AUSTRIAN ACADEMY OF SCIENCES

RICAM – JOHANN RADON INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Mathematical Analysis of the Photo-acoustic Imaging Modality Using Resonating Dielectric Nano-particles: The 2D TM-Model

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RICAM-Report 2022-12

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MATHEMATICAL ANALYSIS OF THE PHOTO-ACOUSTIC IMAGING MODALITY USING RESONATING DIELECTRIC NANO-PARTICLES: THE $2D$ TM-MODEL

AHCENE GHANDRICHE * AND MOURAD SINI‡

Abstract. We deal with the photoacoustic imaging modality using dielectric nano-particles as contrast agents. Exciting the heterogeneous tissue, localized in a bounded domain Ω, with an electromagnetic wave, at a given incident frequency, creates heat in its surrounding which in turn generates an acoustic pressure wave (or fluctuations). The acoustic pressure can be measured in the accessible region $\partial\Omega$ surrounding the tissue of interest. The goal is then to extract information about the optical properties (i.e. the permittivity and conductivity) of this tissue from these measurements. We describe two scenarios. In the first one, we inject single nano-particles while in the second one we inject couples of closely spaced nano-particles (i.e. dimers). From the acoustic pressure measured, before and after injecting the nano-particles (for each scenario), at two single points x_1 and x_2 of $\partial\Omega$ and two single times $t_1 \neq t_2$ such that $t_1, t_2 > \text{diam}(\Omega)$,

(1) we localize the center point z of the single nano-particle and reconstruct the phaseless total field $|u_0|$ on that point z (where u_0 is the total field in the absence of the nano-particles). Hence, we transform the photoacoustic problem into the inversion of phaseless internal electric fields.
(2) we localize the centers z_1 and z_2 of the injected dimers and reconstruct both the permittivity and the conductivity of the tissue on those points.

This is done using *dielectric* nano-particles enjoying high contrasts of their electric permittivity.

These results are possible using frequencies of incidence close to the resonances of the used dielectric nano-particles. These particular frequencies are computable. The error of approximations are given in terms of the scales and the contrasts of the dielectric nano-particles. The results are justified in the 2D TM-model.

1. Introduction and statement of the results

1.1. Motivation and the mathematical models. Imaging using small scaled contrast agents has received in the recent years a considerable attention, see for instance [8, 9, 26]. To motivate it, let us recall that conventional imaging techniques, as microwave imaging, are known to be potentially capable of extracting features in breast cancer, for instance, in case of the relatively high contrast of the permittivity, and conductivity, between healthy tissues and malignant ones, [12]. However, it is observed that in case of benign tissue, the variation of the permittivity is quite low so that such conventional imaging modalities are limited to be used for early detection of such diseases. In these cases, creating such missing contrast is highly desirable. One way to do it is to use micro or nano scaled particles as contrast agents, [8, 9]. There are several imaging modalities using contrast agents as acoustic imaging using gas microbubbles, optical imaging and photoacoustic using dielectric or magnetic nano-particles [8, 12, 22]. The first two modalities are single wave based methods. In this work, we deal with the last imaging modality.

Photoacoustic imaging is a hybrid imaging method which is based on coupling electromagnetic waves with acoustic waves to achieve high-resolution imaging of optical properties of biological tissues, [18, 21].

Date: September 9, 2021.

2010 *Mathematics Subject Classification.* 35R30, 35C20.

Key words and phrases. photo-acoustic imaging, nano-particles, dielectric resonances, inverse problems.

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Precisely, exciting the heterogeneous tissue with an electromagnetic wave, at a certain frequency related to the used small scale particles, creates heat in its surrounding which in turn generates an acoustic pressure wave (or fluctuations). The acoustic wave can be measured in a region surrounding the tissue of interest. The goal is then to extract information about the optical properties of this tissue from these measurements, [18, 21].

A main reason why such a modality is promising is that injecting nano-particles, see [8, 9] for information on its feasibility, with appropriate scales between their sizes and optical properties, in the targeted tissue will create localized contrasts in the tissue and hence amplify the local electromagnetic energy around its location. This amplification can be more pronounced if the used incident electromagnetic wave is sent at frequencies close to resonances. In particular, dielectric or magnetic nano-particles (as gold nano-particles [18]) can exhibit such resonances when its inner electric permittivity or magnetic permeability is tuned appropriately, see below. Our target here is to mathematically analyze this imaging technique when injecting such nano-particles.

