}) = \frac{1}{N}\frac{1}{\binom{N}{n}}\prod_{i\in I}\frac{1}{\sqrt{2\pi}\epsilon_i c}\exp\Big(-\frac{(\epsilon_i'-\epsilon_i)^2}{2\epsilon_i^2 c^2}\Big)$$

and

$$\pi(\mathcal{E}) = \prod_{i=1}^{N}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{(\epsilon_i)^2}{2}\Big).$$

Thus for the Metropolis-step, the acceptance ratio is given by

$$\alpha(\mathcal{E}',\mathcal{E}) = \min\left\{1, \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_1,\Phi_2,\mathcal{E}')}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_1,\Phi_2,\mathcal{E})}\frac{\pi(\mathcal{E}')}{\pi(\mathcal{E})}\frac{q(\mathcal{E}|\mathcal{E}')}{q(\mathcal{E}'|\mathcal{E})}\right\}.$$

Update $\mathbf{\Phi_1}$:

Birth-and-death step: Choose birth-move with probability $p \in (0,1)$. Generate (τ_1, ξ_1), where $\tau_1 \sim \mathcal{U}([0,T])$ and $\xi_1 \sim \mathrm{Ex}(\beta_1)$. Proposal transition kernel $q(\Phi_1, \Phi_1 \cup \{(\tau_1,\xi_1)\})$ has density

$$q(\Phi_1, \Phi_1 \cup \{(\tau_1,\xi_1)\}) = \beta_1^{-1}\exp(-(\beta_1^{-1}-1)\xi_1)$$

with respect to the product of Lebesgue-measure on $[0,T]$ and Ex(1) measure on $(0,\infty)$.

Choose death-move with probability $1-p$. Select a randomly selected point (τ_{1i}, ξ_{1i}) being removed from Φ_1 (if Φ_1 is not empty). The proposal transition kernel with respect to the counting measure is

$$q(\Phi_1, \Phi_1 \setminus \{(\tau_{1i},\xi_{1i})\}) = \frac{1}{N_T^1},$$

where N_T^1 is the number of points in Φ_1 before the death-move.
The acceptance ratio for birth-move from Φ_1 to $\Phi_1 \cup \{(\tau_1,\xi_1)\}$ is then given by

$$\alpha(\Phi_1, \Phi_1 \cup \{(\tau_1,\xi_1)\}) = \min\{1, r(\Phi_1, (\tau_1,\xi_1))\}$$

and the acceptance ratio for the death-move from Φ_1 to $\Phi_1 \setminus \{(\tau_{1i},\xi_{1i})$ is

$$\alpha(\Phi_1, \Phi_1 \setminus \{(\tau_{1i},\xi_{1i})) = \min\left\{1, \frac{1}{r(\Phi_1 \setminus \{(\tau_{1i},\xi_{1i})\}, (\tau_{1i},\xi_{1i}))}\right\},$$

where

$$\begin{aligned} r(\hat{\Phi}_1,(\theta_1,\beta_1)) &= \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_1\cup\{(\tau_1,\xi_1)\},\lambda_{J_2},\Phi_2,\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_1,\lambda_{J_2},\Phi_2,\mathcal{E})}\frac{\pi(\hat{\Phi}_1\cup\{(\tau_1,\xi_1)|\theta_1,\beta_1\})}{\pi(\hat{\Phi}_1|\theta_1,\beta_1\})} \\ &\quad\times\frac{1-p}{p}\frac{1}{(N_T^1+1)q(\hat{\Phi}_1,\hat{\Phi}_1\cup\{(\theta_1,\xi_1)\})} \\ &= \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_1\cup\{(\tau_1,\xi_1)\},\lambda_{J_2},\Phi_2,\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_1,\lambda_{J_2},\Phi_2,\mathcal{E})}\frac{1-p}{p}\frac{T}{\hat{N}_T^1+1}\theta_1. \end{aligned}$$

Local displacement move: Assume that the jump times are ordered, i.e., $\tau_{1,1} < \cdots < \tau_{1,N_T^1}$. Choose randomly one of the jump times, say $\tau_{1,j}$, and generate a new jump time uniformly on $[\tau_{1,j-1}, \tau_{1,j+1}]$. Displace and resize the point $(\tau_{1,j}, \xi_{1,j})$ to (τ_1, ξ_1), where $\xi_1 = e^{-\lambda_{J1}^{-1}(\tau_1-\tau_{1,j})}\xi_{1,j}$. Perform Metropolis-step with acceptance ratio $\alpha(\Phi_1, \Phi_1') = \min\{1, r(\Phi_1, \Phi_1')\}$, where

$$r(\Phi_1,\Phi_1') = \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1',\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1,\mathcal{E})}\frac{e^{-\beta_1^{-1}\xi_1}}{e^{-\beta_1^{-1}\xi_{1,j}}}e^{-\lambda_{J1}(\tau_1-\tau_{1,j})}.$$

Multiplicative jump size update For each jump $(\tau_{1,j},\xi_{1,j})$ propose a new jump size $\xi_{1,j}' = \xi_{1,j}\phi_{1,j}$, where $\log(\phi_{1,j}) \sim \mathcal{N}(0,c_1^2)$ are i.i.d. random variables. The variance c_1^2 is chosen inversely proportional to the current number of jumps. Perform Metropolis-step with acceptance ratio [1]

$$\alpha(\Phi_1,\Phi_1') = \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1',\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1,\mathcal{E})}\exp\left\{-(\beta_1^{-1}-1)\sum_{i=1}^{N_T^1}(\xi_{1,i}'-\xi_{1,i})\right\}\prod_{i=1}^{N_T^1}\frac{\xi_{1,i}'}{\xi_{1,i}}.$$

Update $\mathbf{\Phi_2}$:

Birth-and-death step: Choose birth-move with probability $p \in (0,1)$. Generate (τ_2,ξ_2), where $\tau_2 \sim \mathcal{U}([0,T])$ and $\xi_2 \sim \mathrm{Ex}(\beta_2)$. Proposal transition kernel $q(\Phi_2,\Phi_2\cup\{(\tau_2,\xi_2)\})$ has density

$$q(\Phi_2,\Phi_2\cup\{(\tau_2,\xi_2)\}) = \beta_2^{-1}\exp(-(\beta_2^{-1}-1)\xi_2)$$

with respect to the product of Lebesgue-measure on $[0,T]$ and Ex(1) measure on $(0,\infty)$.

Choose death-move with probability $1-p$. Select a randomly selected point (τ_{2i},ξ_{2i}) being removed from Φ_2 (if Φ_2 is not empty). The proposal transition kernel with respect to the counting measure is

$$q(\Phi_2,\Phi_2\setminus\{(\tau_{2i},\xi_{2i})\}) = \frac{1}{N_T^2},$$

where N_T^2 is the number of points in Φ_2 before the death-move.
The acceptance ratio for birth-move from Φ_2 to $\Phi_2\cup\{(\tau_2,\xi_2)\}$ is then given by

$$\alpha(\Phi_2,\Phi_2\cup\{(\tau_2,\xi_2)\}) = \min\{1, r(\Phi_2,(\tau_2,\xi_2))\}$$

and the acceptance ratio for the death-move from Φ_2 to $\Phi_2\setminus\{(\tau_{2i},\xi_{2i})$ is

$$\alpha(\Phi_2,\Phi_2\setminus\{(\tau_{2i},\xi_{2i})) = \min\left\{1, \frac{1}{r(\Phi_2\setminus\{(\tau_{2i},\xi_{2i})\},(\tau_{2i},\xi_{2i}))}\right\},$$

where

$$\begin{aligned} r(\hat{\Phi}_2,(\theta_2,\beta_2)) &= \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_2\cup\{(\tau_2,\xi_2)\},\lambda_{J_2},\Phi_1,\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_2,\lambda_{J_2},\Phi_1,\mathcal{E})}\frac{\pi(\hat{\Phi}_2\cup\{(\tau_2,\xi_2)|\theta_2,\beta_2\})}{\pi(\hat{\Phi}_2|\theta_2,\beta_2\})} \\ &\quad\times\frac{1-p}{p}\frac{1}{(N_T^2+1)q(\hat{\Phi}_2,\hat{\Phi}_2\cup\{(\theta_2,\xi_2)\})} \\ &= \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_2\cup\{(\tau_2,\xi_2)\},\lambda_{J_2},\Phi_1,\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\hat{\Phi}_1,\lambda_{J_2},\Phi_2,\mathcal{E})}\frac{1-p}{p}\frac{T}{\hat{N}_T^2+1}\theta_2. \end{aligned}$$

Local displacement move: Assume that the jump times are ordered, i.e. $\tau_{2,1} < \cdots < \tau_{2,N_T^2}$. Choose randomly one of the jump times, say $\tau_{2,j}$, and generate a new jump time uniformly on $[\tau_{2,j-1},\tau_{2,j+1}]$. Displace and resize the point $(\tau_{2,j},\xi_{2,j})$ to (τ_2,ξ_2), where $\xi_2 = e^{-\lambda_{J2}^{-1}(\tau_2-\tau_{2,j})}\xi_{2,j}$. Perform Metropolis-step with acceptance ratio $\alpha(\Phi_2,\Phi_2') = \min\{1, r(\Phi_2,\Phi_2')\}$, where

$$r(\Phi_2,\Phi_2') = \frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2',\Phi_1,\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1,\mathcal{E})}\frac{e^{-\beta_2^{-1}\xi_2}}{e^{-\beta_2^{-1}\xi_{2,j}}}e^{-\lambda_{J2}(\tau_2-\tau_{2,j})}.$$

[1]The acceptance ratio depends on the chosen reference measure, see Appendix B.

