On the range of the X-ray transform of symmetric tensors compactly supported in the plane

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ON THE RANGE OF THE X-RAY TRANSFORM OF SYMMETRIC TENSORS COMPACTLY SUPPORTED IN THE PLANE

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ABSTRACT. We find the necessary and sufficient conditions for the Fourier coefficients of a function g on the torus to be in the range of the X-ray transform of a symmetric tensor of compact support in the plane.

1. INTRODUCTION

We revisit the range characterization of the X-ray transform of a real valued symmetric m- tensor of compact support in the Euclidean plane. The most studied case $m = 0$ is the classical Radon transform [34], for which the necessary and sufficient constraints have been long established independently by Gelfand and Graev [11], Helgason [12], and Ludwig [21]; we refer to the result as GGHL characterization. However, due to their practical use in computed tomography, the range characterization problem in the 0-tensor case continues to stimulate the interests of both mathematicians and practitioners [6, 8, 15, 16, 35, 22, 20]. In particular, in [35] the authors introduced the Bukhgeim-Hilbert transform, a Hilbert-like transform associated with A-analytic maps in the sense of Bukhgeim [5], and gave a range characterization of the range of the X-ray transform in terms of it. The latter result was extended to 1-and 2-tensors in [36, 38], and to an arbitrary order in [29].

For tensors of order $m \geq 1$, the non-injectivity of the X-ray transform makes the range characterization problem even more interesting. In the Euclidean plane, the GGHL-characterization was extended to arbitrary symmetric m-tensors in [30]. A partial survey of results on tomography of tensors in the Euclidean plane can be found in [9].

The systematic study of tensor tomography in non-Euclidean spaces originated in [39]. On simple Riemannian surfaces, the range characterization of the geodesic X-ray of compactly supported 0 and 1 has been established in terms of the scattering relation in [33]. The connection between the Euclidean version of the characterization in [33] and the classical GGHL characterization was established in [22]. The more general case of the system of the Momenta X-ray transform has been recently studied in [18].

Models which account for the attenuation have also been considered. For 0-tensors we mention the homogeneous case in [19], and the non-homogeneous case in the breakthrough works [1, 27, 28], and subsequently [26, 4, 3]. For higher order tensors we refer to [23, 2]; see [32] for a survey.

In [37] the authors considered the lines parametrized by points on the torus and gave a new range characterization for compactly supported functions in terms of the Fourier coefficients on the Fourier lattice of the torus. This novel point of view allowed to establish the missing connection

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between the result in [35] and the classical GGHL characterization. Although Xf is a function on the torus, our problem differs from the one in [14], where f itself is on a torus.

In here we extend the result in [37] from 0-order to symmetric tensors of an arbitrary order. While the method of proof uses the characterization in [37] of traces on the unit circle of A-analytic maps in the unit disc, for tensors of higher order, new qualitative relations arise. Apart from the symmetry constraints due to the double parameterization of the lines, of specific interest are the moment conditions. The thrust of this work are the constraints (2.17) and (2.33) replacing the generalized moment conditions in [30] and the sufficiency part in Theorems 2.1 and 2.2 for tensors of finite smoothness.

Unlike the characterization results in [35, 36, 38, 29], the main results here now allow to generate elements in the range of the X-ray without starting from an actual tensor $\mathbf{f}$ and computing the line integrals, but by merely starting from a double indexed sequence satisfying (2.25) for even order tensor (or (2.41) for odd order tensor). This simple and fast way of generating the X-ray data is likely to be used in numerical studies.

This work concerns real valued tensors. For a complex valued tensor $\mathbf{f}$, since $\mathbb{R}\mathrm{e}(X\mathbf{f}) = X(\mathbb{R}\mathrm{e}(\mathbf{f}))$ and $\mathbb{I}\mathrm{m}(X\mathbf{f}) = X(\mathbb{I}\mathrm{m}(\mathbf{f}))$, Theorems 2.1 and 2.2 apply separately to $\mathbb{R}\mathrm{e}(\mathbf{f})$ and to $\mathbb{I}\mathrm{m}(\mathbf{f})$. While the presented method also applies to the attenuated case, we do not consider it here.

All the details establishing notation and the statement of the main results are in Section 2. In Section 3 we briefly recall existing results on A-analytic maps that are used in the proofs. In Section 4 we present the proof of the Theorems 2.1 and 2.2. To improve the readability of the work, some of the claims are proven in the appendix.

2. Preliminaries and statement of main results

Let $\mathbf{f} = (f_{i_1 i_2 \dots i_m})$, with $i_1, ..., i_m \in \{1, 2\}$ be a real valued symmetric m-tensor, with integrable components of compact support in $\mathbb{R}^2$. By scaling and translating, we may assume that all the components have compact support inside the the unit disc $\Omega = \{z \in \mathbb{C} : |z| < 1\}$. The boundary $\varGamma$ of Ω is the unit circle, but we keep this notation to differentiate from the set $\mathbb{S}^1$ of directions. The symmetry refers to the components $f_{i_1 i_2 \dots i_m}$ being invariant under any transposition of indexes.

For $\boldsymbol{\theta}^m := \underbrace{\boldsymbol{\theta} \otimes \boldsymbol{\theta} \otimes \cdots \otimes \boldsymbol{\theta}}_{m} \in \left(\mathbb{S}^1\right)^m$ and $z \in \Omega$, let $\langle \mathbf{f}(z), \boldsymbol{\theta}^m \rangle$ denote the pairing

$$\langle \mathbf{f}(z), \boldsymbol{\theta}^m \rangle = f_{i_1 \cdots i_m}(z) \theta^{i_1} \cdot \theta^{i_2} \cdots \theta^{i_m}, \tag{2.1}$$

where the summation convention is understood over all repeated indexes $(i_1, i_2, \cdots, i_m) \in \{1, 2\}^m$.

The X-ray transform of $\mathbf{f}$ (extended by zero outside Ω) is given by

$$X\mathbf{f}(z, \boldsymbol{\theta}) := \int_{-\infty}^{\infty} \langle \mathbf{f}(z + t\boldsymbol{\theta}), \boldsymbol{\theta}^m \rangle dt, \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbb{S}^1. \tag{2.2}$$

Following directly from its definition

$$X\mathbf{f}(x, -\boldsymbol{\theta}) = (-1)^m X\mathbf{f}(x, \boldsymbol{\theta}), \tag{2.3}$$

so that $\boldsymbol{\theta} \mapsto X\mathbf{f}(x, \boldsymbol{\theta})$ is an even function for $\mathbf{f}$ of an even order m, and an odd function for $\mathbf{f}$ of an odd order m.

Lines $L_{(\beta,\theta)} := \{e^{i\beta} + se^{i\theta} : \ s \in \mathbb{R}\}$ intersecting $\overline{\Omega}$ are parametrized in coordinates $\{(e^{i\beta}, e^{i\theta}) : \beta, \theta \in (-\pi, \pi]\}$ on the torus $\varGamma \times \mathbb{S}^1$, and then

$$X\mathbf{f}(e^{i\beta}, e^{i\theta}) = \int_{-\infty}^{\infty} \langle \mathbf{f}(e^{i\beta} + se^{i\theta}), \boldsymbol{\theta}^m \rangle ds \tag{2.4}$$

is also understood as a function on the torus.

Since $L_{(\beta,\theta)} = L_{(2\theta-\beta-\pi,\theta)} = L_{(\beta,\theta+\pi)} = L_{(2\theta-\beta-\pi,\theta+\pi)}$, the set of lines intersecting $\overline{\Omega}$ are quadruply covered when $(e^{i\beta}, e^{i\theta})$ ranges over the entire torus $\varGamma \times \mathbb{S}^1$. Moreover, the following symmetries are satisfied,

$$X\mathbf{f}(e^{i\beta}, e^{i\theta}) = (-1)^m X\mathbf{f}(e^{i\beta}, e^{i(\theta+\pi)}), \text{ and} \tag{2.5}$$

$$X\mathbf{f}(e^{i\beta}, e^{i\theta}) = (-1)^m X\mathbf{f}(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}), \text{ for } (e^{i\beta}, e^{i\theta}) \in \varGamma \times \mathbb{S}^1; \tag{2.6}$$

see Figure 1 below.

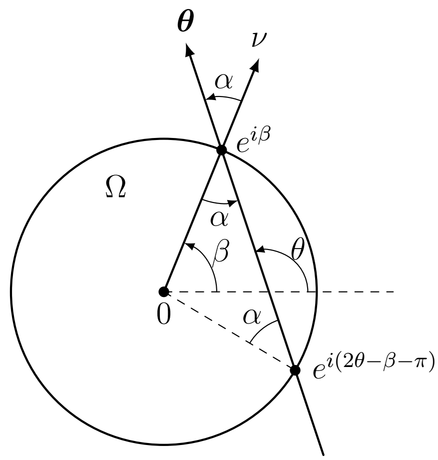

FIGURE 1. Fan-beam coordinates: $e^{i\beta} \in \varGamma$, $e^{i\theta} \in \mathbb{S}^1$, and $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$.

If $\mathbf{f} = (f_{i_1 i_2 \dots i_m})$, with $i_1, ..., i_m \in \{1, 2\}$ is merely integrable in Ω, then $X\mathbf{f}$ may not be integrable on the torus. However, if either

$$\text{supp } f_{i_1 \dots i_m} \subset \Omega, \text{ or } f_{i_1 \dots i_m} \in L^p(\Omega) \text{ for some } p > 2, \tag{2.7}$$

then $X\mathbf{f} \in L^1(\Gamma \times \mathbb{S}^1)$; see Proposition A.1 in the appendix.

We consider the partition of the torus into three parts: the "outflux" part

$$\varGamma_+ := \left\{ (e^{i\beta}, e^{i(\beta+\alpha)}) \in \varGamma \times \mathbb{S}^1 : \beta \in (-\pi, \pi], \ |\alpha| < \frac{\pi}{2} \right\}, \tag{2.8}$$

the "influx" part

$$\varGamma_- := \left\{ (e^{i\beta}, e^{i(\beta+\alpha)}) \in \varGamma \times \mathbb{S}^1 : \beta \in (-\pi, \pi], \ \frac{\pi}{2} < |\alpha| \le \pi \right\}, \tag{2.9}$$

and the (Lebesgue negligible) variety $\varGamma_0 := (\varGamma \times \mathbb{S}^1) \setminus (\varGamma_+ \cup \varGamma_-)$ parameterizing the tangent lines to the circle; see Figure 1.

Our result gives necessary and sufficient conditions for a function $g \in L^1(\varGamma \times \mathbb{S}^1)$ to satisfy

$$g = \begin{cases} X\mathbf{f}, & \text{on } \varGamma_+, \\ -X\mathbf{f}, & \text{on } \varGamma_-. \end{cases} \tag{2.10}$$

The characterization is in terms of the Fourier coefficients

$$g_{n,k} := \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(e^{i\beta}, e^{i\theta}) e^{-in\theta} e^{-ik\beta} d\theta d\beta, \ n, k \in \mathbb{Z} \tag{2.11}$$

on the lattice $\mathbb{Z} \times \mathbb{Z}$.

The two indexes play a different role. Throughout, the first index is the Fourier mode in the angular variable on $\mathbb{S}^1$, and we call it an *angular mode*. The second index is the mode in the boundary variable on $\varGamma$, and we call it a *boundary mode*.

The change of parity in the order of the tensor propagates to the statements of the results. For the sake of clarity we separate the two cases.

2.1. **The case of an even order m-tensor.** If m is even, it is easy to check that g in (2.10) satisfy the symmetry relation

$$g(e^{i\beta}, e^{i\theta}) = g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}), \text{ for a.e. } (e^{i\beta}, e^{i\theta}) \in \varGamma \times \mathbb{S}^1. \tag{2.12}$$

Motivated by this relation, let $L^1_{\text{sym}}(\Omega \times \mathbb{S}^1)$ denote the space of integrable functions g on the torus satisfying the symmetry in (2.12). Since $\left(e^{i\beta}, e^{i\theta}\right)$ and $\left(e^{i(2\theta-\beta-\pi)}, e^{(\theta+\pi)}\right)$ are either both in $\varGamma_+$, or both in $\varGamma_-$, we can consider the spaces $L^1_{\text{sym}}(\varGamma_\pm)$ of integrable functions on the half-tori satisfying (2.12). Clearly, $g \in L^1_{sym}(\varGamma \times \mathbb{S}^1)$ if and only if its restrictions $g|_{\varGamma_\pm} \in L^1_{\text{sym}}(\varGamma_\pm)$.

Moreover, since g in (2.10) is odd with respect to the angular variable:

$$g(e^{i\beta}, e^{i\theta}) = -g(e^{i\beta}, e^{i(\theta+\pi)}), \tag{2.13}$$

let us consider the subspace $L^1_{\text{sym,odd}}(\varGamma \times \mathbb{S}^1)$ of functions in $L^1_{\text{sym}}(\varGamma \times \mathbb{S}^1)$, which, in addition to satisfying (2.12), they also satisfy (2.13).