To give more insight to this, let us briefly recall the photoacoustic model, see [11, 15, 19, 25, 27, 28] for extensive studies and different motivations of this model and related topics. We assume the time harmonic (TM) approximation for the electromagnetic model [1], then the third component of the electric field, that we denote by u, satisfies

$$\Delta u + \omega^2 \boldsymbol{\varepsilon} \mu_\infty u = 0, \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

with

$$u := u^i + u^s \text{ where } u^i := u^i(x, d, \omega) := e^{i\omega\,\sqrt{\epsilon_\infty \mu_\infty}\, d\cdot x},$$

is the incident plane wave, sent at a frequency ω (and) in direction d, $|d| = 1$, and $u^s := u^s(x, \omega)$ is the corresponding scattered wave selected according to the outgoing Sommerfeld radiation condition (S.R.C) at infinity. Here, ϵ_∞ and μ_∞ are the electric permittivity and magnetic permeability of the vacuum, which we assume to be positive real constants, and $\boldsymbol{\varepsilon} := \boldsymbol{\varepsilon}(x)$ is defined as

$$\boldsymbol{\varepsilon}(x) := \begin{cases} \epsilon_\infty, & in \quad \mathbb{R}^2 \setminus \Omega, \\ \epsilon_0(x), & in \quad \Omega \setminus \overset{M}{\underset{m=1}{\cup}} D_m, \\ \epsilon_m, & in \quad D_m, \end{cases} \tag{1.2}$$

where $\epsilon_0 := \epsilon_r + i\frac{\sigma_\Omega}{\omega}$ with ϵ_r as the permittivity and σ_Ω the conductivity of the heterogeneous tissue (i.e. variable functions). The quantity ϵ_m is the permittivity constant of the particle D_m, of radius $a \ll 1$, which is taken to be complex valued, i.e. $\epsilon_m := \epsilon_{m,r} + i\frac{\sigma_m}{\omega}$ where $\epsilon_{m,r}$ is its actual electric permittivity and σ_m its conductivity. The bounded domain Ω models the region of the tissue of interest. We take the nano-particle of dielectric type, meaning that $\dfrac{\epsilon_m}{\epsilon_\infty} \gg 1$ when $a \ll 1$, and hence its relative index of refraction is very large as well. Under particular rates of the ratio $\dfrac{\epsilon_m}{\epsilon_\infty} \gg 1$, resonances can occur, as the dielectric (or Mie-electric) resonances. These regimes will be of particular interest to us. Here, we take the D_m's of the form $D_m := z_m + a\, B_m$ where z_m models its location, a its radius and B_m as a smooth domain of radius 1 containing the origin.

As said above, exciting the tissue with such electromagnetic waves will generate a heat $\boldsymbol{T}$ which in turn generates acoustic pressure p. Under some appropriate conditions, see [5, 28] for instance, this process is modeled by the following system:

[1]Here, we describe the photoacoustic model assuming the TM-approximation of the electromagnetic field. The more realistic model is of course the full Maxwell system.

$$
\begin{cases}
\rho_0 c_p \dfrac{\partial \boldsymbol{T}}{\partial t} - \nabla \cdot \kappa \nabla \boldsymbol{T} = \omega \mathrm{Im}\,(\boldsymbol{\varepsilon})|u|^2 \delta_0(t), \\[2ex]
\dfrac{1}{c_s^2}\dfrac{\partial^2 p}{\partial t^2} - \Delta p = \rho_0 \beta_0 \dfrac{\partial^2 \boldsymbol{T}}{\partial t^2}
\end{cases}
$$

where ρ_0 is the mass density, c_p the heat capacity, κ is the heat conductivity, c_s is the wave speed, β_0 the thermal expansion coefficient and $\delta_0(\cdot)$ is the Dirac delta function. To these two equations, we supplement the homogeneous initial conditions:

$$
\boldsymbol{T} = p = \frac{\partial p}{\partial t} = 0, \text{ at } t = 0.
$$

Under additional assumptions on the smallness of the heat conductivity κ, one can neglect the term $\nabla \cdot \kappa \nabla \boldsymbol{T}$ and hence, we end up with the photoacoustic model linking the electromagnetic field to the acoustic pressure [2]:

$$
\begin{cases}
\dfrac{\partial^2 p}{\partial t^2} - c_s^2 \Delta p = 0, & in \quad \mathbb{R}^2 \times \mathbb{R}_+, \\[2ex]
p(x,0) = \dfrac{\omega \beta_0}{c_p} \mathrm{Im}\,(\boldsymbol{\varepsilon})|u|^2 & in \quad \mathbb{R}^2, \\[2ex]
\dfrac{\partial p}{\partial t}(x,0) = 0, & in \quad \mathbb{R}^2.
\end{cases} \tag{1.3}
$$