Multiplicative jump size update For each jump $(\tau_{2,j}, \xi_{2,j})$ propose a new jump size $\xi'_{2,j} = \xi_{2,j}\phi_{2,j}$, where $\log(\phi_{2,j}) \sim \mathcal{N}(0, c_2^2)$ are i.i.d. random variables. The variance c_2^2 is chosen inversely proportional to the current number of jumps. Perform Metropolis-step with acceptance ratio

$$\alpha(\Phi_2, \Phi'_2) = \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi'_2, \Phi_1, \mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi_1, \mathcal{E})} \exp\left\{-(\beta_2^{-1}-1)\sum_{i=1}^{N_T^2}(\xi'_{2,i} - \xi_{2,i})\right\} \prod_{i=1}^{N_T^2} \frac{\xi'_{2,i}}{\xi_{2,i}}.$$

4. Posterior predictive check

We assess the adequacy of the latent variables involved in our models by calculating p-values for the trajectories of the posterior distribution of the hidden variables, which we obtain throughout our MCMC procedure. As demonstrated in [7], this allows us to perform posterior predictive checks in the sense of [15]. Let m be the total number of iterations of the MCMC algorithm and n be the number of iterations within the burn-in period. For each iteration $n < k \le m$ calculate p-values in the following way:

4.1. **p-values for the Gaussian OU-process Y_1.** For N i.i.d. random variables $\epsilon_{1,j} \sim \mathcal{N}(0,1)$, $j = 1, \dots, N$, the transition densities of the Gaussian OU-process $Y_1^{(k)}$ are given by

$$Y_1^{(k)}(t_j) = Y_1^{(k)}(t_{j-1}) e^{-\left(\lambda_{Y_1}^{(k)}\right)^{-1}\Delta_j} + \left(\frac{\left(\sigma_{Y_1}^{(k)}\right)^2 \lambda_{Y_1}^{(k)}}{2} \left(1 - e^{-2\left(\lambda_{Y_1}^{(k)}\right)^{-1}\Delta_j}\right)\right)^{1/2} \epsilon_{1,j}^{(k)}, \quad j = 1, \dots, N.$$

Take the realisations $\left(y_{1,0}^{(k)}, \dots, y_{1,N}^{(k)}\right)$ of $Y_1^{(k)}$ obtained in the MCMC step and calculate noise data

$$\epsilon_{1,j}^{(k)} = \left(y_{1,j}^{(k)} - y_{1,j-1}^{(k)} e^{-\left(\lambda_{Y_1}^{(k)}\right)^{-1}\Delta_j}\right) \left(\frac{\left(\sigma_{Y_1}^{(k)}\right)^2 \lambda_{Y_1}^{(k)}}{2} \left(1 - e^{-2\left(\lambda_{Y_1}^{(k)}\right)^{-1}\Delta_j}\right)\right)^{-1/2}, \quad j = 1, \dots, N.$$

Use the noise data $\left\{\epsilon_{1,j}^{(k)}\right\}_{j=1,\dots,N}$ as input for a Kolmogorov-Smirnov test for the standard normal distribution, yielding the corresponding p-value $p_{Y_1}^{(k)}$.

4.2. **p-values for the Gaussian OU-process Y_2.** Subject the components $\left\{\epsilon_{2,j}^{(k)}\right\}_{j=1,\dots,N}$ of the hidden variable $\mathcal{E}^{(k)}$ obtained in the k-th step of the MCMC algorithm to a Kolmogorov-Smirnov test for the standard normal distribution, yielding the corresponding p-value $p_{Y_2}^{(k)}$.

4.3. **p-values for the jump processes.** For the jump data $\Phi_i^{(k)}$, $i \in \{1, 2\}$ sampled from the Markov chain in the k-th iteration we have

$$\Phi_i^{(k)} = \left\{\left(\tau_{i,j}^{(k)}, \xi_{i,j}^{(k)}\right)\right\}_{0 \le j \le N_{T_i}^{(k)}}.$$

Subject the jump sizes $\left\{\xi_{i,j}^{(k)}\right\}_{j=0,\dots,N_{T_i}}$ to a Kolmogorov-Smirnov test for the exponential distribution with mean $\beta_i^{(k)}$ to get the p-value $p_{\xi_i}^{(k)}$. For the jump locations $\left\{\tau_{i,j}^{(k)}\right\}_{j=1,\dots,N_{T_i}}$ calculate the set of inter-arrival times $\left\{\Delta\tau_{i,j}^{(k)}\right\}_{j=1,\dots,N_{T_i}}$ with $\Delta\tau_{i,j}^{(k)} = \tau_{i,j}^{(k)} - \tau_{i,j-1}^{(k)}$. Perform a Kolmogorov-Smirnov test for the exponential distribution with mean $\left(\theta_i^{(k)}\right)^{-1}$ on the set of inter-arrival times $\left\{\Delta\tau_{i,j}^{(k)}\right\}_{j=1,\dots,N_{T_i}}$ to obtain the p-value $p_{\theta_i}^{(k)}$.

4.4. **Calculation of posterior predictive p-values.** For each iteration step k, $n < k \leq m$, we obtain 6 p-values

$$\left\{p_{Y_1}^{(k)}, p_{Y_2}^{(k)}, p_{\xi_1}^{(k)}, p_{\theta_1}^{(k)}, p_{\xi_2}^{(k)}, p_{\theta_2}^{(k)}\right\}.$$

The posterior predictive p-values $\{p_{Y_1}, p_{Y_2}, p_{\xi_1}, p_{\theta_1}, p_{\xi_2}, p_{\theta_2}\}$ are now obtained by calculating the mean values

$$\{p_{Y_1}, p_{Y_2}, p_{\xi_1}, p_{\theta_1}, p_{\xi_2}, p_{\theta_2}\} = \left\{\frac{\sum\limits_{k=n+1}^{m} p_{Y_1}^{(k)}}{m-n}, \frac{\sum\limits_{n+1}^{m} p_{Y_2}^{(k)}}{m-n}, \frac{\sum\limits_{n+1}^{m} p_{\xi_1}^{(k)}}{m-n}, \frac{\sum\limits_{n+1}^{m} p_{\theta_1}^{(k)}}{m-n}, \frac{\sum\limits_{n+1}^{m} p_{\xi_2}^{(k)}}{m-n}, \frac{\sum\limits_{n+1}^{m} p_{\theta_2}^{(k)}}{m-n}\right\}.$$

5. Evaluation of the model fit

In this section, we compare the fit of the 3-factor and 4-factor model for three different time periods, namely the pre-crisis period 2018-2021, the crisis 2021-23 and the whole interval 2018-2023. In contrast to the studies carried out in [7], our price data exhibits negative jumps in every time period we investigate. This is the reason why our model features a negative stochastic jump component. The generalization to a model with multiple jump components as described in [7] is straight forward, but was not considered in our study for the sake of computational tractability. To the best of our knowledge, this is the first attempt to fit the superposition model for the extremely volatile data in the period 2021-2023 and can be considered as starting point for future research.

5.1. **Posterior properties of the parameters.** We start with an overview of the posterior properties of the model parameters obtained from the MCMC procedure described in Section 3.5. Later in this section, we present a more detailed analysis of our calibration results.

Fast convergence of the MCMC algorithm is heavily dependent on finding appropriate paths of the latent variables, and hence it is important to update the latent variables in an efficient way. In Table 4, we provide details on the number of update steps, we used within every iteration of the algorithm.

5.2. **Model fit for different time periods.** We now analyse the calibration results within the respective time intervals, providing posterior sample paths of the latent variables as well as simulations based on the calibrated parameters (Table 3). Model adequacy is then assessed by calculating the posterior predictive p-values as described in Section 4.

5.2.1. *2018-21 spot price data:* We calibrate the 3-factor model and the 4-factor model for the spot-price data in the time interval 2018-2021. As can be seen in Figure 3, the long term behavior of the spot price is modeled by a Gaussian OU-process in both the 3-factor and the 4-factor model. The main difference is that in the 4-factor model, the additional Gaussian OU component accounts for a large part of the short term deviations which had previously been modeled by the jump process thus leading to much sparser jump paths (cf. Fig. 3). This is in line with the lower jump intensity rates observed in the 4-factor model (cf. Table 3). The high p-value of the second OU-process indicates a good fit and better adequacy of the 4-factor model in this time period.