In the statements below we adapt the notations in [39]:

$$L^1(\mathbf{S}^m; \Omega) = \left\{ \mathbf{f} = (f_{i_1\cdots i_m}) \in \mathbf{S}^m(\Omega) : f_{i_1\cdots i_m} \in L^1(\Omega) \right\}$$

for the space of real valued, symmetric tensor fields of order m with integrable components. Similarly, $C^\mu(\mathbf{S}^m; \Omega)$, $0 < \mu < 1$, denotes the tensor fields of order m with locally Hölder continuous components, and we use the notation $\langle n \rangle = (1 + |n|^2)^{1/2}$.

Theorem 2.1 (Range characterization for even order tensors)**.** *Let $m = 2q$, $q \geq 0$ be an even integer. (i) Let $\mathbf{f} \in L^1(\mathbf{S}^m; \Omega)$ be a real valued, integrable symmetric m-tensor field satisfying* (2.7) *and $g \in L^1_{\text{sym,odd}}(\varGamma \times \mathbb{S}^1)$, with*

$$g = X\mathbf{f} \text{ on } \varGamma_+ (\text{ and } g = -X\mathbf{f} \text{ on } \varGamma_-).$$

Then the Fourier coefficients $\{g_{n,k}\}_{n,k\in\mathbb{Z}}$ of g satisfy the following conditions:

$$\text{Oddness}: \quad g_{n,k} = 0, \quad \text{for all even } n \in \mathbb{Z}, \text{ and all } k \in \mathbb{Z}; \tag{2.14}$$

$$\text{Conjugacy}: \quad g_{-n,-k} = \overline{g_{n,k}}, \quad \text{for all } n, k \in \mathbb{Z}; \tag{2.15}$$

$$\text{Symmetry}: \quad g_{n,k} = (-1)^{n+k} g_{n+2k,-k}, \quad \text{for all } n, k \in \mathbb{Z}; \tag{2.16}$$

$$\text{Moments}: \quad g_{n,k} = (-1)^k g_{n+2k,-k}, \quad \text{for all odd } n \leq -(2q+1), \text{ and all } k \leq 0. \tag{2.17}$$

(ii) Let $\{g_{n,k}\}$ be given for all odd *$n \leq -1$, and $k \in \mathbb{Z}$ such that*

$$\sum_{\substack{n\leq -1\\ n=\text{odd}}} \langle n \rangle^2 \sum_{k=-\infty}^{\infty} |g_{n,k}| < \infty, \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \langle k \rangle^{1+\mu} \sum_{\substack{n\leq -1\\ n=\text{odd}}} |g_{n,k}| < \infty, \tag{2.18}$$

for some $1/2 < \mu \leq 1$.

If $\{g_{n,k}\}$ *satisfy* (2.16) *and* (2.17)*, then there exists a real valued* $\mathbf{f} \in L^1(\mathbf{S}^m;\Omega)$ *such that the mapping*

$$(\Gamma \times \mathbb{S}^1) \ni (e^{i\beta}, e^{i\theta}) \longmapsto 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{\substack{n\leq -1\\ n=\text{odd}}} \sum_{k\in\mathbb{Z}} g_{n,k} e^{in\theta} e^{ik\beta} \right\} \tag{2.19}$$

defines a function in $L^1_{\text{sym,odd}}(\varGamma \times \mathbb{S}^1)$*, which coincides with* $X\mathbf{f}$ *on* $\varGamma_+$ *(and with* $-X\mathbf{f}$ *on* $\varGamma_-$*).*

Moreover, if $\{g_n\}$ *for odd* $n \leq -1$ *are defined on* $\varGamma$ *by*

$$g_n(e^{i\beta}) := \sum_{k=-\infty}^{\infty} g_{n,k}\, e^{ik\beta},\ \ e^{i\beta} \in \varGamma, \tag{2.20}$$

then, for $q \geq 1$*, the even-order tensor field* $\mathbf{f}$ *is uniquely determined by an element in the class*

$$\Psi_g^{\text{even}} := \left\{ \left(\psi_{-1}, \psi_{-3}, \cdots \psi_{-(2q-1)}\right) \in \left(W^{1,1}(\Omega;\mathbb{C})\right)^q : \psi_{-(2j-1)}\big|_{\varGamma} = g_{-(2j-1)},\ 1 \leq j \leq q \right\}. \tag{2.21}$$

Moreover if $\psi_{-1}, \psi_{-3}, ..., \psi_{-2q+1} \in C^{1,\mu}(\overline{\Omega})$*, then* $\mathbf{f} \in C^{\mu}(\mathbf{S}^m;\Omega)$*.*

If $q = 0$*, the class is empty and* $\mathbf{f}$ *is uniquely determined by the data.*

The traditional non-unique determination of the m-tensor by its X-ray transform (see [40, Theorems 2.4.2 and 3.4.3]) is in terms of a (potential) real valued $(m-1)$-symmetric tensor constant at the boundary. The components of this tensor are m-many arbitrary functions. The class Ψ_g^{even} in (2.21) uses $\dfrac{m}{2}$-many *complex valued*, hence also m-many real valued arbitrary extensions from the boundary.

The oddness and conjugacy constraints in (2.14) and (2.15) are not intrinsic to the X-ray transform. The symmetry constraints (2.16) merely account for each line being doubly parametrized in $\varGamma_+$, and they are shared by any function on the torus satisfying the symmetry (2.12); see Lemma A.2 in the appendix.

The following result is a direct consequence of the algebraic interaction of the range conditions in (2.14), (2.15), (2.16), and (2.17). To illustrate the result of these interactions, let us consider the partition of $\mathbb{Z}^- \times \mathbb{Z}$ as in Figure 2:

For an even integer $m \geq 0$, the white region $W = W^+ \cup W^-$, where

$$\begin{aligned} W^+ &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{odd } n \leq -m-1, \text{ and } 0 \leq k \leq -\frac{n+m+1}{2} \right\}, \\ W^- &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^- \ : \text{odd } n \leq -m-1, \text{ and } k \leq 0 \right\}, \end{aligned} \tag{2.22}$$

and the green region $G = G_L \cup G_R$, where

$$\begin{aligned} G_L &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{odd } n \leq -1, \text{ and } \frac{-n+1}{2} \leq k \leq -n \right\}. \\ G_R &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{odd } n \leq -1, \text{ and } k > -n \right\}. \end{aligned} \tag{2.23}$$

Moreover, for even integer $m \geq 2$, the red region $R = R^+ \cup R^-$, where

$$\begin{aligned} R^+ &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{odd } n \leq -1, \text{ and } -\frac{n+m-1}{2} \leq k \leq -\frac{n+1}{2} \right\}, \\ R^- &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^- \ : n \text{ odd}, -m+1 \leq n \leq -1, \text{ and } k \leq 0 \right\}. \end{aligned} \tag{2.24}$$

If $m = 0$, then the corresponding red region R defined by (2.24) is empty.

Remark 2.1. *The modes in the region R are affected solely by the symmetry* (2.16) *due to the double parametrization of the lines in $\varGamma_+$. The modes in the region G are affected both by the symmetry and the reality of the tensor* (2.15). *The white region contains the modes affected by the nature of the operator (integration) along the line* (2.17) *in combination with the symmetry. The width of the region R defines the lowest order of the symmetric tensor possible. See Figure 2.*

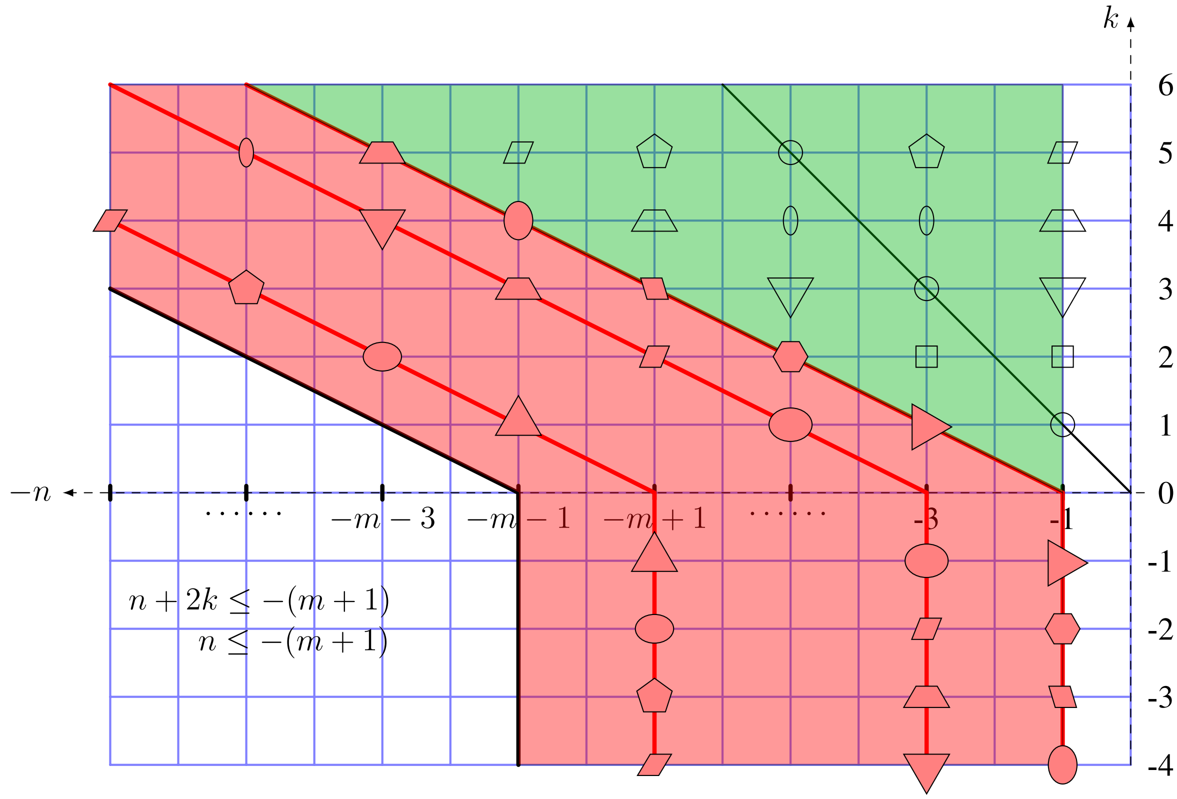

FIGURE 2. An even order m-tensor field $\mathbf{f}$ is determined by the odd negative angular modes on or above the diagonal $k = -n$ (green region), and the odd negative angular modes (marked red) on the $\frac{m}{2}$ red lines $n + 2k = -(m + 1)$ for $k \geq 0$. All the odd non-positive angular modes on and below the line $n + 2k = -(m + 1)$, and left of the line $n = -(m + 1)$ vanish.

Corollary 2.1. *Let $m = 2q$, $q \geq 0$ be an even integer. (i) If $\mathbf{f} \in L^1(\mathbf{S}^m; \Omega)$ is a real valued, integrable symmetric m-tensor field and $g \in L^1_{\text{sym,odd}}(\varGamma \times \mathbb{S}^1)$ coincides with $X\mathbf{f}$ on $\varGamma_+$, then the Fourier coefficients $\{g_{n,k}\}$ of g for all* odd $n \leq -1$ *satisfy*

$$g_{n,k} = \begin{cases} 0, & \text{if } (n,k) \in W, \\ (-1)^{1+k}\overline{g_{-n-2k,k}}, & \text{if } (n,k) \in G_L, \\ (-1)^{1+k} g_{n+2k,-k}, & \text{if } (n,k) \in R, \end{cases} \tag{2.25}$$

see Figure 2.

(ii) Let $\{g_{n,k}\}$ be given for $(n,k) \in R^+ \cup G_L$, such that

$$\sum_{(n,k)\in R^+ \cup G_L} \langle n \rangle^2 |g_{n,k}| < \infty, \quad \text{and} \quad \sum_{(n,k)\in R^+ \cup G_L} \langle k \rangle^{1+\mu} |g_{n,k}| < \infty, \tag{2.26}$$

for some $1/2 < \mu \leq 1$. Extend $g_{n,k}$'s from $R^+ \cup G_L$ to $R \cup G$ via the relations (2.25). *Then there exists a real valued $\mathbf{f} \in L^1(\mathbf{S}^m;\Omega)$, such that the mapping*

$$(\Gamma \times \mathbb{S}^1) \ni (e^{i\beta}, e^{i\theta}) \longmapsto 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{(n,k)\in R\cup G} g_{n,k} e^{in\theta} e^{ik\beta} \right\} \tag{2.27}$$

is precisely $X\mathbf{f}$ on $\varGamma_+$ and $-X\mathbf{f}$ on $\varGamma_-$. For $q \geq 1$, $\mathbf{f}$ is uniquely determined by an element in the class Ψ_g^{even} in (2.21). *If $q = 0$, the zero order tensor is uniquely determined by the data.*

We formulate next the odd-order tensor case.