The imaging problem we wish to focus on is stated in the following terms:

Problem. Reconstruct the coefficient ϵ_0 from the given pressure $p(x,t)$ measured for $(x,t) \in \partial\Omega \times (0,T)$, with some positive time length T,

(1) after injecting single nano-particle located in a sample of points in Ω,

or/and

(2) after injecting closely spaced couples of nano-particles (i.e. dimers) and located in a sample of points in Ω.

It is natural to split this problem into two steps. The first step concerns the acoustic inversion, namely to reconstruct the source term $\mathrm{Im}\,(\boldsymbol{\varepsilon})|u|^2, \quad x \in \Omega$, from the pressure $p(x,t)$ for $(x,t) \in \partial\Omega \times (0,T)$. The second step concerns the electromagnetic inversion, namely to reconstruct ϵ_0 from the internal data $\mathrm{Im}\,(\boldsymbol{\varepsilon})|u|^2$.

1.2. **The acoustic inversion.** We start by recalling the main results related to the model (1.3). More informations about this part can be found in [1] and [16].

For this inversion, there are two cases to distinguish:

Case 1: The speed of propagation c_s is constant everywhere in $\mathbb{R}^2$ and Ω is a disc.

The solution of the problem (1.3) is given by the Poisson formula

$$
p(x,t) = \frac{\omega\, \beta_0}{2\pi c_s c_p} \partial_t \left(\int_{|x-y|<c_s t} \frac{\mathrm{Im}\,(\boldsymbol{\varepsilon})(y)\, |u|^2(y)}{\sqrt{c_s^2 t^2 - |x-y|^2}} dy \right). \tag{1.4}
$$

We denote by $M(f)$ the circular means of f

$$
M(f)(x,r) := \frac{1}{2\pi} \int_{|\xi|=1} f(x + r\xi)\, d\sigma(\xi).
$$

[2]We stated the model in the whole plane $\mathbb{R}^2$. However, we could also state it in a bounded domain supplemented with Dirichlet or Neumann boundary conditions.

The equation (1.4) takes the following form

$$p(x,t) = \frac{\omega\,\beta_0}{c_s c_p}\partial_t\left(\int_0^{c_s t} \frac{r}{\sqrt{c_s^2 t^2 - r^2}} M(\mathrm{Im}\,(\boldsymbol{\varepsilon})\,|u|^2)(x,r)dr\right).$$

The recovery of $Im(\boldsymbol{\varepsilon})\,|u|^2$ from $p(x,t)$, $(x,t) \in \partial\Omega \times [0,T]$, is done in two steps. First, as $\partial\Omega$ is a circle, the circular means can be recovered from the pressure as follows

$$M(Im(\boldsymbol{\varepsilon})\,|u|^2)(x,r) = \frac{2\omega\beta_0}{c_p \pi}\int_0^{c_s r} \frac{p(x,t)}{\sqrt{r^2 - t^2}}dt, \quad x \in \partial\Omega. \tag{1.5}$$

Second, if $Im(\boldsymbol{\varepsilon})\,|u|^2 \in C^\infty(\mathbb{R}^2)$ with $supp(Im(\boldsymbol{\varepsilon})\,|u|^2) \subset \overline{\Omega}$, then, for $x \in \Omega$,

$$Im(\boldsymbol{\varepsilon})(x)\,|u|^2(x) = \frac{1}{2\pi R_0}\int_{\partial\Omega}\int_0^{2R_0}(\partial_r\, r\, \partial_r M(Im(\boldsymbol{\varepsilon})\,|u|^2))(p,r)\,\log(|r^2 - |x-p|^2|)\,dr\,d\sigma(p). \tag{1.6}$$

We can find in [20] and [13] the justification of (1.5) and (1.6) respectively.

