AUSTRIAN ACADEMY OF SCIENCES

RICAM – JOHANN RADON INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

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RICAM-Report 2022-36

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Tractability of L_2-approximation and integration in weighted Hermite spaces of finite smoothness

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September 20, 2022

Abstract

In this paper we consider integration and L_2-approximation for functions over $\mathbb{R}^s$ from weighted Hermite spaces. The first part of the paper is devoted to a comparison of several weighted Hermite spaces that appear in literature, which is interesting on its own. Then we study tractability of the integration and L_2-approximation problem for the introduced Hermite spaces, which describes the growth rate of the information complexity when the error threshold ε tends to 0 and the problem dimension s grows to infinity. Our main results are characterizations of tractability in terms of the involved weights, which model the importance of the successive coordinate directions for functions from the weighted Hermite spaces.

1 Introduction

Weighted integration and approximation of functions over the whole s-dimensional Euclidean space $\mathbb{R}^s$ appear in many practical problems, often with respect to the Gaussian weight φ. From a theoretical point of view Gaussian problems can be studied in a very elegant way in the context of Hermite spaces of functions, which are the major object of interest of this paper. We present several examples of weighted Hermite spaces that appear in literature and discuss relations, similarities but also differences between these spaces. In order to be able to go into more details we briefly introduce the general function space setting.

We consider weighted Hermite spaces of functions with finite smoothness, using a similar notation as in [4]. In particular, for $k \in \mathbb{N}_0$, we denote the k-th Hermite polynomial by

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} \exp(x^2/2) \frac{\mathrm{d}^k}{\mathrm{d}x^k} \exp(-x^2/2).$$

*The authors are supported by the Austrian Science Fund (FWF), Projects F5508-N26 (Leobacher), F5509-N26 (Pillichshammer), and F5506-N26 (Ebert), which are parts of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

For example,

$$H_0(x) = 1,\ H_1(x) = x,\ H_2(x) = \tfrac{1}{\sqrt{2}}(x^2 - 1),\ H_3(x) = \tfrac{1}{\sqrt{6}}(x^3 - 3x), \dots .$$

Here we follow the definition given in [2], but we remark that there are slightly different ways to introduce Hermite polynomials (see, e.g., [21]). We recall the definition of the standard normal density as $\varphi(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ for $x \in \mathbb{R}$. Furthermore, for $s \in \mathbb{N}$, $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ and $\boldsymbol{x} = (x_1, \dots, x_s) \in \mathbb{R}^s$ we define the $\boldsymbol{k}$-th Hermite polynomial by

$$H_{\boldsymbol{k}}(\boldsymbol{x}) := \prod_{j=1}^{s} H_{k_j}(x_j)$$

and additionally set $\varphi_s(\boldsymbol{x}) := \prod_{j=1}^{s} \varphi(x_j)$, i.e., φ_s is the standard normal density on $\mathbb{R}^s$. It is well known, see [2], that the sequence of Hermite polynomials $(H_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{N}_0^s}$ forms an orthonormal basis of the function space $L_2(\mathbb{R}^s, \varphi_s)$, i.e., for all $f \in L_2(\mathbb{R}^s, \varphi_s)$ we have the *Hermite expansion*

$$f \sim \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) H_{\boldsymbol{k}}\,,$$

where $\sim$ denotes convergence in $L_2(\mathbb{R}^s, \varphi_s)$ and where

$$\widehat{f}(\boldsymbol{k}) = \int_{\mathbb{R}^s} f(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}$$

is the $\boldsymbol{k}$-th *Hermite coefficient* of f.

Similar to what has been done in [11], we are now going to define function spaces based on Hermite expansions. These spaces are Hilbert spaces with a *reproducing kernel*. For details on reproducing kernel Hilbert spaces, we refer to the classical treatment [1].

For the time being, let $R : \mathbb{N}_0^s \to \mathbb{R}^+$ be a summable function, i.e., $\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} R(\boldsymbol{k}) < \infty$ (this condition will be slightly relaxed later on in concrete examples). Define a so-called *Hermite kernel* as

$$K_R(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} R(\boldsymbol{k}) H_{\boldsymbol{k}}(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{y}) \qquad \text{for} \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^s \tag{1}$$

and an inner product

$$\langle f, g \rangle_R = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \frac{1}{R(\boldsymbol{k})} \widehat{f}(\boldsymbol{k}) \widehat{g}(\boldsymbol{k})\,. \tag{2}$$

The weight coefficients $R(\boldsymbol{k})$ are sometimes also referred to as *Fourier weights* (see [7, p. 3]). Note that $K_R(\boldsymbol{x}, \boldsymbol{y})$ is well defined for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^s$, since

$$|K_R(\boldsymbol{x}, \boldsymbol{y})| \le \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} R(\boldsymbol{k}) |H_{\boldsymbol{k}}(\boldsymbol{x})|\, |H_{\boldsymbol{k}}(\boldsymbol{y})| \le \frac{1}{\sqrt{\varphi_s(\boldsymbol{x})\varphi_s(\boldsymbol{y})}} \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} R(\boldsymbol{k}) < \infty,$$

where we have used Cramer's bound for Hermite polynomials, see, e.g., [18, p. 324], which states that

$$|H_k(x)| \le \frac{1}{\sqrt{\varphi(x)}} \quad \text{for all } k \in \mathbb{N}_0.$$

Let $\mathcal{H}(K_R)$ be the reproducing kernel Hilbert space corresponding to K_R. Such spaces are typically known as *Hermite spaces* (see [7, Definition 3.4]). The norm in $\mathcal{H}(K_R)$ is given by $\|f\|_R^2 = \langle f, f\rangle_R$. From this we see that the functions in $\mathcal{H}(K_R)$ are characterized by the decay rate of their Hermite coefficients, which is regulated by the function R. Roughly speaking, the faster R decreases as $\boldsymbol{k}$ moves away from the origin, the faster the Hermite coefficients of the elements of $\mathcal{H}(K_R)$ decrease.

It is worth mentioning the similarity of Hermite spaces to Korobov spaces, the elements of which are $\mathbb{C}$-valued continuous periodic functions on the unit interval with a prescribed convergence speed of the Fourier coefficients (see, for example, [5] for detailed information). The norm and kernel on a Korobov space are obtained from their analogs by replacing Hermite coefficients by Fourier coefficients, Hermite polynomials by the functions $x \mapsto \mathrm{e}^{2\pi \mathrm{i} k x}$, $k \in \mathbb{Z}$, and summation over the non-negative integers by summation over all integers. However, usually the term "Korobov spaces" is interpreted in a more narrow sense, where the Fourier weights are of the form

$$R_{\mathrm{Kor},\alpha,\gamma}(k) = \begin{cases} 1 & \text{if } k = 0, \\ \gamma |k|^{-\alpha} & \text{if } k \neq 0, \end{cases}$$

for some non-negative weight γ and a smoothness parameter $\alpha > 1$.

We are interested in integration and L_2-approximation of functions from Hermite spaces. In [11], the case of polynomially decreasing R as well as exponentially decreasing R was considered. In [8, 9, 10] further results were obtained for numerical integration and/or L_2-approximation for exponentially decreasing R. In this case exponential convergence rates can be achieved as well as several notions of tractability which exactly describe a favorable dependence of the errors on the dimension.

Numerical integration for the case of polynomially decaying Fourier weights R is considered further in [4]. The main focus there is in achieving optimal error convergence rates for the worst-case error leaving aside the exact analysis of the dependence of the errors on the dimension s.

In this paper, we continue the work on polynomially decreasing R for L_2-approximation and integration in the worst-case setting, where the focus will be on very high-dimensional problems. The quantity of interest is the information complexity which is the number of information evaluations required in order to push the worst-case error below a given error threshold ε, where $\varepsilon \in (0,1)$. The important question that arises for applications is how this information complexity depends on ε and on the dimension s. This question is the subject of tractability theory (see the trilogy [15, 16, 17] by Novak and Woźniakowski for general information). Tractability is a concept to characterize the growth rate of the information complexity when ε tends to 0 and s grows to infinity. We study tractability for L_2-approximation and integration in weighted Hermite spaces and give conditions for various notions of tractability in terms of the involved weights $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$ that model the "importance" of the successive coordinate directions. Despite the apparent similarity between Hermite- and Korobov spaces, there is much more known about tractability of approximation in the worst-case setting for the latter. See [6] for matching necessary and sufficient conditions for both standard and linear information. Thus our aim here is to close some of the gaps in knowledge about Hermite spaces.

The paper is organized as follows. First, in Section 2, we discuss and compare several possibilities of describing finite smoothness via various choices of Fourier weights R that appear in literature. This section is interesting on its own, since often it is not clear which is the right choice of a Hermite space for a given problem. However, we will see that the proposed spaces are equivalent as normed function spaces. For the main example, the so-called *Gaussian ANOVA space*, we present an integral representation of the reproducing kernel in Theorem 3.

In Section 3 we present the general L_2-approximation and integration problem for Hermite spaces and discuss some general facts and relations.

In Section 4 we will study tractability properties of L_2-approximation for functions from a Hermite space for permissible information class from $\Lambda^{\rm all}$, consisting of arbitrary linear functionals, and from $\Lambda^{\rm std}$, consisting exclusively of functions evaluations. The main results are Theorem 16 and Corollary 17 (for $\Lambda^{\rm all}$) and Theorem 20 (for $\Lambda^{\rm std}$). While for $\Lambda^{\rm all}$ we get a very clear picture of the whole situation, that is, we have both necessary and sufficient conditions for a range of notions of tractability, for $\Lambda^{\rm std}$ necessary conditions remain open problems.

Tractability for the integration problem is discussed in Section 5. Here the main result is Theorem 24, giving sufficient conditions for several notions of tractability.

2 Weighted Hermite spaces of finite smoothness

Like for the case of Sobolev spaces of smooth functions over $[0,1]^s$ (see [15, Appendix A]) there are various possible ways for introducing Hermite spaces of functions with finite smoothness over $\mathbb{R}^s$. We consider the weighted setting and discuss possible choices for the Fourier weights R. Throughout let $\alpha \geq 1$ be a parameter that will describe the smoothness via the decay rate of the Hermite coefficients of a function to zero. If $\alpha \in \mathbb{N}$ in many cases this can be related to the smoothness of functions with respect to the existence and integrability of partial derivatives of functions.

2.1 A Gaussian ANOVA space

Our first example will be our main object of interest. Later on we will study approximation and integration of functions from this space.

Let $\alpha \geq 1$ and let $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$ be a sequence of so-called product weights. We assume throughout that the weights are in $(0,1]$ and that they are in descending order, i.e.,

$1 \geq \gamma_1 \geq \gamma_2 \geq \gamma_3 \geq \ldots > 0$. Then the function space of interest is the reproducing kernel Hilbert space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ with kernel (1) and corresponding inner product (2) determined by $R(\boldsymbol{k}) = r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{s} r_{\alpha,\gamma_j}(k_j)$ with

$$r_{\alpha,\gamma}(k) := \begin{cases} 1 & \text{for } k = 0, \\ \gamma \frac{1}{k!} & \text{for } 1 \leq k < \alpha, \\ \gamma \frac{(k-\alpha)!}{k!} & \text{for } k \geq \alpha. \end{cases}$$

for a generic weight $\gamma \in (0, 1]$. Note that we always have $r_{\alpha,\gamma}(k) \in (0, 1]$.

The space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}} := \mathcal{H}(K_{r_{s,\alpha,\boldsymbol{\gamma}}})$ is a *weighted Hermite space* with smoothness parameter α (see Equation (5) below) and weights $\boldsymbol{\gamma}$. The weights are introduced in order to model the "importance" of the different coordinates for the functions from the space, where weight γ_j is assigned to coordinate direction $j \in \mathbb{N}$, according to an idea of Sloan and Woźniakowski (see [19]). If all weights equal 1, i.e., if $\gamma_j = 1$ for all $j \in \mathbb{N}$, then we speak about the unweighted Hermite space.

The following lemma gives easy bounds on the decay of the function $r_{\alpha,\gamma}$, showing that $r_{\alpha,\gamma}$ has the same decay rate as the corresponding Fourier weights for the classical Korobov space of smoothness α.

Lemma 1. *For all $k \in \mathbb{N}$ we have*

$$\frac{\gamma}{k^\alpha} \leq r_{\alpha,\gamma}(k) \leq \gamma \left(\frac{\alpha}{k}\right)^\alpha.$$

Proof. If $1 \leq k < \alpha$ we have

$$\frac{1}{r_{\alpha,\gamma}(k)} = \frac{1}{\gamma} k! \leq \frac{1}{\gamma} k^k \leq \frac{1}{\gamma} k^\alpha.$$

If $k \geq \alpha$ we have

$$\frac{1}{r_{\alpha,\gamma}(k)} = \frac{1}{\gamma} \frac{k!}{(k-\alpha)!} = \frac{1}{\gamma} k(k-1)\cdots(k-\alpha+1) \leq \frac{1}{\gamma} k^\alpha.$$

Hence we find that

$$\frac{\gamma}{k^\alpha} \leq r_{\alpha,\gamma}(k).$$

In order to show the upper bound we consider the case that $k > \alpha$ first. Then

$$r_{\alpha,\gamma}(k) = \frac{\gamma}{k(k-1)\cdots(k-\alpha+1)} \leq \frac{\gamma}{(k-\alpha+1)^\alpha} = \frac{\gamma}{k^\alpha (1 - \frac{\alpha-1}{k})^\alpha} \leq \gamma \left(\frac{\alpha}{k}\right)^\alpha,$$

because for $k > \alpha$ we have

$$1 - \frac{\alpha-1}{k} \geq 1 - \frac{\alpha-1}{\alpha} = \frac{1}{\alpha}.$$

For $1 \leq k \leq \alpha$ we have $r_{\alpha,\gamma}(k) = \frac{\gamma}{k!}$ and $\left(\frac{\alpha}{k}\right)^\alpha \geq 1$, and hence

$$r_{\alpha,\gamma}(k) \leq \gamma \left(\frac{\alpha}{k}\right)^\alpha.$$

This finishes the proof. $\square$

Note that for $\alpha = 1$ we have $\sum_{k\in\mathbb{N}_0} r_{\alpha,\gamma}(k) = \infty$. Nevertheless, from [4, Lemma 1] we know that for all $\boldsymbol{k} \in \mathbb{N}_0$ and for all $\boldsymbol{x} \in \mathbb{R}^s$ we even have

$$|H_{\boldsymbol{k}}(\boldsymbol{x})\sqrt{\varphi_s(\boldsymbol{x})}| \leq \prod_{j=1}^{s} \min\left(1, \frac{\sqrt{\pi}}{k_j^{1/12}}\right).$$

This is a slight improvement of Cramer's bound mentioned earlier in this paper. From this estimate it follows again that $K_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y})$ is well defined for all $\alpha \geq 1$ and for all $\boldsymbol{x},\boldsymbol{y} \in \mathbb{R}^s$, since

$$\begin{aligned}
|K_{r_{s,\alpha,\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})| &\leq \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})|H_{\boldsymbol{k}}(\boldsymbol{x})H_{\boldsymbol{k}}(\boldsymbol{y})| \\
&\leq \frac{1}{\sqrt{\varphi_s(\boldsymbol{x})\varphi_s(\boldsymbol{y})}} \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \prod_{j=1}^{s} \min\left(1, \frac{\pi}{k_j^{1/6}}\right) \\
&\leq \frac{1}{\sqrt{\varphi_s(\boldsymbol{x})\varphi_s(\boldsymbol{y})}} \prod_{j=1}^{s} \left(1 + \gamma_j \alpha^\alpha \left(\sum_{1<k<\pi^6} \frac{1}{k^\alpha} + \pi \sum_{k\geq\pi^6} \frac{1}{k^{\alpha+1/6}}\right)\right) < \infty.
\end{aligned}$$

Now we explain how the parameter α is related to the smoothness of the functions from the Hermite space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ whenever α is an integer. Let $\alpha \in \mathbb{N}$. For $f \in \mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ we have the Hermite expansion, see [11],

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \widehat{f}(\boldsymbol{k}) H_{\boldsymbol{k}}(\boldsymbol{x}) \qquad \text{for all } \boldsymbol{x} \in \mathbb{R}^s$$

and for any $\boldsymbol{\tau} = (\tau_1,\ldots,\tau_s) \in \mathbb{N}_0^s$ with $\boldsymbol{\tau} \leq \alpha$ we have that

$$\partial_{\boldsymbol{x}}^{\boldsymbol{\tau}} f \sim \sum_{\boldsymbol{k}\geq\boldsymbol{\tau}} \widehat{f}(\boldsymbol{k}) \sqrt{\frac{\boldsymbol{k}!}{(\boldsymbol{k}-\boldsymbol{\tau})!}}\, H_{\boldsymbol{k}-\boldsymbol{\tau}}. \tag{3}$$

For $s \in \mathbb{N}$ we write $[s] := \{1,2,\ldots,s\}$. Observe also the use of the standard multiindex notation $\boldsymbol{\tau}! = \prod_{j=1}^{s} \tau_j!$ and $\boldsymbol{\tau} \leq \alpha$, which means that $\tau_j \leq \alpha$ for all $j \in [s]$ for $\boldsymbol{\tau} \in \mathbb{N}_0^s$ and likewise $\boldsymbol{\tau} \leq \boldsymbol{k}$, which means that $\tau_j \leq k_j$ for all $j \in [s]$, for $\boldsymbol{\tau},\boldsymbol{k} \in \mathbb{N}_0^s$. Then the inner product of the weighted Hermite space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ can be written as

$$\begin{aligned}
\langle f,g\rangle_{r_{s,\alpha,\boldsymbol{\gamma}}} = \sum_{\mathfrak{u}\subseteq[s]} \sum_{\boldsymbol{\tau}_{\mathfrak{u}}\in\{0,\ldots,\alpha-1\}^{|\mathfrak{u}|}} \gamma_{\boldsymbol{\tau}_{\mathfrak{u}}}^{-1} \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{|\mathfrak{u}|}} \partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x}) \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}}) \,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}\right) \\
\times \left(\int_{\mathbb{R}^{|\mathfrak{u}|}} \partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} g(\boldsymbol{x}) \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}}) \,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}\right) \varphi_{s-|\mathfrak{u}|}(\boldsymbol{x}_{-\mathfrak{u}}) \,\mathrm{d}\boldsymbol{x}_{-\mathfrak{u}},
\end{aligned} \tag{4}$$

where $(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}}) \in \mathbb{N}_0^s$ denotes the multiindex for which the j-th component equals α for $j \notin \mathfrak{u}$ and τ_j for $j \in \mathfrak{u}$, and where $\gamma_{\boldsymbol{\tau}_{\mathfrak{u}}}$ is the product of the γ_j over those j for which the

j-th component of $(\boldsymbol{\tau}_{\mathfrak{u}}, \alpha_{-\mathfrak{u}})$ does not equal 0, i.e.,

$$\gamma_{\boldsymbol{\tau}_{\mathfrak{u}}} = \prod_{\substack{j=1 \\ \tau_j \neq 0 \vee j \notin \mathfrak{u}}}^{s} \gamma_j = \left(\prod_{j \in [s] \setminus \mathfrak{u}} \gamma_j \right) \prod_{\substack{j \in \mathfrak{u} \\ \tau_j \neq 0}} \gamma_j ,$$

and $\partial_{\boldsymbol{x}}^{\boldsymbol{\eta}} = \frac{\partial^{\eta_1}}{(\partial x_1)^{\eta_1}} \cdots \frac{\partial^{\eta_s}}{(\partial x_s)^{\eta_s}}$ for $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_s) \in \mathbb{N}_0^s$. Hence we may express the norm in $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ as a certain instance of a Sobolev type norm in the form

$$\|f\|_{r_{s,\alpha,\boldsymbol{\gamma}}}^2 = \sum_{\mathfrak{u} \subseteq [s]} \sum_{\boldsymbol{\tau}_{\mathfrak{u}} \in \{0,\ldots,\alpha-1\}^{|\mathfrak{u}|}} \gamma_{\boldsymbol{\tau}_{\mathfrak{u}}}^{-1} \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{|\mathfrak{u}|}} \partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}}, \alpha_{-\mathfrak{u}})} f(\boldsymbol{x}) \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}}) \,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}} \right)^2 \varphi_{s-|\mathfrak{u}|}(\boldsymbol{x}_{-\mathfrak{u}}) \,\mathrm{d}\boldsymbol{x}_{-\mathfrak{u}} , \tag{5}$$

where $-\mathfrak{u} := [s] \setminus \mathfrak{u}$. We will provide a proof of (5) (and thus of (4)) shortly. In particular, a finite norm for $f \in \mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ requires that all partial mixed derivatives of f of order α in every coordinate direction are square integrable. Actually, as a vector space, $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ is precisely the space of continuous functions on $\mathbb{R}^s$, for which for every $\boldsymbol{\tau} \leq \alpha$ the $\boldsymbol{\tau}$-th mixed weak partial derivative exists and is square integrable.

