The orbit-sum method for higher order equations

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THE ORBIT-SUM METHOD FOR HIGHER ORDER EQUATIONS

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Abstract. The orbit-sum method is an algebraic version of the reflection-principle that was introduced by Bousquet-Mélou and Mishna to solve functional equations that arise in the enumeration of lattice walks with small steps restricted to $\mathbb{N}^2$. Its extension to walks with large steps was started by Bostan, Bousquet-Mélou and Melczer. We continue it here, making use of the primitive element theorem, Gröbner bases and the shape lemma, and the Newton-Puiseux algorithm.

1. Introduction

Many generating functions can be described as solutions of certain functional equations. One important type of such functional equations is the class of discrete differential equations (DDE's). They arise in the context of the enumeration of lattice walks restricted to cones, a systematic study of them was initiated in [14, 7].

Discrete differential equations are equations of the form

$$F = P(x, y) + tQ(x, y, t, \Delta_x^k \Delta_y^l F : k, l \in \mathbb{N}) \tag{1}$$

where $F \in \mathbb{Q}[x, y][[t]]$ is unknown, and $P \in \mathbb{Q}[x, y]$ and $Q \in \mathbb{Q}[x, y, t, v_{kl} : k, l \in \mathbb{N}]$ are given polynomials. The operator Δ_x is the discrete derivative with respect to x, which acts on $\mathbb{Q}[x, y][[t]]$ by

$$F(x, y; t) \mapsto \frac{F(x, y; t) - F(0, y; t)}{x}.$$

The operator Δ_y, the discrete derivative with respect to y, is defined analogously. A DDE is a partial discrete differential equation (PDDE) if it involves discrete derivatives of F with respect to both x and y, and an ordinary discrete differential equations (ODDE) otherwise. The degree of the equation is the total degree of P with respect to the v_{kl}'s. If its degree is at most 1, it is a linear DDE, otherwise it is non-linear. If $k + l$ is maximal among the discrete derivatives $\Delta_x^k \Delta_y^l F$ appearing in equation (1), then $k + l$ is the order of the equation.

A DDE has a unique solution $F \in \mathbb{Q}[x, y][[t]]$ as can be seen from the recurrence relation for $[t^n]F$ that results from extracting the coefficient of t^n from equation (1). To solve a DDE means to decide whether its solution is algebraic, D-finite or D-algebraic, and in case it is, to determine a polynomial or differential equation satisfied by it.

There is a family of methods for solving DDE's [6, 3, 9, 7, 4, 2, 5, 15, 13, 16] that involve only the operations

$$+,\ \cdot,\ \circ \quad \text{and} \quad [x^>] \text{ and } [y^>],$$

that is, the addition, multiplication and composition of series, and the operation of discarding all terms of a series which involve non-positive powers in x and y, respectively. The orbit-sum method [7, 2] is one of them, used to solve linear DDE's. It proceeds in three steps. In the first step, a set of substitutions, the so-called orbit, is determined that can be applied to the given functional equation. In the second step, a linear combination of the various transformed versions of the functional equation is formed to the end of eliminating all the evaluations of $F(x, 0)$ and $F(0, y)$. The resulting equation then only contains the unknown series F and various series obtained from it by substituting the elements of the orbit. In the third step of the method, by means of coefficient extraction, an expression for the unknown series F is obtained.

Example 1. *We solve the equation*

$$F = 1 + t(x + y)F + t\Delta_x F + t\Delta_y F \tag{2}$$

for $F \in \mathbb{Q}[x, y][[t]]$. It is equivalent to

$$xy(1 - tS)F(x, y) = xy - txF(x, 0) - tyF(0, y), \tag{3}$$

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where $S := x + y + \bar{x} + \bar{y}$*, and* $\bar{x} := 1/x$ *and* $\bar{y} := 1/y$*, and* $F(x,y) \equiv F(x,y;t)$*. We exploit the symmetry of* S *and the fact that the unknowns on the right of the equation either do not depend on* x *or on* y*. By iteratively performing the substitutions* $x \mapsto \bar{x}$ *and* $y \mapsto \bar{y}$ *that leave* S *invariant we can derive three additional equations,*

$$\begin{aligned}\bar{x}y(1-tS)F(\bar{x},y) &= \bar{x}y - t\bar{x}F(\bar{x};0) - tyF(0,y),\\ \bar{x}\bar{y}(1-tS)F(\bar{x},\bar{y}) &= \bar{x}\bar{y} - t\bar{x}F(\bar{x};0) - t\bar{y}F(0,\bar{y}),\\ x\bar{y}(1-tS)F(x,\bar{y}) &= x\bar{y} - txF(x;0) - t\bar{y}F(0,\bar{y}),\end{aligned}$$

which, together with equation (3), can be linearly combined to

$$xyF(x,y) - \bar{x}yF(\bar{x},y) + \bar{x}\bar{y}F(\bar{x},\bar{y}) - x\bar{y}F(x,\bar{y}) = \frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1-tS}.$$