2.2. **The case of odd order m-tensors.** If m is odd, it is easy to check that g in (2.10) obeys the skew-symmetry relation

$$g(e^{i\beta}, e^{i\theta}) = -g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}), \text{ for a.e. } (e^{i\beta}, e^{i\theta}) \in \varGamma \times \mathbb{S}^1. \tag{2.28}$$

This motivates to work in the space $L^1_{\text{skew}}(\Omega \times \mathbb{S}^1)$ of integrable functions g on the torus satisfying the skew-symmetry in (2.28). Since $\left(e^{i\beta}, e^{i\theta}\right)$ and $\left(e^{i(2\theta-\beta-\pi)}, e^{(\theta+\pi)}\right)$ are either both in $\varGamma_+$, or both in $\varGamma_-$, we can consider the spaces $L^1_{\text{skew}}(\varGamma_\pm)$ of integrable functions on the half-tori $\varGamma_\pm$ satisfying (2.28). Clearly, $g \in L^1_{skew}(\varGamma \times \mathbb{S}^1)$ if and only if its restrictions $g|_{\varGamma_\pm} \in L^1_{\text{skew}}(\varGamma_\pm)$.

Moreover, since g in (2.10) is even with respect to the angular variable:

$$g(e^{i\beta}, e^{i\theta}) = g(e^{i\beta}, e^{i(\theta+\pi)}), \tag{2.29}$$

we further consider the subspace $L^1_{\text{skew,even}}(\varGamma \times \mathbb{S}^1)$ of functions in $L^1_{\text{skew}}(\varGamma \times \mathbb{S}^1)$, which, in addition to satisfying (2.28), satisfy (2.29).

Our result gives necessary and sufficient conditions for a function $g \in L^1_{\text{skew,even}}(\varGamma \times \mathbb{S}^1)$ to coincide with $X\mathbf{f}$ on $\varGamma_+$ (and implicitly with $-X\mathbf{f}$ on $\varGamma_-$), for some symmetric tensor $\mathbf{f}$ of odd order m.

Theorem 2.2 (Range characterization for odd order tensors)**.** *Let $m = 2q+1$, $q \geq 0$ be an odd integer. (i) Let $\mathbf{f} \in L^1(\mathbf{S}^m;\Omega)$ be a real valued, integrable symmetric m-tensor field satisfying* (2.7)*, and $g \in L^1_{\text{skew,even}}(\varGamma \times \mathbb{S}^1)$, with*

$$g = X\mathbf{f} \textit{ on } \varGamma_+ (\textit{ and } g = -X\mathbf{f} \textit{ on } \varGamma_-).$$

Then the Fourier coefficients $\{g_{n,k}\}_{n,k\in\mathbb{Z}}$ of g satisfy the following conditions:

$$\textit{Evenness}: \quad g_{n,k} = 0, \quad \textit{for all } \text{odd } n \in \mathbb{Z}, \textit{ and all } k \in \mathbb{Z}; \tag{2.30}$$

$$\textit{Conjugacy}: \quad g_{-n,-k} = \overline{g_{n,k}}, \quad \textit{for all } n,k \in \mathbb{Z}; \tag{2.31}$$

$$\textit{Symmetry}: \quad g_{n,k} = -(-1)^{n+k} g_{n+2k,-k}, \quad \textit{for all } n,k \in \mathbb{Z}; \tag{2.32}$$

$$\textit{Moments}: \quad g_{n,k} = (-1)^k g_{n+2k,-k}, \quad \textit{for all } \text{even } n \leq -(2q+2), \textit{ and all } k \leq 0. \tag{2.33}$$

(ii) Let $\{g_{n,k}\}$ be given for all even *$n \leq -2$, and $k \in \mathbb{Z}$ such that*

$$\sum_{\substack{n\leq -2\\ n=\text{even}}} \langle n\rangle^2 \sum_{k=-\infty}^{\infty} |g_{n,k}| < \infty, \quad \textit{and} \quad \sum_{k=-\infty}^{\infty} \langle k\rangle^{1+\mu} \sum_{\substack{n\leq -2\\ n=\text{even}}} |g_{n,k}| < \infty, \tag{2.34}$$

for some $1/2 < \mu \leq 1$.

If $\{g_{n,k}\}$ *satisfy* (2.32) *and* (2.33)*, then there exists a real valued* $\mathbf{f} \in L^1(\mathbf{S}^m;\Omega)$ *such that the mapping*

$$(\Gamma \times \mathbb{S}^1) \ni (e^{i\beta}, e^{i\theta}) \longmapsto 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{\substack{n\leq -2\\ n=\text{even}}} \sum_{k\in\mathbb{Z}} g_{n,k} e^{in\theta} e^{ik\beta} \right\} \tag{2.35}$$

defines a function in $L^1_{\text{skew,even}}(\varGamma \times \mathbb{S}^1)$*, which coincides with* $X\mathbf{f}$ *on* $\varGamma_+$ *(and with* $-X\mathbf{f}$ *on* $\varGamma_-$*).*

Moreover, if $\{g_n\}$ *for even* $n \leq -2$ *are defined on* $\varGamma$ *by*

$$g_n(e^{i\beta}) := \sum_{k=-\infty}^{\infty} g_{n,k}\, e^{ik\beta},\ \ e^{i\beta} \in \varGamma, \tag{2.36}$$

then, for $q \geq 0$*, the odd-order tensor field* $\mathbf{f}$ *is uniquely determined by an element in the class*

$$\Psi^{\text{odd}}_g := \left\{ (\psi_0, \psi_{-2}, \cdots, \psi_{-2q}) \in W^{1,1}(\Omega;\mathbb{R}) \times \left(W^{1,1}(\Omega;\mathbb{C})\right)^q : \psi_{-2j}\big|_{\varGamma} = g_{-2j},\ 0 \leq j \leq q \right\}. \tag{2.37}$$

Moreover if $\psi_0, \psi_{-2}, ..., \psi_{-2q} \in C^{1,\mu}(\overline{\Omega})$*, then* $\mathbf{f} \in C^{\mu}(\mathbf{S}^m;\Omega)$*.*

Similar to the even order tensor case, the non-uniqueness class Ψ^{odd}_g in (2.37) uses $\dfrac{m-1}{2}$-many *complex valued* and one real valued arbitrary functions with specified traces at the boundary. The same count of extensions holds in the traditional non-uniqueness description by a (potential) real valued $(m-1)$-symmetric tensor.

The following result is a direct consequence of the algebraic interaction of the range conditions in (2.30), (2.31), (2.32), and (2.33). To illustrate the result of these interactions, let us consider the partition of $\mathbb{Z}^- \times \mathbb{Z}$ as in Figure 3:

For an odd integer $m \geq 1$, the white region $W = W^+ \cup W^-$, where

$$\begin{aligned} W^+ &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{even } n \leq -m-3,\ \text{and } 0 \leq k \leq -\frac{n+m+3}{2} \right\}, \\ W^- &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^- \ : \text{even } n \leq -m-3,\ \text{and } k \leq 0 \right\}, \end{aligned} \tag{2.38}$$

the green region $G = G_L \cup G_R$, where

$$\begin{aligned} G_L &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{even } n \leq -2,\ \text{and } -\frac{n}{2} \leq k \leq -n \right\}. \\ G_R &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{even } n \leq 0,\ \text{and } k > -n \right\}. \end{aligned} \tag{2.39}$$

and the red region $R = R^+ \cup R^-$, where

$$\begin{aligned} R^+ &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^+ \ : \text{even } n \leq -2,\ \text{and } -\frac{n+m+1}{2} \leq k \leq -\frac{n}{2} \right\}, \\ R^- &:= \left\{ (n,k) \in \mathbb{Z}^- \times \mathbb{Z}^- \ : n \text{ even}, -m-1 \leq n \leq 0,\ \text{and } k \leq 0 \right\}. \end{aligned} \tag{2.40}$$

Note that the slanted line $\{(n,k) : n + 2k = 0\}$ belongs to $G_L \cap R^+$.

Remark 2.2. *Similar to the even order case: The modes in the region* R *are affected solely by the symmetry* (2.32) *due to the double parametrization of the lines in* $\varGamma_+$*. The modes in the region* G *are affected both by the symmetry and the reality of the tensor* (2.31)*. The white region contains the modes affected by the nature of the operator (integration) along the line* (2.33) *in combination with the symmetry. The lowest possible order of the tensor can be read off the width of the red region* R*. See Figure 3.*

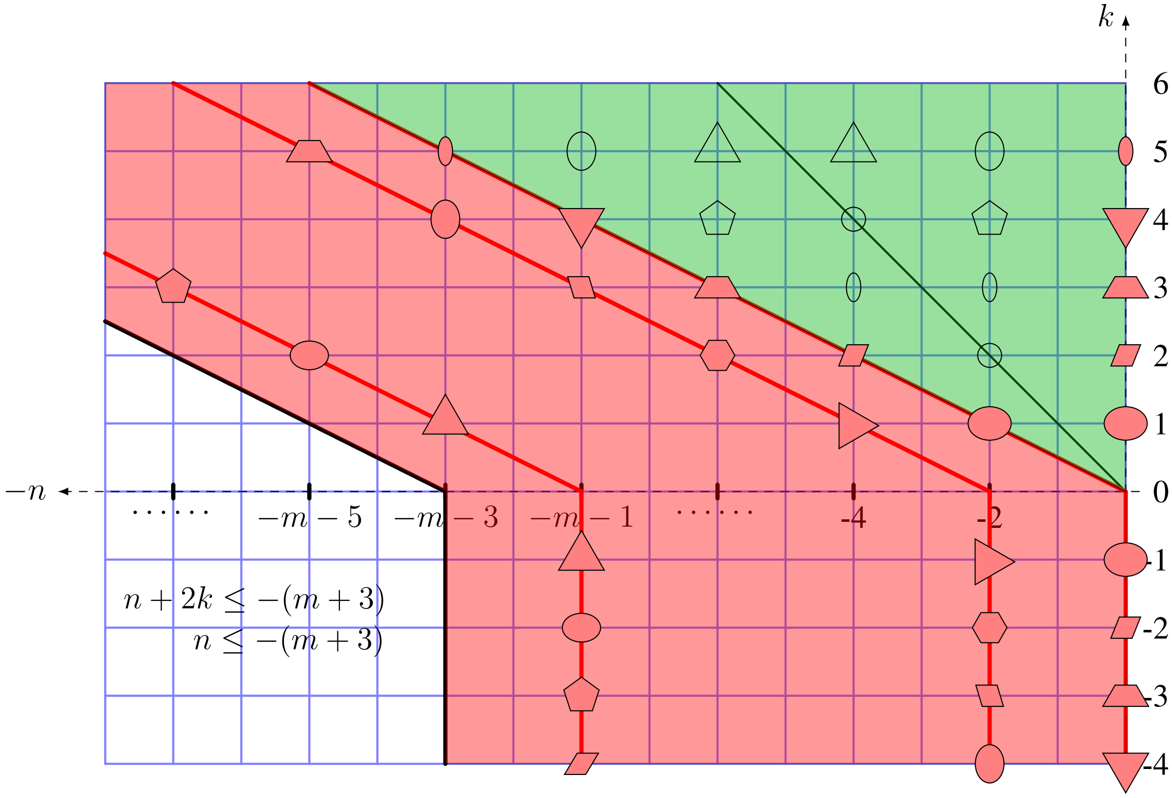

FIGURE 3. An odd order m-tensor field $\mathbf{f}$ is determined by the even negative angular modes on or above the diagonal $k = -n$ (green region), and the even negative angular modes (marked red) on the $\frac{m+1}{2}$ red lines $n + 2k = -(m + 1)$ for $k \geq 0$. All the even non-positive angular modes on and below the line $n+2k = -(m+3)$, and left of the line $n = -(m + 3)$ vanish.

Similarly to the even tensor case, the constraints in the range interact to yield the following result.

Corollary 2.2. *Let $m = 2q + 1$, $q \geq 0$ be an odd integer. (i) If $\mathbf{f} \in L^1(\mathbf{S}^m; \Omega)$ is a real valued, integrable symmetric m-tensor field and $g \in L^1_{\text{skew,even}}(\varGamma \times \mathbb{S}^1)$ coincides with $X\mathbf{f}$ on $\varGamma_+$, then the Fourier coefficients $\{g_{n,k}\}$ of g for all even $n \leq 0$ satisfy*

$$g_{n,k} = \begin{cases} 0, & \text{if } (n,k) \in W, \\ (-1)^k \overline{g_{-n-2k,k}}, & \text{if } (n,k) \in G_L, \\ (-1)^k g_{n+2k,-k}, & \text{if } (n,k) \in R, \end{cases} \tag{2.41}$$

see Figure 3.

(ii) Let $\{g_{n,k}\}$ be given for $(n, k) \in R^+ \cup G_L$, such that

$$\sum_{(n,k)\in R^+\cup G_L} \langle n\rangle^2 |g_{n,k}| < \infty, \quad \textit{and} \quad \sum_{(n,k)\in R^+\cup G_L} \langle k\rangle^{1+\mu} |g_{n,k}| < \infty, \tag{2.42}$$

for some $1/2 < \mu \leq 1$. *Extend* $g_{n,k}$*'s from* $R^+ \cup G_L$ *to* $R \cup G$ *via the relations* (2.41). *Then there exists a real valued* $\mathbf{f} \in L^1(\mathbf{S}^m; \Omega)$, *such that the mapping*

$$(\Gamma \times \mathbb{S}^1) \ni (e^{i\beta}, e^{i\theta}) \longmapsto 2\,\mathbb{R}\mathrm{e}\left\{ \sum_{(n,k)\in R\cup G} g_{n,k} e^{in\theta} e^{ik\beta} \right\} \tag{2.43}$$

is precisely $X\mathbf{f}$ *on* $\varGamma_+$, *and* $-X\mathbf{f}$ *on* $\varGamma_-$. *Moreover,* $\mathbf{f}$ *is uniquely determined by an element in the class* Ψ_g^{odd} *in* (2.37).