In order to have a concrete impression we describe the following special instances.

Example 2. For example, for $s = \alpha = 1$ we have

$$\|f\|_{r_{1,1,\gamma}}^2 = \left(\int_{\mathbb{R}} f(x) \varphi(x) \,\mathrm{d}x \right)^2 + \frac{1}{\gamma} \int_{\mathbb{R}} (f'(x))^2 \varphi(x) \,\mathrm{d}x,$$

for $s = 1$, $\alpha \in \mathbb{N}$ we have

$$\|f\|_{r_{1,\alpha,\gamma}}^2 = \left(\int_{\mathbb{R}} f(x) \varphi(x) \,\mathrm{d}x \right)^2 + \frac{1}{\gamma} \sum_{\tau=1}^{\alpha-1} \left(\int_{\mathbb{R}} f^{(\tau)}(x) \varphi(x) \,\mathrm{d}x \right)^2 + \frac{1}{\gamma} \int_{\mathbb{R}} (f^{(\alpha)}(x))^2 \varphi(x) \,\mathrm{d}x, \tag{6}$$

and for $s = 2$, $\alpha = 1$ we have

$$\begin{aligned} \|f\|_{r_{2,1,\gamma}}^2 = &\left(\int_{\mathbb{R}^2} f(x_1, x_2) \varphi(x_1) \varphi(x_2) \,\mathrm{d}(x_1, x_2) \right)^2 \\ &+ \frac{1}{\gamma_1} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\partial f(x_1, x_2)}{\partial x_1} \varphi(x_2) \,\mathrm{d}x_2 \right)^2 \varphi(x_1) \,\mathrm{d}x_1 \\ &+ \frac{1}{\gamma_2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{\partial f(x_1, x_2)}{\partial x_2} \varphi(x_1) \,\mathrm{d}x_1 \right)^2 \varphi(x_2) \,\mathrm{d}x_2 \\ &+ \frac{1}{\gamma_1 \gamma_2} \int_{\mathbb{R}^2} \left(\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} \right)^2 \varphi(x_1) \varphi(x_2) \,\mathrm{d}(x_1, x_2). \end{aligned}$$

Proof of Equation (5) . We introduce another multiindex notation: $(\boldsymbol{k} - \boldsymbol{\tau})_{\mathfrak{u}} := \boldsymbol{k}_{\mathfrak{u}} - \boldsymbol{\tau}_{\mathfrak{u}}$ resp. $(\boldsymbol{k} - \alpha)_{-\mathfrak{u}} := \boldsymbol{k}_{-\mathfrak{u}} - \alpha_{-\mathfrak{u}}$. With this we write

$$\partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}}, \alpha_{-\mathfrak{u}})} f(\boldsymbol{x}) \sim \sum_{\boldsymbol{k} \geq (\boldsymbol{\tau}_{\mathfrak{u}}, \alpha_{-\mathfrak{u}})} \sqrt{\frac{\boldsymbol{k}!}{(\boldsymbol{k} - (\boldsymbol{\tau}_{\mathfrak{u}}, \alpha_{-\mathfrak{u}}))!}} \widehat{f}(\boldsymbol{k}) H_{\boldsymbol{k} - (\boldsymbol{\tau}_{\mathfrak{u}}, \alpha_{-\mathfrak{u}})}(\boldsymbol{x})$$

$$= \sum_{\boldsymbol{k}_{-\mathfrak{u}} \geq \alpha_{-\mathfrak{u}}} \sum_{\boldsymbol{k}_{\mathfrak{u}} \geq \boldsymbol{\tau}_{\mathfrak{u}}} \sqrt{\frac{\boldsymbol{k}_{-\mathfrak{u}}!}{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}!}} \sqrt{\frac{\boldsymbol{k}_{\mathfrak{u}}!}{(\boldsymbol{k}-\boldsymbol{\tau})_{\mathfrak{u}}!}} \widehat{f}(\boldsymbol{k}) H_{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}}(\boldsymbol{x}_{-\mathfrak{u}}) H_{(\boldsymbol{k}-\boldsymbol{\tau})_{\mathfrak{u}}}(\boldsymbol{x}_{\mathfrak{u}}).$$

Thus

$$\begin{aligned}
&\left(\int_{\mathbb{R}^{|\mathfrak{u}|}} \partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x}) \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}}) d\boldsymbol{x}_{\mathfrak{u}}\right)^2 \\
&= \left(\sum_{\boldsymbol{k}_{-\mathfrak{u}} \geq \alpha_{-\mathfrak{u}}} \sqrt{\frac{\boldsymbol{k}_{-\mathfrak{u}}!}{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}!}} \sqrt{\boldsymbol{\tau}_{\mathfrak{u}}!}\, \widehat{f}(\boldsymbol{\tau}_{\mathfrak{u}}, \boldsymbol{k}_{-\mathfrak{u}}) H_{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}}(\boldsymbol{x}_{-\mathfrak{u}})\right)^2 \\
&= \boldsymbol{\tau}_{\mathfrak{u}}! \sum_{\boldsymbol{k}_{-\mathfrak{u}} \geq \alpha_{-\mathfrak{u}}} \sum_{\boldsymbol{l}_{-\mathfrak{u}} \geq \alpha_{-\mathfrak{u}}} \sqrt{\frac{\boldsymbol{k}_{-\mathfrak{u}}!}{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}!}} \sqrt{\frac{\boldsymbol{l}_{-\mathfrak{u}}!}{(\boldsymbol{l}-\alpha)_{-\mathfrak{u}}!}} \widehat{f}(\boldsymbol{\tau}_{\mathfrak{u}}, \boldsymbol{k}_{-\mathfrak{u}}) H_{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}}(\boldsymbol{x}_{-\mathfrak{u}}) H_{(\boldsymbol{l}-\alpha)_{-\mathfrak{u}}}(\boldsymbol{x}_{-\mathfrak{u}})
\end{aligned}$$

such that, using the orthogonality of the multidimensional Hermite polynomials,

$$\begin{aligned}
&\int_{\mathbb{R}^{s-|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{|\mathfrak{u}|}} \partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x}) \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}}) d\boldsymbol{x}_{\mathfrak{u}}\right)^2 \varphi_{s-|\mathfrak{u}|}(\boldsymbol{x}_{-\mathfrak{u}})\, \mathrm{d}\boldsymbol{x}_{-\mathfrak{u}} \\
&= \boldsymbol{\tau}_{\mathfrak{u}}! \sum_{\boldsymbol{k}_{-\mathfrak{u}} \geq \alpha_{-\mathfrak{u}}} \frac{\boldsymbol{k}_{-\mathfrak{u}}!}{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}!} \widehat{f}(\boldsymbol{\tau}_{\mathfrak{u}}, \boldsymbol{k}_{-\mathfrak{u}})^2.
\end{aligned}$$

With this we finally get

$$\begin{aligned}
&\sum_{\mathfrak{u} \subseteq [s]} \sum_{\boldsymbol{\tau}_{\mathfrak{u}} \in \{0\ldots,\alpha-1\}^{|\mathfrak{u}|}} \boldsymbol{\gamma}_{\boldsymbol{\tau}_{\mathfrak{u}}}^{-1} \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{|\mathfrak{u}|}} \partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x}) \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}}) d\boldsymbol{x}_{\mathfrak{u}}\right)^2 \varphi_{s-|\mathfrak{u}|}(\boldsymbol{x}_{-\mathfrak{u}})\, \mathrm{d}\boldsymbol{x}_{-\mathfrak{u}} \\
&= \sum_{\mathfrak{u} \subseteq [s]} \sum_{\boldsymbol{\tau}_{\mathfrak{u}} \in \{0\ldots,\alpha-1\}^{|\mathfrak{u}|}} \sum_{\boldsymbol{k}_{-\mathfrak{u}} \geq \alpha_{-\mathfrak{u}}} \Bigg(\prod_{\substack{j \in \mathfrak{u} \\ \tau_j \neq 0}} \gamma_j^{-1}\Bigg) \boldsymbol{\tau}_{\mathfrak{u}}! \Bigg(\prod_{\ell \in [s] \setminus \mathfrak{u}} \gamma_\ell^{-1}\Bigg) \frac{\boldsymbol{k}_{-\mathfrak{u}}!}{(\boldsymbol{k}-\alpha)_{-\mathfrak{u}}!} \widehat{f}(\boldsymbol{\tau}_{\mathfrak{u}}, \boldsymbol{k}_{-\mathfrak{u}})^2 \\
&= \sum_{\boldsymbol{k} \in \mathbb{N}^s} \frac{1}{r_{s,\alpha,\gamma}(\boldsymbol{k})} \widehat{f}(\boldsymbol{k})^2 = \|f\|_{r_{s,\alpha,\gamma}}^2.
\end{aligned}$$

$\square$

Equations (4) and (5) show that the Hermite space with the present choice of $r_{s,\alpha,\boldsymbol{\gamma}}$ can be interpreted as a Gaussian ANOVA space on the $\mathbb{R}^s$ or as a Gaussian unanchored Sobolev space of functions on the $\mathbb{R}^s$.

We have an interesting integral representation of the kernel in the one-dimensional case. In the following we use the notation $\Phi(y) := \int_{-\infty}^{y} \varphi(\eta)\, \mathrm{d}\eta$ and

$$\vartheta(x,y) := 1_{(-\infty,x]}(y)\Phi(y) - 1_{(x,\infty)}(y)\Phi(-y)$$

for $x, y \in \mathbb{R}$.

Theorem 3. *For $\alpha \in \mathbb{N}$ and $\gamma > 0$ we have*

$$
\begin{aligned}
&K_{r_{\alpha,\gamma}}(x,y)\\
&= 1+\gamma\sum_{k=1}^{\alpha-1}\frac{H_k(x)H_k(y)}{k!}\\
&\quad +\gamma\int_{\mathbb{R}}\frac{1}{\varphi(s)}\left(\int_{\mathbb{R}^{2\alpha-2}}\vartheta_\alpha(x,\xi_{\alpha-1},\ldots,\xi_1,s)\vartheta_\alpha(y,\eta_{\alpha-1},\ldots,\eta_1,s)\prod_{k=1}^{\alpha-1}\left(\mathrm{d}\xi_k\,\mathrm{d}\eta_k\right)\right)\mathrm{d}s,
\end{aligned}
$$

where $\vartheta_n(z_1,\ldots,z_{n+1}) := \prod_{k=1}^{n}\vartheta(z_k,z_{k+1})$ for $n \in \mathbb{N}$.

A proof for this representation will be given in Appendix A.

2.2 A first variant of the Gaussian ANOVA space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$

In [11], Irrgeher and Leobacher define in a similar way a variant of the Hermite space. They consider the reproducing kernel (1) with corresponding inner product given by $R(\boldsymbol{k}) = \rho_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{s}\rho_{\alpha,\gamma_j}(k_j)$ with

$$\rho_{\alpha,\gamma}(k) := \begin{cases} 1 & \text{for } k=0,\\ \frac{\gamma}{k^\alpha} & \text{for } k\ge 1,\end{cases}$$

for $\alpha \ge 1$ and a generic weight $\gamma \in (0,1]$. Note that these Fourier weights are equal to those of the classical Korobov space of smoothness α. Denote the corresponding reproducing kernel Hilbert space by $\mathcal{H}_{\rho_{s,\alpha,\boldsymbol{\gamma}}} := \mathcal{H}(K_{\rho_{s,\alpha,\boldsymbol{\gamma}}})$.

For the norm $\|\cdot\|_{\rho_{s,\alpha,\boldsymbol{\gamma}}}$ we do not have a representation as a Sobolev type norm like in (5) for the norm $\|\cdot\|_{r_{s,\alpha,\boldsymbol{\gamma}}}$.

Proposition 4. *We have*

$$\|f\|_{r_{s,\alpha,\boldsymbol{\gamma}}} \le \|f\|_{\rho_{s,\alpha,\boldsymbol{\gamma}}} \le \|f\|_{r_{s,\alpha,\boldsymbol{\gamma}/\alpha^\alpha}},$$

where $\boldsymbol{\gamma}/\alpha^\alpha := (\gamma_j/\alpha^\alpha)_{j\ge 1}$. In particular $\mathcal{H}_{\rho_{s,\alpha,\boldsymbol{\gamma}}}$ is continuously embedded in the space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ and the norm of the embedding operator is bounded by 1.

Proof. According to Lemma 1 we have

$$\rho_{\alpha,\gamma}(k) \le r_{\alpha,\gamma}(k) \le \alpha^\alpha \rho_{\alpha,\gamma}(k) = \rho_{\alpha,\alpha^\alpha\gamma}(k) \quad \text{for all } k \in \mathbb{N}$$

and obviously $\rho_{\alpha,\gamma}(0) = r_{\alpha,\gamma}(0) = 1$. Hence

$$\rho_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \le r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \le \alpha^{\alpha|\mathfrak{u}(\boldsymbol{k})|}\rho_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \quad \text{for all } \boldsymbol{k} \in \mathbb{N}_0^s,$$

where for $\boldsymbol{k} \in \mathbb{N}_0^s$ we write $\mathfrak{u}(\boldsymbol{k}) := \{j \in [s] \ : \ k_j \neq 0\}$, and hence

$$\|f\|_{r_{s,\alpha,\boldsymbol{\gamma}}}^2 = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s}\frac{1}{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})}|\widehat{f}(\boldsymbol{k})|^2 \le \sum_{\boldsymbol{k}\in\mathbb{N}_0^s}\frac{1}{\rho_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})}|\widehat{f}(\boldsymbol{k})|^2 = \|f\|_{\rho_{s,\alpha,\boldsymbol{\gamma}}}^2$$

$$
\begin{aligned}
&= \sum_{\mathfrak{u}\subseteq[s]} \sum_{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|}} \frac{1}{\prod_{j\in\mathfrak{u}} \rho_{\alpha,\gamma_j}(k_j)} |\widehat{f}(\boldsymbol{k}_{\mathfrak{u}},0)|^2 \\
&\le \sum_{\mathfrak{u}\subseteq[s]} \sum_{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|}} \frac{1}{\prod_{j\in\mathfrak{u}} (r_{\alpha,\gamma_j}(k_j)/\alpha^\alpha)} |\widehat{f}(\boldsymbol{k}_{\mathfrak{u}},0)|^2 \\
&= \sum_{\mathfrak{u}\subseteq[s]} \sum_{\boldsymbol{k}_{\mathfrak{u}}\in\mathbb{N}^{|\mathfrak{u}|}} \frac{1}{\prod_{j\in\mathfrak{u}} r_{\alpha,\gamma_j/\alpha^\alpha}(k_j)} |\widehat{f}(\boldsymbol{k}_{\mathfrak{u}},0)|^2 = \|f\|^2_{r_{s,\alpha,\boldsymbol{\gamma}/\alpha^\alpha}}.
\end{aligned}
$$

Here for $\boldsymbol{k} = (k_1, \ldots, k_s)$ and $\mathfrak{u} \subseteq [s]$ we write $(\boldsymbol{k}_{\mathfrak{u}}, 0)$ for the s-dimensional vector whose j-th component is k_j if $j \in \mathfrak{u}$ and 0 otherwise. $\square$

2.3 A second variant of the Gaussian ANOVA space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$

In [4] a further Sobolev type norm was considered, namely

$$\|f\|^2 := \sum_{\boldsymbol{\tau}\in\{0,\ldots,\alpha\}^s} \int_{\mathbb{R}^s} (\partial_{\boldsymbol{x}}^{\boldsymbol{\tau}} f(\boldsymbol{x}))^2 \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}.$$

A weighted variant of this is

$$\|f\|^2_{\psi_{s,\alpha,\boldsymbol{\gamma}}} := \sum_{\boldsymbol{\tau}\in\{0,\ldots,\alpha\}^s} \left(\prod_{\substack{j=1\\ \tau_j\neq 0}}^{s} \gamma_j^{-1} \right) \int_{\mathbb{R}^s} (\partial_{\boldsymbol{x}}^{\boldsymbol{\tau}} f(\boldsymbol{x}))^2 \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x}. \tag{7}$$

The meaning of ψ will be explained shortly.

Example 5. As an example, in the univariate case with a generic weight $\gamma > 0$ the squared norm can be written in the form

$$\|f\|^2_{\psi_{1,\alpha,\gamma}} = \int_{\mathbb{R}} (f(x))^2 \varphi(x) \,\mathrm{d}x + \frac{1}{\gamma} \sum_{\tau=1}^{\alpha} \int_{\mathbb{R}} (f^{(\tau)}(x))^2 \varphi(x) \,\mathrm{d}x,$$

which should be compared with (6) in Example 2.