	2018-21				**2021-23**				**2018-23**			
	3-OU		**4-OU**		**3-OU**		**4-OU**		**3-OU**		**4-OU**	
Parameter	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD	Mean	SD
σ_{Y_1}	42.74	9.518	116.94	11.112	357.87	47.435	1.4319	0.3030	79.45	6.4217	124.81	9.2750
σ_{Y_2}	-	-	22.051	6.7	-	-	305.89	0.2160	-	-	70.533	7.8973
λ_{Y_1}	0.2018	3.0381	0.0053	0.0009	0.0099	0.0536	0.0372	0.0167	27.47	771.5	0.0046	0.00077
λ_{Y_2}	-	-	1.882	29.056	-	-	0.9700	0.004	-	-	2.4157	0.8809
λ_{J_1}	0.0057	0.00099	0.0079	0.0026	0.0347	0.0061	0.0202	0.000029	0.0164	0.0017	0.0138	0.0014
λ_{J_2}	0.0020	0.00023	0.0016	0.00028	0.0739	0.0144	0.0039	9.297857e-06	0.0026	0.0003	0.0019	0.00017
θ_1	333.86	82.143	129.84	80.131	178.61	41.323	207.29	10.790	-	-	-	-
θ_2	336.26	59.495	86.999	30.244	132.44	31.951	131.03	8.5646	-	-	-	-
β_1	4.019	0.5241	4.816	0.1306	37.123	4.8553	29.093	1.5189	-	-	-	-
β_2	7.359	0.8767	13.547	3.1564	24.943	3.7563	38.703	2.6341	-	-	-	-
$\theta_1^{(1)}$	-	-	-	-	-	-	-	-	205.56	37.208	6.5369	9.1030
$\theta_2^{(1)}$	-	-	-	-	-	-	-	-	130.35	28.011	92.919	21.170
$\beta_1^{(1)}$	-	-	-	-	-	-	-	-	5.240	0.587	5.7285	295.55
$\beta_2^{(1)}$	-	-	-	-	-	-	-	-	13.385	2.081	17.586	2.9266
$\theta_1^{(2)}$	-	-	-	-	-	-	-	-	373.34	40.119	368.85	47.868
$\theta_2^{(2)}$	-	-	-	-	-	-	-	-	614.17	59.46	510.08	61.584
$\beta_1^{(2)}$	-	-	-	-	-	-	-	-	29.330	2.443	31.157	2.9279
$\beta_2^{(2)}$	-	-	-	-	-	-	-	-	26.523	2.805	30.997	3.5207

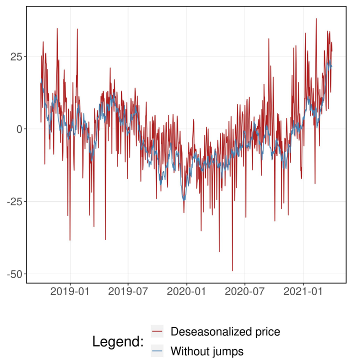
TABLE 3. Posterior properties of the model parameters in different time periods. We present the mean and the standard deviation (SD) for all model parameters.

	2018-21		**2021-23**		**2018-23**	
	3-OU	**4-OU**	**3-OU**	**4-OU**	**3-OU**	**4-OU**
Number of Iterations	10 Mio.	10 Mio.	10 Mio.	10 Mio.	10 Mio.	10 Mio.
Burn-In-Period	9 Mio.	9 Mio.	9 Mio.	9 Mio.	9 Mio.	9 Mio.
Number of Loops for Birth-Death step	5	5	5	1	5	5
Number of Loops for Multiplicative Update	5	5	5	1	5	5
Number of Loops for Latent OU-Variable	-	30	-	200	-	30
Number of New Increments per Loop	-	100	-	1	-	1
Number of Permutations Increments per Loop	-	100	-	2	-	2
Runtime (days)	3.9958	3.3188	4.0901	5.4426	-	8.2958

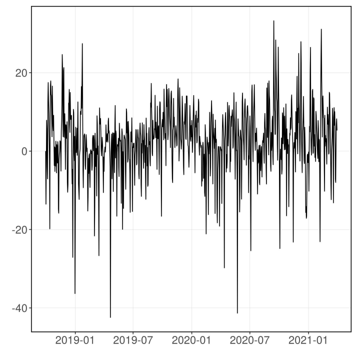
TABLE 4. Information about the updates of the hidden variables.

5.2.2. *2021-23 spot price data.* When calibrating the 4-factor model to the 2021-23 spot price data, it turns out, that the additional Gaussian OU-process is completely insignificant from the modeling point of view (cf. Fig. 6). Thus it is sufficient to calibrate the 3-factor model for the 2021-23 spot price data. As can be seen in Figure 4, the Gaussian OU-process is used to model the long term behavior of the price process, but extreme deviations from the long term mean are now incorporated into the jump processes. A comparison of the average jump sizes in Table 3 for the period 2018-21 with those in 2021-23 confirms this observation. In Figure 5, we showcase the samples generated from the MCMC algorithm for some model parameters. We note that the simulated paths in Figure 3 exhibit the same characteristics as the observed data in this time period and the confidence intervals fit the range of the historic data very well.

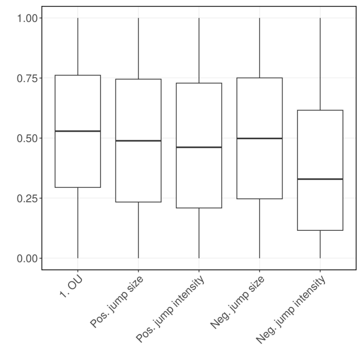
5.2.3. *2018-23 spot price data.* The calibration of the 3-factor and 4-factor model to the whole 2018-23 data yields very low p-values for the jump processes, due to the different jump patterns occurring before and after the start of the energy crisis (cf. Fig. 8). We therefore considered a change-point model, where we have different jump intensities and jump sizes in the respective time intervals (cf. Figure 7). This specification allows a better fit of the jump processes, however, the convergence of the MCMC algorithm is quite slow due the additional parameters. A detailed



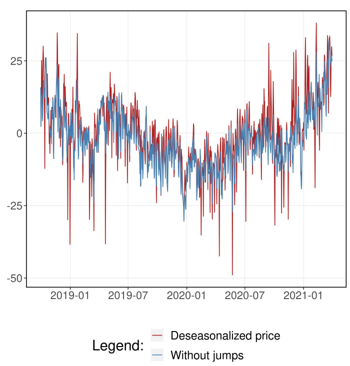
(A) Data and Gaussian residuals after removal of the jumps



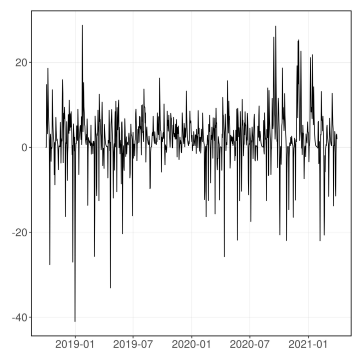
(B) Sample path of the jump process in the 3-factor model



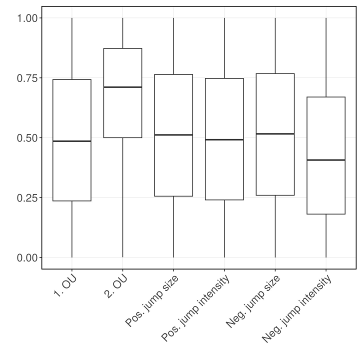
(C) Boxplot for the p-values in the 3-factor model



(D) Data and Gaussian residuals after removal of the jumps



(E) Sample path of the jump process in the 4-factor model



(F) Boxplot for the p-values in the 4-OU model

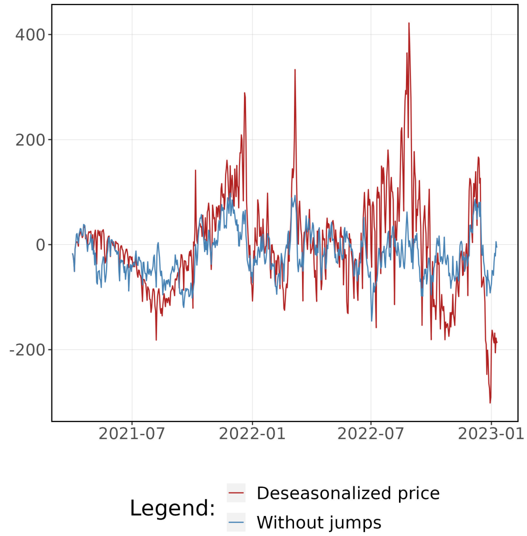
FIGURE 3. Fit of the 3-factor model (top) and 4-factor model (bottom) in the time period 2018-21.

explanation of the jump update procedure in the change point model is presented in Appendix A. Of course, one could think of more advanced models for the spot price on the whole interval 2018-23, for example by introducing stochastic jump intensity rates. However, this would make the MCMC algorithm even more complicated and thus more time-consuming.

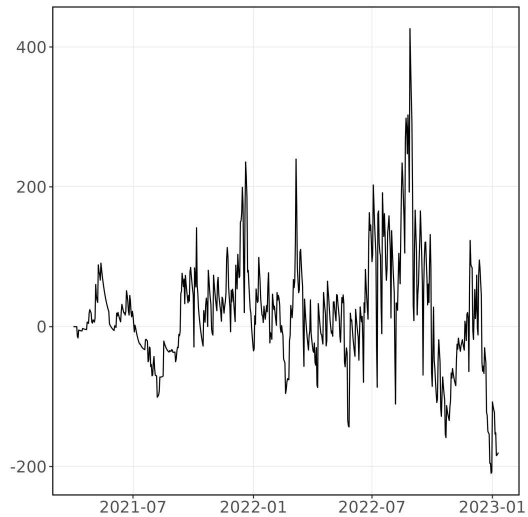
5.3. **Conclusion.** For the pre-crisis data 2018-21, our results suggest that the addition of a second Gaussian component can improve model adequacy, whereas for the crisis data 2021-23 the 4-factor model is hard to calibrate and leads to overfitting. For the whole period 2018-23, the data of 2018-2021 dominates calibration of the Gaussian components, while the behavior in 2021-2023 is mainly explained by jump processes. The very different behavior of the spot price in the time periods 2018-21 and 2021-23 makes it hard to fit the model to the whole data.