Case 2: The speed of propagation is variable in Ω and constant in $\mathbb{R}^2 \setminus \Omega$, with Ω not necessarily a disc. However, the following assumptions are needed, namely (1). $Supp(\mathrm{Im}\,(\boldsymbol{\varepsilon})\,|u|^2)$ is compact in Ω, (2). $c_s(\cdot) > c > 0$ and $Supp(c_s(\cdot) - 1)$ is compact in Ω and (3). the non trapping condition is verified. In $\mathbb{L}^2(\Omega; c_s^{-2}(\cdot)dx)$, we consider the operator given by the differential expression $A = -c_s^{-2}(\cdot)\Delta$ and the Dirichlet boundary condition on $\partial\Omega$. This operator is positive self-adjoint operator, and has discrete spectrum $\{s_k^2\}_{k\geq 1}$ with a basis set of eigenfunctions $\{\psi_k\}_{k\geq 1}$ in $\mathbb{L}^2(\Omega; c_s^{-2}(\cdot)dx)$. Then, the function $\mathrm{Im}\,(\boldsymbol{\varepsilon})(\cdot)\,|u|^2(\cdot)$ can be reconstructed inside Ω from the data p, as the following $\mathbb{L}^2(\Omega)$ convergent series

$$\mathrm{Im}\,(\boldsymbol{\varepsilon})(x)\,|u|^2(x) = \frac{c_p}{\omega\,\beta_0}\sum_k (\mathrm{Im}\,(\boldsymbol{\varepsilon})(x)\,|u|^2)_k \psi_k(x),$$

where the Fourier coefficients $(\mathrm{Im}\,(\boldsymbol{\varepsilon})(x)\,|u|^2)_k$ can be recovered as:

$$(\mathrm{Im}\,(\boldsymbol{\varepsilon})(x)\,|u|^2)_k = s_k^{-2} p_k(0) - s_k^{-3}\int_0^\infty \sin(s_k t)\, p_k''(t)dt,$$

with

$$p_k(t) := \int_{\partial\Omega} p(x,t)\frac{\partial\overline{\psi}_k}{\partial\nu}(x)dx.$$

More details can be found in [1].

In our work, we address the following two situations regarding the type of the used dielectric nano-particles.

(1) *Only the permittivity $\epsilon_{m,r}$ of the nano-particle is contrasting* [3]. For this case, we use the results above on the acoustic inversion to obtain $\mathrm{Im}\,(\boldsymbol{\varepsilon})(x)\,|u|^2(x)$, $x \in \Omega$ and hence $|u|(x)$, $x \in D_m$, as $\mathrm{Im}\,\boldsymbol{\varepsilon} = \dfrac{\sigma_m}{\omega}$ on D_m which is known. With this information, we perform the electromagnetic inversion to reconstruct ϵ_r and σ_Ω.

(2) *Both the permittivity $\epsilon_{m,r}$ and the conductivity σ_m of the nano-particle are contrasting.* In this case, we do not rely on the acoustic inversion results above. Instead, we propose direct approximating formulas to link the measured data $p(x,t)$ for $x \in \partial\Omega$ and $t \in (0,T)$, to $|u|(x)$, $x \in D_m$. Actually, we need only to measure $p(x,t)$ on two single points on $\partial\Omega$ for two distinct times t_1 and t_2. Then, we perform the electromagnetic inversion.

[3] We can also allow the conductivity σ_m to be contrasting. But this is not needed as in the case (2).

1.3. **The electromagnetic inversion and motivation for using nearly resonant incident frequencies.** We start from the model

$$\begin{cases} (\Delta + \omega^2 \boldsymbol{n}^2) u = 0 & in \quad \mathbb{R}^2 \\ u := u^i + u^s & and \quad u^s \quad S.R.C \end{cases} \tag{1.7}$$

where, taking $M = 1$ in (1.2),

$$\boldsymbol{n} := \begin{cases} \sqrt{\epsilon_p \mu_\infty} & in \quad D \\ \boldsymbol{n_0} & in \quad \mathbb{R}^2 \setminus D \end{cases}$$

and

$$\boldsymbol{n_0} := \begin{cases} \sqrt{\epsilon_0 \mu_\infty} & in \quad \Omega \\ \sqrt{\epsilon_\infty \mu_\infty} & in \quad \mathbb{R}^2 \setminus \Omega. \end{cases}$$