Likewise, the norm (7) can be represented as a Hermite-type norm and this will explain the ψ in our notation. Using (3) we have

$$\int_{\mathbb{R}^s} (\partial_{\boldsymbol{x}}^{\boldsymbol{\tau}} f(\boldsymbol{x}))^2 \varphi_s(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \sum_{\boldsymbol{k}\ge\boldsymbol{\tau}} \frac{\boldsymbol{k}!}{(\boldsymbol{k}-\boldsymbol{\tau})!} (\widehat{f}(\boldsymbol{k}))^2.$$

Hence

$$\|f\|^2_{\psi_{s,\alpha,\boldsymbol{\gamma}}} = \sum_{\boldsymbol{\tau}\in\{0,\ldots,\alpha\}^s} \left(\prod_{\substack{j=1\\ \tau_j\neq 0}}^{s} \gamma_j^{-1} \right) \sum_{\boldsymbol{k}\ge\boldsymbol{\tau}} \frac{\boldsymbol{k}!}{(\boldsymbol{k}-\boldsymbol{\tau})!} (\widehat{f}(\boldsymbol{k}))^2$$

$$= \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \left(\sum_{\substack{\boldsymbol{\tau}\in\{0,\dots,\alpha\}^s \\ \boldsymbol{\tau}\le\boldsymbol{k}}} \left(\prod_{\substack{j=1 \\ \tau_j\neq 0}}^{s} \gamma_j^{-1} \right) \frac{\boldsymbol{k}!}{(\boldsymbol{k}-\boldsymbol{\tau})!} \right) (\widehat{f}(\boldsymbol{k}))^2$$

$$= \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \left(\prod_{j=1}^{s} \left(1 + \frac{1}{\gamma_j} \sum_{\substack{\tau=1 \\ \tau\le k_j}}^{\alpha} \frac{k_j!}{(k_j-\tau)!} \right) \right) (\widehat{f}(\boldsymbol{k}))^2$$

$$= \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \left(\prod_{j=1}^{s} \left(1 + \frac{1}{\gamma_j} \sum_{\tau=1}^{\alpha} \beta_\tau(k_j) \right) \right) (\widehat{f}(\boldsymbol{k}))^2,$$

where for $k \in \mathbb{N}_0$,

$$\beta_\tau(k) := \begin{cases} \frac{k!}{(k-\tau)!} & \text{if } k \geq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Setting, for $k \in \mathbb{N}_0$ and a generic weight $\gamma > 0$,

$$\psi_{\alpha,\gamma}(k) := \left(1 + \frac{1}{\gamma} \sum_{\tau=1}^{\alpha} \beta_\tau(k_j) \right)^{-1}$$

and for $\boldsymbol{k} = (k_1, \dots, k_s) \in \mathbb{N}_0$, $\psi_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) := \prod_{j=1}^{s} \psi_{\alpha,\gamma_j}(k_j)$, then

$$\|f\|_{\psi_{s,\alpha,\boldsymbol{\gamma}}}^2 = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \frac{1}{\psi_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})} (\widehat{f}(\boldsymbol{k}))^2.$$

Thus, via the norm $\|\cdot\|_{\psi_{s,\alpha,\boldsymbol{\gamma}}}$ we obtain a Hermite space $\mathcal{H}_{\psi_{s,\alpha,\boldsymbol{\gamma}}}$ with reproducing kernel of the form (1) with Fourier weights $R(\boldsymbol{k}) = \psi_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})$.

Remark 6. Using the method of Thomas-Agnan [22] the kernel $K_{\psi_{1,1,\gamma}}$ ($s = 1$ and $\alpha = 1$) can be expressed by means of solutions of the second order differential equation

$$g''(y) = y g'(y) + \gamma g(y)$$

with certain boundary conditions. We omit the details of this observation.

Proposition 7. *We have*

$$\|f\|_{r_{s,\alpha,\boldsymbol{\gamma}}} \leq \|f\|_{\psi_{s,\alpha,\boldsymbol{\gamma}}} \leq \|f\|_{r_{s,\alpha,\boldsymbol{\gamma}/(2\alpha^\alpha)}},$$

where $\boldsymbol{\gamma}/(2\alpha^\alpha) := (\gamma_j/(2\alpha^\alpha))_{j\geq 1}$. *In particular,* $\mathcal{H}_{\psi_{s,\alpha,\boldsymbol{\gamma}}}$ *is continuously embedded in* $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ *and the norm of the embedding operator is bounded by 1.*

Proof. Using the Cauchy-Schwarz inequality we obtain

$$\int_{\mathbb{R}^{s-|\mathfrak{u}|}} \left(\int_{\mathbb{R}^{|\mathfrak{u}|}} \partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x}) \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}}) \,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}} \right)^2 \varphi_{s-|\mathfrak{u}|}(\boldsymbol{x}_{-\mathfrak{u}}) \,\mathrm{d}\boldsymbol{x}_{-\mathfrak{u}}$$

$$\leq \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \int_{\mathbb{R}^{|\mathfrak{u}|}} \left(\partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x})\right)^2 \varphi_{|\mathfrak{u}|}(\boldsymbol{x}_{\mathfrak{u}})\,\mathrm{d}\boldsymbol{x}_{\mathfrak{u}}\,\varphi_{s-|\mathfrak{u}|}(\boldsymbol{x}_{-\mathfrak{u}})\,\mathrm{d}\boldsymbol{x}_{-\mathfrak{u}}$$
$$= \int_{\mathbb{R}^s} \left(\partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x})\right)^2 \varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

such that $\|f\|_{r_{s,\alpha,\boldsymbol{\gamma}}} \leq \|f\|_{\psi_{s,\alpha,\boldsymbol{\gamma}}}$ for all $f \in \mathcal{H}_{\psi_{s,\alpha,\boldsymbol{\gamma}}}$.

On the other hand, for $k \in \mathbb{N}$ we have

$$\sum_{\tau=0}^{\alpha} \beta_\tau(k) \leq 2\,k^\alpha,$$

because:

- If $k > \alpha$, then

$$\sum_{\tau=0}^{\alpha} \beta_\tau(k) = \sum_{\tau=0}^{\alpha} \frac{k!}{(k-\tau)!} \leq \sum_{\tau=0}^{\alpha} k^\tau = \frac{k^{\alpha+1}-1}{k-1} \leq 2\,k^\alpha.$$

- If $1 \leq k \leq \alpha$, then

$$\sum_{\tau=0}^{\alpha} \beta_\tau(k) = \sum_{\tau=0}^{k} \frac{k!}{(k-\tau)!} = k! \sum_{\tau=0}^{k} \frac{1}{\tau!} \leq 2\,k^k \leq 2\,k^\alpha.$$

Therefore and with Lemma 1, for $k \in \mathbb{N}$ we obtain

$$\psi_{\alpha,\gamma}(k) \geq \frac{\gamma}{\sum_{\tau=0}^{\alpha} \beta_\tau(k)} \geq \frac{\gamma}{2\,k^\alpha} \geq \frac{1}{2\,\alpha^\alpha}\, r_{\alpha,\gamma}(k).$$

Again, $\psi_{\alpha,\gamma}(0) = 1 = r_{\alpha,\gamma}(0)$. Hence

$$\psi_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \geq \left(\frac{1}{2\,\alpha^\alpha}\right)^{|\mathfrak{u}(\boldsymbol{k})|} r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \quad \text{for all } \boldsymbol{k} \in \mathbb{N}_0^s.$$

Like in the proof of Proposition 4, this implies that

$$\|f\|_{\psi_{s,\alpha,\boldsymbol{\gamma}}} \leq \|f\|_{r_{s,\alpha,\boldsymbol{\gamma}/(2\alpha^\alpha)}}.$$

□

2.4 An anchored space of Sobolev type

For the sake of completeness we mention also an anchored variant of the ANOVA norm (5) with anchor $\mathbf{0} = (0,\ldots,0)$, which is given by

$$\|f\|_{\pitchfork,s,\alpha,\boldsymbol{\gamma}}^2 := \sum_{\mathfrak{u}\subseteq[s]} \sum_{\boldsymbol{\tau}_{\mathfrak{u}}\in\{0,\ldots,\alpha-1\}^{|\mathfrak{u}|}} \gamma_{\boldsymbol{\tau}_{\mathfrak{u}}}^{-1} \int_{\mathbb{R}^{s-|\mathfrak{u}|}} \left(\partial_{\boldsymbol{x}}^{(\boldsymbol{\tau}_{\mathfrak{u}},\alpha_{-\mathfrak{u}})} f(\boldsymbol{x}_{-\mathfrak{u}},0)\right)^2 \varphi_{s-|\mathfrak{u}|}(\boldsymbol{x}_{-\mathfrak{u}})\,\mathrm{d}\boldsymbol{x}_{-\mathfrak{u}}.$$

Example 8. For $s = 1$ and a generic weight $\gamma > 0$ we have

$$\|f\|^2_{\pitchfork,1,\alpha,\gamma} = (f(0))^2 + \frac{1}{\gamma}\sum_{k=1}^{\alpha-1}(f^{(k)}(0))^2 + \frac{1}{\gamma}\int_{\mathbb{R}}(f^{(\alpha)}(y))^2\varphi(y)\,\mathrm{d}y.$$

Denote the corresponding function space by $\mathcal{H}_{\pitchfork,s,\alpha,\boldsymbol{\gamma}}$. Also this space, the so-called anchored space is a reproducing kernel Hilbert space of tensor product form. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^s$ the reproducing kernel is

$$K_{\pitchfork,s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x}, \boldsymbol{y}) := \prod_{j=1}^{s} K_{\pitchfork,\alpha,\gamma_j}(x_j, y_j),$$

where the kernel $K_{\pitchfork,\alpha,\gamma}$ in the case $s = 1$ ist given in the following proposition.

Proposition 9. *For $x, y \in \mathbb{R}$ and a generic weight $\gamma > 0$ we have*

$$\begin{aligned} K_{\pitchfork,\alpha,\gamma}(x,y) &= 1 + \gamma\sum_{\ell=1}^{\alpha-1}\frac{(xy)^\ell}{(\ell!)^2} \\ &\quad +\gamma\, 1_{[0,\infty)}(x\,y)\int_0^\infty \frac{1}{\varphi(s)}\,\frac{(|x|-s)_+^{\alpha-1}(|y|-s)_+^{\alpha-1}}{((\alpha-1)!)^2}\,\mathrm{d}s. \end{aligned} \tag{8}$$

We omit the proof of this formula. For a similar space and kernel we refer to [16, Sec. 12.5.1].

We see from Proposition 9 that the part

$$L(x,y) := 1_{[0,\infty)}(x\,y)\int_0^\infty \frac{1}{\varphi(s)}\,\frac{(|x|-s)_+^{\alpha-1}(|y|-s)_+^{\alpha-1}}{((\alpha-1)!)^2}\,\mathrm{d}s \tag{9}$$

of the kernel $K_{\pitchfork,\alpha,\gamma}$ is decomposable at 0, meaning that $L(x, y) = 0$ whenever $x < 0 < y$ or $y < 0 < x$.

Remark 10. In general the anchored space $\mathcal{H}_{\pitchfork,s,\alpha,\boldsymbol{\gamma}}$ is not a Hermite space in the sense of the definition in Section 1. To see this, write (for $s = 1$) $\mathcal{H}_{\pitchfork,\alpha,\boldsymbol{\gamma}} = \mathcal{H}_1 + \mathcal{H}_2$, where $\mathcal{H}_1$ is the closed subspace of all polynomials of degree smaller than α and $\mathcal{H}_2$ is the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{H}_{\pitchfork,\alpha,\boldsymbol{\gamma}}$.

Using property (7) from [1, Section 2], we see that $K_{\pitchfork,\alpha,\gamma} = K_1 + K_2$, where K_j is a reproducing kernel for $\mathcal{H}_j$, $j \in \{1, 2\}$. Using the representation of $K_{\pitchfork,\alpha,\gamma}$ from Proposition 9 gives $K_1(x, y) = 1 + \gamma\sum_{\ell=1}^{\alpha-1}\frac{(xy)^\ell}{(\ell!)^2}$.

Now assume, in order to reach a contradiction, that $\mathcal{H}_{\pitchfork,3,1}$ *is* a Hermite space, and therefore there exists R with $K_{\pitchfork,3,1}(x, y) = \sum_{k=0}^{\infty} R(k)H_k(x)H_k(y)$. But then $K_1(x, y) = \sum_{k=0}^{2} R(k)H_k(x)H_k(y)$, so

$$\begin{aligned} 1 + xy + \frac{(xy)^2}{2} &= K_1(x,y) = 1 + R(1)xy + \frac{1}{2}R(2)(x^2-1)(y^2-1) \\ &= 1 + \frac{1}{2}R(2) + R(1)xy + \frac{1}{2}R(2)x^2y^2 - \frac{1}{2}R(2)x^2 - \frac{1}{2}R(2)y^2 \end{aligned}$$

But now comparing coefficients yields $R(2) = 0$ and $R(2) = 1$, the desired contradiction.

Remark 11. It is worth noting that, while $\mathcal{H}_{\pitchfork,s,\alpha,\boldsymbol{\gamma}}$ and $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ are certainly equivalent as Banach spaces, in general the norm of neither space is dominated by that of the other. To see this, let $s=1$, $\alpha=3$, $\gamma=1$, $a,b\in\mathbb{R}$ and consider the function $f\colon\mathbb{R}\to\mathbb{R}$ with $f(x)=a+\frac{b}{2}x^2$. Then $f^{(3)}\equiv 0$, so that $\int_{\mathbb{R}}|f^{(3)}(y)|^2\varphi(y)\,\mathrm{d}y=0$. Now

$$\begin{aligned}\|f\|^2_{\pitchfork,1,3,1} &= (f(0))^2+(f'(0))^2+(f''(0))^2 = a^2+b^2\\ \|f\|^2_{r_{1,3,1}} &= \left(\int_{\mathbb{R}}\left(a+\frac{b}{2}y^2\right)\varphi(y)\,\mathrm{d}y\right)^2+\left(\int_{\mathbb{R}} by\varphi(y)\,\mathrm{d}y\right)^2+\left(\int_{\mathbb{R}} b\varphi(y)\,\mathrm{d}y\right)^2\\ &= \left(a+\frac{b}{2}\right)^2+b^2\,.\end{aligned}$$

Thus, by choosing $a=1$, $b=2$ we get $\|f\|^2_{\pitchfork,1,3,1}=5<8=\|f\|^2_{r_{1,3,1}}$, while by choosing $a=1$, $b=-2$ we find $\|f\|^2_{\pitchfork,1,3,1}=5>4=\|f\|^2_{r_{1,3,1}}$.

3 Integration and L_2-approximation in Hermite spaces

We consider integration and L_2-approximation for functions from a weighted Hermite space H_R where our main focus will be on $R=r_{s,\alpha,\boldsymbol{\gamma}}$. Throughout we assume that $R(\mathbf{0})=1$ and $0<R(\boldsymbol{h})\le 1$ for all $\boldsymbol{h}\in\mathbb{N}_0^s$.

The integration problem. The multivariate integration problem is given by $\mathrm{INT}_R:\mathcal{H}_R\to\mathbb{R}$,

$$\mathrm{INT}_R(f)=\int_{\mathbb{R}^s} f(\boldsymbol{x})\,\varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}=\widehat{f}(\mathbf{0})\quad\text{for}\quad f\in\mathcal{H}_R.$$

In order to approximate INT_R we use linear algorithms of the form

$$A^{\mathrm{int}}_{n,s}(f):=\sum_{i=1}^n w_i f(\boldsymbol{x}_i) \tag{10}$$

with nodes $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\in\mathbb{R}^s$ and integration weights $w_1,\ldots,w_s\in\mathbb{R}$. The quality of the algorithm is measured in terms of the worst-case error which is defined by

$$e^{\mathrm{int}}(A^{\mathrm{int}}_{n,s}):=\sup_{\substack{f\in\mathcal{H}_R\\ \|f\|_R\le 1}}\left|\mathrm{INT}_R(f)-A^{\mathrm{int}}_{n,s}(f)\right|.$$

The n-th minimal error for integration in $\mathcal{H}_R$ is defined as

$$e(n,\mathrm{INT}_R):=\inf_{A^{\mathrm{int}}_{n,s}} e^{\mathrm{int}}(A^{\mathrm{int}}_{n,s}),$$

where the infimum is extended over all linear algorithms of the form (10) using n function evaluations and integration weights, respectively.

The initial (integration) error is $e^{\mathrm{int}}(0, \mathrm{INT}_R) = \|\mathrm{INT}_R\| = 1$, because

$$\|\mathrm{INT}_R\| = \sup_{0 \neq f \in \mathcal{H}_R} \frac{|\widehat{f}(\mathbf{0})|}{\|f\|_R} = \sup_{0 \neq f \in \mathcal{H}_R} \frac{|\widehat{f}(\mathbf{0})|}{\sqrt{\sum_{\boldsymbol{h} \in \mathbb{N}_0^s} R^{-1}(\boldsymbol{h})\, |\widehat{f}(\boldsymbol{h})|^2}} \leq \sup_{0 \neq f \in \mathcal{H}_R} \frac{|\widehat{f}(\mathbf{0})|}{\sqrt{|\widehat{f}(\mathbf{0})|^2}} = 1$$

and for $g = 1 \in \mathcal{H}_R$ we have that

$$\frac{|\widehat{g}(\mathbf{0})|}{\|g\|_R} = \frac{\int_{\mathbb{R}^s} \varphi_s(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}}{\int_{\mathbb{R}^s} \varphi_s(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}} = 1.$$

The L_2-approximation problem. The L_2-approximation of functions from the Hermite space $\mathcal{H}_R$ is given by the embedding operator $\mathrm{APP}_R : \mathcal{H}_R \to L_2(\mathbb{R}^s, \varphi_s)$ with

$$\mathrm{APP}_R(f) = f \quad \text{for} \quad f \in \mathcal{H}_R.$$

In order to approximate APP_R with respect to the norm $\|\cdot\|_{L_2}$ we will employ linear algorithms $A_{n,s}^{\mathrm{app}}$ that use n information evaluations and are of the form

$$A_{n,s}^{\mathrm{app}}(f) = \sum_{i=1}^{n} L_i(f)\, g_i \quad \text{for } f \in \mathcal{H}_R \tag{11}$$

with functions $g_i \in L_2(\mathbb{R}^s, \varphi_s)$ and bounded linear functionals $L_i \in \mathcal{H}_R^*$ for $i \in \{1, 2, \ldots, n\}$ (see [15, Theorem 4.8] or [23]). If, for an algorithm $A_{n,s}^{\mathrm{app}}$ as in (11) all L_i are from the same information class $\Lambda \subseteq \mathcal{H}_R^*$, then we simply write with some abuse of notation $A_{n,s}^{\mathrm{app}} \in \Lambda$.

In this paper we consider two classes of permissible information, namely the class Λ^{all} consisting of all continuous linear functionals, i.e., $\Lambda^{\mathrm{all}} = \mathcal{H}_R^*$, and the class Λ^{std} consisting exclusively of point evaluation functionals. Since $\mathcal{H}_R$ is a reproducing kernel Hilbert space it is clear that point evaluation functionals are continuous and hence $\Lambda^{\mathrm{std}} \subseteq \Lambda^{\mathrm{all}}$.

We remark that the embedding operator APP_R is continuous for all $s \in \mathbb{N}$, which can be seen as follows. We have for all $f \in \mathcal{H}_R$ that

$$\|\mathrm{APP}_R(f)\|_{L_2}^2 = \|f\|_{L_2}^2 = \sum_{\boldsymbol{h} \in \mathbb{N}_0^s} |\widehat{f}(\boldsymbol{h})|^2 \leq \sum_{\boldsymbol{h} \in \mathbb{N}_0^s} \frac{1}{R(\boldsymbol{h})} |\widehat{f}(\boldsymbol{h})|^2 = \|f\|_R^2 < \infty,$$

where we used Parseval's identity and the fact that $0 < R(\boldsymbol{h}) \leq 1$ for all $\boldsymbol{h} \in \mathbb{N}_0^s$. By considering the choice $f \equiv 1$, it follows that the above inequality is sharp, such that the operator norm of APP_R is given by

$$\|\mathrm{APP}_R\| = 1.$$

Remark 12. Note that it does not make sense to study L_∞-approximation for the Hermite space $\mathcal{H}_R$ since this problem is not well defined because

$$K_R(\boldsymbol{x}, \boldsymbol{x}) = \sum_{\boldsymbol{h} \in \mathbb{N}_0^s} R(\boldsymbol{h}) (H_{\boldsymbol{k}}(\boldsymbol{x}))^2 \geq 1 + R(\mathbf{1}) x_1^2 \cdots x_s^2$$

and hence (see [13, Section 2])

$$\sup_{\substack{f\in\mathcal{H}_R\\ \|f\|_R\le 1}} \|f\|_{L_\infty} = \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^s} \sup_{\substack{f\in\mathcal{H}_R\\ \|f\|_R\le 1}} |f(\boldsymbol{x})| = \operatorname*{ess\,sup}_{\boldsymbol{x}\in\mathbb{R}^s} \sqrt{K_R(\boldsymbol{x},\boldsymbol{x})} = \infty.$$

The worst-case error of an algorithm $A_{n,s}^{\mathrm{app}}$ of the form (11) is defined by

$$e^{\mathrm{app}}(A_{n,s}^{\mathrm{app}}) := \sup_{\substack{f\in\mathcal{H}_R\\ \|f\|_R\le 1}} \|\mathrm{APP}_R(f) - A_{n,s}^{\mathrm{app}}(f)\|_{L_2(\mathbb{R}^s,\varphi_s)}$$

and the n-th minimal worst-case error w.r.t. the information class Λ is given by

$$e(n,\mathrm{APP}_R;\Lambda) := \inf_{A_{n,s}^{\mathrm{app}}\in\Lambda} e^{\mathrm{app}}(A_{n,s}^{\mathrm{app}}).$$

Since $\Lambda^{\mathrm{std}}\subseteq\Lambda^{\mathrm{all}}$ it follows that

$$e(n,\mathrm{APP}_R;\Lambda^{\mathrm{all}}) \le e(n,\mathrm{APP}_R;\Lambda^{\mathrm{std}}). \tag{12}$$

A relation between integration and L_2-approximation.