Since $xyF(x,y)$ *involves only positive powers of* x *and* y*, and because all the other terms on the left-hand side of this equation in* $\mathbb{Q}[x,y,\bar{x},\bar{y}][[t]]$ *involve a negative power of* x *or a negative power in* y*, we find that*

$$xyF(x,y) = [x^{>}y^{>}]\frac{xy - \bar{x}y + \bar{x}\bar{y} - x\bar{y}}{1-tS}. \tag{4}$$

Consequently, F *is D-finite, as rational functions are D-finite and the class of D-finite functions is closed under applying* $[x^{>}y^{>}]$ *[11].*

For linear partial discrete differential equations of higher order an algorithm for determining the orbit was presented in [2, Sec. 3]. The substitutions determined by this algorithm are algebraic functions given by their minimal polynomials. Algebraic functions cause difficulties in the second and third step of the orbit sum method. In the present paper, we discuss these difficulties. In order to carry out the second step algorithmically (Sect. 2), we need to construct an algebraic function field that contains all the algebraic functions appearing in the orbit. This step can be done on the level of "formal" algebraic extensions. The third step however crucially depends on series interpretations of the algebraic functions, so in order to carry it out algorithmically (Sect. 3), we will need to embed the algebraic function field into suitably chosen fields of series. The question is then whether for the equation at hand there exists an embedding that allows the orbit sum method to conclude. To answer this question, we offer a sufficient and algorithmic condition.

2. Orbits, Orbit Equations, and the Orbit-Sum

The substitutions we used to solve equation (2) had the following property: for every substitution (x', y'), there were other substitutions (x'', y'') and (x''', y''') such that $x' = x''$ and $S(x', y') = S(x'', y'')$, and $y' = y'''$ and $S(x', y') = S(x''', y''')$. They allowed us to modify equation (2) without altering $S(x,y)$, and without altering one of the unknown evaluations of $F(x,0)$ and $F(0,y)$. These observations are captured by the definition of the orbit of a polynomial [2, Definition 1].

Definition 1. *Given* $p \in \mathbb{Q}[x,y,\bar{x},\bar{y}]$*, let* $\sim$ *be the relation on* $\overline{\mathbb{Q}(x,y)}^2$ *defined by*

$$(u_1,u_2) \sim (v_1,v_2) \quad :\Longleftrightarrow \quad u_1 = v_1 \text{ or } u_2 = v_2, \text{ and } p(u_1,u_2) = p(v_1,v_2),$$

and let $\approx$ *be the equivalence relation resulting from taking its transitive closure. The orbit of* p *is the set of elements of* $\overline{\mathbb{Q}(x,y)}^2$ *which are equivalent to* (x,y)*.*

The elements of an orbit can be represented by their minimal polynomials. A (semi-) algorithm that determines them was presented in [2, Section 3.2]. It takes as input a Laurent polynomial and outputs, if the orbit is finite, the minimal polynomials of its elements. If the orbit is finite, then the splitting field of the minimal polynomials (of the components) of its elements is a finite field extension of $\mathbb{Q}(x,y)$. Using a constructive version of the primitive element theorem, we can do computations in this field.

Theorem 1. *(Primitive Element Theorem) Let* K *be a field of characteristic* 0*, and let* L/K *be a finite field extension. Then there is an* $\alpha \in L$ *such that* $L = K(\alpha)$*. If* $m(X) \in K[X]$ *is the minimal polynomial of* α*, then*

$$L \cong K[X]/\langle m(X)\rangle.$$

Given the minimal polynomial $m(X)$ of a primitive element α of the splitting field of a set of polynomials $m_1(X), \ldots, m_n(X)$ over $\mathbb{Q}(x,y)$, computations just amount to polynomial arithmetic in $\mathbb{Q}(x,y)[X]/\langle m(X)\rangle$, that is, adding and multiplying polynomials over $\mathbb{Q}(x,y)$, performing division with remainder and computing modular inverses using the extended Euclidean algorithm. It remains to clarify how the minimal polynomial of a primitive element can be found, and how elements of the splitting field can be expressed in terms of the primitive element. Gröbner bases and the shape lemma [10, Theorem 3.7.25] provide an answer.