We call the dielectric (or *Mie-electric*) resonances the possible eigenvalues of (1.7), i.e. the possible solutions (ω^2, u^s) of (1.7) when $u^i = 0$. Using the Lippmann-Schwinger equation (L.S.E), such eigenvalues are also characterized by the equation

$$u(x) = -\omega^2 \int_D (\boldsymbol{n_0}^2 - \boldsymbol{n}^2) G_k(x,y)\, u(y) dy, \quad x \in \mathbb{R}^2, \tag{1.8}$$

where G_k is the Green's function satisfying $(\Delta + \omega^2 \boldsymbol{n_0}^2) G_k = -\delta$ with the S.R.C, and

$$k := \omega\, \boldsymbol{n_0} \tag{1.9}$$

is the wave number. As ϵ_p is constant in D and, for simplicity of the exposition here, we assume ϵ_0 to be constant in Ω then, setting

$$\tau := \epsilon_p - \epsilon_0 \tag{1.10}$$

where ϵ_p is defined by

$$\epsilon_p := \epsilon_{p,r} + i \, \frac{\sigma_p}{\omega}, \tag{1.11}$$

we have

$$\boldsymbol{n}^2 - \boldsymbol{n_0}^2 = \begin{cases} \mu_\infty \, \tau & in \quad D \\ 0 & in \quad \mathbb{R}^2 \setminus D, \end{cases}$$

and then we get from (1.8)

$$u(x) \frac{1}{\omega^2 \mu_\infty \tau} = \int_D G_k(x,y) u(y) dy, \quad x \in \mathbb{R}^2. \tag{1.12}$$

It is known from the scattering theory, precisely Rellich's lemma, that those eigenvalues belong to the lower complex plane $\mathbb{C}_-$. However, as $\tau \gg 1$, and $a \ll 1$, their imaginary parts tend to zero, see [4] for instance. To solve (1.12), it is enough to find and compute eigenvalues $w_n(k)$ of the volumetric potential operator A_k defined as

$$A_k(u)(x) := \int_D G_k(x,y) u(y) dy, \;\; u \in \mathbb{L}^2(D). \tag{1.13}$$

Then combining (1.12) and (1.13), we can write $A_k(u) = \dfrac{1}{\omega^2 \mu_\infty \tau}\, u$ and then solve in ω, and recalling that $k = \omega\, \boldsymbol{n_0}$, the dispersion equation

$$w_n(k) = \frac{1}{\omega^2 \mu_\infty \tau}. \tag{1.14}$$

Let us now recall that the operator LP, called the Logarithmic Potential operator, defined by[4]

$$LP(u)(\eta) := \int_B -\frac{1}{2\pi} \log|\eta - \xi| \; u(\xi) \; d\xi, \; u \in \mathbb{L}^2(B), \; \eta \in B,$$

[4] LP is the potential operator corresponding to the Laplacian.

has a countable sequence of eigenvalues with the corresponding eigenfunctions as a basis of $\mathbb{L}^2(B)$. For more details see [14] and [7]. Correspondingly, we define A_0 to be

$$A_0(u)(x) := \int_D -\frac{1}{2\pi} \log|x-y| \, u(y) \, dy, \; u \in \mathbb{L}^2(D), \; x \in D. \tag{1.15}$$

Rescaling, we have

$$A_0(u)(x) = a^2 LP\left(\tilde{u}\right)(\xi) - \frac{a^2 \log(a)}{2\pi} \int_B \tilde{u}(\xi) d\xi, \quad \xi := \frac{x-z}{a}.$$

Hence the eigenvalue problem $A_0(u) = \lambda_n u$, on D, becomes

$$LP\left(\tilde{u}\right) - \frac{\log(a)}{2\pi} \int_B \tilde{u}(\eta) d\eta = \frac{\lambda_n}{a^2} \tilde{u}, \quad \text{on } B.$$

We observe that the spectrum $Spect(A_0|_{\mathbb{L}^2_0(D)})$ of A_0, restricted to $\mathbb{L}^2_0(D) := \{v \in \mathbb{L}^2(D), \; \int_D v \, dx = 0\}$, is characterized by $Spect(A_0|_{\mathbb{L}^2_0(D)}) = a^{-2} Spect(LP|_{\mathbb{L}^2_0(B)})$. However, as we see it later, the important eigenvalues are those for which the corresponding eigenfunctions are not average-zero. Therefore, we need to handle the other part of the spectrum of A_0 as well. As $\mathbb{L}^2_0(D)$ is not invariant under the action of A_0, the natural decomposition $\mathbb{L}^2(D) = \mathbb{L}^2_0(D) \oplus 1$ does not decompose it.

The following properties are needed in the sequel and we state them as hypotheses to keep a higher generality.

Hypotheses 1. *The particles D, of radius a, $a \ll 1$, are taken such that the spectral problem $A_0 u = \lambda \, u$, in D, has eigenvalues λ_n and corresponding eigenfunctions, e_n, satisfying the following properties:*

(1) $\int_D e_n(x) dx \neq 0, \; \forall a \ll 1$.