Proposition 13. *For the space $\mathcal{H}_R$ we have*

$$e(n,\mathrm{INT}_R) \le e(n,\mathrm{APP}_R;\Lambda^{\mathrm{std}}).$$

Proof. Recall that $\|\mathrm{INT}_R\|=1$. Using Parseval's identity, we have for any algorithm of the form $A_{n,s}^{\mathrm{app}}(f)=\sum_{i=1}^n g_i f(\boldsymbol{x}_i)$ with $\boldsymbol{x}_i\in\mathbb{R}^s$ and $g_i\in L_2(\mathbb{R}^s,\varphi_s)$ for $i\in\{1,2,\ldots,n\}$ that

$$\|\mathrm{APP}_R(f) - A_{n,s}^{\mathrm{app}}(f)\|_{L_2(\mathbb{R}^s,\varphi_s)}^2 = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \left|\widehat{f}(\boldsymbol{k}) - \widehat{A_{n,s}^{\mathrm{app}}(f)}(\boldsymbol{k})\right|^2,$$

where $\widehat{A_{n,s}^{\mathrm{app}}(f)}(\boldsymbol{k})$ is the $\boldsymbol{k}$-th Hermite coefficient given by

$$\widehat{A_{n,s}^{\mathrm{app}}(f)}(\boldsymbol{k}) = \sum_{i=1}^n f(\boldsymbol{x}_i)\int_{\mathbb{R}^s} g_i(\boldsymbol{x})H_{\boldsymbol{k}}(\boldsymbol{x})\varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

This gives

$$\begin{aligned}
\|\mathrm{APP}_R(f) - A_{n,s}^{\mathrm{app}}(f)\|_{L_2(\mathbb{R}^s,\varphi_s)}^2 &= \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \left|\widehat{f}(\boldsymbol{k}) - \widehat{A_{n,s}^{\mathrm{app}}(f)}(\boldsymbol{k})\right|^2 \\
&\ge \left|\widehat{f}(\boldsymbol{0}) - \widehat{A_{n,s}^{\mathrm{app}}(f)}(\boldsymbol{0})\right|^2 \\
&= \left|\mathrm{INT}_R(f) - \sum_{i=1}^n f(\boldsymbol{x}_i)\int_{\mathbb{R}^s} g_i(\boldsymbol{x})\varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\right|^2
\end{aligned}$$

$$= \left| \mathrm{INT}_R(f) - \sum_{i=1}^{n} w_i f(\boldsymbol{x}_i) \right|^2$$
$$= \left| \mathrm{INT}_R(f) - A_{n,s}^{\mathrm{int}}(f) \right|^2$$

where
$$w_i := \int_{\mathbb{R}^s} g_i(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \quad \text{for } i \in \{1, 2, \ldots, n\}$$
and
$$A_{n,s}^{\mathrm{int}}(f) := \sum_{i=1}^{n} w_i f(\boldsymbol{x}_i).$$
Thus, for every linear approximation algorithm $A_{n,s}^{\mathrm{app}}$ we can find a linear integration algorithm $A_{n,s}^{\mathrm{int}}$ such that
$$e^{\mathrm{int}}(A_{n,s}^{\mathrm{int}}) \le e^{\mathrm{app}}(A_{n,s}^{\mathrm{app}}).$$
From this we conclude that
$$e(n, \mathrm{INT}_R) \le e(n, \mathrm{APP}_R; \Lambda^{\mathrm{std}}).$$
□

The next proposition provides some relations between worst-case errors for different but related Hermite spaces.

Proposition 14. *Let $R_1, R_2 : \mathbb{N}_0^s \to \mathbb{R}$ be two Fourier weights for Hermite spaces $\mathcal{H}_{R_1}$ and $\mathcal{H}_{R_2}$ such that for the corresponding norms we have*
$$\|f\|_{R_1} \le \|f\|_{R_2} \quad \textit{for all } f \in \mathcal{H}_{R_2}.$$
Then for all $n \in \mathbb{N}$ we have
$$e(n, \mathrm{INT}_{R_2}) \le e(n, \mathrm{INT}_{R_1})$$
and
$$e(n, \mathrm{APP}_{R_2}; \Lambda) \le e(n, \mathrm{APP}_{R_1}; \Lambda) \quad \textit{for } \Lambda \in \{\Lambda^{\mathrm{all}}, \Lambda^{\mathrm{std}}\}.$$

We omit the easy proof of this result and refer to [5, Proposition 7.5].

Remark 15. Under our assumption that R_1 and R_2 vanish nowhere on $\mathbb{N}_0^s$, it is not hard to check that,
$$\|f\|_{R_1} \le \|f\|_{R_2} \Leftrightarrow \left[R_1(\boldsymbol{k}) \ge R_2(\boldsymbol{k}) \text{ for all } \boldsymbol{k} \in \mathbb{N}_0^s \right].$$

Tractability. We are interested in how the worst-case errors of algorithms $A^{\bullet}_{n,s}$, $\bullet \in \{\mathrm{int}, \mathrm{app}\}$, depend on the number n of information evaluations used and on the problem dimension s. To this end, we define the so-called information complexity as

$$n(\varepsilon, S; \Lambda) := \min\{n \in \mathbb{N}_0 \ : \ e(n, S; \Lambda) \leq \varepsilon\}$$

where $S \in \{\mathrm{INT}_R, \mathrm{APP}_R\}$, with $\varepsilon \in (0,1)$ and $s \in \mathbb{N}$. Note that here we do not need to distinguish between the absolute and the normalized error criterion since in the present case the related initial errors equal 1. If $S = \mathrm{INT}_R$ it obviously makes only sense to consider the class Λ^{std} and hence we just write $n(\varepsilon, \mathrm{INT}_R)$.

Obviously, (12) implies that

$$n(\varepsilon, \mathrm{APP}_R; \Lambda^{\mathrm{all}}) \leq n(\varepsilon, \mathrm{APP}_R; \Lambda^{\mathrm{std}}) \tag{13}$$

and Proposition 13 implies

$$n(\varepsilon, \mathrm{INT}_R) \leq n(\varepsilon, \mathrm{APP}_R; \Lambda^{\mathrm{std}}). \tag{14}$$

In order to characterize the dependency of the information complexity on the dimension s and the error threshold ε, we will study several notions of tractability which are given in the following definition.

Definition 1. Let $S \in \{\mathrm{INT}_R, \mathrm{APP}_R\}$. We say we have:

(a) Polynomial tractability (PT) if there exist non-negative numbers τ, σ, C such that

$$n(\varepsilon, S; \Lambda) \leq C\, \varepsilon^{-\tau} s^{\sigma} \quad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0,1).$$

(b) Strong polynomial tractability (SPT) if there exist non-negative numbers τ, C such that

$$n(\varepsilon, S; \Lambda) \leq C\, \varepsilon^{-\tau} \quad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0,1).$$

In that case we define the *exponent of SPT* as

$$\inf\{\tau \colon \exists C > 0 \text{ such that } n(\varepsilon, S; \Lambda) \leq C\, \varepsilon^{-\tau}\ \forall s \in \mathbb{N}, \varepsilon \in (0,1)\}.$$

(c) Weak tractability (WT) if

$$\lim_{s+\varepsilon^{-1} \to \infty} \frac{\log n(\varepsilon, S; \Lambda)}{s + \varepsilon^{-1}} = 0.$$

(d) Quasi-polynomial tractability (QPT) if there exist non-negative numbers τ, C such that

$$n(\varepsilon, S; \Lambda) \leq C\, \exp(\tau(1 + \log s)(1 + \log \varepsilon^{-1})) \quad \text{for all} \quad s \in \mathbb{N}, \varepsilon \in (0,1).$$

In that case we define the *exponent of QPT* as

$$\inf\{\tau \colon \exists C > 0 \text{ such that } n(\varepsilon, S; \Lambda) \leq C\, \exp(\tau(1+\log s)(1+\log \varepsilon^{-1}))\ \forall s \in \mathbb{N}, \varepsilon \in (0,1)\}.$$

(e) (σ, τ)-weak tractability ((σ,τ)-WT) if there exist positive σ, τ such that

$$\lim_{s+\varepsilon^{-1}} \frac{\log n(\varepsilon, S; \Lambda)}{s^{\sigma} + \varepsilon^{-\tau}} = 0.$$

(f) Uniform weak tractability (UWT) if (σ,τ)-weak tractability holds for all $\sigma, \tau \in (0,1]$.

4 L_2-approximation in weighted Hermite spaces

In this section we present results about tractability of L_2-approximation for Hermite spaces $\mathcal{H}_R$ with Fourier weights $R \in \{r_{s,\alpha,\boldsymbol{\gamma}}, \rho_{s,\alpha,\boldsymbol{\gamma}}, \psi_{s,\alpha,\boldsymbol{\gamma}}\}$. From our examples in Section 2 we mainly concentrate on the most comprising weighted Gaussian ANOVA space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ from Section 2.1. Via embedding we then can derive corresponding results also for the other cases. Throughout we consider the smoothness parameter α and the weights $\boldsymbol{\gamma}$ as fixed. With this in mind, we often simplify the notation by just writing APP_s instead of APP_R or $\mathrm{APP}_{r_{s,\alpha,\boldsymbol{\gamma}}}$.

4.1 Tractability for the class Λ^{all}

In order to characterize tractability properties of the approximation problem we introduce the following figures: For a weight sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1}$ we will use the infimum

$$\boldsymbol{\gamma}_I := \inf_{j\geq 1} \gamma_j$$

and the so-called sum exponent

$$s_{\boldsymbol{\gamma}} := \inf \left\{ \kappa > 0 \ : \ \sum_{j=1}^{\infty} \gamma_j^{\kappa} < \infty \right\}, \tag{15}$$

with the convention that $\inf \emptyset := \infty$.

First we state the exact "if and only if" characterization for tractability of L_2-approximation in $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ for the information class Λ^{all}.

Theorem 16. *Let $\alpha \geq 1$ and $\boldsymbol{\gamma}$ be a sequence of weights. Consider the L_2-approximation problem* $\mathrm{APP} = (\mathrm{APP}_s)_{s\geq 1}$ *for the weighted Hermite spaces $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ for $s \in \mathbb{N}$ and for the information class Λ^{all}. Then we have the following exact characterizations of tractability:*

1. *SPT holds if and only if $s_{\boldsymbol{\gamma}} < \infty$. In this case the exponent of SPT is*

$$\tau^*(\Lambda^{\mathrm{all}}) = 2 \max\left(\frac{1}{\alpha}, s_{\boldsymbol{\gamma}}\right).$$

2. *SPT and PT are equivalent.*

3. *QPT, UWT and WT are equivalent and hold if and only if $\boldsymbol{\gamma}_I < 1$. In this case the exponent of QPT is*

$$t^*(\Lambda^{\mathrm{all}}) = \begin{cases} 2 \max\left(\frac{1}{\alpha}, \frac{1}{\ln \boldsymbol{\gamma}_I^{-1}}\right) & \text{if } \boldsymbol{\gamma}_I \neq 0, \\ 2\frac{1}{\alpha} & \text{if } \boldsymbol{\gamma}_I = 0. \end{cases}$$

4. *For $\sigma > 1$, (σ, τ)-WT holds for all weights $1 \geq \gamma_1 \geq \gamma_2 \geq \ldots > 0$.*

From this theorem we can derive the following consequences.

Corollary 17. *Let $\alpha \geq 1$ and $\boldsymbol{\gamma}$ be a sequence of weights. Consider the L_2-approximation problem* $\mathrm{APP} = (\mathrm{APP}_s)_{s\geq 1}$ *for the weighted Hermite spaces $\mathcal{H}_R$, $R \in \{\rho_{s,\alpha,\boldsymbol{\gamma}}, \psi_{s,\alpha,\boldsymbol{\gamma}}\}$ for $s \in \mathbb{N}$ and for the information class Λ^{all}. Then we have:*

1. *SPT holds if and only if $s_{\boldsymbol{\gamma}} < \infty$. In this case the exponent of SPT is*

$$\tau^*(\Lambda^{\mathrm{all}}) = 2\max\left(\frac{1}{\alpha}, s_{\boldsymbol{\gamma}}\right).$$

2. *SPT and PT are equivalent.*

3. *If $\boldsymbol{\gamma}_I < \infty$, then we have QPT. In this case the exponent of QPT is*

$$t^*(\Lambda^{\mathrm{all}}) = \begin{cases} 2\max\left(\frac{1}{\alpha}, \frac{1}{\ln \boldsymbol{\gamma}_I^{-1}}\right) & \text{if } \boldsymbol{\gamma}_I \neq 0, \\ 2\frac{1}{\alpha} & \text{if } \boldsymbol{\gamma}_I \neq 0. \end{cases}$$

4. *For $\sigma > 1$, (σ, τ)-WT holds for all weights $1 \geq \gamma_1 \geq \gamma_2 \geq \ldots > 0$.*

We start with some preparation for the proof of Theorem 16.

It is commonly known that the n-th minimal worst-case errors $e(n, \mathrm{APP}_s; \Lambda^{\mathrm{all}})$ are directly related to the eigenvalues of the self-adjoint operator

$$W_s := \mathrm{APP}_s^* \mathrm{APP}_s : \mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}} \to \mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}. \tag{16}$$

Denote these eigenvalues by $1 = \lambda_{s,1} \geq \lambda_{s,2} \geq \lambda_{s,3} \geq \ldots$. Then we have (see [15, Corollary 4.12]) that

$$e(n, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) = \lambda_{s,n+1}^{1/2} \tag{17}$$

and hence

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) = \min\{n \;:\; \lambda_{s,n+1} \leq \varepsilon^2\}. \tag{18}$$

In the following lemma, we derive the eigenpairs of the operator W_s. For this purpose, we define for all $\boldsymbol{k} \in \mathbb{N}_0^s$, the vectors $e_{\boldsymbol{k}} = e_{\boldsymbol{k},\alpha,\boldsymbol{\gamma}} := \sqrt{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})}\, H_{\boldsymbol{k}}$. Note that $\|e_{\boldsymbol{k}}\|_{r_{s,\alpha,\boldsymbol{\gamma}}} = 1$.

Lemma 18. *The sequence of eigenpairs of the operator W_s is $(r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}), e_{\boldsymbol{k}})_{\boldsymbol{k}\in\mathbb{N}_0^s}$.*

Proof. We find that for any $f, g \in \mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ we have

$$\langle \mathrm{APP}_s(f), \mathrm{APP}_s(g)\rangle_{L_2(\mathbb{R}^s,\varphi_s)} = \langle f, \mathrm{APP}_s^*\mathrm{APP}_s(g)\rangle_{s,\alpha,\boldsymbol{\gamma}} = \langle f, W_s(g)\rangle_{s,\alpha,\boldsymbol{\gamma}}$$

and hence, due to the orthonormality of the Hermite basis functions,

$$\begin{aligned}\langle e_{\boldsymbol{k}}, W_s(e_{\boldsymbol{h}})\rangle_{s,\alpha,\boldsymbol{\gamma}} = \langle e_{\boldsymbol{k}}, e_{\boldsymbol{h}}\rangle_{L_2(\mathbb{R}^s,\varphi_s)} &= \sqrt{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})}\sqrt{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})}\,\langle H_{\boldsymbol{k}}, H_{\boldsymbol{h}}\rangle_{L_2(\mathbb{R}^s,\varphi_s)} \\ &= \sqrt{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})\, r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})}\,\delta_{\boldsymbol{k},\boldsymbol{h}}.\end{aligned}$$

For $\boldsymbol{k} = \boldsymbol{h}$ this gives $\langle e_{\boldsymbol{k}}, W_s(e_{\boldsymbol{k}})\rangle_{s,\alpha,\boldsymbol{\gamma}} = r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})$ which in turn implies that

$$W_s(e_{\boldsymbol{h}}) = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} \langle W_s(e_{\boldsymbol{h}}), e_{\boldsymbol{k}}\rangle_{s,\alpha,\boldsymbol{\gamma}}\, e_{\boldsymbol{k}} = r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})\, e_{\boldsymbol{h}}$$

and thus proves the lemma. □

In order to exploit the relationship between the eigenvalues of W_s and the information complexity, we define the set

$$\mathcal{A}(\varepsilon, s) := \{\boldsymbol{k}\in\mathbb{N}_0^s \;:\; r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) > \varepsilon^2\}. \tag{19}$$

Then we obtain from (18) and Lemma 18 that

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) = |\mathcal{A}(\varepsilon, s)|, \tag{20}$$

which motivates us to examine the set $\mathcal{A}(\varepsilon, s)$ more closely in the following lemma, which is inspired by [12, Lemma 1].

Lemma 19. *Let $s\in\mathbb{N}$, $\varepsilon\in(0,1)$ and let the weights satisfy $1\ge\gamma_1\ge\gamma_2\ge\ldots>0$. If $q\in\mathbb{R}$ with $q>1/\alpha$, then*

$$|\mathcal{A}(\varepsilon, s)| \le \varepsilon^{-2q}\prod_{j=1}^{s}\left(1+\alpha^{\alpha q}\,\zeta(\alpha q)\,\gamma_j^q\right), \tag{21}$$

where ζ denotes the Riemann zeta function.

Proof. We prove the result by induction on s.