Definition 2. *Let $I \subseteq K[x_1, \dots, x_n]$ be a zero-dimensional ideal. It is said to be in normal x_i-position, $i \in \{1, \dots, n\}$, if any two zeros $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ of I in $\overline{K}^n$ satisfy $a_i \neq b_i$.*

Theorem 2. *(Shape Lemma) Let K be a field of characteristic 0, and let $I \subseteq K[x_1, \dots, x_n]$ be a 0-dimensional radical ideal in normal x_n-position. Then I has a Gröbner basis with respect to lex order which is of the form*

$$\{x_1 - g_1, \dots, x_{n-1} - g_{n-1}, g_n\}$$

for some $g_1, \dots, g_n \in K[x_n]$. In particular, the set $\mathrm{Z}(I)$ of zeros of I is

$$\mathrm{Z}(I) = \{(g_1(a), \dots, g_{n-1}(a), a) \in K^n : g_n(a) = 0\}.$$

Assume that $m_1(X), \dots, m_n(X) \in \mathbb{Q}(x,y)[X]$ are irreducible and pairwise distinct, and let I be the ideal generated by $m_i(X_{ij})$ and $1 - Y \prod_{ij \neq kl}(X_{ij} - X_{kl})$ and $Z - \sum_{ij} a_{ij} X_{ij}$, where the X_{ij}'s and Y and Z are variables and $a_{ij} \in \mathbb{Q}$, for $i = 1, \dots, n$ and $j = 1, \dots, \deg_X(m_i)$. It is no restriction to assume that all the assumptions of the shape lemma are satisfied as we can choose the a_{ij}'s such that I is in normal Z-position [10, Definition 3.7.21] and replace I by its radical $\sqrt{I}$ without altering the set of its zeros [10, Corollary 3.7.16]. The shape lemma implies that the Gröbner basis of I gives rise to a polynomial $m(X) \in \mathbb{Q}(x,y)[X]$ whose roots α are primitive elements of the splitting field of $\{m_1(X), \dots, m_n(X)\}$ over $\mathbb{Q}(x,y)$ and polynomials $p_{ij}(X) \in \mathbb{Q}(x,y)[X]$ such that the roots of $m_1(X), \dots, m_n(X)$ are given by the $p_{ij}(\alpha)$'s.

Let $Q \in \mathbb{Q}[x,y]$ and $P_{kl} \in \mathbb{Q}[x,y]$ be polynomials, and let

$$F = Q(x,y) + t \sum_{k,l} P_{kl}(x,y) \Delta_x^k \Delta_y^l F \tag{5}$$

be a linear discrete differential equation for $F \in \mathbb{Q}[x,y][[t]]$. The kernel polynomial of the equation is the Laurent polynomial that appears as the coefficient of $F(x,y)$ when all the terms involving it are collected on the left hand side of the equation. The orbit of the equation is the orbit of its kernel polynomial. If it is finite, we can now assume that there is some $\alpha \in \overline{\mathbb{Q}(x,y)}$ such that its elements are of the form $(p_1(\alpha), p_2(\alpha))$ and given in terms of $p_1(X), p_2(X) \in \mathbb{Q}(x,y)[X]$ and the minimal polynomial $m(X) \in \mathbb{Q}(x,y)[X]$ of α. The orbit equations result from replacing (x,y) in equation (5) by the elements of the orbit, and an orbit-sum is any $\mathbb{Q}(x,y)[\alpha]$-linear combination of the orbit equations that does not involve any of the sections $F(\cdot, 0)$ and $F(0, \cdot)$. Computing a basis of the vector space of such equations amounts to making an ansatz with undetermined coefficients for the linear combination, setting the coefficients of the sections equal to zero, and solving a system of linear equations over the field $\mathbb{Q}(x,y)[X]/\langle m(X) \rangle$.

3. Positive-Part-Extraction

In the previous section we recalled how the minimal polynomials of the algebraic substitutions required by the orbit-sum method are determined and explained how Gröbner bases and the shape lemma allow to reduce computations in their splitting field to polynomial arithmetic. As a consequence the first two steps of the orbit-sum method can be performed algorithmically, the result of the computations being a basis of the vector space of section-free orbit equations whose elements are of the form

$$F(x,y) + \sum_{(p_1,p_2,p_3)} p_3(\alpha) F(p_1(\alpha), p_2(\alpha)) = p(\alpha), \tag{6}$$

where $F \in \mathbb{Q}[x,y][[t]]$ is unknown, α is an element of $\overline{\mathbb{Q}(x,y)}$, given by its minimal polynomial over $\mathbb{Q}[x,y]$, and $p_1(\alpha), p_2(\alpha)$ and $p_3(\alpha)$ are polynomials in α over $\mathbb{Q}(x,y)$, and $p(\alpha)$ is a polynomial in α over $\mathbb{Q}(x,y,t)$. The purpose of this section is to give a meaning to