(2) $\lambda_n \sim a^2 |\log(a)|, \; \forall a \ll 1$.

In the appendix, see section 5.1, we show that for particles of general shapes, the first eigenvalue and the corresponding eigenfunctions satisfy **Hypotheses** 1. In addition, we characterize the properties of the eigenvalues for the case when D is a disc.

Since, the dominant part of the operator A_k defined in (1.13) is A_0, we can write[5]

$$w_n(k) = \lambda_n + \mathcal{O}(a^2). \tag{1.16}$$

Combining (1.16), (5.1) and (1.14), we get $\lambda_n = \dfrac{1}{\omega^2 \, \mu_\infty \, \tau} + \mathcal{O}(a^2) \overset{1.10}{=} \dfrac{1}{\omega^2 \, \mu_\infty \, \epsilon_p} + \mathcal{O}(a^2)$ or, with help of (1.11), $\omega^2 = \dfrac{1}{\mu_\infty \, \epsilon_p \, \lambda_n} + \mathcal{O}(|\log(a)|^{-1})$. This means that (1.7) has a sequence of eigenvalues that can be approximated by

$$\frac{1}{\mu_\infty \, \epsilon_p \, \lambda_n} + \mathcal{O}(|\log(a)|^{-1}).$$

The dominating term is finite if the contrast of the used nano-particle's permittivity behaves as $\tau \sim \lambda_n^{-1} \sim a^{-2} |\log(a)|^{-1}$ for $a \ll 1$.

We distinguish two cases as related to our imaging problem.

[5]More precisely, using the expansion and the scales of the fundamental solution, we show that an eigenvalue of A_k can be written as

$$a^2 \left(\int_B \int_B \frac{-1}{2\pi} \log|\eta - \xi| \; \overline{e}_n(\eta) \, \overline{e}_n(\xi) \, d\eta \, d\xi + \left(\frac{1}{2\,\pi} |\log(a)| + \varrho \right) \left(\int_B \overline{e}_n(\xi) d\xi \right)^2 \right) + \mathcal{O}\left(a^3\right)$$

where $\overline{e}_n := \frac{\tilde{e}_n}{\|\tilde{e}_n\|}$ and $\tilde{e}_n(\cdot) := e_n\left(\frac{\cdot - z}{a}\right)$.

(1) Injecting one nano-particle and then sending incident plane waves at real frequencies ω close to the values

$$\omega_n := (\mu_\infty \, \epsilon_p \, \lambda_n)^{-1/2}, \tag{1.17}$$

we can excite, approximately, the sequence of eigenvalues described above. As a consequence, see the justification later, if we excite with incident frequencies near ω_n, $n \in \mathbb{N}$, the total field u solution of (1.7), restricted to D will be dominated by $\int_D u_0(x) \, e_n(x) \, dx \;\; e_n(x)$, which is, in turn, dominated by $u_0(z) \, e_n(x) \int_D e_n(x) \, dx$ where u_0 is the wave field in the absence of the nano-particles, i.e. $(\Delta + \omega^2 \boldsymbol{n}_0^2) u_0 = 0, \quad u_0 = u^i + u_0^s$ and u_0^s satisfies the S.R.C. Hence from the acoustic inversion, i.e. from the knowledge of $\mathrm{Im}\,(\boldsymbol{\varepsilon})(x)|u|(x)$, $x \in \Omega$, and hence $|u|(x)$, $x \in D$, as, for $x \in D$, $\mathrm{Im}\,(\boldsymbol{\varepsilon}) = \mathrm{Im}\,(\epsilon_p) = \dfrac{\sigma_p}{\omega}$ is known, we can reconstruct

$$|u_0(z)| \; |e_n(z)| \; \left| \int_D e_n(x) \, dx \right|.$$

As e_n and D are in principle known, then we can recover the total field $|u_0(z)|$. Taking a sampling of points z in Ω, we get at hand the phaseless internal total field $|u_0(z)|$, $z \in \Omega$.

(2) Now, we inject a dimer, meaning a couple of close nano-particles, instead of only single nano-particle, with prescribed high contrasts of the relative permittivity or/and conductivity. Sending incident plane wave at frequencies close to the dielectric resonances, we recover also the amplitude of the field generated by the first interaction of the two nano-particles. Indeed, based on point-approximation expansions, this field can be approximated by the Foldy-Lax field. This field describes the one due to multiple interactions between the nano-particles. We show that the acoustic inversion approximately reconstructs the first multiple interaction field (i.e. the Neumann series cut at the first, and not the zero, order term). From this last field, we recover the values of (real part) of the Green's function, G_k, on the location points of the dimer. Finally, from this Green's function, we reconstruction, at the expense of numerical differentiations, both the permittivity ϵ_r and the conductivity σ_Ω inside Ω.