Let $s=1$. Using the upper estimate in Lemma 1, for any $k\in\mathcal{A}(\varepsilon,1)\setminus\{0\}$ we have

$$\varepsilon^2 < r_{\alpha,\boldsymbol{\gamma}}(k) \le \gamma_1\left(\frac{\alpha}{k}\right)^{\alpha},$$

which in turn gives that $1\le k\le \alpha\left(\frac{\gamma_1}{\varepsilon^2}\right)^{1/\alpha}$. Hence we find that

$$|\mathcal{A}(\varepsilon,1)| \le 1 + \left|\left\{1,\ldots,\left\lfloor \alpha\left(\frac{\gamma_1}{\varepsilon^2}\right)^{1/\alpha}\right\rfloor\right\}\right| = 1 + \left\lfloor \alpha\left(\frac{\gamma_1}{\varepsilon^2}\right)^{1/\alpha}\right\rfloor. \tag{22}$$

Now assume first that $\alpha\left(\gamma_1/\varepsilon^2\right)^{1/\alpha}\ge 1$. Then we obtain from (22) that

$$\begin{aligned}|\mathcal{A}(\varepsilon,1)| &\le 1+\alpha\left(\frac{\gamma_1}{\varepsilon^2}\right)^{1/\alpha} \le 1+\left(\alpha\left(\frac{\gamma_1}{\varepsilon^2}\right)^{1/\alpha}\right)^{\alpha q} = 1+\alpha^{\alpha q}\left(\frac{\gamma_1}{\varepsilon^2}\right)^{q}\\ &= \left(\varepsilon^{2q}+\alpha^{\alpha q}\gamma_1^q\right)\varepsilon^{-2q} \le (1+\alpha^{\alpha q}\,\zeta(\alpha q)\gamma_1^q)\,\varepsilon^{-2q}\end{aligned}$$

for all $q>1/\alpha$, where we used that $\zeta(x)>1$ for all $x>1$. If, on the other hand, we assume that $\alpha\left(\gamma_1/\varepsilon^2\right)^{1/\alpha}<1$, then we trivially have

$$|\mathcal{A}(\varepsilon,1)| \le 1+\left\lfloor \alpha\left(\frac{\gamma_1}{\varepsilon^2}\right)^{1/\alpha}\right\rfloor = 1 \le (1+\alpha^{\alpha q}\,\zeta(\alpha q)\gamma_1^q)\,\varepsilon^{-2q}.$$

Thus the result is shown for $s = 1$.

Now assume that the statement holds true for $s \in \mathbb{N}$ and arbitrary ε. First we show the recurrence

$$|\mathcal{A}(\varepsilon, s+1)| \leq |\mathcal{A}(\varepsilon, s)| + \sum_{k_{s+1}=1}^{\infty} \left| \mathcal{A}\left(\frac{\varepsilon}{\sqrt{\gamma_{s+1}}} \left(\frac{k_{s+1}}{\alpha} \right)^{\alpha/2}, s \right) \right|. \tag{23}$$

To this end, assume that $\boldsymbol{k} = (k_1, \ldots, k_{s+1}) \in \mathcal{A}(\varepsilon, s+1)$. Then we have

$$r_{s+1,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}_{[s]})\, r_{\alpha,\gamma_{s+1}}(k_{s+1}) > \varepsilon^2.$$

If $k_{s+1} = 0$, then $r_{\alpha,\gamma_{s+1}}(k_{s+1}) = 1$ and so $r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}_{[s]}) > \varepsilon^2$, that is, $\boldsymbol{k}_{[s]} \in \mathcal{A}(\varepsilon, s)$. If on the other hand $k_{s+1} > 0$, we see that

$$r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}_{[s]}) > r^{-1}_{\alpha,\gamma_{s+1}}(k_{s+1})\, \varepsilon^2 \geq \frac{\varepsilon^2}{\gamma_{s+1}} \left(\frac{k_{s+1}}{\alpha} \right)^{\alpha}$$

by Lemma 1. Combining both observations yields

$$\begin{aligned}
&\mathcal{A}(\varepsilon, s+1) \\
&\quad \subseteq \ \big\{(\boldsymbol{k}, 0) \ : \ \boldsymbol{k} \in \mathcal{A}(\varepsilon, s)\big\} \,\dot{\cup} \bigcup_{k_{s+1}=1}^{\infty} \left\{ (\boldsymbol{k}, k_{s+1}) \ : \ \boldsymbol{k} \in \mathcal{A}\left(\frac{\varepsilon}{\sqrt{\gamma_{s+1}}} \left(\frac{k_{s+1}}{\alpha} \right)^{\alpha/2}, s \right) \right\},
\end{aligned}$$

where $\dot{\cup}$ indicates a disjoint union. From here (23) follows immediately.

Now, using the recurrence formula (23) and the induction hypothesis we obtain

$$\begin{aligned}
&|\mathcal{A}(\varepsilon, s+1)| \\
&\quad \leq \ |\mathcal{A}(\varepsilon, s)| + \sum_{k=1}^{\infty} \left| \mathcal{A}\left(\frac{\varepsilon}{\sqrt{\gamma_{s+1}}} \left(\frac{k}{\alpha} \right)^{\alpha/2}, s \right) \right| \\
&\quad \leq \ \varepsilon^{-2q} \prod_{j=1}^{s} \left(1 + \alpha^{\alpha q}\, \zeta(\alpha q)\, \gamma_j^q\right) + \prod_{j=1}^{s} \left(1 + \alpha^{\alpha q} \zeta(\alpha q)\, \gamma_j^q\right) \sum_{k=1}^{\infty} \left[\frac{\varepsilon}{\sqrt{\gamma_{s+1}}} \left(\frac{k}{\alpha} \right)^{\alpha/2} \right]^{-2q} \\
&\quad = \ \varepsilon^{-2q} \prod_{j=1}^{s} \left(1 + \alpha^{\alpha q}\, \zeta(\alpha q)\, \gamma_j^q\right) \left(1 + \gamma_{s+1}^q \alpha^{\alpha q} \sum_{k=1}^{\infty} \frac{1}{k^{\alpha q}} \right) \\
&\quad = \ \varepsilon^{-2q} \prod_{j=1}^{s+1} \left(1 + \alpha^{\alpha q}\, \zeta(\alpha q)\, \gamma_j^q\right).
\end{aligned}$$

This finishes the proof. $\square$

Proof of Theorem 16. We prove the necessary and sufficient conditions for each of the listed notions of tractability.

1. In order to give a necessary and sufficient condition for SPT for Λ^{all} we use a criterion from [15, Section 5.1]. From [15, Theorem 5.2] we find that the problem APP is SPT for Λ^{all} if and only if there exists a $\tau > 0$ such that

$$\sup_{s\in\mathbb{N}} \left(\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} (r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^\tau \right)^{1/\tau} < \infty \tag{24}$$

and then

$$\tau^*(\Lambda^{\mathrm{all}}) = \inf\{2\tau \ : \ \tau \text{ satisfies (24)}\}.$$

Assume that $s_{\boldsymbol{\gamma}} < \infty$. Then take τ such that $\tau > \max(s_{\boldsymbol{\gamma}}, 1/\alpha)$ and thus $\sum_{j=1}^\infty \gamma_j^\tau$ is finite. Note that $\alpha\tau > 1$ and hence $\zeta(\alpha\tau) < \infty$. For the sum in (24) we then obtain, making use of the upper estimate in Lemma 1,

$$\begin{aligned}\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} (r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^\tau &= \prod_{j=1}^s \left(\sum_{k=0}^\infty (r_{\alpha,\gamma_j}(k))^\tau \right) \le \prod_{j=1}^s \left(1 + \sum_{k=1}^\infty \gamma_j^\tau \left(\frac{\alpha}{k}\right)^{\alpha\tau} \right) \\ &\le \prod_{j=1}^s \left(1 + \gamma_j^\tau \alpha^{\alpha\tau}\zeta(\alpha\tau)\right) \le \exp\left(\alpha^{\alpha\tau}\zeta(\alpha\tau) \sum_{j=1}^\infty \gamma_j^\tau \right) < \infty.\end{aligned}$$

This implies that we have SPT and that

$$\tau^*(\Lambda^{\mathrm{all}}) \le 2\max\left(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}\right). \tag{25}$$

On the other hand, assume we have SPT. Then there exists a finite τ such that (24) holds true. Using the lower bound in Lemma 1, we have that

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} (r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^\tau = \prod_{j=1}^s \left(\sum_{k=0}^\infty (r_{\alpha,\gamma_j}(k))^\tau \right) \ge \prod_{j=1}^s \left(1 + \gamma_j^\tau \sum_{k=1}^\infty \frac{1}{k^{\alpha\tau}} \right).$$

Since (24) holds true we obviously have $\tau > 1/\alpha$. Then

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^s} (r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^\tau \ge \prod_{j=1}^s (1 + \gamma_j^\tau \zeta(\alpha\tau)) \ge \zeta(\alpha\tau) \sum_{j=1}^s \gamma_j^\tau.$$

Again, since (24) holds true, we also have that $\sum_{j=1}^\infty \gamma_j^\tau < \infty$ and hence $s_{\boldsymbol{\gamma}} < \tau < \infty$. Combining both results yields that $\tau > \max(s_{\boldsymbol{\gamma}}, 1/\alpha)$ and hence also

$$\tau^*(\Lambda^{\mathrm{all}}) \ge 2\max\left(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}\right). \tag{26}$$

Equations (25) and (26) then imply that

$$\tau^*(\Lambda^{\mathrm{all}}) = 2\max\left(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}\right).$$

2. In order to prove the equivalence of SPT and PT it suffices to prove that PT implies SPT. So let us assume that APP is polynomially tractable, i.e., there exist reals $C, p > 0$ and $q \geq 0$ such that

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \leq C s^q \varepsilon^{-p} \quad \text{for all } \varepsilon \in (0,1) \text{ and for all } s \in \mathbb{N}.$$

Without loss of generality we may assume that q is an integer. Take $s \in \mathbb{N}$ such that $s \geq q+1$ and set

$$B_s := \{\boldsymbol{h} \in \{0,1\}^s \colon \text{precisely } q+1 \text{ components of } \boldsymbol{h} \text{ are equal to } 1\}.$$

Now choose $\varepsilon_* = \frac{1}{2}\gamma_s^{(q+1)/2}$. Choose $\boldsymbol{h} \in B_s$ and let $\mathfrak{u} \subseteq [s]$ be the set of indices of $\boldsymbol{h}$ which are equal to 1. Then we have

$$r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{h}) \geq \prod_{j \in \mathfrak{u}} \gamma_j \geq \gamma_s^{q+1} > \varepsilon_*^2.$$

Hence $B_s \subseteq \mathcal{A}(\varepsilon_*, s)$ and this implies

$$|\mathcal{A}(\varepsilon_*, s)| \geq |B_s| = \binom{s}{q+1} \geq \frac{(s-q)^{q+1}}{(q+1)!} \geq \frac{s^{q+1}}{(q+1)!(q+1)^{q+1}} =: s^{q+1} c_q\,,$$

where we used $s \geq q+1$ for the third inequality.

This now yields

$$s^{q+1} c_q \leq |\mathcal{A}(\varepsilon_*, s)| = n(\varepsilon^*, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \leq C s^q \varepsilon_*^{-p} = 2^p C s^q \gamma_s^{-(q+1)p/2}.$$

This implies that

$$\gamma_s^{(q+1)p/2} \ll_{p,q} \frac{1}{s},$$

where $\ll_{p,q}$ means that there is an implied factor which only depends on p and q, and hence

$$\gamma_s \ll_{p,q} \frac{1}{s^{2/((q+1)p)}}.$$

This estimate holds for all $s \geq q+1$. Hence the sum exponent $s_{\boldsymbol{\gamma}}$ of the sequence $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$ is finite, $s_{\boldsymbol{\gamma}} < \infty$, and this implies by the first statement that we have SPT.

3. We use the following criterion for QPT taken from [17, Sec. 23.1.1], which states that QPT holds if and only if there exists $\tau > 0$ such that

$$C := \sup_{s \in \mathbb{N}} \frac{1}{s^2} \left(\sum_{j=1}^{\infty} \lambda_{s,j}^{\tau(1+\log s)} \right)^{1/\tau} < \infty, \tag{27}$$

where $\lambda_{s,j}$ is the j-th eigenvalue of the operator W_s from (16) in non-increasing order.

Assume that $\boldsymbol{\gamma}_I < 1$. For the considered Hermite space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ we have

$$\begin{aligned}
&\sum_{j=1}^{\infty} \lambda_{s,j}^{\tau(1+\log s)} \\
&= \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} (r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^{\tau(1+\log s)} \\
&= \prod_{j=1}^{s} \left(1+\sum_{k=1}^{\infty} (r_{\alpha,\gamma_j}(k))^{\tau(1+\log s)}\right) \\
&= \prod_{j=1}^{s} \left(1+\gamma_j^{\tau(1+\log s)} \left(\sum_{k=1}^{\alpha-1}\left(\frac{1}{k!}\right)^{\tau(1+\log s)} + \sum_{k=\alpha}^{\infty}\left(\frac{(k-\alpha)!}{k!}\right)^{\tau(1+\log s)}\right)\right) \qquad (28)\\
&\leq \prod_{j=1}^{s} \left(1+\gamma_j^{\tau(1+\log s)} \left(\sum_{k=1}^{\alpha-1}\left(\frac{1}{k!}\right)^{\tau(1+\log s)} + \sum_{k=\alpha}^{\infty}\left(\frac{1}{(k-\alpha+1)^{\alpha}}\right)^{\tau(1+\log s)}\right)\right) \\
&= \prod_{j=1}^{s} \left(1+\gamma_j^{\tau(1+\log s)} \left(\sum_{k=1}^{\alpha-1}\left(\frac{1}{k!}\right)^{\tau(1+\log s)} + \sum_{k=1}^{\infty}\left(\frac{1}{k^{\alpha}}\right)^{\tau(1+\log s)}\right)\right) \\
&= \prod_{j=1}^{s} \left(1+\gamma_j^{\tau(1+\log s)} \left(\sum_{k=1}^{\alpha-1}\left(\frac{1}{k!}\right)^{\tau(1+\log s)} + \zeta(\alpha\tau(1+\log s))\right)\right).
\end{aligned}$$

Put $\zeta_s := \zeta(\alpha\tau(1+\log s))$. From now on we assume, without loss of generality, that $\tau > 1/\alpha$, so that $\zeta_s < \infty$.

Next we have

$$\begin{aligned}
\sum_{k=1}^{\infty}\left(\frac{1}{k!}\right)^{\tau(1+\log s)} &= 1+\frac{1}{2^{\tau(1+\log s)}}+\frac{1}{6^{\tau(1+\log s)}}+\sum_{k=4}^{\infty}\left(\frac{1}{k!}\right)^{\tau(1+\log s)} \\
&\leq 1+\frac{1}{2^{\tau(1+\log s)}}+\frac{1}{6^{\tau(1+\log s)}}+\sum_{k=4}^{\infty}\left(\frac{1}{2^{\tau(1+\log s)}}\right)^{k} \\
&= 1+\frac{1}{2^{\tau(1+\log s)}}+\frac{1}{6^{\tau(1+\log s)}}+\frac{1}{2^{4\tau(1+\log s)}}\frac{1}{1-1/2^{\tau(1+\log s)}} \\
&= 1+\frac{1}{2^{\tau(1+\log s)}}+\frac{1}{6^{\tau(1+\log s)}}+\frac{1}{2^{4\tau(1+\log s)}-2^{3\tau(1+\log s)}} \\
&\leq 1+\frac{1}{2^{\tau(1+\log s)}}+\frac{1}{6^{\tau(1+\log s)}}+\frac{1}{2^{3\tau\log s}}\;\frac{1}{2^{4\tau}-2^{3\tau}} \\
&\leq 1+\frac{1}{2^{\tau\log s}}\left(2+\frac{1}{2^{4\tau}-2^{3\tau}}\right) \\
&= 1+\frac{c_\tau}{s^{\tau\log 2}},
\end{aligned}$$

where we put $c_\tau := 2 + 1/(2^{4\tau} - 2^{3\tau})$. This in turn gives that

$$\begin{aligned}
\frac{1}{s^2}\left(\sum_{j=1}^{\infty} \lambda_{s,j}^{\tau(1+\log s)}\right)^{1/\tau} &\leq \frac{1}{s^2}\left(\prod_{j=1}^{s}\left(1+\gamma_j^{\tau(1+\log s)}\left(\frac{c_\tau}{s^\tau \log 2}+\zeta_s\right)\right)\right)^{1/\tau} \\
&= \exp\left(\frac{1}{\tau}\sum_{j=1}^{s}\log\left(1+\gamma_j^{\tau(1+\log s)}\left(\zeta_s+\frac{c_\tau}{s^\tau \log 2}\right)\right) - 2\log s\right) \\
&\leq \exp\left(\frac{1}{\tau}\left(\zeta_s+\frac{c_\tau}{s^\tau\log 2}\right)\sum_{j=1}^{s}\gamma_j^{\tau(1+\log s)} - 2\log s\right),
\end{aligned}$$

where we used that $\log(1+x) \leq x$ for all $x \geq 0$. Now we use the well-known fact that $\zeta(x) \leq 1 + \frac{1}{x-1}$ for all $x > 1$ and thus

$$\zeta_s \leq 1 + \frac{1}{(\alpha\tau - 1) + \alpha\tau \log s}.$$

Then we obtain

$$\begin{aligned}
&\frac{1}{s^2}\left(\sum_{j=1}^{\infty} \lambda_{s,j}^{\tau(1+\log s)}\right)^{1/\tau} \\
&\leq \exp\left(\frac{1}{\tau}\left(1+\frac{1}{(\alpha\tau-1)+\alpha\tau\log s}+\frac{c_\tau}{s^\tau\log 2}\right)\sum_{j=1}^{s}\gamma_j^{\tau(1+\log s)} - 2\log s\right).
\end{aligned}$$

Now we distinguish two cases:

- Case $\boldsymbol{\gamma}_I = 0$: Then $\lim_{j\to\infty}\gamma_j = 0$ and hence, for every $\varepsilon > 0$ there is a positive integer $J = J(\varepsilon)$ such that $\gamma_j \leq \varepsilon$ for all $j \geq J$. Then

  $$\sum_{j=1}^{s}\gamma_j^{\tau(1+\ln s)} \leq \sum_{j=1}^{J-1} 1 + \sum_{j=J}^{s}\varepsilon^{\tau \ln s} \leq J - 1 + s^{1-\tau\ln\varepsilon^{-1}}.$$

  Choosing $\varepsilon = \exp(-1/\tau)$ we obtain

  $$\sum_{j=1}^{s}\gamma_j^{\tau(1+\ln s)} \leq J.$$

  Note that the last J depends on τ, but it is finite for every fixed τ. Thus, if $\tau > 1/\alpha$ and $\boldsymbol{\gamma}_I = 0$ we have

  $$\frac{1}{s^2}\left(\sum_{j=1}^{\infty}\lambda_{s,j}^{\tau(1+\ln s)}\right)^{1/\tau}$$

$$\leq \exp\left(\frac{1}{\tau}\left(1+\frac{1}{(\alpha\tau-1)+\alpha\tau\log s}+\frac{c_\tau}{s^{\tau\log 2}}\right)J-2\log s\right)$$
$$\to 0 \quad \text{if } s\to\infty.$$

By the characterization in (27), this implies QPT.

- Case $\boldsymbol{\gamma}_I \in (0,1)$: then for every real $\gamma_* \in (\boldsymbol{\gamma}_I, 1)$ there exists a $j_0 = j_0(\gamma_*) \in \mathbb{N}$ such that
$$\gamma_j \leq \gamma_* < 1 \quad \text{for all } j > j_0.$$
Then we obtain for every $s \in \mathbb{N}$ that
$$\sum_{j=1}^{s} \gamma_j^{\tau(1+\ln s)} \leq j_0 + \gamma_*^{\tau(1+\ln s)} \max(s-j_0,0)$$
$$= j_0 + \frac{\gamma_*^{\tau}\max(s-j_0,0)}{s^{\tau\ln\gamma_*^{-1}}} \leq j_0+1,$$
as long as $\tau \geq (\ln\gamma_*^{-1})^{-1}$. Thus, if $\tau > 1/\alpha$ and $\tau \geq (\ln\gamma_*^{-1})^{-1}$ we have
$$\frac{1}{s^2}\left(\sum_{j=1}^{\infty}\lambda_{s,j}^{\tau(1+\ln s)}\right)^{1/\tau}$$
$$\leq \exp\left(\frac{1}{\tau}\left(1+\frac{1}{(\alpha\tau-1)+\alpha\tau\log s}+\frac{c_\tau}{s^{\tau\log 2}}\right)(j_0+1)-2\log s\right)$$
$$\to 0 \quad \text{if } s\to\infty.$$
Again, by the characterization in (27), this implies QPT.