6. Futures prices

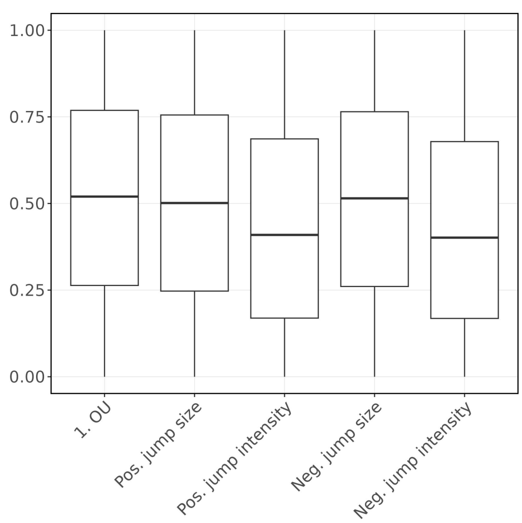
In this section, we give an explicit formula for the price of a futures contract with maturity T in the 4-factor model. In our model, there are six sources of uncertainty, two of them are generated by the diffusion factors of the respective Gaussian OU processes and two generated by each of the jump processes, namely jump-intensity risk and jump-size risk. We assume that the market price of risk arising from the diffusion of the Gaussian OU processes is constant, moreover we make the simplifying assumption that all the jump risk premia are captured by jump-size risk (cf. [18]). Thus, if θ^* denotes the jump-intensity under the risk neutral measure, we assume that $\theta^* = \theta$. Therefore, under the risk neutral measure $\mathbb{Q}$, the spot price process is



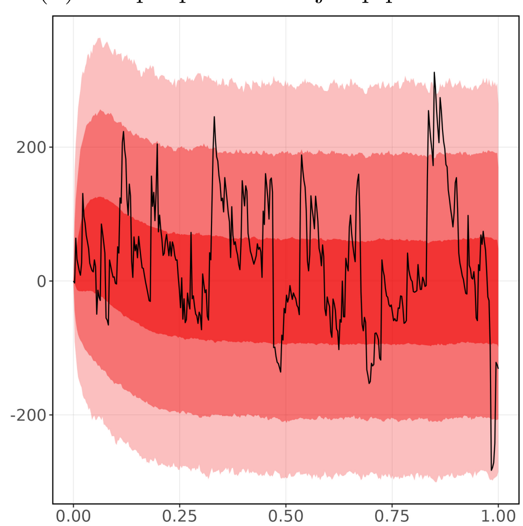
(A) Data and Gaussian residuals after removal of the jumps



(B) Sample path of the jump process



(C) Boxplot of the p-values



(D) One simulated path and quantiles for 10000 simulations (dark to light red: 25-75%, 5-95%, 1-99%)

FIGURE 4. Fit of the 3-factor model in the time period 2021-23.

of the form

$$P_t = f(t) + Y_1(t) + Y_2(t) + J_1(t) - J_2(t), \tag{6.1}$$

with

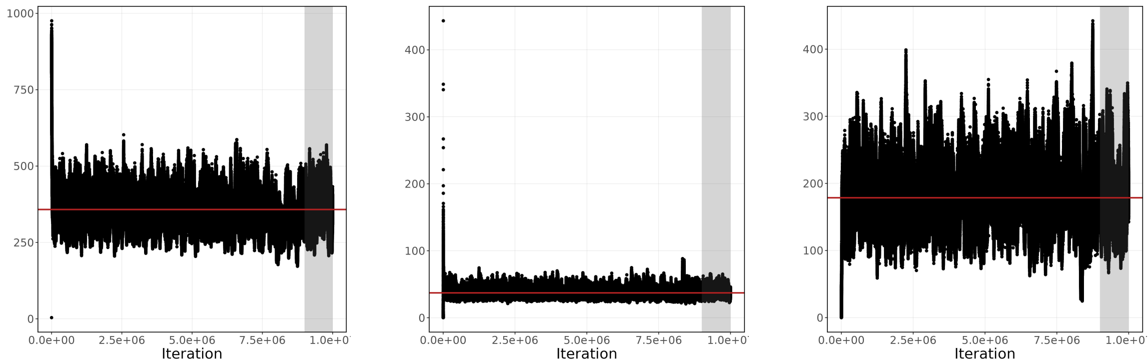
$$dY_1(t) = (-\lambda_{Y_1}^{-1} Y_1(t) - \phi_{Y_1})dt + \sigma_{Y_1} dW_{Y_1}^{\mathbb{Q}}(t), \tag{6.2}$$

$$dY_2(t) = (-\lambda_{Y_2}^{-1} Y_2(t) - \phi_{Y_2})dt + \sigma_{Y_2} dW_{Y_2}^{\mathbb{Q}}(t), \tag{6.3}$$

$$dJ_1(t) = -\lambda_{J_1}^{-1} J_1(t)dt + d\Pi_1(\theta_1, \beta_1^*), \tag{6.4}$$

$$dJ_2(t) = -\lambda_{J_2}^{-1} J_2(t)dt + d\Pi_2(\theta_2, \beta_2^*), \tag{6.5}$$

where ϕ_L and ϕ_S are the risk premia for the long term and short term uncertainty and β_1^*, β_2^* are the risk neutral jump size parameters. The following result provides an explicit formula for the price of a futures contract in the 4-factor model.



(A) Volatility σ_{Y_1} (B) Positive jump size β_1 (C) Positive jump intensity θ_1

FIGURE 5. Samples from the posterior distribution for different parameters in the 3-factor model (2021-23).

Theorem 6.1. *If the electricity spot price P is modelled by the 4-factor model* (6.1) *with risk neutral dynamics* (6.2)-(6.5)*, the price of a future contract on the spot price with maturity T at time t is given as*

$$\begin{aligned} F(t,T,P) &= \mathbb{E}^{\mathbb{Q}}[P_T|\mathcal{F}_t] \\ &= f(T) + Y_1(t)e^{-\lambda_{Y_1}^{-1}(T-t)} - \frac{\phi_{Y_1}}{\lambda_{Y_1}^{-1}}(1-e^{-\lambda_{Y_1}^{-1}(T-t)}) + Y_2(t)e^{-\lambda_{Y_2}^{-1}(T-t)} - \frac{\phi_{Y_2}}{\lambda_{Y_2}^{-1}}(1-e^{-\lambda_{Y_2}^{-1}(T-t)}) \\ &\quad + J_1(t)e^{-\lambda_{J_1}^{-1}(T-t)} + \frac{\theta_1}{\beta_1^*}\lambda_{J_1}(1-e^{-\lambda_{J_1}^{-1}(T-t)}) - J_2(t)e^{-\lambda_{J_2}^{-1}(T-t)} - \frac{\theta_2}{\beta_2^*}\lambda_{J_2}(1-e^{-\lambda_{J_2}^{-1}(T-t)}). \end{aligned}$$

Proof: By the linearity of the conditional expectation, we obtain

$$\mathbb{E}^{\mathbb{Q}}[P_T|\mathcal{F}_t] = f(T) + \mathbb{E}^{\mathbb{Q}}[Y_1(T)|\mathcal{F}_t] + \mathbb{E}^{\mathbb{Q}}[Y_2(T)|\mathcal{F}_t] + \mathbb{E}^{\mathbb{Q}}[J_1(T)|\mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}}[J_2(T)|\mathcal{F}_t].$$

Under the risk neutral measure $\mathbb{Q}$, equation (6.2) has the solution

$$Y_1(T) = Y_1(t)e^{-\lambda_{Y_1}^{-1}(T-t)} - \frac{\phi_{Y_1}}{\lambda_{Y_1}^{-1}}(1-e^{-\lambda_{Y_1}^{-1}(T-t)}) + \sigma_{Y_1}\int_t^T e^{-\lambda_{Y_1}^{-1}(T-s)}dW_{Y_1}^{\mathbb{Q}}(s).$$

Thus we get

$$\mathbb{E}_t^{\mathbb{Q}}[Y_1(T)] := \mathbb{E}^{\mathbb{Q}}[Y_1(T)|\mathcal{F}_t] = Y_1(t)e^{-\lambda_{Y_1}^{-1}(T-t)} - \frac{\phi_{Y_1}}{\lambda_{Y_1}^{-1}}(1-e^{-\lambda_{Y_1}^{-1}(T-t)}).$$

Analogously we have

$$\mathbb{E}_t^{\mathbb{Q}}[Y_2(T)] = Y_2(t)e^{-\lambda_{Y_2}^{-1}(T-t)} - \frac{\phi_{Y_2}}{\lambda_{Y_2}^{-1}}(1-e^{-\lambda_{Y_2}^{-1}(T-t)}).$$