$$[x^{\geq} y^{\geq}] p_3(\alpha) F(p_1(\alpha), p_2(\alpha)), \tag{7}$$

and to present a sufficient and effective condition for equation (6) to imply that

$$F(x,y) = [x^{\geq} y^{\geq}] p(\alpha).$$

This requires to interpret elements of $\overline{\mathbb{Q}(x,y)}$ as series in x and y. The positive part is then the series which results from discarding all terms which involve a non-positive power of x or y, respectively. We did not stress this point in Example 1 because the right hand side of equation (4) can unambiguously be understood as an element of $\mathbb{Q}[x,y,\bar{x},\bar{y}][[t]]$ whose positive part with respect to x and y is well-defined. In general, however, more care is necessary.

Example 2. *It is ambiguous to speak of the positive part of the series solution Y of*

$$(1-x)Y - 1 = 0.$$

The solution of the equation depends on the field of Laurent series over which it is solved. While in $\mathbb{Q}((x))$ it is $Y = \sum_{k=0}^{\infty} x^k$, in $\mathbb{Q}((\bar{x}))$ it is $Y = -\sum_{k=1}^{\infty} \bar{x}^k$, and depending on which of them we choose, we have $[x^>]Y = \sum_{k=1}^{\infty} x^k$ or $[x^>]Y = 0$.

To give a meaning to expression (7) we embed the splitting field $\mathbb{Q}(x,y)[X]/\langle m(X)\rangle$ into a field of Puiseux series and derive information about the support of the series that correspond to α and $p_1(\alpha)$, $p_2(\alpha)$ and $p_3(\alpha)$. Our reasoning is based on [1], an exposition of a theory of Laurent series in several variables, and on [12, 8], a discussion of a (generalized) Newton-Puiseux algorithm. For details, in particular for proofs, we refer to these references.

Definition 3. *A subset C of $\mathbb{R}^n$ is called a cone if $\lambda C = C$ for every $\lambda \in \mathbb{R}_{\geq 0}$. It is called a polyhedral cone if there are $v_1, \ldots, v_k \in \mathbb{R}^n$ such that $C = \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_k$, and rational if $v_1, \ldots, v_k$ can be chosen to be elements of $\mathbb{Q}^n$. A cone C is called convex if $\lambda v + (1-\lambda) w \in C$ for all $v, w \in C$ and all $\lambda \in [0,1]$, and strictly convex if, in addition, $C \cap (-C) = \{0\}$. The dual C^* of C is $C^* = \{u \in \mathbb{R}^n \mid \langle u, C\rangle \leq 0\}$.*

Let $\preceq$ be an additive total order on $\mathbb{Q}^2$ and denote by $\mathbb{C}_{\preceq}((x,y))$ the set of series

$$\phi = \sum_{(i,j)\in\mathbb{Q}^2} a_{ij} x^i y^j$$

such that

$$\operatorname{supp}(\phi) \subseteq (v + C) \cap \frac{1}{k}\mathbb{Z}^2$$

for some $v \in \mathbb{R}^2$, some strictly convex rational cone $C \subseteq \mathbb{R}^2$ which has a maximal element with respect to $\preceq$, and some positive integer $k \in \mathbb{Z}$. The proof of [1, Theorem 15] shows that $\mathbb{C}_{\preceq}((x,y))$ is a field, and by [12] it is algebraically closed.

Any $w \in \mathbb{R}^2$ whose components are linearly independent over $\mathbb{Q}$ defines an additive total order $\preceq$ on $\mathbb{Q}^2$ by

$$\alpha \preceq \beta \quad :\Longleftrightarrow \quad \langle \alpha, w\rangle \leq \langle \beta, w\rangle.$$

In case it exists, the maximal element of a rational polyhedral set $P \subseteq \mathbb{R}^2$ with respect to any such total order is a vertex of P. The next theorem [17, Theorem 4] implies that this is also true for any other additive total order. We will therefore restrict ourselves to total orders induced by elements of $\mathbb{R}^2$ whose components are linearly independent over $\mathbb{Q}$.

Definition 4. *Let $w \in \mathbb{R}^n$. The rational dimension of w, denoted by $\mathrm{d}(w)$, is the dimension of the $\mathbb{Q}$-vector space generated by the components of w.*

Theorem 3. *For any additive total order $\preceq$ on $\mathbb{Q}^n$, there exist non-zero pairwise orthogonal vectors $u_1, \ldots, u_s \in \mathbb{R}^n$ such that $\mathrm{d}(u_1) + \cdots + \mathrm{d}(u_s) = n$ and*

$$\iota : (\mathbb{Q}^n, \preceq) \to (\mathbb{R}^n, \preceq_{lex}) \quad \textit{defined by} \quad \iota(v) = (v \cdot u_1, \ldots, v \cdot u_s)$$

is an injective order homomorphism.