3. L^2-ANALYTIC MAPS AND THEIR TRACE CHARACTERIZATION

The method of proof of Theorems 2.1 and 2.2 is based on the characterization in [35] of traces of A-analytic maps in the sense of Bukhgeim [5]. In this section we summarize those existing results used in the proof.

Bukhgeim's original theory in [5] considers the sequence valued maps

$$\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), ...\rangle, \tag{3.1}$$

and solution of the Beltrami-like equation

$$\overline{\partial}\mathbf{u}(z) + \mathcal{L}\partial\mathbf{u}(z) = 0, \quad z \in \Omega, \tag{3.2}$$

where $\mathcal{L}\mathbf{u}(z) = \mathcal{L}(u_0(z), u_{-1}(z), u_{-2}(z), ...) := (u_{-1}(z), u_{-2}(z), ...)$ denotes the left translation. These solutions are called $\mathcal{L}$-analytic.

Similar to classical analytic maps, the solution of (3.2) satisfy a Cauchy-like integral formula,

$$\mathbf{u}(z) = \mathcal{B}[\mathbf{u}|_\varGamma](z), \quad z \in \Omega, \tag{3.3}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator acting on $\mathbf{u}|_\varGamma$ defined component-wise [10] for $n \geq 0$ by

(3.4)
$$(\mathcal{B}\mathbf{u})_{-n}(z) := \frac{1}{2\pi i}\int_\varGamma \frac{u_{-n}(\zeta)}{\zeta - z}d\zeta + \frac{1}{2\pi i}\int_\varGamma \left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}\sum_{j=1}^{\infty} u_{-n-j}(\zeta)\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j, \ z \in \Omega.$$

The traces of $\mathcal{L}$-analytic maps on the boundary must satisfy some constraints, which can be expressed in terms of a corresponding Hilbert-like transform introduced in [35]. More precisely, the Bukhgeim-Hilbert transform $\mathcal{H}$ acting on g,

$$\varGamma \ni z \mapsto (\mathcal{H}\mathbf{g})(z) = \langle (\mathcal{H}\mathbf{g})_0(z), (\mathcal{H}\mathbf{g})_{-1}(z), (\mathcal{H}\mathbf{g})_{-2}(z), ...\rangle \tag{3.5}$$

is defined component-wise for $n \geq 0$ by

$$(\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi}\int_\varGamma \frac{g_{-n}(\zeta)}{\zeta - z}d\zeta + \frac{1}{\pi}\int_\varGamma \left\{\frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}}\right\}\sum_{j=1}^{\infty} g_{-n-j}(\zeta)\left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\right)^j, z \in \varGamma. \tag{3.6}$$

The theorems below comprise some results in [35, 36, 37]. For $0<\mu<1$, $p=1,2$, we consider the Banach spaces:

$$(3.7)\qquad \begin{aligned} l^{1,p}_{\infty}(\varGamma) &:= \left\{ \mathbf{g}=\langle g_0, g_{-1}, g_{-2}, ...\rangle \ : \|\mathbf{g}\|_{l^{1,p}_{\infty}(\varGamma)} := \sup_{\xi\in\varGamma}\sum_{j=0}^{\infty}\langle j\rangle^p |g_{-j}(\xi)| <\infty \right\},\\ C^{\mu}(\varGamma;l_1) &:= \left\{ \mathbf{g}=\langle g_0, g_{-1}, g_{-2}, ...\rangle : \sup_{\xi\in\varGamma}\|\mathbf{g}(\xi)\|_{l_1} + \sup_{\substack{\xi,\eta\in\varGamma\\ \xi\neq\eta}} \frac{\|\mathbf{g}(\xi)-\mathbf{g}(\eta)\|_{l_1}}{|\xi-\eta|^{\mu}} <\infty \right\},\\ Y_{\mu}(\varGamma) &:= \left\{ \mathbf{g}: \mathbf{g}\in l^{1,2}_{\infty}(\varGamma) \text{ and } \sup_{\substack{\xi,\eta\in\varGamma\\ \xi\neq\eta}} \sum_{j=0}^{\infty}\langle j\rangle \frac{|g_{-j}(\xi)-g_{-j}(\eta)|}{|\xi-\eta|^{\mu}} <\infty \right\}, \end{aligned}$$

where, for brevity, we use the notation $\langle j\rangle = (1+|j|^2)^{1/2}$. Similarly, we consider $C^{\mu}(\overline{\Omega};l_\infty)$, and $C^{\mu}(\Omega;l_\infty)=\bigcup_{0<r<1} C^{\mu}(\overline{\Omega}_r;l_\infty)$, where for $0<r<1$, $\Omega_r=\{z\in\mathbb{C}:|z|<r\}$.

Theorem 3.1. *Let $0<\mu<1$. Let $\mathbf{g}=\langle g_0, g_{-1}, g_{-2}, ...\rangle$ be a sequence valued map defined on the boundary $\varGamma$ and $\mathcal{B}$ be the Bukhgeim-Cauchy operator acting on $\mathbf{g}$ as in* (3.4).
(i) If $\mathbf{g}\in l^{1,1}_{\infty}(\varGamma)\cap C^{\mu}(\varGamma;l_1)$, then $\mathbf{u}:=\mathcal{B}\mathbf{g}\in C^{1,\mu}(\Omega;l_\infty)\cap C(\overline{\Omega};l_\infty)$ is $\mathcal{L}$-analytic in Ω.
(ii) Moreover, if $\mathbf{g}\in Y_{\mu}(\varGamma)$ for $\mu>1/2$, then $\mathcal{B}\mathbf{g}\in C^{1,\mu}(\Omega;l_1)\cap C^{\mu}(\overline{\Omega};l_1)$.

For the proof of Theorem 3.1 (i) we refer to [35, Theorem 3.1], and for part (ii) we refer to [36, Proposition 2.3].

The following result characterize the traces of $\mathcal{L}$-analytic maps.

Theorem 3.2. *Let $0<\mu<1$, and let $\mathcal{H}$ be the Bukhgeim-Hilbert transform in* (3.6).
(i) If $\mathbf{g}\in l^{1,1}_{\infty}(\varGamma)\cap C^{\mu}(\varGamma;l_1)$ is the boundary value of an $\mathcal{L}$-analytic function, then $\mathcal{H}\mathbf{g}\in C^{\mu}(\varGamma;l_\infty)$ and satisfies

$$(I+i\mathcal{H})\mathbf{g}=\mathbf{0}. \tag{3.8}$$

(ii) If $\mathbf{g}\in Y_{\mu}(\varGamma)$ for $\mu>1/2$, satisfies (3.8)*, then $\mathcal{L}$-analytic function $\mathbf{u}:=\mathcal{B}\mathbf{g}\in C^{1,\mu}(\Omega;l_1)\cap C^{\mu}(\overline{\Omega};l_1)$, satisfy*

$$\mathbf{u}|_{\varGamma}=\mathbf{g}. \tag{3.9}$$

For the proof of Theorem 3.2 we refer to [35, Proposition 3.1, Theorem 3.2, Corollary 4.1, and Proposition 4.2].

The results above need Ω be a strictly convex domain, but not necessarily a unit disk. However, the following result [37, Theorem 4.1] uses Ω be the unit disk, and $\varGamma$ be its unit circle boundary. Given $\mathbf{g}=\langle g_0, g_{-1}, g_{-2}, ...\rangle\in l_\infty(\mathbb{N};L^1(\varGamma))$, we consider the Fourier coefficients of its components

$$g_{-n,k}:=\frac{1}{2\pi}\int_{-\pi}^{\pi} g_{-n}\left(e^{i\beta}\right)e^{-ik\beta}d\beta, \text{ for all } n\geq 0, \text{ and } k\in\mathbb{Z}. \tag{3.10}$$

Theorem 3.3. [37, Theorem 4.1] *Let $\mathbf{g}=\langle g_0, g_{-1}, g_{-2}...\rangle\in l^{1,1}_{\infty}(\varGamma)\cap C^{\mu}(\varGamma;l_1)$, $0<\mu<1$, and $g_{-n,k}$ be the Fourier coefficients of its components as in* (3.10)*. Let $\mathcal{H}\mathbf{g}=\langle(\mathcal{H}\mathbf{g})_0,(\mathcal{H}\mathbf{g})_{-1},(\mathcal{H}\mathbf{g})_{-2},...\rangle$ be the Bukhgeim-Hilbert transform acting on $\mathbf{g}$ as defined in* (3.6)*. Then $\mathcal{H}\mathbf{g}\in C^{\mu}(\varGamma;l_\infty)$, and* *the Fourier coefficients* $(\mathcal{H}\mathbf{g})_{-n,k} := \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi}(\mathcal{H}\mathbf{g})_{-n}\left(e^{i\beta}\right)e^{-ik\beta}d\beta$*, for* $n\geq 0, k\in\mathbb{Z}$*, of its components satisfy*

$$(-i)(\mathcal{H}\mathbf{g})_{-n,k} = \begin{cases} g_{-n,k} & \text{if } k\geq 0,\\ -g_{-n,k}+2(-1)^k g_{-n+2k,-k} & \text{if } k\leq -1. \end{cases} \tag{3.11}$$

4. Proof of Theorems 2.1 and 2.2

Since $\mathbf{f}$ is symmetric, for any m-tuple $(i_1,\cdots,i_m)\in\{1,2\}^m$ such that 2 occurs exactly k times (and 1 occurs $m-k$ times), the component $f_{i_1\dots i_m}$ satisfies

$$f_{i_1\dots i_m} = f_{\underbrace{1\cdots 1}_{m-k}\underbrace{2\cdots 2}_{k}} =: \tilde{f}_k. \tag{4.1}$$

Since there are $\binom{m}{k}$ many m-tuples $(i_1,i_2,\cdots,i_m)$ that contain exactly k many $2's$, we get

$$\begin{aligned}\langle\mathbf{f}(z),\boldsymbol{\theta}^m\rangle &= f_{i_1\cdots i_m}(z)\theta^{i_1}\cdot\theta^{i_2}\cdots\theta^{i_m} \overset{(4.1)}{=\!=} \sum_{k=0}^{m}\binom{m}{k}\tilde{f}_k(z)\,(\cos\theta)^{(m-k)}(\sin\theta)^k\\ &= (e^{-\mathrm{i}\theta})^m\sum_{k=0}^{m}\frac{(-\mathrm{i})^k}{2^m}\binom{m}{k}\tilde{f}_k(z)\,Q_{m,k}(e^{2\mathrm{i}\theta}),\end{aligned}$$

where $Q_{m.k}(t) = (t+1)^{m-k}(t-1)^k$.

Since any polynomial of degree m is uniquely represented as a linear combination of $\{Q_{m,k}(t)\}_{k=0}^m$,

$$\langle\mathbf{f},\boldsymbol{\theta}^m\rangle = \begin{cases}\displaystyle\sum_{k=0}^{q} f_{2k}e^{-\mathrm{i}(2k)\theta}+\sum_{k=1}^{q} f_{-2k}e^{\mathrm{i}(2k)\theta}, & \text{if } m=2q,\\ \displaystyle\sum_{k=0}^{q} f_{2k+1}e^{-\mathrm{i}(2k+1)\theta}+f_{-(2k+1)}e^{\mathrm{i}(2k+1)\theta}, & \text{if } m=2q+1,\end{cases} \tag{4.2}$$

where f_k's are in a one -to-one correspondence to $\tilde{f}_k$, and thus with $f_{i_1\cdots i_m}$. We refer to the Lemma A.1 in the appendix for details on this one-to-one correspondence in (4.2).