Both steps are justified using incident frequencies close to the dielectric resonance of the nano-particles. This would not be possible using incident frequencies away from these resonances.

1.4. **Statement of the results.** We recall that the mathematical model of the photoacoustic imaging modality is (1.1), (1.2) and (1.3).

Next, we set $u := u_j$, $j = 0, 1, 2$, the solution of (1.1) and (1.2) when there is no nano-particle injected, there is one or two nano-particles, respectively (i.e take $M = 0, 1$ or 2 in (1.2)).

To keep the technicalities to the minimum, we deal only with the case when the electromagnetic properties of the injected nano-particles are the same i.e,

$$\epsilon_1 = \cdots = \epsilon_M =: \epsilon_p. \tag{1.18}$$

1.4.1. *Imaging using dielectric nano-particles with permittivity contrast only.* Let the permittivity $\epsilon_0(\cdot)$, of the medium, be $W^{1,\infty}-$smooth in Ω and the permeability μ_∞ to be constant and positive. Let also the injected nano-particles D satisfy **Hypotheses** 1. We assume these nano-particles to be characterized with moderate magnetic permeability and their permittiviy and conductivity are such that $\epsilon_{p,r} \sim a^{-2} \, |\log(a)|^{-1}$ while $\sigma_p \sim 1$ as $a \ll 1$. The frequency of the incidence ω is chosen close to the dielectric resonance ω_{n_0}

$$\omega_{n_0}^2 := (\mu_\infty \, \epsilon_p \, \lambda_{n_0})^{-1}, \tag{1.19}$$

as follows

$$\omega^2 = \text{Re}\,\left(\omega^2_{n_0}\right)\left(1 \pm |\log(a)|^{-h}\right), \quad \frac{1}{2} < h < 1. \text{ [6]}$$

Here, we assume that the acoustic inversion is already performed using one of the methods given in section 1.2. Hence, we have at hands

$$|u_j|(x),\, x \in D,\, j = 1, 2.$$

Theorem 1.1. *We have the following expansion of u_j, $j = 1, 2$:*

(1) Injecting a single nano-particle at once. In this case, we use the data $|u_1|(x)$, $x \in D$. We have the following approximation

$$\int_D |u_1|^2(x)dx = \frac{|u_0|^2(z)(\int_D e_{n_0}(x)dx)^2}{|1 - \omega^2 \mu_\infty \epsilon_p \lambda_{n_0}|^2} + \mathcal{O}\left(a^2\,|\log(a)|^{3h-1}\right). \tag{1.20}$$

(2) Injecting two closely spaced nano-partilces. These two nano-particles are distant to each other as

$$d := |z_1 - z_2| \sim a^{|\log(a)|^{-h}},\; a \ll 1,$$

where z_1 and z_2 are the location points of the nano-particles.
In this case, we use as data $|u_j|(x)$, $x \in D$, $j = 1, 2$, where D is any one of the two nano-particles. The following expansion is valid

$$\text{Re}\,\left(G_k(z_1, z_2)\right) = \Phi_0 + \frac{1}{2\boldsymbol{C}}\,\frac{A_1 - (1 - \boldsymbol{C}\Phi_0)^2}{A_1 - 2(1 - \boldsymbol{C}\Phi_0)} + \mathcal{O}\left(|\log(a)|^{\max(h-1;1-2h)}\right), \tag{1.21}$$

where

$$A_1 := \frac{\int_D |u_1|^2(x)dx}{\int_D |u_2|^2(x)dx}, \quad \Phi_0 := \frac{-1}{2\pi}\log|z_1 - z_2|$$

and

$$\boldsymbol{C} := \int_D \left[\frac{1}{\omega^2 \mu_\infty \epsilon_p} I - A_0\right]^{-1}(1)(x)dx = \frac{\omega^2 \mu_\infty \epsilon_p}{1 - \omega^2 \mu_\infty \epsilon_p \lambda_{n_0}}\left(\int_D e_{n_0}(x)dx\right)^2 + \mathcal{O}\left(|\log(a)|^{-1}\right).$$

1. Using solely single nanoparticles, from the formula (1.20), we can derive an estimate of the total field in the absence of the nano-particles, i.e. $|u_0|(x)$, $x \in \Omega$, by repeating the same experiment scanning the targeted tissue located in Ω by injecting single nano-particles. Hence, we transform the photo-acoustic inverse problem to the reconstruction of ϵ_0 in the equation $(\Delta + \omega^2 \mu_\infty \epsilon_0)u_0 = 0$, in $\mathbb{R}^2$, from the phaseless internal data $|u_0|(x)$, $x \in \Omega$.