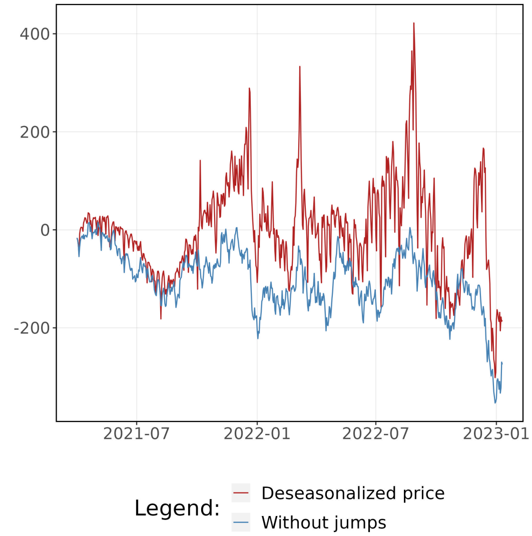
Let $N(t,T)$ denote the number of jumps of the Poisson process in the interval (t,T) and let $\tau_1^* < \cdots < \tau_{N(t,T)}^*$ be the jump arrival times with corresponding jump sizes $\xi_1^*, \ldots, \xi_{N(t,T)}^*$. Then we have

$$J_T = J_t e^{-\lambda_J^{-1}(T-t)} + \sum_{i=1}^{N(t,T)} e^{-\lambda_J^{-1}(T-\tau_i^*)}\xi_i^*,$$

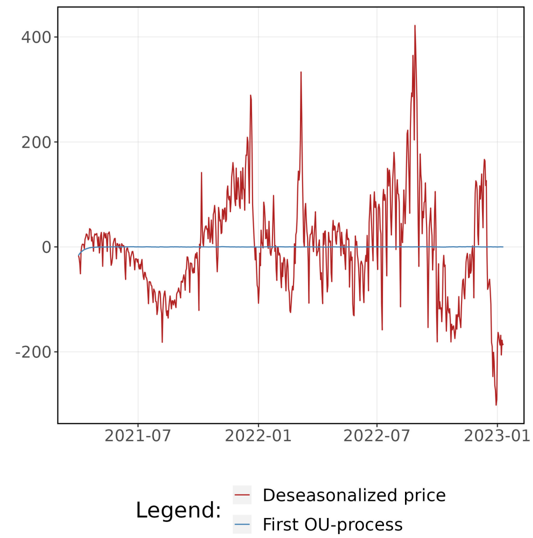
where the jump sizes are exponentially distributed with parameter β^* and the jump intensity $\theta^* = \theta$ does not change under the risk neutral measure. For the conditional expectation we get

$$\mathbb{E}_t^{\mathbb{Q}}[J_T] = \mathbb{E}_t^{\mathbb{Q}}\Big[J_t e^{-\lambda_J^{-1}(T-t)} + \sum_{i=1}^{N(t,T)} e^{-\lambda_J^{-1}(T-\tau_i^*)}\xi_i^*\Big] = J_t e^{-\lambda_J^{-1}(T-t)} + \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} e^{-\lambda_J^{-1}(T-\tau_i^*)}\xi_i^*\Big],$$

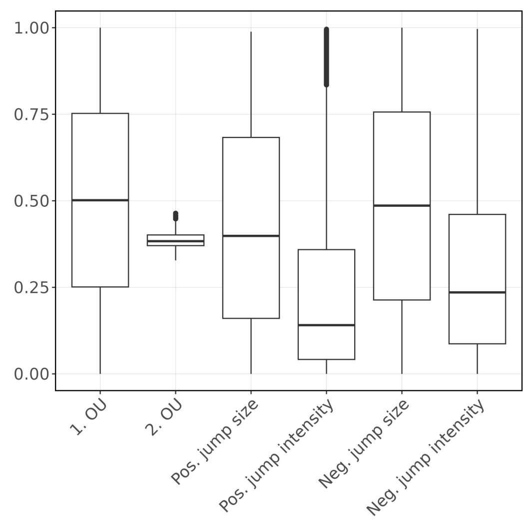
and since



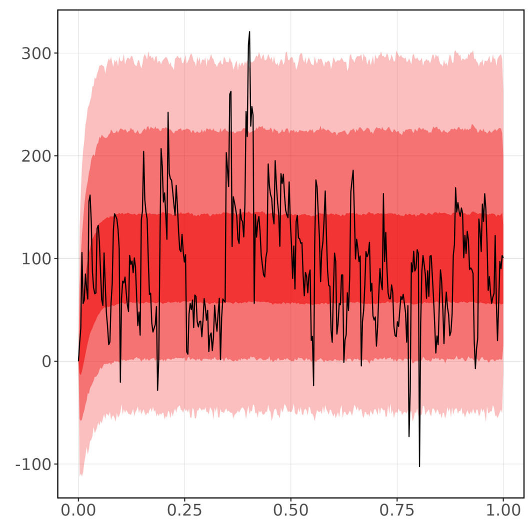
(A) Data and Gaussian residuals after removal of the jumps



(B) Sample path of the second OU-process



(C) Boxplot of the p-values

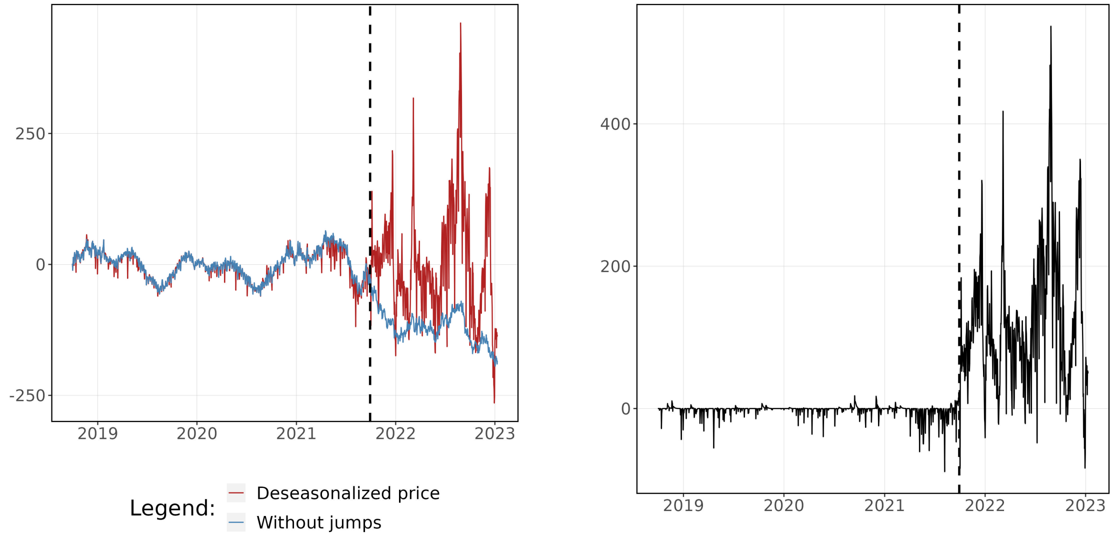


(D) One simulated path and quantiles for 10000 simulations (dark to light red: 25-75%, 5-95%, 1-99%)

FIGURE 6. Fit of the 4-factor model in the time period 2021-23.

$$\mathbb{E}_t^{\mathbb{Q}}[J_T] - J_t e^{-\lambda_J^{-1}(T-t)} = \mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} e^{-\lambda_J^{-1}(T-\tau_i^*)}\xi_i^*\Big] = e^{-\lambda_J^{-1}T}\mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} e^{\lambda_J^{-1}\tau_i^*}\xi_i^*\Big]$$
$$= e^{-\lambda_J^{-1}T}\mathbb{E}_t^{\mathbb{Q}}\Big[\mathbb{E}^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} e^{\lambda_J^{-1}\tau_i^*}\xi_i^*|N(t,T)\Big]\Big] = e^{-\lambda_J^{-1}T}\mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} \mathbb{E}^{\mathbb{Q}}[e^{\lambda_J^{-1}\tau_i^*}\xi_i^*]\Big]$$
$$= e^{-\lambda_J^{-1}T}\mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} \mathbb{E}^{\mathbb{Q}}[e^{\lambda_J^{-1}\tau_i^*}]\mathbb{E}^{\mathbb{Q}}[\xi_i^*]\Big] = e^{-\lambda_J^{-1}T}\frac{1}{\beta^*}\mathbb{E}_t^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} \mathbb{E}^{\mathbb{Q}}[e^{\lambda_J^{-1}\tau_i^*}]\Big],$$

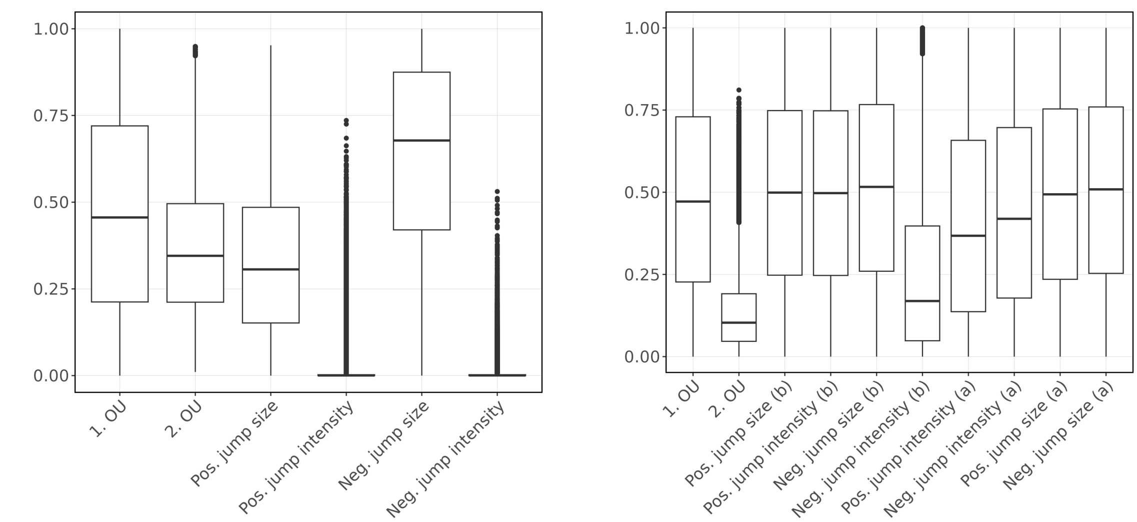
and



(A) Gaussian residuals

(B) Jumps

FIGURE 7. 4-factor model with change point. The jump parameters are calibrated separately for the different jump regimes before and after the change point. The change point is given by the dashed vertical line.