We approach the range characterization via the well-known connection with the transport model, where the unique solution $u(z,\boldsymbol{\theta})$ to the boundary value problem

$$\boldsymbol{\theta}\cdot\nabla u(z,\boldsymbol{\theta}) = 2\langle\mathbf{f}(z),\boldsymbol{\theta}^m\rangle \tag{4.3a}$$

$$u|_{\varGamma_-} = -X\mathbf{f} \tag{4.3b}$$

has the trace $u|_{\varGamma\times\mathbb{S}^1}= g$, with g in (2.10), i.e.

$$u|_{\varGamma\times\mathbb{S}^1} = \begin{cases} X\mathbf{f}, & \text{on } \varGamma_+,\\ -X\mathbf{f}, & \text{on } \varGamma_-.\end{cases} \tag{4.4}$$

Proposition 4.1. *(a) If* $m=2q,\ q\geq 0$*, then the solution* u *to the boundary value problem* (4.3) *is an odd function of* $\boldsymbol{\theta}$*,*

$$u(z,\boldsymbol{\theta}) = -u(z,-\boldsymbol{\theta}).$$

(b) If $m=2q+1,\ q\geq 0$*, then the solution* u *to the boundary value problem* (4.3) *is an even function of* $\boldsymbol{\theta}$*,*

$$u(z,\boldsymbol{\theta}) = u(z,-\boldsymbol{\theta}).$$

Proof. (a) We note that for even $m \geq 0$, $(-\boldsymbol{\theta})^m = \boldsymbol{\theta}^m$.
If $u^{\text{odd}}(z,\boldsymbol{\theta}) := \frac{1}{2}\left[u(z,\boldsymbol{\theta}) - u(z,-\boldsymbol{\theta})\right]$ denotes the angularly odd part of u, then

$$\boldsymbol{\theta}\cdot\nabla u^{\text{odd}}(z,\boldsymbol{\theta}) = \frac{1}{2}\left[\boldsymbol{\theta}\cdot\nabla u(z,\boldsymbol{\theta}) + (-\boldsymbol{\theta})\cdot\nabla u(z,-\boldsymbol{\theta})\right] = 2\langle \mathbf{f}(z), \boldsymbol{\theta}^m\rangle,$$

and $u^{\text{even}} := u - u^{\text{odd}}$ solves

$$\boldsymbol{\theta}\cdot\nabla u^{\text{even}}(z,\boldsymbol{\theta}) = 0, \quad (z,\boldsymbol{\theta}) \in \Omega\times\mathbb{S}^1, \tag{4.5a}$$

$$u^{\text{even}}|_{\varGamma_-} = 0. \tag{4.5b}$$

Since (4.5) has the unique solution $u^{\text{even}} \equiv 0$, $u = u^{\text{odd}}$ in $\overline{\Omega}\times\mathbb{S}^1$.
(b) We note that for odd $m \geq 1$, $(-\boldsymbol{\theta})^m = -\boldsymbol{\theta}^m$.
If $u^{\text{even}}(z,\boldsymbol{\theta}) := \frac{1}{2}\left[u(z,\boldsymbol{\theta}) + u(z,-\boldsymbol{\theta})\right]$ denotes the angularly even part of u, then

$$\boldsymbol{\theta}\cdot\nabla u^{\text{even}}(z,\boldsymbol{\theta}) = \frac{1}{2}\left[\boldsymbol{\theta}\cdot\nabla u(z,\boldsymbol{\theta}) - (-\boldsymbol{\theta})\cdot\nabla u(z,-\boldsymbol{\theta})\right] = 2\langle \mathbf{f}(z), \boldsymbol{\theta}^m\rangle,$$

and $u^{\text{odd}} := u - u^{\text{even}}$ solves

$$\boldsymbol{\theta}\cdot\nabla u^{\text{odd}}(z,\boldsymbol{\theta}) = 0, \quad (z,\boldsymbol{\theta}) \in \Omega\times\mathbb{S}^1, \tag{4.6a}$$

$$u^{\text{odd}}|_{\varGamma_-} = 0. \tag{4.6b}$$

Since (4.6) has the unique solution $u^{\text{odd}} \equiv 0$, $u = u^{\text{even}}$ in $\overline{\Omega}\times\mathbb{S}^1$. □

We present in detail the proof for the even tensor $m = 2q$ (Theorem 2.1). For the odd tensor case (Theorem 2.2), the proof is essentially the same, where small changes occur due to the change in parity.

(i) **Proof of necessity in Theorem 2.1:** Since g is angularly odd, (2.14) holds. Since g is real valued, (2.15) holds. The identities (2.16) follow by direct calculation, see Lemma A.2.

We will first prove (2.17) for an even order $m = 2q$ tensor $\mathbf{f} = (f_{i_1 i_2 \dots i_m})$ with smooth components $f_{i_1 i_2 \dots i_m} \in C_0^2(\Omega)$. The result for components $f_{i_1 i_2 \dots i_m} \in L^1(\Omega)$ follows by a density argument.

By using the notations $\overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2$, $\partial = (\partial_{x_1} - i\partial_{x_2})/2$, and $\theta = \arg\boldsymbol{\theta} \in (-\pi,\pi]$, the transport equation (4.3a) becomes

$$[e^{-i\theta}\overline{\partial} + e^{i\theta}\partial]u(z,\boldsymbol{\theta}) = \sum_{n=-q}^{q} f_{2n}(z)e^{-\mathrm{i}(2n)\theta}, \quad (z,\boldsymbol{\theta}) \in \Omega\times\mathbb{S}^1. \tag{4.7}$$

By Proposition 4.1 (a), the solution u to (4.7) is an odd function of $\boldsymbol{\theta}$, thus all of its Fourier coefficients (in the angular variable) of even order vanish,

$$u(z,\boldsymbol{\theta}) = \sum_{\substack{n\in\mathbb{Z}\\ n=\text{odd}}} u_n(z)e^{in\theta}.$$

By identifying the Fourier modes of the same order, the equation (4.7) reduces to the system:

$$\overline{\partial}u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = f_{2n}(z), \qquad 0 \leq n \leq q, \tag{4.8}$$

$$\overline{\partial}u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = 0, \qquad n \geq q+1. \tag{4.9}$$

Since $\mathbf{f}$ is real valued, the solution $u(z,\boldsymbol{\theta})$ of (4.7) is also real valued, and its Fourier modes in the angular variable occur in conjugates, $u_{-n} = \overline{u_n}$. Thus, it suffices to consider the non-positive odd Fourier modes of $u(z,\cdot)$: Let $\mathbf{u}$ be the sequence valued map

$$\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_{-1}(z), u_{-3}(z), u_{-5}(z), ...\rangle. \tag{4.10}$$

Since all the components $f_{i_1 i_2 \dots i_m} \in C_0^2(\Omega)$, $X\mathbf{f} \in C^2(\varGamma_-)$, and, thus, the solution u to the transport problem (4.7) is in $C^2(\overline{\Omega}\times\mathbb{S}^1)$. Moreover, its trace $u|_{\varGamma\times\mathbb{S}^1} \in C^2(\varGamma\times\mathbb{S}^1)$.

For $z = e^{\mathrm{i}\beta} \in \varGamma$, we use the negative odd Fourier modes of the trace $g(e^{i\beta},\cdot) = u|_{\varGamma\times\mathbb{S}^1}(e^{\mathrm{i}\beta},\cdot)$ to define the sequence valued map

$$\mathbf{g}(e^{i\beta}) := \langle g_{-1}(e^{i\beta}), g_{-3}(e^{i\beta}), ...\rangle, \text{ with } g_{-2n-1}(e^{i\beta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\beta}e^{i\theta})e^{i(2n+1)\theta}d\theta. \tag{4.11}$$

Since $u \in C^2(\overline{\Omega}\times\mathbb{S}^1)$, $\mathbf{u} \in C^1(\overline{\Omega}; l_1)$, and, thus,

$$\mathbf{g} = \mathbf{u}|_{\varGamma} \in C^1(\varGamma; l_1) \subset l^{1,1}_{\infty}(\varGamma) \cap C^{\mu}(\varGamma; l^1), \mu > 1/2. \tag{4.12}$$

By (4.9), $\mathcal{L}^q\mathbf{u} = \langle u_{-(2q+1)}, u_{-(2q+3)}, \cdots\rangle$ is $\mathcal{L}$-analytic in Ω and its trace $\mathcal{L}^q\mathbf{g} = \mathcal{L}^q\mathbf{u}|_{\varGamma}$ is the boundary value of an $\mathcal{L}$-analytic map.

Recall that $\mathcal{H}$ is the Bukhgeim-Hilbert operator in (3.6). By the necessity part in Theorem 3.2, we have $\mathcal{H}(\mathcal{L}^q\mathbf{g}) \in C^{\mu}(\varGamma; l_\infty)$ and $(I + i\mathcal{H})(\mathcal{L}^q\mathbf{g}) = \mathbf{0}$.

Since $\mathcal{H}$ commutes with the left translation $\mathcal{L}$, we obtained

$$\mathcal{L}^q(I + i\mathcal{H})(\mathbf{g}) = \mathbf{0}. \tag{4.13}$$

In particular, for all odd $n \le -2q-1$, and $k \in \mathbb{Z}$, we obtained

$$([I + i\mathcal{H}]\mathbf{g})_{n,k} = 0. \tag{4.14}$$

By Theorem 3.3, the Fourier coefficients $(\mathcal{H}\mathbf{g})_{n,k}$, for odd $n \le -2q-1$ and $k \in \mathbb{Z}$, satisfy (3.11), and thus

$$([I + i\mathcal{H}]\mathbf{g})_{n,k} = \begin{cases} 0 & \text{if } k \ge 0, \\ 2g_{n,k} - 2(-1)^k g_{n+2k,-k} & \text{if } k \le -1. \end{cases}$$

In conjunction with (4.14), the Fourier coefficients of $\mathbf{g}$ must satisfy (2.17), i.e.,

$$g_{n,k} = (-1)^k g_{n+2k,-k}, \text{ for all odd } n \le -2q-1, \text{ and } k \le -1.$$

Equation (2.17) for $k = 0$ is trivially satisfied.

The proof for $\mathbf{f} \in L^1(\mathbf{S}^m;\Omega)$ follows from the density of $C_0^2(\Omega)$ in $L^1(\Omega)$.

(ii) **Proof of sufficiency in Theorem 2.1:** Recall $m = 2q$, $q \ge 0$. Given the double sequence $\{g_{n,k}\}$ for all odd $n \le -2q-1$, and $k \in \mathbb{Z}$, we construct a real valued symmetric m- tensor $\mathbf{f}$ in Ω such that the map on the torus $\begin{cases} X\mathbf{f} \text{ on } \varGamma_+, \\ -X\mathbf{f} \text{ on } \varGamma_- \end{cases}$ has the Fourier coefficients matching the $\{g_{n,k}\}$'s.

Recall the construction in (2.20),

$$g_{-n}(e^{i\beta}) := \sum_{k=-\infty}^{\infty} g_{-n,k}\, e^{ik\beta}, \textit{ for odd } n \le -2q-1, \quad e^{i\beta} \in \varGamma, \tag{4.15}$$

and define the sequence valued map on $\varGamma$

$$\mathbf{g}^{\text{odd}}(e^{i\beta}) := \langle g_{-(2q+1)}(e^{i\beta}), g_{-(2q+3)}(e^{i\beta}), g_{-(2q+5)}(e^{i\beta}), \cdots\rangle. \tag{4.16}$$

By the decay assumption (2.18) and [37, Lemma A.4.], $\mathbf{g}^{\text{odd}} \in l^{1,2}_{\infty}(\varGamma) \cap C^{1,\mu}(\varGamma; l^1)$. In particular, $\mathbf{g}^{\text{odd}} \in Y_\mu(\varGamma)$ for $\mu > 1/2$.

We use the Bukhgeim-Cauchy integral formula (3.4) to construct the sequence valued map $\mathbf{u}^{\text{odd}}(z)$ inside Ω:

$$\mathbf{u}^{\text{odd}}(z) = \langle u_{-(2q+1)}(z), u_{-(2q+3)}(z), \cdots \rangle := \mathcal{B}\left[\mathbf{g}^{\text{odd}}\right](z), \quad z \in \Omega. \tag{4.17}$$

By Theorem 3.1 (ii), the constructed $\mathbf{u}^{\text{odd}} \in C^{1,\mu}(\Omega; l^1) \cap C^{\mu}(\overline{\Omega}; l^1)$ is $\mathcal{L}$-analytic in Ω,

$$\overline{\partial} u_n + \partial u_{n-2} = 0, \quad \text{for all odd } n \leq -2q-1. \tag{4.18}$$

While $\mathbf{u}^{\text{odd}}$ constructed in (4.17) is $\mathcal{L}$-analytic, in general, its trace $\mathbf{u}^{\text{odd}}|_{\varGamma}$ need not be equal to $\mathbf{g}^{\text{odd}}$. It is at this point that the constraints (2.17) come into play. By using the hypothesis (2.17),

$$g_{n,k} = (-1)^k g_{n+2k,-k}, \quad \text{for odd } \; n \leq -2q-1, \text{ and } k \leq -1,$$

and Theorem 3.3, we obtain

$$([I + i\mathcal{H}]\mathbf{g}^{\text{odd}})_{n,k} = 0, \text{ for all odd } n \leq -2q-1, \text{ and } k \in \mathbb{Z}.$$

Thus, $[I + i\mathcal{H}]\mathbf{g}^{\text{odd}} = \mathbf{0}$, and the sufficiency part of Theorem 3.2 applies to yield

$$\mathbf{u}^{\text{odd}}|_{\varGamma} = \mathbf{g}^{\text{odd}}. \tag{4.19}$$

All of the positive Fourier modes u_n, g_n for odd $n \geq 2q+1$ are constructed by conjugation,

$$u_n := \overline{u_{-n}}, \quad \text{in } \Omega, \tag{4.20}$$

$$g_n := \overline{g_{-n}}, \quad \text{on } \varGamma. \tag{4.21}$$

Also, by conjugating (4.18) we note that the positive Fourier modes satisfy

$$\overline{\partial} u_{n+2} + \partial u_n = 0, \quad \text{for all odd } n \geq 2q+1.$$

Moreover, using (4.19) they extend continuously to $\varGamma$ and

$$u_n|_{\varGamma} = \overline{u_{-n}}|_{\varGamma} = \overline{g_{-n}} = g_n, \quad \text{odd } n \geq 2q+1.$$

In summary, we have shown that

$$\overline{\partial} u_n + \partial u_{n-2} = 0, \quad \text{for all odd integers } |n| \geq 2q+3, \tag{4.22}$$

$$u_n|_{\varGamma} = g_n, \qquad \text{for all odd integers } |n| \geq 2q+1. \tag{4.23}$$

In the case of the 0-tensor, $\mathbf{f} = f_0$, which is defined directly from u_{-1} by $f_0 = \partial u_{-1} + \overline{\partial u_{-1}}$.
We consider next the case $q \geq 1$ of tensors of order 2 or higher.
Recall the construction (2.20),

$$g_{-2n+1} := \sum_{k=-\infty}^{\infty} g_{-2n+1,k}\, e^{ik\beta}, \text{ for } 1 \leq n \leq q, \tag{4.24}$$

and define $g_1, g_3, ..., g_{2q-1}$ by conjugation

$$g_{2n-1} := \overline{g_{-2n+1}}, 1 \leq n \leq q. \tag{4.25}$$

Also recall the non-uniqueness class Ψ_g^{even} in (2.21).