Having chosen a total order $\preceq$ on $\mathbb{Q}^2$ we can identify the field of rational functions $\mathbb{C}(x,y)$ with a subfield of $\mathbb{C}_{\preceq}((x,y))$: the series in $\mathbb{C}_{\preceq}((x,y))$ associated with a rational function $p/q \in \mathbb{C}(x,y)$ is

$$\frac{p}{\mathrm{lt}_{\preceq}(q)} \sum_{k\geq 0} \left(1 - \frac{q}{\mathrm{lt}_{\preceq}(q)}\right)^k.$$

Note that this series depends on the total order $\preceq$ only to the extent of what the leading term $\mathrm{lt}_{\preceq}(q)$ of q with respect to it is. Viewing $\mathbb{C}(x,y)$ as a subfield of $\mathbb{C}_{\preceq}((x,y))$, any series root $\phi \in \mathbb{C}_{\preceq}((x,y))$ of $m(X)$ induces an embedding

$$p(X) + \langle m(X)\rangle \quad \mapsto \quad p(\phi)$$

of $\mathbb{Q}(x,y)[X]/\langle m(X)\rangle$ into $\mathbb{C}_{\preceq}((x,y))$. This embedding allows us to study equation (6) in the form

$$F(x,y) + \sum_{(p_1,p_2,p_3)} p_3(\phi) F(p_1(\phi), p_2(\phi)) = p(\phi), \tag{8}$$

which involves only series to which $[x^{\geq} y^{\geq}]$ can be applied. The question how such series roots, and hence such embeddings, can be constructed is answered by the Newton-Puiseux algorithm. We only state the specification of the algorithm here, for a detailed discussion we refer to [12, 8].

Algorithm 1 (Newton-Puiseux Algorithm)**.** *Input: A square-free and non-constant polynomial* $p \in \mathbb{Q}[\mathbf{x}, y]$*, an element* $w \in \mathbb{R}^n$ *inducing a total order on* $\mathbb{Q}^n$*, and an integer* k*.*
Output: A list of $\deg_y(p)$ *many pairs* $(c_1\mathbf{x}^{\alpha_1} + \cdots + c_N\mathbf{x}^{\alpha_N}, C)$ *with* $c_1\mathbf{x}^{\alpha_1}, \ldots, c_N\mathbf{x}^{\alpha_N}$ *being the first* N *terms of a series solution* $\phi \in \mathbb{C}_{\preceq}((x, y))$ *of* $p(\mathbf{x}, \phi) = 0$*, ordered with respect to* w*, and* C *being a strictly convex rational cone such that* $\operatorname{supp}(\phi) \subseteq \{\alpha_1, \ldots, \alpha_{N-1}\} \cup (\alpha_N + C)$*, where* $N \geq k$ *is minimal such that the series solutions can be distinguished by their first* N *terms.*

The Newton-Puiseux algorithm is not only useful for constructing series solutions of polynomial equations but also for encoding these series by a finite amount of data, performing effective arithmetic on the level of these encodings, and deriving information about the convex hull of their supports [8, Sec. 5]. Important for us is that for $p_i(\phi)$ in equation (8) we can compute the (finitely many) vertices of the convex hull of its support, and that for each of these vertices v we can determine a cone C such that $\operatorname{supp}(p_i(\phi)) \subseteq v + C$. Though these cones are strictly convex and rational, it is an open problem [8, Problem 1] how to find the vertex cones, i.e. the cones that are minimal. Another open problem is whether for any polynomial $p \in \mathbb{C}[\mathbf{x}, y]$ there are only finitely many series solutions ϕ of $p(\mathbf{x}, \phi) = 0$ and to which extent they depend on the total order given to the Newton-Puiseux algorithm.

It remains to clarify how to derive an estimate for the support of $F(p_1(\phi), p_2(\phi))$. The following theorem [1, Theorem 17] gives a sufficient condition for the composition of Puiseux series to be well-defined, and in case it is, it provides a cone that contains its support.