Of course, QPT implies UWT and this in turn implies WT.

So it suffices to show that WT implies $\boldsymbol{\gamma}_I < 1$. Assume on the contrary that $\boldsymbol{\gamma}_I = 1$, i.e., $\gamma_j = 1$ for all $j \in \mathbb{N}$. Then for all $\boldsymbol{k} \in \{0,1\}^s$ we have $r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = 1$. This means that $\{0,1\}^s \subseteq \mathcal{A}(\varepsilon, s)$, where $\mathcal{A}(\varepsilon, s)$ is defined in (19), and hence, according to (20), $n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \geq 2^s$. This means that the approximation problem suffers from the curse of dimensionality and, in particular, we cannot have WT. This proves the first assertion of item 3.

It remains to show the result about the exponent of QPT. Again from [17, Sec. 23.1.1] we know that the exponent of QPT is
$$t^*(\Lambda^{\mathrm{all}}) = 2\inf\{\tau \ : \ \tau \text{ such that (27) holds}\}.$$
From the first part of the proof of item 3. it follows that τ satisfies (27) as long as
$$\tau > \begin{cases} \max(\frac{1}{\alpha}, (\ln\boldsymbol{\gamma}_I^{-1})^{-1}) & \text{if } \boldsymbol{\gamma}_I \neq 0, \\ \frac{1}{\alpha} & \text{if } \boldsymbol{\gamma}_I = 0. \end{cases}$$

Therefore,

$$t^*(\Lambda^{\rm all}) \leq 2 \max\left(\frac{1}{\alpha}, \frac{1}{\ln \boldsymbol{\gamma}_I^{-1}}\right).$$

Assume now that we have QPT. Then (27) holds true for some $\tau > 0$. Considering the special instance $s = 1$ this means

$$C \geq \left(\sum_{j=1}^{\infty} \lambda_{1,j}^{\tau}\right)^{1/\tau}.$$

According to (28) we then have

$$C \geq \left(1 + \gamma_1^{\tau}\left(\sum_{k=\alpha+1}^{\infty}\left(\frac{(k-\alpha)!}{k!}\right)^{\tau}\right)\right)^{1/\tau} \geq \gamma_1 \left(\sum_{k=\alpha+1}^{\infty} \frac{1}{k^{\alpha\tau}}\right)^{1/\tau}$$

and hence we must have $\tau > 1/\alpha$. This already implies the result $t^*(\Lambda^{\rm all}) = 2/\alpha$, whenever $\boldsymbol{\gamma}_I = 0$.

It remains to study the case $\boldsymbol{\gamma}_I > 0$. Now, again according to (27) and (28), there exists a $\tau > 1/\alpha$ such that for all $s \in \mathbb{N}$ we have

$$C \geq \frac{1}{s^2}\left(\prod_{j=1}^{s}\left(1 + \gamma_j^{(\tau(1+\log s)}\right)\right)^{1/\tau} = \exp\left(\frac{1}{\tau}\sum_{j=1}^{s} \log\left(1 + \gamma_j^{\tau(1+\log s)}\right) - 2\log s\right).$$

Taking the logarithm leads to

$$\log C \geq \frac{1}{\tau}\sum_{j=1}^{s} \log\left(1 + \gamma_j^{\tau(1+\log s)}\right) - 2\log s \geq \frac{s}{\tau}\log\left(1 + \boldsymbol{\gamma}_I^{\tau(1+\log s)}\right) - 2\log s$$

for all $s \in \mathbb{N}$. Since $\boldsymbol{\gamma}_I \in (0,1)$ and since $\log(1+x) \geq x \log 2$ for all $x \in [0,1]$ it follows that for all $s \in \mathbb{N}$ we have

$$\log C \geq \frac{s \log 2}{\tau} \boldsymbol{\gamma}_I^{\tau(1+\log s)} - 2\log s = \frac{\boldsymbol{\gamma}_I^{\tau} s \log 2}{\tau\, s^{\tau \log \boldsymbol{\gamma}_I^{-1}}} - 2\log s.$$

This implies that $\tau \geq (\log \boldsymbol{\gamma}_I^{-1})^{-1}$. Therefore, we also have that

$$t^*(\Lambda^{\rm all}) \geq 2\max\left(\frac{1}{\alpha}, \frac{1}{\log \boldsymbol{\gamma}_I^{-1}}\right).$$

Hence item 3. is proven .

4. We know from (20) that $n(\varepsilon, \mathrm{APP}_s) = |\mathcal{A}(\varepsilon, s)|$. Fixing some $q > 1/\alpha$, we get from Lemma 19

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\rm all}) \quad = \quad |\mathcal{A}(\varepsilon, s)|$$

$$\leq \quad \varepsilon^{-2q} \prod_{j=1}^{s} \left(1 + \alpha^{\alpha q}\, \zeta(\alpha q)\, \gamma_j^q\right)$$
$$\leq \quad \varepsilon^{-2q} \left(1 + \alpha^{\alpha q}\, \zeta(\alpha q)\right)^s .$$

Write $c := 1 + \alpha^{\alpha q}\, \zeta(\alpha q)$ which is larger than 1. Then we have

$$\log n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \leq 2q \log \varepsilon^{-1} + s \log c.$$

Hence, for $\sigma > 1$

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}})}{s^\sigma + \varepsilon^{-\tau}} \leq \lim_{s+\varepsilon^{-1}\to\infty} \frac{2q \log \varepsilon^{-1} + s \log c}{s^\sigma + \varepsilon^{-\tau}} = 0.$$

This implies (σ, τ)-WT.

□

Proof of Corollary 17. From Proposition 4 and 7 in conjunction with Proposition 14 we find that for $R \in \{\rho_{s,\alpha,\boldsymbol{\gamma}}, \psi_{s,\alpha,\boldsymbol{\gamma}}\}$ and for all $n \in \mathbb{N}$ we have

$$e(n, \mathrm{APP}_R; \Lambda^{\mathrm{all}}) \leq e(n, \mathrm{APP}_{r_{s,\alpha,\boldsymbol{\gamma}}}; \Lambda^{\mathrm{all}})$$

and hence

$$n(\varepsilon, \mathrm{APP}_R; \Lambda^{\mathrm{all}}) \leq n(\varepsilon, \mathrm{APP}_{r_{s,\alpha,\boldsymbol{\gamma}}}; \Lambda^{\mathrm{all}}) \quad \text{for all } \varepsilon \in (0,1).$$

Thus the sufficient conditions follow directly from Theorem 16.

Let $R \in \{\rho_{s,\alpha,\boldsymbol{\gamma}}, \psi_{s,\alpha,\boldsymbol{\gamma}}\}$ and assume that we have (S)PT for L_2-approximation in $\mathcal{H}_R$ for the class Λ^{all}. Then it follows from Proposition 4 and 7 in conjunction with Proposition 14 that we have (S)PT for L_2-approximation in $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}/(t\alpha^\alpha)}}$ for the class Λ^{all}, where $t = 1$ if $R = \rho_{s,\alpha,\boldsymbol{\gamma}}$ and $t = 2$ if $R = \psi_{s,\alpha,\boldsymbol{\gamma}}$. From Theorem 16 we now obtain $s_{\boldsymbol{\gamma}/(t\alpha^\alpha)} < \infty$. Since $s_{\boldsymbol{\gamma}} = s_{\boldsymbol{\gamma}/(t\alpha^\alpha)}$ this implies $s_{\boldsymbol{\gamma}} < \infty$. □

4.2 Tractability for the class Λ^{std}

The next theorem states sufficient conditions for tractability of L_2-approximation for the class Λ^{std}.

Theorem 20. *Let $\alpha > 1$ and $\boldsymbol{\gamma}$ be a sequence of weights. Consider multivariate approximation $\mathrm{APP} = (\mathrm{APP}_s)_{s\geq 1}$ for the weighted Hermite spaces $\mathcal{H}_R$, $R \in \{r_{s,\alpha,\boldsymbol{\gamma}}, \rho_{s,\alpha,\boldsymbol{\gamma}}, \psi_{s,\alpha,\boldsymbol{\gamma}}\}$ for $s \in \mathbb{N}$ and for the information class Λ^{std}. Then we have the following sufficient conditions:*

1. *SPT holds if*

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

In this case the exponent of SPT satisfies

$$\tau^*(\Lambda^{\mathrm{std}}) = 2 \max\left(\frac{1}{\alpha}, s_{\boldsymbol{\gamma}}\right). \tag{29}$$

2. PT holds if

$$\limsup_{s\to\infty} \frac{1}{\ln s}\sum_{j=1}^{s} \gamma_j < \infty. \tag{30}$$

3. WT holds if

$$\lim_{s\to\infty} \frac{1}{s}\sum_{j=1}^{s} \gamma_j = 0. \tag{31}$$

4. For $\sigma \in (0,1]$ (σ,τ)-WT holds if

$$\lim_{s\to\infty} \frac{1}{s^{\sigma}}\sum_{j=1}^{s} \gamma_j = 0. \tag{32}$$

5. UWT holds if

$$\lim_{s\to\infty} \frac{1}{s^{\sigma}}\sum_{j=1}^{s} \gamma_j = 0 \quad \textit{for all } \sigma \in (0,1]. \tag{33}$$

It suffices to prove the result for $R = r_{s,\alpha,\boldsymbol{\gamma}}$. Our analysis will be based on relations between the minimal errors of Λ^{std} and Λ^{all}, in particular on [3, Theorem 1] and on [24, Theorem 1] (see also [17, Theorem 26.10]). These results provide that the trace of the operator W_s from (16) is finite. Recall that the trace of W_s is given by the sum of its eigenvalues, that is,

$$\begin{aligned}
\mathrm{trace}(W_s) &= \sum_{j=1}^{\infty} \lambda_{s,j} = \sum_{\boldsymbol{k}\in\mathbb{N}_0^s} r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{s}\left(1+\sum_{k=1}^{\infty} r_{\alpha,\gamma_j}(k)\right) \\
&= \prod_{j=1}^{s}\left(1+\gamma_j\left(\sum_{k=1}^{\alpha-1}\frac{1}{k!}+\sum_{k=\alpha}^{\infty}\frac{(k-\alpha)!}{k!}\right)\right) \\
&\le \prod_{j=1}^{s}\left(1+\gamma_j\left(\mathrm{e}-1+\sum_{k=1}^{\infty}\frac{1}{k^{\alpha}}\right)\right) \\
&= \prod_{j=1}^{s}\Big(1+\gamma_j\big(\mathrm{e}-1+\zeta(\alpha)\big)\Big),
\end{aligned}$$

which is finite provided that $\alpha > 1$. Using Lemma 1 we obtain in a similar way that

$$\mathrm{trace}(W_s) \ge \prod_{j=1}^{s}\big(1+\gamma_j\zeta(\alpha)\big)$$

and hence $\mathrm{trace}(W_s)$ is infinite if and only if $\alpha = 1$. Note that in general there is no relation between the power of Λ^{all} and Λ^{std} whenever the trace of W_s is infinite. For a discussion of this issue we refer to [17, Section 26.3].

However, if $\alpha > 1$ we obtain that there exists a positive constant $c(\alpha) \in [\zeta(\alpha), \mathrm{e} - 2 + \zeta(\alpha)]$ such that the trace of W_s equals

$$\operatorname{trace}(W_s) = \prod_{j=1}^{s} \left(1 + \gamma_j \, c(\alpha)\right) \tag{34}$$

and is finite for all $s \in \mathbb{N}$.

Proof of Theorem 20. According to Proposition 4 and 7 in conjunction with Proposition 14 it suffices to proof the result for the Fourier weights $R = r_{s,\alpha,\boldsymbol{\gamma}}$.

Since $\alpha > 1$ we know that $\operatorname{trace}(W_s)$ is finite for all $s \in \mathbb{N}$.

1. For the proof we use [3, Theorem 1] from which we know that there exists a universal constant $c \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ we have

$$e(c\,n, \mathrm{APP}_s; \Lambda^{\mathrm{std}})^2 \le \frac{1}{n} \sum_{k=n}^{\infty} e(k, \mathrm{APP}_s; \Lambda^{\mathrm{all}})^2. \tag{35}$$

Assume that $\sum_{j=1}^{\infty} \gamma_j < \infty$. Then, obviously, the sum exponent of the weight sequence $\boldsymbol{\gamma}$ satisfies $s_{\boldsymbol{\gamma}} \le 1$. Assume first that $s_{\boldsymbol{\gamma}} < 1$. Then, according to Theorem 16 we have SPT for Λ^{all} with exponent

$$\tau^*(\Lambda^{\mathrm{all}}) = 2 \max\left(\frac{1}{\alpha}, s_{\boldsymbol{\gamma}}\right) < 2.$$

Hence for every $\tau > \tau^*(\Lambda^{\mathrm{all}})$ there exists a $C > 0$ such that $n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \le C\varepsilon^{-\tau}$ and from this we deduce

$$e(k, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \le \frac{C}{k^{1/\tau}}.$$

Inserting into (35) yields

$$\begin{aligned}
e(c\,n, \mathrm{APP}_s; \Lambda^{\mathrm{std}})^2 &\le \frac{C}{n} \sum_{k=n}^{\infty} \frac{1}{k^{2/\tau}} \\
&\le \frac{C}{n} \int_{n-1}^{\infty} \frac{1}{x^{2/\tau}} \,\mathrm{d}x \\
&= \frac{C}{n} \frac{\tau}{2-\tau} \frac{1}{(n-1)^{2/\tau - 1}} \\
&\le \frac{C\,\tau}{2-\tau} \frac{1}{(n-1)^{2/\tau}}.
\end{aligned}$$

Hence there exists a number $a_\tau > 0$ such that

$$e(c\,n, \mathrm{APP}_s; \Lambda^{\mathrm{std}}) \le \frac{a_\tau}{n^{1/\tau}}.$$

This implies that

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{std}}) \le \left\lceil c\, a_\tau^\tau\, \varepsilon^{-\tau} \right\rceil$$

and hence, since $\tau > \tau^*(\Lambda^{\mathrm{std}})$ was arbitrary, we have SPT with exponent

$$\tau^*(\Lambda^{\mathrm{std}}) = 2 \max\left(\frac{1}{\alpha}, s_{\boldsymbol{\gamma}}\right).$$

(Note that trivially $\tau^*(\Lambda^{\mathrm{std}}) \ge \tau^*(\Lambda^{\mathrm{all}}) = 2\max(1/\alpha, s_{\boldsymbol{\gamma}})$.)

Now assume that $s_{\boldsymbol{\gamma}} = 1$. From (35) and (17) we obtain

$$e(c\, n, \mathrm{APP}_s; \Lambda^{\mathrm{std}})^2 \le \frac{1}{n} \sum_{k=n}^{\infty} \lambda_{s,k+1} \le \frac{1}{n} \sum_{j=1}^{\infty} \lambda_{s,j} = \frac{\mathrm{trace}(W_s)}{n}. \tag{36}$$

Now we use (34). For $\sum_{j=1}^\infty \gamma_j < \infty$ and $\alpha > 1$ we have

$$\mathrm{trace}(W_s) = \exp\left(\sum_{j=1}^{s} \ln(1+\gamma_j c(\alpha))\right) \le \exp\left(c(\alpha) \sum_{j=1}^{\infty} \gamma_j\right) =: \Gamma < \infty.$$

Hence, inserting into (36) gives

$$e(c\, n, \mathrm{APP}_s; \Lambda^{\mathrm{std}})^2 \le \frac{\Gamma}{n}.$$

From this we obtain in the same way as above SPT with exponent

$$\tau^*(\Lambda^{\mathrm{std}}) = 2 = 2 \max\left(\frac{1}{\alpha}, s_{\boldsymbol{\gamma}}\right).$$

2. We will use [17, Theorem 26.13]. Assume that the weights satisfy (30). This implies that there exists a finite, positive M such that $\frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < M$ for all s. Then we have

$$\mathrm{trace}(W_s) \le \exp\left(c(\alpha) \sum_{j=1}^{s} \gamma_j\right) \le \exp\left(c(\alpha) M \ln s\right) = s^{c(\alpha) M}.$$

Furthermore, assumption (30) implies that

$$\frac{s\, \gamma_s}{\ln s} \le \frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < M \quad \text{for all} \quad s \in \mathbb{N}$$

and therefore $\gamma_j = \mathcal{O}(j^{-1} \ln j)$ and in particular $s_{\boldsymbol{\gamma}} = 1$. By the characterization in Theorem 16 this implies that approximation is (S)PT for the class in Λ^{all}, i.e., there exist positive C^{all} and p^{all} such that

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \le C^{\mathrm{all}} \varepsilon^{-p^{\mathrm{all}}} \quad \text{for all } \varepsilon \in (0,1) \text{ and } s \in \mathbb{N}.$$

Now [17, Theorem 26.13] implies the existence of a positive $C^{\rm std}$ such that

$$n(\varepsilon, {\rm APP}_s; \Lambda^{\rm std}) \le C^{\rm std} \varepsilon^{-p^{\rm std}} s^{q^{\rm std}} \qquad \text{for all } \varepsilon \in (0,1) \text{ and } s \in \mathbb{N},$$

where

$$p^{\rm std} = p^{\rm all} + 2 \quad \text{and} \quad q^{\rm std} = c(\alpha) M.$$

Hence we have PT also for the class $\Lambda^{\rm std}$.

3.-5. We prove the three statements in one combined argument. If any of the three conditions (31), (32) or (33) holds, then this implies that the weights $(\gamma_j)_{j\ge 1}$ (which we assumed to be non-increasing) have to become less than 1 eventually since otherwise, for every $\sigma \in (0,1]$,

$$\lim_{s\to\infty} \frac{1}{s^\sigma} \sum_{j=1}^s \gamma_j = \lim_{s\to\infty} \frac{s}{s^\sigma} = \lim_{s\to\infty} s^{1-\sigma} \ge 1.$$

(Actually we even have $\boldsymbol{\gamma}_I = 0$.) Therefore, we have by Theorem 16 that UWT (and even QPT) holds for the class $\Lambda^{\rm all}$. Furthermore, we observe that

$$\frac{\ln({\rm trace}(W_s))}{s^\sigma} = \frac{1}{s^\sigma} \ln\left(\prod_{j=1}^s (1+\gamma_j\, c(\alpha))\right) = \frac{1}{s^\sigma} \sum_{j=1}^s \ln(1+\gamma_j\, c(\alpha)) \le \frac{c(\alpha)}{s^\sigma} \sum_{j=1}^s \gamma_j$$

and thus if $\frac{1}{s^\sigma}\sum_{j=1}^s \gamma_j$ converges to 0 as s goes to infinity, with $\sigma \in (0,1]$, then

$$\lim_{s\to\infty} \frac{\ln({\rm trace}(W_s))}{s^\sigma} \le \lim_{s\to\infty} \frac{c(\alpha)}{s^\sigma} \sum_{j=1}^s \gamma_j = 0. \tag{37}$$

Now we obtain with the same arguments as in the proof of [17, Theorem 26.11] that (31) implies WT for the class $\Lambda^{\rm std}$. The proof for the other two notions of WT can be obtained analogously by appropriately modifying the argument used in the proof of [17, Theorem 26.11].