For $\left(\psi_{-1}, \psi_{-3}, \cdots \psi_{-(2q-1)}\right) \in \Psi_g^{\text{even}}$ arbitrary, define the modes $u_{\pm 1}, u_{\pm 3}, ..., u_{\pm(2q-1)}$ in Ω by

$$u_{-(2n-1)} := \psi_{-(2n-1)} \text{ and } u_{2n-1} := \overline{\psi}_{-(2n-1)}, \quad 1 \leq n \leq q. \tag{4.26}$$

By the definition of the class (2.21) and (4.25),

$$\begin{aligned} u_{-(2n-1)}|_{\varGamma} &= g_{-(2n-1)}, \quad 1 \leq n \leq q, \; q \geq 1, \quad \text{and} \\ u_{2n-1}|_{\varGamma} &= \overline{g}_{-(2n-1)} = g_{2n-1}, \quad 1 \leq n \leq q, \; q \geq 1. \end{aligned} \tag{4.27}$$

The components of the m-tensor $\mathbf{f}$ are defined via the one-to-one correspondence between $\{\tilde{f}_{2n}: -q \leq n \leq q\}$ in (4.1) and the functions $\{f_{2n}: \; -q \leq n \leq q\}$ as follows.

We define f_{2q} by using $\psi_{-(2q-1)}$ from the non-uniqueness class, and $u_{-(2q+1)}$ from the Bukhgeim-Cauchy formula (4.17),

$$2f_{2q} := \overline{\partial}\psi_{-(2q-1)} + \partial u_{-(2q+1)}. \tag{4.28}$$

Then, define $\{f_{2n}: \; 0 \leq n \leq q-1\}$ solely from the information in the non-uniqueness class via

$$2f_{2n} := \overline{\partial}\psi_{-(2n-1)} + \partial\psi_{-(2n+1)}, \quad 0 \leq n \leq q-1, \tag{4.29}$$

Note that f_0 is real valued. Finally, define $\{f_{-2n}: \; 1 \leq n \leq q\}$ by conjugation

$$f_{-2n} := \overline{f_{2n}}, \quad 1 \leq n \leq q. \tag{4.30}$$

By construction, $f_{2n} \in L^1(\Omega)$, $-q \leq n \leq q$. Moreover, if $\psi_{-1}, \cdots, \psi_{-2q+1} \in C^{1,\mu}(\Omega)$, then $f_{2n} \in C^{\mu}(\Omega)$.

In the remaining of the proof we check that, for the tensor $\mathbf{f}$ defined above, the Fourier coefficients of the map $\begin{cases} X\mathbf{f} \text{ on } \varGamma_+, \\ -X\mathbf{f} \text{ on } \varGamma_- \end{cases}$ match the given $g_{n,k}$'s for all odd $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

With u_n defined in (4.17) for odd $n \leq -2q-1$ (and by conjugation for odd $n \geq 2q+1$), and with ψ_{-2n+1} given by the non-uniqueness class (and ψ_{2n-1} given by conjugation) for $0 \leq n \leq q$, we define the following two functions:

$$u^{\text{odd}}(z, e^{i\theta}) := \sum_{|n|\geq q} u_{2n+1} e^{\mathrm{i}(2n+1)\theta} + \sum_{n=1}^{q} \psi_{-(2n-1)} e^{-\mathrm{i}(2n-1)\theta} + \sum_{n=1}^{q} \overline{\psi}_{-(2n-1)} e^{\mathrm{i}(2n-1)\theta} \tag{4.31}$$

and

$$g(z, e^{i\theta}) := \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} g_n(z) e^{in\theta}, \quad (z,\theta) \in \varGamma \times \mathbb{S}^1. \tag{4.32}$$

Since $\mathbf{u}^{\text{odd}} \in C^{1,\mu}(\Omega; l^1) \cap C^{\mu}(\overline{\Omega}; l^1)$, we employ [35, Corollary 4.1] and [35, Proposition 4.1 (iii)] to infer the regularity $u^{\text{odd}} \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$.

In particular, for each $e^{i\theta} \in \mathbb{S}^1$, the trace of $u^{\text{odd}}(\cdot, e^{i\theta})$ on $\varGamma$ satisfies

$$u^{\text{odd}}(\cdot, e^{i\theta})|_{\varGamma} = \left. \left(\sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} u_n e^{in\theta} \right) \right|_{\varGamma} = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} (u_n|_{\varGamma})\, e^{in\theta} = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} g_n e^{in\theta} = g(\cdot, e^{i\theta}), \tag{4.33}$$

where the third equality above uses (4.23) and (4.27). Note that, since $u^{\text{odd}} \in C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$, we can conclude now that g defined in (4.32) lies in $C^{\mu}(\varGamma \times \mathbb{S}^1)$.

Since the term by term differentiation in (4.31) is now justified,

$$\begin{aligned}
\boldsymbol{\theta} \cdot \nabla u^{\text{odd}} &= \sum_{n=0}^{q-1} (\overline{\partial}\psi_{-(2n-1)} + \partial\psi_{-(2n+1)}) e^{-\mathrm{i}(2n)\theta} + \sum_{n=1}^{q-1} (\overline{\partial}\, \overline{\psi}_{-(2n+1)} + \partial\overline{\psi}_{-(2n-1)}) e^{\mathrm{i}(2n)\theta} \\
&\quad + e^{-\mathrm{i}(2q)\theta} (\overline{\partial}\psi_{-(2q-1)} + \partial u_{-(2q+1)}) + e^{\mathrm{i}(2q)\theta} (\partial\overline{\psi}_{-(2q-1)} + \overline{\partial}\, \overline{u}_{-(2q+1)}) \\
&= 2\sum_{k=-q}^{q} f_{2k}(z) e^{-\mathrm{i}(2k)\theta} = 2\langle \mathbf{f}, \boldsymbol{\theta}^{2q} \rangle
\end{aligned}$$

where the cancellation uses (4.22), and the second equality uses the definition of f_{2k}'s in (4.28), (4.29), and (4.30).

We remark next two properties of the function g defined in (4.32) which are needed later: Since only the odd modes in the angular variable are used, $g(z,\cdot)$ is an odd function, and thus the odd condition (2.13) is satisfied. In addition, we claim that g also satisfies the symmetry condition (2.12). Indeed,

$$\begin{aligned}
g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}) &= \sum_{\substack{m=-\infty\\ m=\text{odd}}}^{\infty} \sum_{p\in\mathbb{Z}} g_{m,p} e^{im(\theta+\pi)} e^{ip(2\theta-\beta-\pi)} \\
&= \sum_{\substack{m=-\infty\\ m=\text{odd}}}^{\infty} \sum_{p\in\mathbb{Z}} (-1)^{m+p} g_{m,p} e^{i(2p+m)\theta} e^{-ip\beta} \\
&\xlongequal{m=n+2k,\ p=-k} \sum_{\substack{n=-\infty\\ n=\text{odd}}}^{\infty} \sum_{k\in\mathbb{Z}} (-1)^{n+k} g_{n+2k,-k} e^{in\theta} e^{ik\beta} \\
&\xlongequal{(2.16)} \sum_{\substack{n=-\infty\\ n=\text{odd}}}^{\infty} \sum_{k\in\mathbb{Z}} g_{n,k} e^{in\theta} e^{ik\beta} = g(e^{i\beta}, e^{i\theta}).
\end{aligned} \tag{4.34}$$

We have all the ingredients needed to show that g coincides with $X\mathbf{f}$ on $\varGamma_+$. In the following calculation we let $l(\beta,\theta) = |e^{i\beta} - e^{i(2\theta-\beta-\pi)}|$ denote the length of the chord in Figure 1, and use the geometric equality $e^{i\beta} - l(\beta,\theta)e^{i\theta} = e^{i(2\theta-\beta-\pi)}$. For each $(e^{i\beta}, e^{i\theta}) \in \varGamma_+$,

$$\begin{aligned}
2g(e^{i\beta}, e^{i\theta}) &= g(e^{i\beta}, e^{i\theta}) - g(e^{i\beta}, e^{i(\theta+\pi)}) \\
&= g(e^{i\beta}, e^{i\theta}) - g(e^{i(2\theta-\beta-\pi)}, e^{i\theta}) \\
&= u^{\text{odd}}(e^{i\beta}, e^{i\theta}) - u^{\text{odd}}(e^{i(2\theta-\beta-\pi)}, e^{i\theta}) \\
&= \int_{-l(\beta,\theta)}^{0} \boldsymbol{\theta}\cdot\nabla u^{\text{odd}}(e^{i\beta} + te^{i\theta}, e^{i\theta})dt \\
&= \int_{-l(\beta,\theta)}^{0} 2\langle \mathbf{f}(e^{i\beta} + te^{i\theta}), \boldsymbol{\theta}^m\rangle dt = 2[X\mathbf{f}](e^{i\beta}, e^{i\theta}),
\end{aligned} \tag{4.35}$$

where the first equality uses $g(e^{i\beta},\cdot)$ is angularly odd, the second equality uses the symmetry relation (4.34) with θ replaced by $\theta+\pi$, the third equality uses (4.33), the fourth equality is the fundamental theorem of calculus, the fifth equality uses (4.7), and the last equality uses the support of $\mathbf{f}$ in Ω.

Therefore $g = X\mathbf{f}$ on $\varGamma_+$, and, since g is angularly odd, $g = -X\mathbf{f}$ on $\varGamma_-$.

The equation (4.35) also shows that $z \mapsto \langle \mathbf{f}(z), \boldsymbol{\theta}^m\rangle$ is integrable in Ω , for every $\boldsymbol{\theta} \in \mathbb{S}^1$. An application of the identity (4.2) yields that all of the constructed f_{2n}'s and, by the one-to-one correspondence, all the components of $\mathbf{f}$ are integrable in Ω.

This finishes the proof of the even order tensor case.

□

The analytical reasoning in the proof of Theorem 2.2 (the odd order tensor case) is the same as above. The change in parity of the order of the tensor merely modifies the algebraic statements. We sketch them below for the sake of completeness.

(i) **Sketch of proof of necessity in Theorem 2.2:** Since g is angularly even, (2.30) holds. Since g is real valued, (2.31) holds. The identities (2.32) follow by direct calculation, see Lemma A.2.

As in the even case, it suffices to prove (2.33) for odd $m = (2q+1)$-tensor with components in $C_0^2(\Omega)$.

The transport equation (4.3a) has the right hand side modified as

$$[e^{-i\theta}\overline{\partial} + e^{i\theta}\partial]u(z,\boldsymbol{\theta}) = \sum_{n=0}^{q} \left(f_{2n+1}e^{-\mathrm{i}(2n+1)\theta} + f_{-(2n+1)}e^{\mathrm{i}(2n+1)\theta} \right), \quad (z,\boldsymbol{\theta}) \in \Omega \times \mathbb{S}^1. \tag{4.36}$$

By Proposition 4.1 (b), the solution u to (4.36) is an even function of $\boldsymbol{\theta}$, thus all of its Fourier coefficients (in the angular variable) of odd order vanish,

$$u(z,\boldsymbol{\theta}) = \sum_{\substack{n\in\mathbb{Z} \\ n=\text{ even}}} u_n(z)e^{in\theta}.$$

By identifying the Fourier modes of the same order, the equation (4.7) reduces to the system:

$$\overline{\partial}u_{-2n}(z) + \partial u_{-(2n+2)}(z) = f_{2n+1}(z), \qquad 0 \le n \le q,\ q \ge 0, \tag{4.37}$$

$$\overline{\partial}u_{-2n}(z) + \partial u_{-(2n+2)}(z) = 0, \qquad n \ge q+1,\ q \ge 0, \tag{4.38}$$

Since $\mathbf{f}$ is real valued, the solution $u(z,\boldsymbol{\theta})$ of (4.7) is also real valued, and its Fourier modes in the angular variable occur in conjugates, $u_{-n} = \overline{u_n}$. Thus, it suffices to consider the non-positive even Fourier coefficients of $u(z,\cdot)$: Let $\mathbf{u}$ be the sequence valued map

$$\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_{-2}(z), u_{-4}(z), ...\rangle. \tag{4.39}$$

Since all the components $f_{i_1 i_2 \dots i_m} \in C_0^2(\Omega)$, $X\mathbf{f} \in C^2(\varGamma_-)$, and, thus, the solution u to the transport problem (4.36) is in $C^2(\overline{\Omega}\times\mathbb{S}^1)$. Moreover, its trace $u|_{\varGamma\times\mathbb{S}^1} \in C^2(\varGamma\times\mathbb{S}^1)$.