Theorem 4. *Let* $C \subseteq \mathbb{R}^n$ *be a strictly convex cone and* $F(x_1, \ldots, x_n)$ *a series such that* $\operatorname{supp}(F) \subseteq C$*, let* $\preceq$ *be an additive order on* $\mathbb{Z}^m$ *and* $g_1, \ldots, g_n \in \mathbb{C}_{\preceq}((y_1, \ldots, y_m)) \setminus \{0\}$*. Furthermore, let* $M \in \mathbb{Z}^{m\times n}$ *be the matrix whose i-th column consists of the leading exponent of* $g_i(y_1, \ldots, y_m)$ *with respect to* $\preceq$*, and let* C' *be a cone that contains the image of* C *under* M *and* $\operatorname{supp}(g_i/\operatorname{lt}(g_i))$ *for* $i = 1, \ldots, n$*. If* $C \cap \ker(M) = \{0\}$ *and if* C' *is strictly convex, then* $F(g_1, \ldots, g_n)$ *is well-defined and* $\operatorname{supp}(F(g_1, \ldots, g_n)) \subseteq C'$*.*

We are interested in applying Theorem 4 when $F \in \mathbb{C}[x, y][[t]]$ satisfies $[t^0]F = 1$, and $g_1, g_2 \in \mathbb{C}_{\preceq}((x, y))$ and $g_3 = t$. The next lemma states that in this case the assumptions of the theorem are always fulfilled.

Lemma 1. *Let* C *be a strictly convex cone in* $\mathbb{R}^3$ *such that* $C \cap \big(\mathbb{R}^2 \times \{0\}\big) = \{0\}$*, and let* $F \in \mathbb{C}[x, y][[t]]$ *be such that* $\operatorname{supp}(F) \subseteq C$*. Let* $\preceq$ *be an additive total order on* $\mathbb{Q}^3$ *and* $g_1, g_2 \in \mathbb{C}_{\preceq}((x, y))$*, and let* M *be the matrix whose columns are the leading exponents of* g_1, g_2 *and* t*. Then* $C \cap \ker(M) = \{0\}$*, and the cone generated by* MC *and* $\operatorname{supp}(g_i/\operatorname{lt}(g_i))$ *for* $i \in \{1, 2\}$ *is strictly convex.*

Proof. The series g_1 and g_2 do not depend on t, therefore $\ker(M) \subseteq \mathbb{R}^2 \times \{0\}$, and so $C \cap \ker(M) = \{0\}$, by assumption on C. Since g_1 and g_2 are elements of $\mathbb{C}_{\preceq}((x, y))$, the cone generated by the support of $g_1/\operatorname{lt}(g_1)$ and $g_2/\operatorname{lt}(g_2)$ is strictly convex, and because g_1 and g_2 are independent of t, it is contained in $\mathbb{R}^2 \times \{0\}$. The shape of M implies $MC \cap \big(\mathbb{R}^2 \times \{0\}\big) = M\left(C \cap \big(\mathbb{R}^2 \times \{0\}\big)\right) = \{0\}$. To finish the proof of the lemma, it is therefore sufficient to show that MC is strictly convex. Assume that there is a $v \neq 0$ such that $v \in MC$ and $-v \in MC$. Then there are $u_1, u_2 \in C$ such that $Mu_1 = v$ and $Mu_2 = -v$. But then $M(u_1 + u_2) = 0$, i.e. $u_1 + u_2 \in \ker(M)$. Together with $u_1 + u_2 \in C$ and $C \cap \ker(M) = \{0\}$ this implies that $u_1 + u_2 = 0$. Since C is strictly convex, $u_1 = 0 = u_2$, and therefore $v = 0$. So MC is strictly convex as well. □

To summarize, we can construct series roots ϕ of $m(X)$ to embed $\mathbb{C}(x, y)[X]/\langle m(X)\rangle$ into fields $\mathbb{C}_{\preceq}((x, y))$ of Puiseux series, for each such embedding we can determine the vertices of the convex hull of the support of $p_i(\phi)$, and for each of these vertices v we can compute a strictly convex rational cone C_v such that $\operatorname{supp}(p_i(\phi)) \subseteq v + C_v$, and finally, we are able to find a strictly convex cone C such that $\operatorname{supp}(F(p_1(\phi), p_2(\phi); t)) \subseteq C$. The support of $p_3(\phi)F(p_1(\phi), p_2(\phi); t)$ is then contained in $v + C_v + C$. If $\big(\mathbb{Q}^2_{\geq 0} \times \mathbb{Q}\big) \cap (v + C_v + C) = \emptyset$, then $[x^{\geq}y^{\geq}]p_3(\phi)F(p_1(\phi), p_2(\phi); t) = 0$.