The proof is complete. □

Remark 21. It is obvious from (13) that the sufficient conditions for tractability for information from the class $\Lambda^{\rm std}$ are not weaker than the sufficient conditions for the respective notion of tractability for information from the class $\Lambda^{\rm all}$. For example SPT for the class $\Lambda^{\rm all}$ holds if $s_{\boldsymbol{\gamma}} < \infty$, whereas the sufficient condition for SPT for the class $\Lambda^{\rm std}$ is $\sum_{j=1}^\infty \gamma_j < \infty$, which can be re-formulated in an equivalent way as $s_{\boldsymbol{\gamma}} \le 1$.

Remark 22. Again from (13) it follows that every necessary condition for tractability for information from the class $\Lambda^{\rm all}$ is also necessary for the respective notion of tractability for information from the class $\Lambda^{\rm std}$. Unfortunately these conditions do not match the sufficient conditions obtained from Theorem 20. However, it follows from the argument used in item 3 of the proof of Theorem 16 that in the unweighted case, i.e., $\gamma_j = 1$ for all $j \in \mathbb{N}$, we have

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{std}}) \geq n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \geq 2^s$$

and hence for the unweighted case the L_2-approximation problem for information from Λ^{std} suffers from the curse of dimensionality.

Remark 23. While we have a very clear picture of tractability of L_2-approximation for the Hermite space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ for the information class Λ^{all} there remain several open questions concerning Λ^{std}. In the first place, matching necessary conditions for the respective notions of tractability are still missing. Furthermore, we neither have sufficient nor necessary conditions for quasi-polynomial tractability beyond the sufficient condition for polynomial tractability which obviously also implies quasi-polynomial tractability. Finally, our results require a smoothness parameter α bigger than 1. Similar results for $\alpha = 1$ are still missing.

5 Integration in weighted Hermite spaces

Now we consider the integration problem. The next theorem states sufficient conditions for tractability of integration. Obviously, the information class Λ^{all} makes this problem trivial. For this reason we restrict to the class Λ^{std}.

Theorem 24. *Let $\alpha > 1$ and $\boldsymbol{\gamma}$ be a sequence of weights. Consider multivariate integration $\mathrm{INT} = (\mathrm{INT}_s)_{s \geq 1}$ for the weighted Hermite spaces $\mathcal{H}_R$, $R \in \{r_{s,\alpha,\boldsymbol{\gamma}}, \rho_{s,\alpha,\boldsymbol{\gamma}}, \psi_{s,\alpha,\boldsymbol{\gamma}}\}$ for $s \in \mathbb{N}$. Then we have the following sufficient conditions:*

1. *SPT holds if*
$$\sum_{j=1}^{\infty} \gamma_j < \infty$$
(which is equivalent to $s_{\boldsymbol{\gamma}} \leq 1$). In this case the exponent of SPT satisfies
$$\tau^*(\Lambda^{\mathrm{std}}) \leq 2 \max\left(\frac{1}{\alpha}, s_{\boldsymbol{\gamma}}\right)$$
2. *PT holds if*
$$\limsup_{s \to \infty} \frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < \infty.$$
3. *WT holds if*
$$\lim_{s \to \infty} \frac{1}{s} \sum_{j=1}^{s} \gamma_j = 0.$$
4. *For $\sigma \in (0,1]$ (σ,τ)-WT holds if*
$$\lim_{s \to \infty} \frac{1}{s^{\sigma}} \sum_{j=1}^{s} \gamma_j = 0.$$

5. *UWT holds if*

$$\lim_{s\to\infty} \frac{1}{s^{\sigma}} \sum_{j=1}^{s} \gamma_j = 0 \quad \textit{for all } \sigma \in (0,1].$$

Proof. Using (14) one can transfer our results about tractability of the L_2-approximation problem for standard information to the integration problem in $\mathcal{H}_R$. □

Remark 25. Note that item 1 of the theorem yields an improvement over the upper bound on the exponent of SPT in [11] from 2 to $2\max(1/\alpha, s_{\boldsymbol{\gamma}})$.

Remark 26. Like for the approximation problem using exclusively standard information, also for the integration problem some questions remain open. These comprise of the quest for necessary conditions for the respective notions of tractability, for necessary and sufficient conditions for QPT and results for the case of smoothness $\alpha = 1$.

If we restrict ourselves to linear algorithms of the form

$$A_{n,s}^{\mathrm{int}}(f) = \sum_{i=1}^{n} w_i f(\boldsymbol{x}_i) \tag{38}$$

with $n \in \mathbb{N}$, nodes $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n$ in $\mathbb{R}^s$ and *non-negative* integration weights $w_1, w_2, \ldots, w_n$ we can show that the sufficient conditions for tractability are even necessary. This method has been used by Sloan and Woźniakowski in [20] in the context of numerical integration in Korobov spaces.

We introduce a restricted version of the information complexity by taking into account only linear algorithms with non-negative weights. Define, for $\varepsilon \in (0,1)$ and $s \in \mathbb{N}$, the quantity

$$\begin{aligned} n^{\mathrm{lin,pos}}(\varepsilon, \mathrm{INT}_s) := \min\{n \in \mathbb{N} \ : \ & \exists A_{n,s}^{\mathrm{int}} \text{ of the form (38) with non-negative weights,} \\ & \text{such that } \ e^{\mathrm{int}}(A_{n,s}^{\mathrm{int}}, \mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}) \le \varepsilon\}. \end{aligned}$$

Obviously, $n(\varepsilon, \mathrm{INT}_s) \le n^{\mathrm{lin,pos}}(\varepsilon, \mathrm{INT}_s)$.

Theorem 27. *Let $\alpha \ge 1$ and $\boldsymbol{\gamma}$ be a sequence of weights. Consider multivariate integration* $\mathrm{INT} = (\mathrm{INT}_s)_{s\ge 1}$ *for the weighted Hermite spaces $\mathcal{H}_R$, $R \in \{r_{s,\alpha,\boldsymbol{\gamma}}, \rho_{s,\alpha,\boldsymbol{\gamma}}, \psi_{s,\alpha,\boldsymbol{\gamma}}\}$ for $s \in \mathbb{N}$, but restrict to the class of linear algorithms of the form* (38) *with non-negative weights. Then we have the following necessary conditions:*

1. *SPT implies*

$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

2. *PT implies*

$$\limsup_{s\to\infty} \frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < \infty.$$

3. WT implies

$$\lim_{s\to\infty}\frac{1}{s}\sum_{j=1}^{s}\gamma_j=0.$$

4. For $\sigma\in(0,1]$ (σ,τ)-WT implies

$$\lim_{s\to\infty}\frac{1}{s^\sigma}\sum_{j=1}^{s}\gamma_j=0.$$

5. UWT implies

$$\lim_{s\to\infty}\frac{1}{s^\sigma}\sum_{j=1}^{s}\gamma_j=0 \quad \textit{for all } \sigma\in(0,1].$$

The proof of Theorem 27 is based on the following proposition, which is an analogy to [20, Theorem 4], that applies to Korobov spaces.

Proposition 28. *For every linear algorithm $A_{n,s}^{\rm int}$ of the form (38) with non-negative integration weights we have*

$$\left(e^{\rm int}(A_{n,s}^{\rm int},\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}})\right)^2\geq 1-\frac{n}{\prod_{j=1}^{s}(1+\gamma_j c_\omega)},$$

where $c_\omega:=(1-\sqrt{1-\omega^2})/\sqrt{1-\omega^2}>0$ and $\omega:=3^{-\alpha/3}$. In particular,

$$n^{\rm lin,pos}(\varepsilon,{\rm INT}_s)\geq(1-\varepsilon^2)\prod_{j=1}^{s}(1+\gamma_j c_\omega).$$

Proof. We define a further reproducing kernel Hilbert space based on Hermite polynomials. For $\omega\in(0,1)$ we let

$$K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y}):=\sum_{\boldsymbol{k}\in\mathbb{N}_0^s}\phi_{s,\omega,\boldsymbol{\gamma}}(\boldsymbol{k})H_{\boldsymbol{k}}(\boldsymbol{x})H_{\boldsymbol{k}}(\boldsymbol{y}),$$

where now the used Fourier weights are $R(\boldsymbol{k})=\phi_{s,\omega,\boldsymbol{\gamma}}(\boldsymbol{k}):=\prod_{j=1}^{s}\phi_{\omega,\gamma_j}(k_j)$ with

$$\phi_{\omega,\gamma}(k):=\begin{cases}1 & \text{for } k=0,\\ \gamma\omega^k & \text{for } k\geq 1.\end{cases}$$

Let $\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ denote the corresponding reproducing kernel Hilbert space with inner product and norm

$$\langle f,g\rangle_{\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}}:=\sum_{\boldsymbol{k}\in\mathbb{N}_0^s}\frac{1}{\phi_{s,\omega,\boldsymbol{\gamma}}(\boldsymbol{k})}\widehat{f}(\boldsymbol{k})\,\widehat{g}(\boldsymbol{k}) \quad\text{and}\quad \|f\|_{\phi_{s,\omega,\boldsymbol{\gamma}}}=\sqrt{\langle f,f\rangle_{\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}}},$$

respectively. The space $\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ and integration therein has been studied already in [11]. In particular, in [11, Proposition 3.7] it is shown that the functions from the space $\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ are analytic functions.

For $\alpha \geq 1$ choose $\omega = \omega(\alpha) \in (0,1)$ such that we have

$$\frac{1}{k^\alpha} \geq \omega^k \quad \text{for all } k \in \mathbb{N}.$$

For example $\omega := \min_{k\geq 1} k^{-\alpha/k} = 3^{-\alpha/3}$ is a suitable choice. Then we have

$$r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \geq \phi_{s,\omega,\gamma}(\boldsymbol{k}) \quad \text{for all } \boldsymbol{k} \in \mathbb{N}_0^s$$

and hence, for every $f \in \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ we have

$$\|f\|_{r_{s,\alpha,\boldsymbol{\gamma}}} \leq \|f\|_{\phi_{s,\omega,\boldsymbol{\gamma}}}.$$

This shows that the space $\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ is continuously embedded in the space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$ and the norm of the embedding operator is at most 1. Therefore, integration in $\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ is not harder than in the space $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$. This implies that for every algorithm, and we restrict ourselves to linear algorithms $A_{n,s}^{\rm int}$ with non-negative integration weights like in (38) in the following, the integration errors in $\mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ and in $\mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}$, respectively, are related as

$$e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}}) \leq e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{r_{s,\alpha,\boldsymbol{\gamma}}}).$$

Now we consider $e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}})$. Using a well-known formula for the squared integration error of linear algorithms in reproducing kernel Hilbert spaces (see, e.g., [14, Exercise 3.15]) we obtain

$$\begin{aligned}
\left(e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}})\right)^2 \;=\; & \int_{\mathbb{R}^s}\int_{\mathbb{R}^s} K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})\varphi_s(\boldsymbol{x})\varphi_s(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\
& -2\sum_{i=1}^n w_i \int_{\mathbb{R}^s} K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{x}_i)\varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& +\sum_{i,\ell=1}^n w_i w_\ell K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x}_i,\boldsymbol{x}_\ell).
\end{aligned}$$

It is easy to see (or consult [11, p. 191]) that

$$\int_{\mathbb{R}^s}\int_{\mathbb{R}^s} K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{y})\varphi_s(\boldsymbol{x})\varphi_s(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} = 1$$

and

$$\int_{\mathbb{R}^s} K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x},\boldsymbol{x}_i)\varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1 \quad \text{for all } i \in \{1,2,\ldots,n\}.$$

Therefore we obtain

$$\left(e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}})\right)^2 = 1 - 2\sum_{i=1}^n w_i + \sum_{i,\ell=1}^n w_i w_\ell K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x}_i,\boldsymbol{x}_\ell). \tag{39}$$

From Mehler's formula (see [21]), which states that for every $x, y \in \mathbb{R}$ and every $\omega \in (-1, 1)$ we have

$$\sum_{k=0}^{\infty} H_k(x) H_k(y) \omega^k = \frac{1}{\sqrt{1-\omega^2}} \exp\left(\frac{\omega x y}{1+\omega} - \frac{\omega^2 (x-y)^2}{2(1-\omega^2)}\right),$$

one can derive that

$$K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x}, \boldsymbol{y}) = \prod_{j=1}^{s} \left(1 - \gamma_j + \gamma_j \frac{1}{\sqrt{1-\omega^2}} \exp\left(\frac{\omega x_j y_j}{1+\omega} - \frac{\omega^2 (x_j - y_j)^2}{2(1-\omega^2)}\right)\right).$$

This shows, in particular, that the kernel $K_{\phi_{s,\omega,\boldsymbol{\gamma}}}$ is non-negative. Since also the integration weights w_i are non-negative, we deduce from (39) by neglecting the non-diagonal terms in the double-sum that

$$\left(e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}})\right)^2 \ge 1 - 2\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i^2 K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x}_i, \boldsymbol{x}_i). \tag{40}$$

Now, for $i \in \{1, 2, \ldots, n\}$ we have

$$\begin{aligned} K_{\phi_{s,\omega,\boldsymbol{\gamma}}}(\boldsymbol{x}_i, \boldsymbol{x}_i) &= \prod_{j=1}^{s} \left(1 - \gamma_j + \gamma_j \frac{1}{\sqrt{1-\omega^2}} \exp\left(\frac{\omega x_{i,j}^2}{1+\omega}\right)\right) \\ &\ge \prod_{j=1}^{s} \left(1 - \gamma_j + \gamma_j \frac{1}{\sqrt{1-\omega^2}}\right) \\ &= \prod_{j=1}^{s} (1 + \gamma_j c_\omega), \end{aligned}$$

where $x_{i,j}$ is the j-th component of $\boldsymbol{x}_i$ and $c_\omega := (1 - \sqrt{1-\omega^2})/\sqrt{1-\omega^2} > 0$. Inserting this estimate into (40) we get

$$\left(e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}})\right)^2 \ge 1 - 2\sum_{i=1}^{n} w_i + \sum_{i=1}^{n} w_i^2 \prod_{j=1}^{s} (1 + \gamma_j c_\omega). \tag{41}$$

Next, set $\beta := (\sum_{i=1}^{n} w_i^2)^{1/2}$ and observe that by the Cauchy-Schwarz inequality we have $\sum_{i=1}^{n} w_i \le \sqrt{n}\beta$. Thus we my conclude from (41) that

$$\left(e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}})\right)^2 \ge 1 - 2\sqrt{n}\beta + \beta^2 \prod_{j=1}^{s} (1 + \gamma_j c_\omega). \tag{42}$$

Minimizing the expression on the right-hand side of (42) with respect to β we obtain that

$$\left(e^{\rm int}(A_{n,s}^{\rm int}, \mathcal{H}_{\phi_{s,\omega,\boldsymbol{\gamma}}})\right)^2 \ge 1 - \frac{n}{\prod_{j=1}^{s} (1 + \gamma_j c_\omega)}.$$

From here the upper bound on $n^{\rm lin,pos}(\varepsilon, {\rm INT}_s)$ follows immediately. $\square$

Now we can give the proof of Theorem 27.

Proof of Theorem 27. Again it suffices to prove the result for $R = r_{s,\alpha,\boldsymbol{\gamma}}$. We will use Proposition 28 and arguments from [20, Proof of Theorem 5].

Assume that the weights are bounded from below by some positive number γ_*, i.e. $\gamma_j \geq \gamma_* > 0$ for all $j \in \mathbb{N}$. Then it follows from Proposition 28 that

$$n^{\mathrm{lin,pos}}(\varepsilon, \mathrm{INT}_s) \geq (1-\varepsilon^2)(1+\gamma_* c_\omega)^s.$$

Thus $n^{\mathrm{lin,pos}}(\varepsilon, \mathrm{INT}_s)$ grows exponentially fast in s and hence we cannot have any form of tractability. Thus, if we have some form of tractability, then we must also have $\lim_{j\to\infty} \gamma_j = 0$.

Now suppose that we have $\lim_{j\to\infty}\gamma_j = 0$ but $\sum_{j=1}^{\infty}\gamma_j = \infty$. For $\lim_{j\to\infty}\gamma_j = 0$ it is a well-known fact that

$$\prod_{j=1}^{s}(1+\gamma_j c_\omega) = \Theta\left(\exp\left(c_\omega \sum_{j=1}^{s}\gamma_j\right)\right). \tag{43}$$

Then it follows from Proposition 28 and Equation (43) that $\lim_{s\to\infty} n^{\mathrm{lin,pos}}(\varepsilon, \mathrm{INT}_s) = \infty$ and this contradicts SPT. Thus $\sum_{j=1}^{\infty}\gamma_j < \infty$ is a necessary condition for SPT.

Suppose next that we have $\lim_{j\to\infty}\gamma_j = 0$ but $\limsup_{s\to\infty}(1/\log s)\sum_{j=1}^{s}\gamma_j = \infty$. Since

$$\prod_{j=1}^{s}(1+\gamma_j c_\omega) = \Theta\left(s^{c_\omega (1/\log s)\sum_{j=1}^{s}\gamma_j}\right),$$

it follows from Proposition 28 that $n^{\mathrm{lin,pos}}(\varepsilon, \mathrm{INT}_s)$ goes to infinity faster than any power of s and this contradicts PT. Thus $\limsup_{s\to\infty}(1/\log s)\sum_{j=1}^{s}\gamma_j < \infty$ is a necessary condition for PT.

Finally, assume that for $\sigma \in (0,1]$ we have

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n^{\mathrm{lin,pos}}(\varepsilon, \mathrm{INT}_s)}{s^\sigma + \varepsilon^{-\tau}} = 0.$$

Then it follows from Proposition 28 and Equation (43) that

$$\lim_{s\to\infty}\frac{1}{s^\sigma}\sum_{j=1}^{s}\gamma_j = 0.$$

This implies the necessary conditions for the three WT notions. □

6 Remarks on integration in the anchored space

In [25] Wasilkowski and Woźniakowski studied L_2-approximation and integration over unbounded domains. The underlying function space in this work is a more general version of the reproducing kernel Hilbert space with kernel L from (9). Choosing $\psi = \varphi^{1/2}$ and $\omega = \varphi$ in [25] corresponds exactly to the setting of the present work.

Unfortunately, the results from [25] concerning tractability of L_2-approximation cannot be transferred to our setting here (see Section 2.4), since [25, Theorems 1 and 2] require the assumption [25, Eq. (17)] which is $\int_{\mathbb{R}}(\sqrt{\omega(x)}/\psi(x))^{1/\alpha}\,\mathrm{d}x < \infty$, but which is obviously not satisfied in our case where $\psi = \varphi^{1/2}$ and $\omega = \varphi$.

The results about integration in [25] do not require this assumption. This means we can transfer them directly into our setting in order to obtain "if and only if"-conditions for numerical integration in the anchored space $\mathcal{H}_{⫛,s,\alpha,\boldsymbol{\gamma}}$. This has already been done in [16, Sec. 12.5.1] (in a slightly different but equivalent formulation). The following result is basically [16, Corollary 12.8] (which we extend by results about (σ,τ)-WT and UWT). We stress that here we also have necessary conditions thanks to the fact that the kernel $K_{⫛,\alpha,\gamma}$ contains the decomposable part L.