For $z = e^{\mathrm{i}\beta} \in \varGamma$, we use the negative even Fourier modes of the trace $g(e^{i\beta},\cdot) = u|_{\varGamma\times\mathbb{S}^1}(e^{i\beta},\cdot)$ to define the sequence valued map

$$\mathbf{g}(e^{i\beta}) := \langle g_{-2}(e^{i\beta}), g_{-4}(e^{i\beta}), ...\rangle, \text{ with } g_{-2n}(e^{i\beta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\beta}e^{i\theta})e^{i(2n)\theta}d\theta. \tag{4.40}$$

Since $u \in C^2(\overline{\Omega}\times\mathbb{S}^1)$, $\mathbf{u} \in C^1(\overline{\Omega}; l_1)$, and, thus,

$$\mathbf{g} = \mathbf{u}|_{\varGamma} \in C^1(\varGamma; l_1) \subset l^{1,1}_{\infty}(\varGamma) \cap C^{\mu}(\varGamma; l^1), \mu > 1/2. \tag{4.41}$$

By (4.38), $\mathcal{L}^q\mathbf{u} = \langle u_{-(2q+2)}, u_{-(2q+4)}, \cdots\rangle$ is $\mathcal{L}$-analytic in Ω and its trace $\mathcal{L}^q\mathbf{g} = \mathcal{L}^q\mathbf{u}|_{\varGamma}$ is the boundary value of an $\mathcal{L}$-analytic map.

By the necessity part in Theorem 3.2, we have $\mathcal{H}(\mathcal{L}^q\mathbf{g}) \in C^{\mu}(\varGamma; l_\infty)$ and $(I + i\mathcal{H})(\mathcal{L}^q\mathbf{g}) = \mathbf{0}$. Since $\mathcal{H}$ commutes with the left translation $\mathcal{L}$, we obtained

$$\mathcal{L}^q(I + i\mathcal{H})(\mathbf{g}) = \mathbf{0}. \tag{4.42}$$

In particular, for all even $n \le -2q-2$, and $k \in \mathbb{Z}$, we obtained

$$([I + i\mathcal{H}]\mathbf{g})_{n,k} = 0, \tag{4.43}$$

By Theorem 3.3, the Fourier coefficients $(\mathcal{H}\mathbf{g})_{n,k}$, for even $n \le -2q$, and $k \in \mathbb{Z}$, satisfy (3.11), and thus

$$([I + i\mathcal{H}]\mathbf{g})_{n,k} = \begin{cases} 0 & \text{if } k \ge 0, \\ 2g_{n,k} - 2(-1)^k g_{n+2k,-k} & \text{if } k \le -1. \end{cases}$$

In conjunction with (4.43), the Fourier coefficients of $\mathbf{g}$ must satisfy (2.33).

The proof for $\mathbf{f} \in L^1(\mathbf{S}^m;\Omega)$ follows from the density of $C_0^2(\Omega)$ in $L^1(\Omega)$.

(ii) **Sketch of proof of sufficiency in Theorem 2.2:** Recall $m = 2q+1$, $q \geq 0$. Given the double sequence $\{g_{n,k}\}$ for all even $n \leq -2q$, and $k \in \mathbb{Z}$, we construct below a real valued symmetric m-tensor $\mathbf{f}$ in Ω such that the map on the torus $\begin{cases} X\mathbf{f} \text{ on } \varGamma_+, \\ -X\mathbf{f} \text{ on } \varGamma_- \end{cases}$ has the Fourier coefficients matching the $\{g_{n,k}\}$'s.

Recall the construction in (2.36),

$$g_{-n}(e^{i\beta}) := \sum_{k=-\infty}^{\infty} g_{-n,k}\, e^{ik\beta}, \textit{ for even } n \leq -2q, \quad e^{i\beta} \in \varGamma, \tag{4.44}$$

and define the sequence valued map on $\varGamma$

$$\mathbf{g}^{\text{odd}}(e^{i\beta}) := \langle g_{-(2q)}(e^{i\beta}), g_{-(2q+2)}(e^{i\beta}), \cdots \rangle. \tag{4.45}$$

By the decay assumption (2.18) and [37, Lemma A.4.], $\mathbf{g}^{\text{even}} \in l^{1,2}_{\infty}(\varGamma) \cap C^{1,\mu}(\varGamma; l^1)$. In particular, $\mathbf{g}^{\text{even}} \in Y_\mu(\varGamma)$ for $\mu > 1/2$.

We use the Bukhgeim-Cauchy integral formula (3.4) to construct the sequence valued map $\mathbf{u}^{\text{even}}(z)$ inside Ω:

$$\mathbf{u}^{\text{even}}(z) = \langle u_{-2q}(z), u_{-(2q+2)}(z), \cdots \rangle := \mathcal{B}\left[\mathbf{g}^{\text{even}}\right](z), \quad z \in \Omega. \tag{4.46}$$

By Theorem 3.1 (ii), the constructed $\mathbf{u}^{\text{even}} \in C^{1,\mu}(\Omega; l^1) \cap C^{\mu}(\overline{\Omega}; l^1)$ is $\mathcal{L}$-analytic in Ω,

$$\overline{\partial} u_n + \partial u_{n-2} = 0, \quad \text{for all even } n \leq -2q. \tag{4.47}$$

By using the hypothesis (2.33), and Theorem 3.3, we obtain $[I + i\mathcal{H}]\mathbf{g}^{\text{even}} = \mathbf{0}$. The sufficiency part of Theorem 3.2 now applies to yield

$$\mathbf{u}^{\text{even}}|_{\varGamma} = \mathbf{g}^{\text{even}}. \tag{4.48}$$

All of the positive Fourier modes u_n, g_n for even $n \geq 2q, q \geq 0$, are constructed by conjugation,

$$u_n := \overline{u_{-n}}, \text{ in } \Omega, \text{ and } g_n := \overline{g_{-n}}, \text{ on } \varGamma. \tag{4.49}$$

As in the even order tensor case, it is easy to check that

$$\overline{\partial} u_n + \partial u_{n-2} = 0, \quad \text{for all even integers } |n| > 2q, q \geq 0 \tag{4.50}$$

$$u_n|_{\varGamma} = g_n, \quad \text{for all even integers } |n| \geq 2q, q \geq 0. \tag{4.51}$$

Recall the construction (2.36),

$$g_{-2n} := \sum_{k=-\infty}^{\infty} g_{-2n,k}\, e^{ik\beta}, \text{ for } 0 \leq n \leq q, \tag{4.52}$$

and define $g_2, g_4, ..., g_{2q}$ by conjugation

$$g_{2n} := \overline{g_{-2n}}, \quad 1 \leq n \leq q. \tag{4.53}$$

Also recall the non-uniqueness class Ψ_g^{odd} in (2.37).

For $(\psi_0, \psi_{-2}, \cdots \psi_{-2q}) \in \Psi_g^{\text{odd}}$ arbitrary, define the modes $u_0, u_{\pm 2}, ..., u_{\pm 2q}$ in Ω by

$$u_{-2n} := \psi_{-2n} \text{ and } u_{2n} := \overline{\psi}_{-2n}, \quad 0 \leq n \leq q. \tag{4.54}$$

By the definition of the class (2.37) and (4.53),

$$u_{-2n}|_{\varGamma} = g_{-2n}, \quad -q \leq n \leq q. \tag{4.55}$$

The components of the m-tensor $\mathbf{f}$ are defined via the one-to-one correspondence between $\{\tilde{f}_{\pm(2n+1)} : \ 0 \leq n \leq q\}$ in (4.1) and the functions $\{f_{\pm(2n+1)} : \ 0 \leq n \leq q\}$ as follows.

Define the function f_{2q+1} by using ψ_{-2q} from the non-uniqueness class, and $u_{-(2q+2)}$ from the Bukhgeim-Cauchy formula (4.46), via

$$2f_{2q+1} := \overline{\partial}\psi_{-2q} + \partial u_{-2(q+1)}. \tag{4.56}$$

Then define $\{f_{2n+1} : \ 0 \leq n \leq q-1\}$ solely from the information in the non-uniqueness class via

$$2f_{2n+1} := \overline{\partial}\psi_{-2n} + \partial \psi_{-(2n+2)}, \ \ 0 \leq n \leq q-1. \tag{4.57}$$

Finally, define $\{f_{-2n-1} : \ 0 \leq n \leq q\}$ by conjugation

$$f_{-(2n+1)} := \overline{f_{2n+1}}, \quad 0 \leq n \leq q. \tag{4.58}$$

By construction, $f_{\pm(2n+1)} \in L^1(\Omega)$, $0 \leq n \leq q$. Moreover, if $\psi_0, \cdots, \psi_{-2q} \in C^{1,\mu}(\Omega)$, then $f_{\pm(2n+1)} \in C^{\mu}(\Omega)$.

In the remaining of the proof we check that, for the tensor $\mathbf{f}$ defined above, the Fourier coefficients of the map $\begin{cases} X\mathbf{f} \text{ on } \varGamma_+, \\ -X\mathbf{f} \text{ on } \varGamma_- \end{cases}$ match the given $g_{n,k}$'s for all even $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$.

As in the even order tensor case, we define

$$u^{\text{even}}(z, e^{i\theta}) := \sum_{|n| \geq q+1} u_{2n}(z) e^{\mathrm{i}(2n)\theta} + \sum_{n=0}^{q} \psi_{-2n}(z) e^{-\mathrm{i}(2n)\theta} + \sum_{n=0}^{q} \overline{\psi}_{-2n}(z) e^{\mathrm{i}(2n)\theta} \tag{4.59}$$

and

$$g(z, e^{i\theta}) := \sum_{\substack{n=-\infty \\ n=\text{even}}}^{\infty} g_n(z) e^{in\theta}, \quad (z, \theta) \in \varGamma \times \mathbb{S}^1, \tag{4.60}$$

and the corresponding trace identity $u^{\text{even}}(\cdot, e^{i\theta})|_{\varGamma} = g(\cdot, e^{i\theta})$ holds.

Since $\mathbf{u}^{\text{even}} \in C^{1,\mu}(\Omega; l^1) \cap C^{\mu}(\overline{\Omega}; l^1)$, we employ [35, Corollary 4.1] and [35, Proposition 4.1 (iii)] to infer the regularity $u^{\text{even}} \in C^{1,\mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1)$.

Term by term differentiation in (4.59), and an application of (4.50), (4.54), (4.56), (4.57), and (4.58) yields

$$\boldsymbol{\theta} \cdot \nabla u^{\text{even}} = 2\langle \mathbf{f}, \boldsymbol{\theta}^{2q+1} \rangle.$$

Since only even modes in the angular variable are used, $g(z, \cdot)$ is an even function, and thus the even condition (2.29) is satisfied. In addition, we claim that g also satisfies the symmetry condition

(2.28). Indeed,

$$
\begin{aligned}
g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}) &= \sum_{\substack{m=-\infty \\ m=\text{ even}}}^{\infty} \sum_{p\in\mathbb{Z}} g_{m,p} e^{im(\theta+\pi)} e^{ip(2\theta-\beta-\pi)} \\
&= \sum_{\substack{m=-\infty \\ m=\text{ even}}}^{\infty} \sum_{p\in\mathbb{Z}} (-1)^{m+p} g_{m,p} e^{i(2p+m)\theta} e^{-ip\beta} \\
&\overset{m=n+2k,\ p=-k}{=\!=\!=} \sum_{\substack{n=-\infty \\ n=\text{ even}}}^{\infty} \sum_{k\in\mathbb{Z}} (-1)^{n+k} g_{n+2k,-k} e^{in\theta} e^{ik\beta} \\
&\overset{(2.32)}{=\!=} -\sum_{\substack{n=-\infty \\ n=\text{ even}}}^{\infty} \sum_{k\in\mathbb{Z}} g_{n,k} e^{in\theta} e^{ik\beta} = -g(e^{i\beta}, e^{i\theta}). \qquad (4.61)
\end{aligned}
$$

For each $(e^{i\beta}, e^{i\theta}) \in \Gamma_+$,

$$
\begin{aligned}
2g(e^{i\beta}, e^{i\theta}) &= g(e^{i\beta}, e^{i\theta}) + g(e^{i\beta}, e^{i(\theta+\pi)}) \\
&= g(e^{i\beta}, e^{i\theta}) - g(e^{i(2\theta-\beta-\pi)}, e^{i\theta}) \\
&= u^{\text{even}}(e^{i\beta}, e^{i\theta}) - u^{\text{even}}(e^{i(2\theta-\beta-\pi)}, e^{i\theta}) \\
&= \int_{-l(\beta,\theta)}^{0} \boldsymbol{\theta} \cdot \nabla u^{\text{even}}(e^{i\beta} + te^{i\theta}, e^{i\theta}) dt \\
&= \int_{-l(\beta,\theta)}^{0} 2\langle \mathbf{f}(e^{i\beta} + te^{i\theta}), \boldsymbol{\theta}^m \rangle dt = 2[X\mathbf{f}](e^{i\beta}, e^{i\theta}), \qquad (4.62)
\end{aligned}
$$

where the first equality uses $g(e^{i\beta}, \cdot)$ is angularly even, the second equality uses the symmetry relation (4.61) with θ replaced by $\theta + \pi$, the fifth equality uses (4.36), and the last equality uses the support of $\mathbf{f}$ in Ω.