Example 3. *We solve the system of discrete differential equations*

$$\begin{aligned} F_0 &= 1 + tF_1 + t\Delta_x\Delta_y F_1 \\ F_1 &= t(1 + x + y)F_0 + ty\Delta_x F_0 \end{aligned} \tag{9}$$

for $F_0, F_1 \in \mathbb{Q}[x, y][[t]]$ *and show that their solution is D-finite. We begin with eliminating* $F_1(x, y; t)$ *from the first of these equations and continue working with*

$$(1 - t^2 S_0 S_1)F_0 = 1 - t\bar{x}\bar{y}(F_1(x, 0) + F_1(0, y) - F_1(0, 0)) - t^2(\bar{x}\bar{y} + 1)\bar{x}yF_0(0, y), \tag{10}$$

where

$$S_0 := \bar{x}y + y + x + 1 \quad \textit{and} \quad S_1 := \bar{x}\bar{y} + 1.$$

The Laurent polynomial S_0S_1 has a finite orbit. Its elements are (x,y), $(x,\bar{y})$, $(p_1(\alpha),y)$ and $(p_{-1}(\alpha),y)$, and $(p_1(\alpha),\bar{y})$ and $(p_{-1}(\alpha),\bar{y})$, where

$$p_i(X) = \frac{x+y+xy+xy^2+iX}{2x^2y} \quad \text{and} \quad \alpha = \sqrt{4x^3y^2+(x+y+xy+xy^2)^2}.$$

We consider their components as elements of the extension of $\mathbb{C}(x,y)$ by a root α of

$$m(X) = X^2 - 4x^3y^2 - (x+y+xy+xy^2)^2.$$

Plugging the elements of the orbit into equation (10), forming a linear combination of the resulting equations with undetermined coefficients, and equating the coefficients of the sections of F_0 and F_1 to zero results in a linear system over $\mathbb{C}(x,y)[\alpha]$. The vector space of solutions is 1-dimensional, and so is the vector space of section-free orbit equations. The latter is generated by the equation

$$F_0(x,y) - \bar{y}^2F_0(x,\bar{y}) - \sum_{i,j=\pm 1} c_{ij}(\alpha)F_0(p_i(\alpha),y^j) = \frac{(-1+y^2)(2y-x^3y+x(1+y+y^2))}{x^3y^3(1-t^2S_0S_1)}.$$

The coefficients $c_{ij}(\alpha)$ in $\mathbb{C}(x,y)[\alpha]$ are

$$c_{ij}(\alpha) = i\frac{x+2y+xy+xy^2}{2x^3y} + j\frac{\alpha(2y^2+2x^3y^2+3xy(1+y+y^2)+x^2(1+y+y^2)^2)}{2x^3y(y^2+4x^3y^2+2xy(1+y+y^2)+x^2(1+y+y^2)^2)}.$$

Let $\preceq$ be the total order on $\mathbb{Q}^2$ defined by $w=(\sqrt{2},1/2)$ and let ϕ be the series solution of $m(X)=0$ in $\mathbb{C}_{\preceq}((x,y))$ whose first term is $2x^{3/2}y$. We identify $p_i(\alpha)$ and $c_{ij}(\alpha)$ with $p_i(\phi)$ and $c_{ij}(\phi)$ in $\mathbb{C}_{\preceq}((x,y))$ and show that the only term on the left hand side of the equation that remains when applying $[x^{\geq}y^{\geq}]$ is $F_0(x,y)$. Consequently, F_0 is the non-negative part of a rational function, and therefore D-finite, and so is F_1 by the second of the equations in (9). Obviously, $[x^{\geq}y^{\geq}]F_0(x,y) = F_0(x,y)$ and $[x^{\geq}y^{\geq}]\bar{y}^2F_0(x,\bar{y}) = 0$. Using the Newton-Puiseux algorithm, one can show that

$$\operatorname{supp}(p_i(\phi)) \subseteq (-1/2,0,0) + \langle(-1,2,0),(-1,-2,0)\rangle$$

and

$$\operatorname{supp}(c_{ij}(\phi)) \subseteq (-3/2,-1+j,0) + \langle(-1,2,0),(-1,-2,0)\rangle.$$

Theorem 4 then implies that

$$\operatorname{supp}(F_0(p_i(\phi),y^j)) \subseteq \langle(0,0,1),(0,j,1),(-1,2,0),(-1,-2,0)\rangle.$$