(A) Boxplot of p-values fot the model without change point

(B) Boxplot of p-values fot the model with change point

FIGURE 8. Fit of the 4-factor model in the time period 2018-23.

$$\sum_{i=1}^{N(t,T)} \mathbb{E}^{\mathbb{Q}}\Big[e^{\lambda_J^{-1}\tau_i^*}\Big] = \mathbb{E}^{\mathbb{Q}}\Big[\sum_{i=1}^{N(t,T)} e^{\lambda_J^{-1}\tau_i^*}|N(t,T)\Big] = N(t,T)\mathbb{E}^{\mathbb{Q}}\Big[e^{\lambda_J^{-1}\tau}\Big]$$

where $\tau \sim \mathcal{U}(t,T)$ and the last equality follows by the fact that $\tau_1^*, \ldots, \tau_{N(t,T)}^*$ are the jump times of a Poisson process with $N(t,T)$ jumps in the interval (t,T), i.e., they are distributed like the order statistics of $N(t,T)$ independent random variables that are uniformly distributed on the interval (t,T). Hence we end up with

$$\begin{aligned}\mathbb{E}_t^{\mathbb{Q}}[J_T] - J_t e^{-\lambda_J^{-1}(T-t)} &= e^{-\lambda_J^{-1}T}\frac{1}{\beta^*}\mathbb{E}_t^{\mathbb{Q}}\Big[N(t,T)\mathbb{E}^{\mathbb{Q}}[e^{\lambda_J^{-1}\tau}]\Big] = e^{-\lambda_J^{-1}T}\frac{1}{\beta^*}\mathbb{E}_t^{\mathbb{Q}}[N(t,T)]\mathbb{E}^{\mathbb{Q}}[e^{\lambda_J^{-1}\tau}] \\ &= e^{-\lambda_J^{-1}T}\frac{1}{\beta^*}\theta(T-t)\frac{1}{T-t}\int_t^T e^{\lambda_J^{-1}s}ds = \frac{\theta}{\beta^*}\lambda_J(1-e^{-\lambda_J^{-1}(T-t)}).\end{aligned}$$

Thus

$$\mathbb{E}_t^{\mathbb{Q}}[J_T] = J_t e^{-\lambda_J^{-1}(T-t)} + \frac{\theta}{\beta^*}\lambda_J(1-e^{-\lambda_J^{-1}(T-t)}).$$

□

Acknowledgements

This research paper was inspired by a real-world industrial application. We therefore wish to express our gratitude to LINZ STROM GAS WÄRME GmbH – Geschäftsbereich Energiemanagement.

F. Aichinger is supported by the Austrian Science Fund (FWF) Project P34808. For the purpose of open access, the authors have applied a CC BY public copyright licence to any author accepted manuscript version arising from this submission.

Appendix A: Jump update with change point

In this appendix, we give a detailed explanation for the update steps of the jump processes in case that jump sizes and intensity change over time. We assume that there are two different jump regimes on the interval $[0,T]$, the change point is denoted by T_C. In the interval $[0,T_C]$, the jump arrival times have intensity $\theta_1^{(1)}$ and the jump size is $\beta_1^{(1)}$, whereas in the interval $(T_C,T]$, the jump intensity is $\theta_1^{(2)}$ and the size is $\beta_1^{(2)}$.

Update $\boldsymbol{\theta_1}^{(\mathbf{1})}$:
With the choice of an $\mathrm{Ga}(a_{\theta_1^{(1)}}, b_{\theta_1^{(1)}})$ distributed prior for $\theta_1^{(1)}$, for the posterior we get

$$\theta_1^{(1)}|\Phi_1 \sim \mathrm{Ga}\left(a_{\theta_1^{(1)}} + N_{T_C}^1, b_{\theta_1^{(1)}} + T_C^1\right).$$

Update $\boldsymbol{\theta_1}^{(\mathbf{2})}$:
With the choice of an $\mathrm{Ga}(a_{\theta_1^{(2)}}, b_{\theta_1^{(2)}})$ distributed prior for $\theta_1^{(2)}$, for the posterior we get

$$\theta_1^{(2)}|\Phi_1 \sim \mathrm{Ga}\left(a_{\theta_1^{(2)}} + (N_T^1 - N_{T_C}^1), b_{\theta_1^{(2)}} + (T - T_C^1)\right).$$

Update $\boldsymbol{\beta_1}^{(\mathbf{1})}$
With the choice of an $\mathrm{IG}(a_{\beta_1^{(1)}}, b_{\beta_1^{(1)}})$ distributed prior for $\beta_1^{(1)}$, for the posterior we get

$$\beta_1^{(1)}|\Phi_1 \sim \mathrm{IG}\left(a_{\beta_1^{(1)}} + N_{T_C}^1, b_{\beta_1^{(1)}} + \sum_{i=1}^{N_{T_C}^1}\xi_{1i}\right).$$

Update $\boldsymbol{\beta_1}^{(\mathbf{2})}$

With the choice of an $\mathrm{IG}(a_{\beta_1^{(2)}}, b_{\beta_1^{(2)}})$ distributed prior for $\beta_1^{(2)}$, for the posterior we get

$$\beta_1^{(2)}|\Phi_1 \sim \mathrm{IG}\left(a_{\beta_1^{(2)}} + (N_T^1 - N_{T_C}^1), b_{\beta_1^{(2)}} + \sum_{N_{T_C}^1+1}^{N_T^1} \xi_{1i}\right).$$

Update Φ_1
Birth-and-death step:
Choose birth-move with probability $p \in (0,1)$. Generate (τ_1, ξ_1) by the following procedure. With probability

$$\frac{\theta_1^{(1)} T_C}{\theta_1^{(1)} T_C + \theta_1^{(2)}(T - T_C)} \text{ draw } \tau_1 \sim \mathcal{U}([0, T_C]) \text{ and } \xi_1 \sim \mathrm{Ex}(\beta_1^1),$$

$$\frac{\theta_1^{(2)}(T - T_C)}{\theta_1^{(1)} T_C + \theta_1^{(2)}(T - T_C)} \text{ draw } \tau_1 \sim \mathcal{U}((T_C, T]) \text{ and } \xi_1 \sim \mathrm{Ex}(\beta_1^2).$$

The proposal transition kernel $q(\Phi_1, \Phi_1 \cup \{(\tau_1, \xi_1)\})$ thus has density

$$q(\Phi_1, \Phi_1 \cup \{(\tau_1, \xi_1)\}) = \begin{cases} (\beta_1^{(1)})^{-1} \exp(-((\beta_1^{(1)})^{-1} - 1)\xi_1) \frac{\theta_1^{(1)}}{\theta_1^{(1)} T_C + \theta_1^{(2)}(T - T_C)}, & \tau_1 \in [0, T_C] \\ (\beta_1^{(2)})^{-1} \exp(-((\beta_1^{(2)})^{-1} - 1)\xi_1) \frac{\theta_1^{(2)}}{\theta_1^{(1)} T_C + \theta_1^{(2)}(T - T_C)}, & \tau_1 \in (T_C, T] \end{cases}$$

with respect to the product of Lebesgue-measure on $[0, T]$ and $\mathrm{Ex}(1)$ measure on $[0, \infty)$.

Choose death-move with probability $1 - p$. Select a randomly selected point (τ_{1i}, ξ_{1i}) being removed from Φ_1 (if Φ_1 is not empty). The proposal transition kernel with respect to the counting measure is

$$q(\Phi_1, \Phi_1 \setminus \{(\tau_{1i}, \xi_{1i})\}) = \frac{1}{N_T^1},$$

where N_T^1 is the number of points in Φ_1 before the death-move.