Therefore $g = X\mathbf{f}$ on Γ_+, and, since g is angularly even, $g = -X$ on Γ_-.

The equation (4.62) also shows that $z \mapsto \langle \mathbf{f}(z), \boldsymbol{\theta}^m \rangle$ is integrable in Ω , for every $\boldsymbol{\theta} \in \mathbb{S}^1$. An application of the identity (4.2) yields that all of the constructed f_{2n+1}'s and, thus, all the components of $\mathbf{f}$ are integrable in Ω.

□

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APPENDIX A. ELEMENTARY RESULTS

To improve the readability, we moved the proof of the more elementary claims to this section. The presentation follows the order of their occurrence.

Lemma A.1. *Let $m, k \geq 0$ be integers with $0 \leq k \leq m$ and $Q_{m.k}(t) = (t+1)^{m-k}(t-1)^k$. Any polynomial of degree m is uniquely represented as a linear combination of $\{Q_{m,k}(t)\}_{k=0}^m$.*

Proof. Given $h_0, h_1, \cdots, h_m$, we show that there are unique $g_0, g_1, \cdots, g_m$ such that

$$\sum_{k=0}^{m} g_k (t-1)^k (t+1)^{m-k} = \sum_{k=0}^{m} h_k t^k. \tag{A.1}$$

We argue by induction in m. For $m = 1$, $g_0 = (h_1 + h_0)/2$ and $g_1 = (h_1 - h_0)/2$.

Assume next that a polynomial of degree $m-1$ is represented as a linear combination of $\{Q_{m-1,k}(t)\}_{k=0}^{m-1}$. To simplify notation, let $a_k := \binom{m}{k} = \frac{m!}{k!(m-k)!}$, for $0 \leq k \leq m$. Note that $\displaystyle\sum_{k=0}^{m} a_k = \sum_{k=0}^{m} \binom{m}{k} = 2^m$.

The left hand side of (A.1) rewrites

$$\begin{aligned}
\sum_{k=0}^{m} g_k (t-1)^k (t+1)^{m-k} &= (t+1) \sum_{k=0}^{m-1} g_k (t-1)^k (t+1)^{m-1-k} + g_m (t-1)^m \\
&\overset{\text{induction hypothesis}}{=\!=\!=} (t+1) \sum_{k=0}^{m-1} \gamma_k t^k + \sum_{k=0}^{m} (-1)^{m-k} a_k g_m t^k \\
&= \sum_{k=0}^{m-1} \gamma_k t^{k+1} + \sum_{k=0}^{m-1} \gamma_k t^k + \sum_{k=0}^{m} (-1)^{m-k} a_k g_m t^k \\
&= (\gamma_0 + (-1)^m a_0 g_m) + \sum_{k=1}^{m-1} \left(\gamma_{k-1} + \gamma_k + (-1)^{m-k} a_k g_m \right) t^k + \left(\gamma_{m-1} + a_m g_m \right) t^m.
\end{aligned}$$

The identity (A.1) yields that $\gamma_0, \gamma_1, \cdots, \gamma_{m-1}, g_m$ solve the $(m+1) \times (m+1)$ linear system:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & (-1)^m a_0 \\
1 & 1 & 0 & 0 & \cdots & 0 & 0 & (-1)^{m-1} a_1 \\
0 & 1 & 1 & 0 & \cdots & 0 & 0 & (-1)^{m-2} a_2 \\
0 & 0 & 1 & 1 & \cdots & 0 & 0 & (-1)^{m-3} a_3 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & 1 & -a_{m-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & a_m
\end{bmatrix}
\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \vdots \\ \gamma_{m-1} \\ g_m \end{bmatrix}
=
\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{m-1} \\ h_m \end{bmatrix}. \tag{A.2}$$

The determinant of the matrix above is calculated by expanding it along the last column: Since all the $m \times m$- cofactor matrices have determinant 1, the $(m+1) \times (m+1)$ determinant is $\displaystyle\sum_{k=0}^{m} a_k = 2^m > 0$.

In addition to g_m being determined, the determination of $\gamma_0, ..., \gamma_{m-1}$ together with the induction hypothesis uniquely determine $g_0, ... g_{m-1}$. In fact this argument can be construed in a recursive computation as follows. Since $g_m = 2^{-m} \sum_{k=0}^{m} (-1)^k h_k$, the unknowns γ_{m-i} can be determined recursively for $i = 1, ..., m$, and thus the problem is reduced to the $m \times m$ case.

□

Recall $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ is the complex unit disc, $\varGamma = \{z \in \mathbb{C} : |z| = 1\}$ is its boundary, and $\mathbb{S}^1$ is the set of unit directions.

Proposition A.1. *Let* $\mathbf{f} \in L^1(S^m;\Omega)$ *be a symmetric* m*-tensor with integrable components. Assume that each of its components satisfies one of the conditions:*

$$\text{supp } f_{i_1\dots i_m} \subset \{z : |z| \leq \sqrt{1-\delta^2}\},\ 0<\delta<1, \quad \text{or} \quad f_{i_1\dots i_m} \in L^p(\Omega),\ p>2, \tag{A.3}$$

then $X\mathbf{f} \in L^1(\Gamma \times \mathbb{S}^1)$.

Proof. To fix ideas, we consider the even tensor case, when $m = 2l$. The odd tensor case follows similarly.

From the one-to-one linear combination correspondence between the components $f_{i_1\dots i_m}$ of $\mathbf{f}$, and the functions f_{2k}'s, $-l \leq k \leq l$ in the identity (4.2),

$$\langle \mathbf{f}, \boldsymbol{\theta}^m \rangle = \sum_{k=-l}^{l} f_{2k} e^{-\mathrm{i}(2k)\theta},$$

we have that each of the f_{2k}'s also satisfies $f_{2k} \in L^1(\Omega)$ and

$$\text{supp } f_{2k} \subset \{z : |z| \leq \sqrt{1-\delta^2}\},\ 0<\delta<1, \quad \text{or} \quad f_{2k} \in L^p(\Omega),\ p>2. \tag{A.4}$$

For $e^{i\beta} \in \varGamma$ and $e^{i\theta} \in \mathbb{S}^1$, the X-ray transform of $\mathbf{f}$ (with components extended by 0 outside Ω) is given by

$$X\mathbf{f}(e^{i\beta}, e^{i\theta}) = \sum_{k=-l}^{l} \int_{-\infty}^{\infty} f_{2k}(e^{i\beta} + te^{i\theta}) e^{-i2k\theta} dt.$$

We will show that each term in the sum above is integrable on the torus. To simplify notation, we drop the index notation from the function and show that if f satisfies (A.4), then

$$X_k f(e^{i\beta}, e^{i\theta}) := \int_{-\infty}^{\infty} f(e^{i\beta} + te^{i\theta}) e^{-i2k\theta} dt \text{ lies in } L^1(\varGamma \times \mathbb{S}^1). \tag{A.5}$$

Rewrite $X_k f(e^{i\beta}, e^{i\theta}) := \displaystyle\int_{-\infty}^{\infty} f_\theta(e^{i(\beta-\theta)} + t) e^{-i2k\theta} dt$, where $f_\theta(z) := f(ze^{i\theta})$ is obtained from f by a rotation of the domain by angle θ. Note that f_θ preserves the L^p-norm of f for any $p \geq 1$.

We estimate

$$\begin{aligned}
\|X_k f\|_{L^1(\Gamma\times\mathbb{S}^1)} &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |X_k f(e^{i\beta}, e^{i\theta})| d\beta d\theta \\
&\leq \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} |f_\theta(e^{i(\beta-\theta)} + t)| dt d\beta d\theta \\
&\overset{\alpha=\beta-\theta}{=\!=} 2 \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} |f_\theta(e^{i\alpha} + t)| dt d\alpha d\theta \\
&\overset{s=\sin\alpha}{=\!=} \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-1}^{1} \frac{|f_\theta(\sqrt{1-s^2} + t + is)|}{\sqrt{1-s^2}} ds dt d\theta \\
&\overset{u=t+\sqrt{1-s^2}}{=\!=} \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-1}^{1} \frac{|f_\theta(u+is)|}{\sqrt{1-s^2}} ds du d\theta \\
&= \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-1}^{1} \int_{-1}^{1} \frac{|f_\theta(u+is)|}{\sqrt{1-s^2}} ds du d\theta,
\end{aligned} \tag{A.6}$$

where the last equality uses $\text{supp} f_\theta \subset \Omega$ for any $\theta \in (-\pi, \pi]$.

On the one hand, if supp$f \subset \{z : |z| \leq \sqrt{1-\delta^2}\}$, then

$$\|X_k f\|_{L^1(\Gamma\times\mathbb{S}^1)} \leq \frac{1}{2\pi^2}\int_{-\pi}^{\pi}\int_{-1}^{1}\int_{-\sqrt{1-\delta^2}}^{\sqrt{1-\delta^2}} \frac{|f_\theta(u+is)|}{\sqrt{1-s^2}} ds du d\theta$$
$$\leq \frac{1}{2\pi^2}\frac{1}{\delta}\int_{-\pi}^{\pi} \|f_\theta\|_{L^1(\Omega)} d\theta = \frac{1}{\pi\delta}\|f\|_{L^1(\Omega)}.$$

On the other hand, if $f \in L^p(\Omega)$, $p > 2$, let $T := (-1,1)\times(-1,1)$ denote the unit square. Since $\Omega \subset T$, for every $\theta \in (-\pi, \pi]$, $f_\theta \in L^p(T)$, and

$$\|f_\theta\|_{L^p(T)} = \|f\|_{L^p(\Omega)}. \tag{A.7}$$

Let $q = \frac{p}{p-1}$ be the conjugate index of p. If $p > 2$, then $q < 2$ and the map

$$T \ni (u,s) \mapsto \frac{1}{\sqrt{1-s^2}} \text{ lies in } L^q(T).$$

Moreover, since it is constant in one coordinate, $\left\|\frac{1}{\sqrt{1-(\cdot)^2}}\right\|_{L^q(T)} = 2\left\|\frac{1}{\sqrt{1-(\cdot)^2}}\right\|_{L^q(-1,1)}$.

An application of the Hölder's inequality in (A.6) and (A.7) yield

$$\|X_k f\|_{L^1(\Gamma\times\mathbb{S}^1)} \leq \frac{1}{\pi^2}\int_{-\pi}^{\pi}\left\|\frac{1}{\sqrt{1-(\cdot)^2}}\right\|_{L^q(T)}\|f_\theta\|_{L^p(T)} d\theta = \frac{2}{\pi}\left\|\frac{1}{\sqrt{1-(\cdot)^2}}\right\|_{L^q(-1,1)}\|f\|_{L^p(\Omega)}.$$

□

Lemma A.2. *Let $m \geq 0$ be an integer. Let g be an integrable function on the torus satisfying the symmetry relation*

$$g(e^{i\beta}, e^{i\theta}) = (-1)^m g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}), \text{ for } (e^{i\beta}, e^{i\theta}) \in \mathit{\Gamma}\times\mathbb{S}^1, \tag{A.8}$$

and $g_{n,k}$'s be its Fourier coefficients. Then

$$g_{n,k} = (-1)^m(-1)^{n+k} g_{n+2k,-k}, \text{ for all } n,k \in \mathbb{Z}. \tag{A.9}$$

Proof. Indeed,

$$\begin{aligned}
g_{n,k} &= \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(e^{i\beta}, e^{i\theta}) e^{-in\theta} e^{-ik\beta} d\theta d\beta \\
&\overset{\text{(A.8)}}{=\!=} (-1)^m \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}) e^{-in\theta} e^{-ik\beta} d\theta d\beta \\
&\overset{\gamma=\theta+\pi}{=\!=} (-1)^m(-1)^n \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{0}^{2\pi} g(e^{i(2\gamma-\beta-3\pi)}, e^{i\gamma}) e^{-in\gamma} e^{-ik\beta} d\gamma d\beta \\
&\overset{\alpha=2\gamma-\beta-\pi}{=\!=} (-1)^m(-1)^n \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{2\gamma}^{2\gamma-2\pi} g(e^{i\alpha}, e^{i\gamma}) e^{-ik(2\gamma-\alpha-\pi)} e^{-in\gamma}(-d\alpha) d\gamma \\
&= (-1)^m(-1)^{n+k} \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{2\gamma-2\pi}^{2\gamma} g(e^{i\alpha}, e^{i\gamma}) e^{-i(n+2k)\gamma} e^{ik\alpha} d\alpha d\gamma \\
&= (-1)^m(-1)^{n+k} g_{n+2k,-k}.
\end{aligned}$$

□

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