Therefore,

$$\operatorname{supp}(c_{ij}(\phi)F_0(p_i(\phi),y^j)) \subseteq (-3/2,-1+j,0) + \langle(0,0,1),(0,j,1),(-1,2,0),(-1,-2,0)\rangle,$$

and so

$$[x^{\geq}y^{\geq}]c_{ij}(\phi)F_0(p_i(\phi),y^j) = 0.$$

We saw before that for every rational function $p/q \in \mathbb{C}((x,y))$ there are only finitely many ways to consider it as a series, although there are infinitely many different fields $\mathbb{C}_{\preceq}((x,y))$ of Puiseux series such a series is an element of. As a consequence we will not work with a single total order but families of them, and we will describe them by convex cones. A convex cone $C \subseteq \mathbb{R}^2$ will encode the family of total orders induced by elements of the dual C^* whose components are independent over $\mathbb{Q}$. For instance, the cone $C := \langle(1,0),(0,1)\rangle \subseteq \mathbb{R}^2$ represents the family of total orders $\preceq$ on $\mathbb{Q}^2$ with respect to which $1/(1-x-y) \in \mathbb{C}_{\preceq}((x,y))$ is given by $\sum_{k,l\geq 0}\binom{k}{k-l}x^ky^{k-l}$. Equivalently, C is the smallest convex cone such that for any total order $\preceq$ defined by an element of C^* the support of $\sum_{k,l\geq 0}\binom{k}{k-l}x^ky^{k-l}$ has a maximal element with respect to it. We say that C is the order cone of the series.

We give a sufficient and effective condition for the application of $[x^{\geq}y^{\geq}]$ to the orbit-sum (6) to result in an expression of F as the non-negative part of an algebraic function.

Algorithm 2. *Input: An irreducible polynomial $m(X)$ over $\mathbb{Q}(x,y)$, a list L_0 whose elements are tuples $(p_1(X),p_2(X),p_3(X))$ of polynomials over $\mathbb{Q}(x,y)$, and a cone $C_0 \subseteq \mathbb{R}^3$ that contains the support of a series $F \in \mathbb{Q}[x,y][[t]]$ such that $C_0 \cap (\mathbb{R}^2 \times \{0\}) = \{0\}$.*
Output: True or Failed, with the output being True only if there is a series root ϕ of $m(X)$ such that $[x^{\geq}y^{\geq}]p_3(\phi)F(p_1(\phi),p_2(\phi)) = 0$ for all $(p_1(X),p_2(X),p_3(X))$ in L.
1 *For each series root ϕ of $m(X)$, do:*
2 *Compute an estimate C_1 of the order cone of ϕ.*
3 *Determine the maximal list L_1 of minimal cones C such that for every polynomial $p(X)$ which appears as a component of an element of L_0 its series expansion in $\mathbb{C}_{\preceq}(x,y)[X]$ does only depend on the cone C but not on the specific total order induced by an element of C^*.*

4 *For each $C \in L_1$ such that $C + C_1$ is strictly convex, do:*

5 *Choose any total order $\preceq$ on $\mathbb{Q}^2$ induced by some element of $(C + C_1)^*$, and determine for each $p(X)$ which appears as a component of an element of L_0 a list L_p of pairs (v_p, C_{v_p}) such that v_p is a vertex of the convex hull of the support of $p(\phi)$ in $\mathbb{C}_{\preceq}((x,y))$ and C_{v_p} is an estimate of the corresponding vertex cone.*

6 *If for each $(p_1(X), p_2(X), p_3(X))$ in L_0 there are $(v_{p_i}, C_{v_{p_i}})$ in L_{p_i} such that for the cone C' computed from C_0 and $(v_{p_1}, C_{v_{p_1}})$ and $(v_{p_2}, C_{v_{p_2}})$ using Theorem 4 we have*

$$\left(\mathbb{Q}^2_{\geq 0} \times \mathbb{Q}\right) \cap \left(v_{p_3} + C_{v_{p_3}} + C'\right) = \emptyset,$$

then return True.

7 *Return Failed.*

4. Conclusion

We have extended the applicability of the orbit-sum method for linear DDE's of higher order. However, there remain many equations where the method fails, and there are basically two reasons for that. First, there are equations which do not admit a solution by the orbit-sum method, simply because the shape of the equation does not allow the method to conclude. In some cases, for instance, the orbit is not finite and in others the orbit is finite but there is no section-free orbit equation, and again for others the section-free orbit equations only have a zero orbit-sum. Second, there are equations for which the orbit-sum method as presented here fails because we have not addressed some of the problems that can arise. For instance, if there is essentially more than one section-free orbit equation it is not clear which of them should be chosen to extract the non-negative part. It is natural to ask whether it can happen that Algorithm 2 returns Failed although the only term on the left-hand side of the orbit equation that remains when applying $[x^{\geq}y^{\geq}]$ is F. In [2, Proposition 24] it was shown that it can happen that there are two terms $p_3(\alpha)F(p_1(\alpha), p_2(\alpha))$ whose expansions involve terms with non-negative powers in x or y, although their sum does not. The answer to the question whether this is the only reason is certainly no, if the estimates of the support of F and of the order cones and vertex cones are too big.

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