Acceptance ratio for birth-move from Φ_1 to $\Phi_1 \cup \{(\tau_1, \xi_1)\}$ is

$$\alpha(\Phi_1, \Phi_1 \cup \{(\tau_1, \xi_1)\}) = \min\{1, r(\Phi_1, (\tau_1, \xi_1))\}.$$

Acceptance ratio for the death-move from Φ_1 to $\Phi_1 \setminus \{(\tau_{1i}, \xi_{1i})$ is

$$\alpha(\Phi_1, \Phi_1 \setminus \{(\tau_{1i}, \xi_{1i})) = \min\{1, \frac{1}{r(\Phi_1 \setminus \{(\tau_{1i}, \xi_{1i})\}, (\tau_{1i}, \xi_{1i}))}\}$$

where
Case $\tau_1 \in [0, T_C]$:

$$\begin{aligned} &r(\hat{\Phi}_1, (\tau_1, \xi_1)) \\ &= \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1 \cup \{(\tau_1, \xi_1)\}, \lambda_{J_2}, \Phi_2, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1, \lambda_{J_2}, \Phi_2, \mathcal{B})} \frac{\pi(\hat{\Phi}_1 \cup \{(\tau_1, \xi_1)|\theta_1^{(1)}, \theta_1^{(2)}, \beta_1^{(1)}, \beta_1^{(2)}\})}{\pi(\hat{\Phi}_1|\theta_1^{(1)}, \theta_1^{(2)}, \beta_1^{(1)}, \beta_1^{(2)}\})} \\ &\quad \times \frac{1-p}{p} \frac{1}{(N_T^1 + 1) q(\hat{\Phi}_1, \hat{\Phi}_1 \cup \{(\theta_1, \xi_1)\})} \\ &= \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1 \cup \{(\tau_1, \xi_1)\}, \lambda_{J_2}, \Phi_2, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1, \lambda_{J_2}, \Phi_2, \mathcal{B})} \frac{1-p}{p} \frac{\theta_1^{(1)} T_C + \theta_1^{(2)}(T - T_C)}{\hat{N}_T^1 + 1}. \end{aligned}$$

Case $\tau_1 \in (T_C, T]$**:**

$$\begin{aligned}
&r(\hat{\Phi}_1, (\tau_1, \xi_1)) \\
&= \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1 \cup \{(\tau_1, \xi_1)\}, \lambda_{J_2}, \Phi_2, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1, \lambda_{J_2}, \Phi_2, \mathcal{B})} \frac{\pi(\hat{\Phi}_1 \cup \{(\tau_1, \xi_1)|\theta_1^{(1)}, \theta_1^{(2)}, \beta_1^{(1)}, \beta_1^{(2)}\})}{\pi(\hat{\Phi}_1|\theta_1^{(1)}, \theta_1^{(2)}, \beta_1^{(1)}, \beta_1^{(2)}\})} \\
&\quad \times \frac{1-p}{p} \frac{1}{(N_T^1 + 1) q(\hat{\Phi}_1, \hat{\Phi}_1 \cup \{(\theta_1, \xi_1)\})} \\
&= \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1 \cup \{(\tau_1, \xi_1)\}, \lambda_{J_2}, \Phi_2, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \hat{\Phi}_1, \lambda_{J_2}, \Phi_2, \mathcal{B})} \frac{1-p}{p} \frac{\theta_1^{(1)} T_C + \theta_1^{(2)} (T - T_C)}{\hat{N}_T^1 + 1}.
\end{aligned}$$

Multiplicative jump size update

For each jump $(\tau_{1,j}, \xi_{1,j})$ propose a new jump size $\xi'_{1,j} = \xi_{1,j} \phi_{1,j}$, where $\log(\phi_{1,j}) \sim \mathcal{N}(0, c_1^2)$ are i.i.d. random variables. The variance c_1^2 is chosen inversely proportional to the current number of jumps. Perform Metropolis-step with acceptance ratio

$$\begin{aligned}
\alpha(\Phi_1, \Phi'_1) = \min\Big\{1, &\frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi'_1, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi_1, \mathcal{B})} \prod_{i=1}^{N_T^1} \frac{\xi'_{1,i}}{\xi_{1,i}} \\
&\times \exp\Big\{ -((\beta_1^{(1)})^{-1} - 1) \sum_{i=1}^{N_C^1} (\xi'_{1,i} - \xi_{1,i})\Big\} \exp\Big\{ -((\beta_1^{(2)})^{-1} - 1) \sum_{i=N_C^1+1}^{N_T^1} (\xi'_{1,i} - \xi_{1,i})\Big\}\Big\}.
\end{aligned}$$

Local displacement move

Assume that the jump times are ordered, i.e. $\tau_{1,1} < \cdots < \tau_{1,N_T^1}$. Choose randomly one of the jump times, say $\tau_{1,j}$, and generate a new jump time τ_1 uniformly on $[\tau_{1,j-1}, \tau_{1,j+1}]$. Displace and resize the point $(\tau_{1,j}, \xi_{1,j})$ to (τ_1, ξ_1), where $\xi_1 = e^{-\lambda_{J1}^{-1}(\tau_1 - \tau_{1,j})} \xi_{1,j}$. Perform Metropolis-step with acceptance ratio $\alpha(\Phi_1, \Phi'_1) = \min\{1, r(\Phi_1, \Phi'_1)\}$, where

Case $\tau_1, \tau_{1,j} \in [0, T_C]$:

$$r(\Phi_1, \Phi'_1) = \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi'_1, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi_1, \mathcal{B})} \frac{e^{-(\beta_1^{(1)})^{-1} \xi_1}}{e^{-(\beta_1^{(1)})^{-1} \xi_{1,j}}} e^{-\lambda_{J1}(\tau_1 - \tau_{1,j})},$$

Case $\tau_1 \in [0, T_C], \tau_{1,j} \in (T_C, T]$:

$$r(\Phi_1, \Phi'_1) = \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi'_1, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi_1, \mathcal{B})} \frac{\theta_1^{(1)}}{\theta_1^{(2)}} \frac{e^{-(\beta_1^{(1)})^{-1} \xi_1}}{e^{-(\beta_1^{(2)})^{-1} \xi_{1,j}}} e^{-\lambda_{J1}(\tau_1 - \tau_{1,j})},$$

Case $\tau_1 \in (T_C, T], \tau_{1,j} \in [0, T_C]$:

$$r(\Phi_1, \Phi'_1) = \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi'_1, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi_1, \mathcal{B})} \frac{\theta_1^{(2)}}{\theta_1^{(1)}} \frac{e^{-(\beta_1^{(2)})^{-1} \xi_1}}{e^{-(\beta_1^{(1)})^{-1} \xi_{1,j}}} e^{-\lambda_{J1}(\tau_1 - \tau_{1,j})},$$

Case $\tau_1, \tau_{1,j} \in (T_C, T]$:

$$r(\Phi_1, \Phi'_1) = \frac{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi'_1, \mathcal{B})}{l(\mathcal{X}|\lambda_{Y_1}, \sigma_{Y_1}, \lambda_{Y_2}, \sigma_{Y_2}, \lambda_{J_1}, \lambda_{J_2}, \Phi_2, \Phi_1, \mathcal{B})} \frac{e^{-(\beta_1^{(2)})^{-1} \xi_1}}{e^{-(\beta_1^{(2)})^{-1} \xi_{1,j}}} e^{-\lambda_{J1}(\tau_1 - \tau_{1,j})}.$$

Appendix B: Poisson Processes

Let Π be a Poisson process on $[0,T]$ with constant intensity θ. For each point τ of the random set Π, we associate an $\mathrm{Ex}(\beta)$ distributed random variable ξ_τ (the mark of τ), taking values in $\mathbb{R}^+$. The pair (τ, ξ_τ) can then be regarded as a random point in the product space $[0,T]\times\mathbb{R}^+$. Then by the Marking Theorem [10], the random set

$$\Pi^* = \{(\tau,\xi_\tau) | \tau \in \Pi\}$$

is a Poisson process on $[0,T]\times\mathbb{R}^+$ and its intensity measure is given by

$$\Lambda(C) = \int\int_{(t,x)\in C} \theta(dt) p_\xi(dx) = \int\int_{(t,x)\in C} \theta \frac{1}{\beta} e^{-\frac{1}{\beta}x} dtdx.$$

Thus the sets $\Phi_1 = \{(\tau_{1,j},\xi_{1,j})\}_{1\le j\le N_{T_1}}$ and $\Phi_2 = \{(\tau_{2,j},\xi_{2,j})\}_{1\le j\le N_{T_2}}$ of jump times and corresponding jump sizes can be interpreted as realisations of marked Poisson processes taking values in $[0,T]\times\mathbb{R}^+$. We are now interested in the likelihood $l(\Phi_i|\theta_i,\beta_i)$ with respect to a so called dominating measure. Let therefore $\mathcal{X} = [0,T]\times\mathbb{R}^+$ and denote by $\mathcal{M}$ the set of σ-finite measures defined on $(\mathcal{X},\mathcal{B}(\mathcal{X})$ and by $\mathcal{M}_0$ the subspace of integer-valued point measures $X = \sum_i \epsilon_{x_i}$, where $x_i \in \mathcal{X}$ and ϵ_x is the point mass located in x. By $\mathcal{B}(\mathcal{M}_0)$ we denote the smallest σ-algebra of subsets of $\mathcal{M}$ with respect to which all mappings

$$\Pi_B : \mathcal{M}_0 \to \{0,1,2,\ldots,\infty\} \text{ with } \Pi_B(X) = X(B),\ B\in\mathcal{B}(\mathcal{X}),$$

are measurable. Let now Π_0 be a Poisson process taking values in $\mathcal{X}$ with constant intensity θ_0 on $[0,T]$ and $\mathrm{Ex}(\beta_0)$ distributed jump sizes. Then Π_0 induces a measure P_0 on $(\mathcal{M}_0,\mathcal{B}(\mathcal{M}_0))$ via

$$P_0(X\in\mathcal{M}_0 : X(B) = x) = \mathbb{P}(\omega\in\Omega : \Pi_0(\omega)(B) = x).$$