Theorem 29. *Let $\alpha > 1$ and $\boldsymbol{\gamma}$ be a sequence of weights. Consider multivariate integration* $\mathrm{INT} = (\mathrm{INT}_s)_{s\geq 1}$ *for the weighted anchored spaces $\mathcal{H}_{⫛,s,\alpha,\boldsymbol{\gamma}}$ for $s \in \mathbb{N}$. Then we have:*

1. *SPT holds if and only if*
$$\sum_{j=1}^{\infty} \gamma_j < \infty.$$

2. *PT holds if and only if*
$$\limsup_{s\to\infty} \frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < \infty.$$

3. *WT holds if and only if*
$$\lim_{s\to\infty} \frac{1}{s} \sum_{j=1}^{s} \gamma_j = 0.$$

4. *For $\sigma \in (0,1]$ (σ,τ)-WT holds if and only if*
$$\lim_{s\to\infty} \frac{1}{s^\sigma} \sum_{j=1}^{s} \gamma_j = 0.$$

5. *UWT holds if and only if*
$$\lim_{s\to\infty} \frac{1}{s^\sigma} \sum_{j=1}^{s} \gamma_j = 0 \quad \text{for all } \sigma \in (0,1].$$

A Appendix: The proof of Theorem 3

For the Gaussian ANOVA space we know the Hermite expansion of the reproducing kernel, namely, in dimension 1 and for a generic weight $\gamma > 0$,

$$K_{r_{\alpha,\gamma}}(x,y) = \sum_{k\in\mathbb{N}_0} r_{\alpha,\gamma}(k) H_k(x) H_k(y).$$

Now we derive the integral representation presented in Theorem 3.

The starting point is the weighted Gaussian ANOVA norm from (6) given by

$$\|f\|^2_{r_{1,\alpha,\gamma}} = \left(\int_{\mathbb{R}} f(y)\varphi(y)\,\mathrm{d}y\right)^2 + \frac{1}{\gamma}\sum_{k=1}^{\alpha-1}\left(\int_{\mathbb{R}} f^{(k)}(y)\varphi(y)\,\mathrm{d}y\right)^2 + \frac{1}{\gamma}\int_{\mathbb{R}} (f^{(\alpha)}(y))^2\varphi(y)\,\mathrm{d}y$$

on the space of functions

$$\mathcal{H}_\alpha = \left\{ f\colon \mathbb{R}\to\mathbb{R}\colon f^{(\alpha-1)} \text{ exists and is abs. continuous}, \int_{\mathbb{R}} |f^{(\alpha)}(y)|^2\varphi(y)\,\mathrm{d}y < \infty \right\}.$$

The space $\mathcal{H}_\alpha$ decomposes into the orthogonal subspaces

$$\begin{aligned}
\mathcal{H}_{1,\alpha} &:= \left\{ f\colon \mathbb{R}\to\mathbb{R}\colon f^{(\alpha-1)} \text{ exists and is abs. continuous}, \int_{\mathbb{R}} |f^{(\alpha)}(y)|^2\varphi(y)\,\mathrm{d}y = 0 \right\},\\
\mathcal{H}_{2,\alpha} &:= \Bigg\{ f\colon \mathbb{R}\to\mathbb{R}\colon f^{(\alpha-1)} \text{ exists and is abs. continuous}, \int_{\mathbb{R}} |f^{(\alpha)}(y)|^2\varphi(y)\,\mathrm{d}y < \infty,\\
&\qquad\qquad \int_{\mathbb{R}} f^{(k)}(y)\varphi(y)\,\mathrm{d}y = 0,\ k\in\{0,\dots,\alpha-1\} \Bigg\},
\end{aligned}$$

and therefore, using property (7) from [1, Section 2], the reproducing kernel K_α of the space $(\mathcal{H}_\alpha, \|\cdot\|_{r_{1,\alpha,\gamma}})$ is of the form $K_\alpha = K_{1,\alpha} + K_{2,\alpha}$, where $K_{j,\alpha}$ is a reproducing kernel for $\mathcal{H}_{j,\alpha}$, $j\in\{1,2\}$. Clearly, $\mathcal{H}_{1,\alpha}$ consists precisely of the polynomials of degree smaller than α and therefore every $f_1 \in \mathcal{H}_{1,\alpha}$ can be written as

$$f_1(x) = \sum_{k=0}^{\alpha-1} \widehat{f}_1(k) H_k(x),$$

such that, for $j\in\{0,1,\dots,\alpha-1\}$,

$$\int_{\mathbb{R}} f_1^{(j)}(y)\varphi(y)\,\mathrm{d}y = \sum_{k=j}^{\alpha-1}\sqrt{\frac{k!}{(k-j)!}}\,\widehat{f}_1(k)\int_{\mathbb{R}} H_{k-j}(y)\varphi(y)\,\mathrm{d}y = \sqrt{j!}\,\widehat{f}_1(j)$$

and therefore

$$f_1(x) = \sum_{j=0}^{\alpha-1} (j!)^{-1/2}\int_{\mathbb{R}} f^{(j)}(y)\,\varphi(y)\,\mathrm{d}y\, H_j(x)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} f(y)\varphi(y)\,\mathrm{d}y \ \int_{\mathbb{R}} K_{1,\alpha}(x,y)\varphi(y)\,\mathrm{d}y \\
&\quad + \frac{1}{\gamma}\sum_{j=1}^{\alpha-1}\int_{\mathbb{R}} f^{(j)}(y)\,\varphi(y)\,\mathrm{d}y \ \int_{\mathbb{R}}\left(\frac{\partial^j}{\partial y^j}K_{1,\alpha}(x,y)\right)\varphi(y)\,\mathrm{d}y
\end{aligned}
$$

with $K_{1,\alpha}(x,y) = 1 + \sum_{k=1}^{\alpha-1}\frac{\gamma}{k!}H_k(x)H_k(y)$. Here we used that the j-th derivative of H_k equals

$$
H_k^{(j)}(y) = \begin{cases} \sqrt{\frac{k!}{(k-j)!}}\,H_{k-j}(y) & \text{if } k \geq j, \\ 0 & \text{otherwise,} \end{cases}
$$

from which we obtain that

$$
H_j(x) = \sqrt{j!}\int_{\mathbb{R}}\left(\frac{\partial^j}{\partial y^j}K_{1,\alpha}(x,y)\right)\varphi(y)\,\mathrm{d}y.
$$

We proceed to compute $K_{2,\alpha}$. Here and in the following we write $\Phi(y) := \int_{-\infty}^{y}\varphi(\eta)\,\mathrm{d}\eta$ and

$$
\vartheta(x,y) := 1_{(-\infty,x]}(y)\Phi(y) - 1_{(x,\infty)}(y)\Phi(-y)\,.
$$

Recall that for $y \leq -1$ we have

$$
0 \leq \Phi(y) = \int_{-\infty}^{y}\varphi(\eta)\,\mathrm{d}\eta \leq \int_{-\infty}^{y}(-\eta)\varphi(\eta)\,\mathrm{d}\eta = \int_{-\infty}^{y}\varphi'(\eta)\,\mathrm{d}\eta = \varphi(y)
$$

and that therefore also $0 \leq \Phi(-y) \leq \varphi(-y) = \varphi(y)$ for $y > 1$ so that

$$
\int_{\mathbb{R}} 1_{(-\infty,x]}(y)\Phi(y)\,\mathrm{d}y \quad \text{and} \quad \int_{\mathbb{R}} 1_{(x,\infty)}(y)\Phi(-y)\,\mathrm{d}y
$$

are real numbers.

Lemma 30. *Let $h\colon \mathbb{R} \to \mathbb{R}$ be measurable with $\int_{\mathbb{R}}(h(y))^2\varphi(y)\,\mathrm{d}y < \infty$. Then $g\colon \mathbb{R} \to \mathbb{R}$ with*

$$
g(x) := \int_{\mathbb{R}} h(y)\vartheta(x,y)\,\mathrm{d}y \qquad \text{for } x \in \mathbb{R},
$$

is the unique absolutely continuous function with $g' = h$ a.e. and $\int_{\mathbb{R}} g(y)\varphi(y)\,\mathrm{d}y = 0$.

Proof. Since we may write

$$
g(x) := \int_{\mathbb{R}} h(y)\vartheta(x,y)\,\mathrm{d}y = \int_{-\infty}^{x} h(y)\Phi(y)\,\mathrm{d}y - \int_{x}^{\infty} h(y)\Phi(-y)\,\mathrm{d}y,
$$

and the integrals exist since

$$
\int_{-\infty}^{-1}|h(y)|\Phi(y)\,\mathrm{d}y \leq \int_{-\infty}^{-1}|h(y)|\varphi(y)\,\mathrm{d}y \leq \left(\int_{\mathbb{R}}(h(y))^2\varphi(y)\,\mathrm{d}y\right)^{1/2} < \infty,
$$

it is clear that g is absolutely continuous. Differentiating gives a.e.

$$g'(x) = h(x)\Phi(x) + h(x)\Phi(-x) = h(x)\Phi(x) + h(x)(1 - \Phi(x)) = h(x)\,.$$

Next we integrate g with respect to the weight φ and use Fubini's theorem to get

$$\begin{aligned}
\int_{\mathbb{R}} g(x)\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}}\int_{-\infty}^{x} h(y)\Phi(y)\,\mathrm{d}y\,\varphi(x)\,\mathrm{d}x - \int_{\mathbb{R}}\int_{x}^{\infty} h(y)\Phi(-y)\,\mathrm{d}y\,\varphi(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}}\int_{y}^{\infty} h(y)\Phi(y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}}\int_{\infty}^{y} h(y)\Phi(-y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{\mathbb{R}} h(y)\Phi(y)\Phi(-y)\,\mathrm{d}y - \int_{\mathbb{R}} h(y)\Phi(-y)\Phi(y)\,\mathrm{d}y = 0.
\end{aligned}$$

Finally let g_1 be an arbitrary absolutely continuous function with $g_1' = h$ a.e. and $\int_{\mathbb{R}} g'(y)\varphi(y)\,\mathrm{d}y = 0$. Then $g_1(x) = \int_0^x h(y)\,\mathrm{d}y + c_1$ a.e. On the other hand, $g(x) = \int_0^x h(y)\,\mathrm{d}y + c_2$, so $g_1(x) = g(x) + c_3$ a.e. But since $0 = \int_{\mathbb{R}} g_1(x)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} g(x)\varphi(x)\,\mathrm{d}x + c_3 = c_3$, we have $g_1 = g$ a.e. $\square$

Next we compute the reproducing Kernel $K_{2,\alpha}$ for $\alpha = 1$. For every $f \in \mathcal{H}_{2,1}$, i.e., with

$$\int_{\mathbb{R}} f(y)\varphi(y)\,\mathrm{d}y = 0 \quad \text{and} \quad \int_{\mathbb{R}} (f'(y))^2\varphi(y)\,\mathrm{d}y < \infty$$

we have from the reproducing property of the kernel $K_{2,1}$ that

$$f(x) = \frac{1}{\gamma}\int_{\mathbb{R}} f'(y)\frac{\partial}{\partial y}K_{2,1}(x,y)\varphi(y)\,\mathrm{d}y.$$

On the other hand we know from Lemma 30 that $f(x) = \int_{\mathbb{R}} f'(y)\vartheta(x,y)\,\mathrm{d}y$, so

$$\int_{\mathbb{R}} f'(y)\left(\frac{1}{\gamma}\frac{\partial}{\partial y}K_{2,1}(x,y)\varphi(y) - \vartheta(x,y)\right)\mathrm{d}y = 0.$$

Since this holds in particular if f' is the indicator function of an arbitrary measurable set, we conclude

$$\frac{\partial}{\partial y}K_{2,1}(x,y) = \gamma\varphi(y)^{-1}\vartheta(x,y)$$

for a.e. $y \in \mathbb{R}$. Since $y \mapsto K_{2,1}(x,y)$ is an element of $\mathcal{H}_{2,1}$, we need to have $\int_{\mathbb{R}} K_{2,1}(x,y)\varphi(y)\,\mathrm{d}y = 0$. Again from Lemma 30 we derive the integral representation

$$K_{2,1}(x,y) = \int_{\mathbb{R}} \frac{\partial}{\partial y}K_{2,1}(x,\eta)\vartheta(y,\eta)\,\mathrm{d}\eta = \gamma\int_{\mathbb{R}} \varphi(\eta)^{-1}\vartheta(x,\eta)\vartheta(y,\eta)\,\mathrm{d}\eta\,,$$

Which finishes the proof of Theorem 3 for the case $\alpha = 1$. Note that

$$\begin{aligned}
\vartheta(x,\eta)\vartheta(y,\eta) &= \left(1_{(-\infty,x]}(\eta)\Phi(\eta) - 1_{(x,\infty)}(\eta)\Phi(-\eta)\right)\left(1_{(-\infty,y]}(\eta)\Phi(\eta) - 1_{(y,\infty)}(\eta)\Phi(-\eta)\right) \\
&= 1_{(-\infty,\min(x,y)]}(\eta)(\Phi(\eta))^2 + 1_{(\max(x,y),\infty)}(\eta)(\Phi(-\eta))^2
\end{aligned}$$

$$- 1_{(\min(x,y),\max(x,y)]}(\eta)\Phi(\eta)\Phi(-\eta).$$

So $K_{2,1}$ can be written in terms of the primitive functions of

$$\varphi^{-1}\Phi^2,\ \varphi^{-1}(1-\Phi)^2,\ \text{and}\ \varphi^{-1}\Phi(1-\Phi).$$

Next we compute $K_{2,\alpha}$ for $\alpha = 2$. If $f \in \mathcal{H}_{2,2}$, we obtain from the reproducing property of the kernel $K_{2,2}$ that

$$f(x) = \frac{1}{\gamma}\int_{\mathbb{R}} f''(y)\frac{\partial^2}{\partial y^2}K_{2,2}(x,y)\,\mathrm{d}y$$

so that

$$f'(x) = \frac{1}{\gamma}\int_{\mathbb{R}} f''(y)\frac{\partial^3}{\partial x\partial y^2}K_{2,2}(x,y)\varphi(y)\,\mathrm{d}y.$$

On the other hand, if $f \in \mathcal{H}_{2,2}$ then $f' \in \mathcal{H}_{2,1}$, and therefore

$$f'(x) = \frac{1}{\gamma}\int_{\mathbb{R}} f''(y)\frac{\partial}{\partial y}K_{2,1}(x,y)\varphi(y)\,\mathrm{d}y.$$

Thus we have

$$\frac{1}{\gamma}\int_{\mathbb{R}} f''(y)\left(\frac{\partial^3}{\partial x\partial y^2}K_{2,2}(x,y) - \frac{\partial}{\partial y}K_{2,1}(x,y)\right)\varphi(y)\,\mathrm{d}y = 0$$

from which we obtain $\frac{\partial^3}{\partial x\partial y^2}K_{2,2} = \frac{\partial}{\partial y}K_{2,1}$. Now

$$\frac{\partial^2}{\partial x\partial y}K_{2,2}(x,y) = K_{2,1}(x,y) + c_1(x)$$

and, since $K_{2,2}$ is symmetric and $\frac{\partial^2}{\partial x\partial y}K_{2,2}$ is continuous, $\frac{\partial^2}{\partial x\partial y}K_{2,2} = \frac{\partial^2}{\partial y\partial x}K_{2,2}$, so $\frac{\partial^2}{\partial x\partial y}K_{2,2}$ is also symmetric. From this it follows that c_1 is actually constant, $c_1(x) = c_1$.

Since the function $y \mapsto K_{2,2}(x,y)$ is an element of $\mathcal{H}_{2,2}$ for every $x \in \mathbb{R}$, we get

$$\int_{\mathbb{R}} \frac{\partial}{\partial y}K_{2,2}(x,y)\varphi(y)\,\mathrm{d}y = 0 \quad \text{for every } x \in \mathbb{R},$$

so that also

$$\begin{aligned}
0 &= \frac{\partial}{\partial x}\int_{\mathbb{R}} \frac{\partial}{\partial y}K_{2,2}(x,y)\varphi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}} \frac{\partial^2}{\partial x\partial y}K_{2,2}(x,y)\varphi(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}} (K_{2,1}(x,y) + c_1)\varphi(y)\,\mathrm{d}y \\
&= c_1.
\end{aligned}$$

So, actually $c_1 = 0$, and hence

$$\frac{\partial^2}{\partial x \partial y} K_{2,2}(x,y) = K_{2,1}(x,y).$$

Now, integrating with respect to x another time, we get using Lemma 30 once more that

$$\frac{\partial}{\partial y} K_{2,2}(x,y) = \int_{\mathbb{R}} K_{2,1}(\xi,y)\vartheta(x,\xi)\,\mathrm{d}\xi + c_2(y), \tag{44}$$

so

$$\int_{\mathbb{R}} \frac{\partial}{\partial y} K_{2,2}(x,y)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}} K_{2,1}(\xi,y) \int_{\mathbb{R}} \vartheta(x,\xi)\varphi(x)\,\mathrm{d}x\,\mathrm{d}\xi + c_2(y).$$

But

$$\begin{aligned}\int_{\mathbb{R}} \vartheta(x,y)\varphi(x)\,\mathrm{d}x &= \int_{\mathbb{R}} 1_{(-\infty,x]}(y)\varphi(x)\,\mathrm{d}x\,\Phi(y) - \int_{\mathbb{R}} 1_{(x,\infty)}(y)\varphi(x)\,\mathrm{d}x\,\Phi(-y)\\ &= \Phi(-y)\Phi(y) - \Phi(y)\Phi(-y) = 0\end{aligned}$$

and

$$\int_{\mathbb{R}} \frac{\partial}{\partial y} K_{2,2}(x,y)\varphi(x)\,\mathrm{d}x = \frac{\partial}{\partial y}\int_{\mathbb{R}} K_{2,2}(x,y)\varphi(x)\,\mathrm{d}x = 0$$

by symmetry of $K_{2,2}$ so $c_2(y) = 0$. Now, integrating (44) (with $c_2(y) = 0$) with respect to y gives,

$$K_{2,2}(x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{2,1}(\xi,\eta)\vartheta(x,\xi)\vartheta(y,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + c_3(x),$$

and we see that

$$0 = \int_{\mathbb{R}} K_{2,2}(x,y)\varphi(y)\,\mathrm{d}y = \int_{\mathbb{R}}\int_{\mathbb{R}} K_{2,1}(\xi,\eta)\vartheta(x,\xi)\int_{\mathbb{R}} \vartheta(y,\eta)\varphi(y)\,\mathrm{d}y\,\mathrm{d}\xi\,\mathrm{d}\eta + c_3(x) = c_3(x).$$

Thus we have found

$$\begin{aligned}K_{2,2}(x,y) &= \int_{\mathbb{R}}\int_{\mathbb{R}} K_{2,1}(\xi,\eta)\vartheta(x,\xi)\vartheta(y,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\ &= \gamma\int_{\mathbb{R}^3} \frac{1}{\varphi(s)}\vartheta(\xi,s)\vartheta(\eta,s)\vartheta(x,\xi)\vartheta(y,\eta)\,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}\eta.\end{aligned}$$

By induction we get the integral representation of the general kernel

$$K_{2,\alpha}(x,y) = \gamma\int_{\mathbb{R}^{2\alpha-1}} \frac{1}{\varphi(s)}\vartheta_\alpha(x,\xi_{\alpha-1},\dots,\xi_1,s)\vartheta_\alpha(y,\eta_{\alpha-1},\dots,\eta_1,s)\,\mathrm{d}s\prod_{k=1}^{\alpha-1}(\,\mathrm{d}\xi_k\,\mathrm{d}\eta_k)$$

with $\vartheta_n(z_1,\dots,z_{n+1}) := \prod_{k=1}^n \vartheta(z_k,z_{k+1})$ for $n \in \mathbb{N}$. This finishes the proof of Theorem 3. $\square$

Acknowledgment. We thank David Krieg for a valuable discussion, in particular for his help in proving the exponent of SPT in (29).

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