The measure P_0 will now be our reference measure on $\mathcal{B}(\mathcal{M}_0)$ and will be referred to as the dominating measure. For convenience, the dominating measure is often chosen as the one induced by a Poisson process with unit intensity on $[0,T]$ and exponential jump sizes with parameter 1, however, the choices of θ_0 and β_0 can also be different [14]. For another Poisson process Π taking values in $\mathcal{X}$ with constant intensity θ on $[0,T]$ and $\mathrm{Ex}(\beta)$ distributed jump sizes, the density of P with reference to the dominating measure P_0 can now be calculated by [11, Theorem 1.3] as

$$\frac{dP}{dP_0}(X) = \exp\left(\int_{\mathcal{X}} \ln(S(x))X(dx) - \int_{\mathcal{X}}(S(x)-1)\Lambda_0(dx)\right), \quad X\in\mathcal{M}_0$$

where

$$S(x) := \frac{d\Lambda}{d\Lambda_0}((x_1,x_2)) = \frac{\theta}{\theta_0}\frac{\beta_0}{\beta}\exp\left(-\left(\frac{1}{\beta}-\frac{1}{\beta_0}\right)x_2\right).$$

Thus for a set $\Phi = \{(\tau_1,\xi_1),\ldots,(\tau_{N_T},\xi_{N_T})\}$, $\tau_i\in[0,T]$, $\xi_i\in\mathbb{R}^+$ we can now calculate the likelihood of Φ with respect to the dominating measure as

$$\begin{aligned}
l(\Phi|\theta,\beta) = \frac{dP}{dP_0}(\Phi) &= \exp\left(\int_{\mathcal{X}} \ln(S(x))\Phi(dx) - \int_{\mathcal{X}}(S(x)-1)\Lambda_0(dx))\right) \\
&= \exp\left(\sum_{i=1}^{N_T}\ln(S(\tau_i,\xi_i)) - \left(\int_{\mathcal{X}}\Lambda(dx) - \int_{\mathcal{X}}\Lambda_0(dx)\right)\right) \\
&= \prod_{i=1}^{N_T} S(\tau_i,\xi_i)\exp\left(\Lambda_0(\mathcal{X}) - \Lambda(\mathcal{X})\right) \\
&= \prod_{i=1}^{N_T}\frac{\theta}{\theta_0}\frac{\beta_0}{\beta}\exp\left(-\left(\frac{1}{\beta}-\frac{1}{\beta_0}\right)\xi_i\right)\cdot\exp\left(\theta_0 T - \theta T\right) \\
&= \left(\frac{\theta}{\theta_0}\right)^{N_T}\left(\frac{\beta}{\beta_0}\right)^{-N_T}\exp\left(-\left(\frac{1}{\beta}-\frac{1}{\beta_0}\right)\sum_{i=1}^{N_T}\xi_i\right)\cdot\exp\left(-(\theta-\theta_0)T\right).
\end{aligned}$$

If the dominating measure is chosen as a Poisson process with unit intensity on $[0,T]$ and exponential jump sizes with parameter 1 (cf. [7, Section 3.1]), Φ has likelihood

$$l(\Phi|\theta,\beta) == \theta^{N_T}\beta^{-N_T}\exp\left(-\left(\frac{1}{\beta}-1\right)\sum_{i=1}^{N_T}\xi_i\right)\cdot\exp\left(-(\theta-1)T\right).$$

Note: The acceptance ratio in the multiplicative update step of the jump process

$$\alpha(\Phi_1,\Phi_1')=\frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1',\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1,\mathcal{E})}\frac{l(\Phi_1'|\beta_1,\theta_1)}{l(\Phi_1|\beta_1,\theta_1)}\frac{q(\Phi_1,\Phi_1')}{q(\Phi_1',\Phi_1)}.$$

in Section 3.5 depends on the chosen reference measure. If the jump size parameter of the Poisson process inducing the dominating measure is 1, the acceptance ratio is given as

$$\alpha(\Phi_1,\Phi_1')=\frac{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1',\mathcal{E})}{l(\mathcal{X}|\lambda_{Y_1},\sigma_{Y_1},\lambda_{Y_2},\sigma_{Y_2},\lambda_{J_1},\lambda_{J_2},\Phi_2,\Phi_1,\mathcal{E})}\exp\left\{-(\beta_1^{-1}-1)\sum_{i=1}^{N_T^1}(\xi_{1,i}'-\xi_{1,i})\right\}\prod_{i=1}^{N_T^1}\frac{\xi_{1,i}'}{\xi_{1,i}},$$

whereas, if the jump size parameter is changed to (arbitrarily large) c, the term $\frac{l(\Phi_1'|\beta_1,\theta_1)}{l(\Phi_1|\beta_1,\theta_1)}$ evaluates to $\exp\left\{-(\beta_1^{-1}-c^{-1})\sum_{i=1}^{N_T^1}(\xi_{1,i}'-\xi_{1,i})\right\}$, where c^{-1} can be arbitrarily close to 0.

References

[1] S. Brooks, A. Gelman, G. Jones and X. Meng, Handbook of Markov Chain Monte Carlo, (2011)

[2] O.E. Barndorff-Nielsen and N. Shephard, Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics, Journal of the Royal Statistical Society. Series B. Statistical Methodology, 63 (2001), pp. 167–241.

[3] F. Benth, J. Kallsen and T. Meyer-Brandis, A non-Gaussian Ornstein-Uhlenbeck process for electricity spot price modeling and derivatives pricing, Applied Mathematical Finance, 14 (2007), pp. 153–169.

[4] S. Frühwirth-Schnatter and L. Sögner, Bayesian estimation of stochastic volatility models based on OU processes with marginal Gamma law, Annals of the Institute of Statistical Mathematics, 61 (2009), pp. 159–179.

[5] A. Gelman, J. Carlin, H. Stern, D. Dunson, A. Vehtari and D. Rubin, Bayesian Data Analysis, (2013)

[6] H. Geman and A. Roncoroni, Understanding the fine structure of electricity prices, The Journal of Business, 79 (2006), pp. 1225–1261.

[7] J. Gonzalez, J. Moriarty and J. Palczewski, Bayesian calibration and number of jump components in electricity spot price models, Energy Economics, 65 (2017), pp. 375–388.

[8] R. Green and M. Nossman, Markov chain Monte Carlo estimation of a multi-factor jump diffusion model for power prices, The Journal of Energy Markets, 1 (2008), pp. 65–91.

[9] W.J. Hinderks and A. Wagner, Factor models in the German electricity market: Stylized facts, seasonality, and calibration, Energy Economics, 85 (2020), pp. 1043–1051.

[10] J.F. Kingman, Poisson processes, 3 (1992)

[11] Y. A. Kutoyants, Statistical inference for spatial Poisson processes, 134 (2012)

[12] J. Lucia and E. Schwartz, Electricity prices and power derivatives: Evidence from the Nordic Power Exchange, Review of derivatives research, 5 (2002), pp. 5–50.

[13] T. Meyer-Brandis and P. Tankov, Multi-factor jump-diffusion models of electricity prices, International Journal of Theoretical and Applied Finance, 11 (2008), pp. 503–528.

[14] G. O. Roberts, O. Papaspiliopoulos and P. Dellaportas, Bayesian inference for non-Gaussian Ornstein-Uhlenbeck stochastic volatility processes, Journal of the Royal Statistical Society. Series B. Statistical Methodology, 66 (2004), pp. 369–393.

[15] D.B. Rubin, Bayesianly justifiable and relevant frequency calculations for the applied statistician, The Annals of Statistics, 12 (1984), pp. 1151–1172.

[16] E. Schwartz and J. Smith, Short-term variations and long-term dynamics in commodity prices, Management Science, 46 (2000), pp. 893–911.

[17] J. Seifert and M. Uhrig-Homburg, Modelling jumps in electricity prices: theory and empirical evidence, Review of Derivatives Research, 10 (2007), pp. 59–85.

[18] P. Villaplana, Pricing power derivatives: A two-factor jump-diffusion approach, Available at SSRN 493943 (2003)

(C. Laudagé) Rheinland-Pfälzische Technische Universität Kaiserslautern-Landau, DE-67663, Kaiserslautern, Germany
Email address: christian.laudage@rptu.de

(F. Aichinger) Johann Radon Institute for Computational and Applied Mathematics (RICAM), Austrian Academy of Sciences, AT-4040 Linz, Austria/ Institute for Financial Mathematics and Applied Number Theory, University of Linz, AT-4040 Linz, Austria
Email address: florian.aichinger@ricam.oeaw.ac.at

(S. Desmettre) Institute for Financial Mathematics and Applied Number Theory, University of Linz, AT-4040 Linz, Austria
Email address: sascha.desmettre@jku.at