2. Using dimers, we deduce from (1.20) and (1.21), the values of the Green's function $\text{Re}\, G_k(z_1, z_2)$ for z_1 and z_2 in Ω (after scanning Ω with such dimers). At the expense of numerical differentiation, we recover from $\text{Re}\, G_k$ the coefficient $\epsilon_0(\cdot)$ in Ω. The details are provided in section 2.3.

1.4.2. *Imaging using dielectric nano-particles with both permittivity and conductivity contrasts.* As in section 1.4.1, let the permittivity $\epsilon_0(\cdot)$, of the medium, be $W^{1,\infty}-$smooth in Ω and the permeability μ_∞ to be constant and positive. Let also the injected nano-particles D satisfy **Hypotheses** 1. Here, we assume that $\epsilon_{m,r} \sim a^{-2}|\log(a)|^{-1}$and $\sigma_m \sim a^{-2}|\log(a)|^{-1-h-s}$, $s > 0$.[7] The frequency of the incidence ω is chosen close to the dielectric resonance ω_{n_0}, where $\omega^2_{n_0} := (\mu_\infty\, \epsilon_p\, \lambda_{n_0})^{-1}$, as follows

$$\omega^2 = \omega^2_\pm := \text{Re}\,\left(\omega^2_{n_0}\right)\left(1 \pm |\log(a)|^{-h}\right), \quad \frac{1}{2} < h < 1. \tag{1.22}$$

Theorem 1.2. *Let $x \in \partial\Omega$ and $t \geq diam(\Omega)$. We have the following expansions of the pressure:*

[6]Choosing + or - does not make a difference for the results in Theorem 1.1.

[7]Observe that the ratio $\frac{\sigma_m}{\epsilon_{m,r}} \ll 1$, as $a \ll 1$ which is compatible with the properties of dielectric nano-particles.

(1) *Injecting a single nano-particle at once. In this case, we have the expansion*

$$
\begin{aligned}
(p^+ + p^- - 2p_0)(t,x) \quad &= \quad \frac{-t\,\omega\,\beta_0}{c_p\,(t^2-|x-z|^2)^{\frac{3}{2}}}\,\frac{2\text{Im}\,(\epsilon_p)|u_0(z)|^2}{|1-\omega^2\mu_\infty\lambda_{n_0}\epsilon_p|^2}\left(\int_D e_{n_0}(x)dx\right)^2 \\
&+ \quad \mathcal{O}\left(|\log(a)|^{\max(-h-s,2h-2,h-1-2s)}\right),
\end{aligned} \tag{1.23}
$$

under the condition $0<s<1-h$, where p^+ and p^- correspond to the pressure after injecting the nano-particle and exciting with frequencies of incidence (1.22), with the sign $+$ and $-$ respectively, while p_0 is the pressure in the absence of the nano-particles.

(2) *Injecting two close dielectric nano-particles. We set*

$$\tilde{p}(t,x) := \left(p^+ - p_0\right)(t,x) + \Theta\left(p^- - p_0\right)(t,x),$$

where

$$\Theta := 2\,\frac{\left|\omega_\pm^2-\omega_{n_0}^2\right|\frac{1}{2\pi}\,|\log(a)|}{\left|\omega_\pm^2-\omega_{n_0}^2\right|\frac{1}{2\pi}\,|\log(a)|+\Phi_0\left(z_1,z_2\right)\,Re\left(\omega_{n_0}^2\right)}.$$

Then, under the condition $0<s<1-h$, we have the following expansions.

(a) *When the two nano-particles are distant to each other as $d=\mathcal{O}\left(a^{|\log(a)|^{-h}}\right)$, we have* [8]

$$\tilde{p}(t,x) = \frac{\omega\,\beta_0}{c_p}\,\frac{-2\,\Theta\,t\,Im\,(\epsilon_p)}{(t^2-|x-z_2|^2)^{\frac{3}{2}}}\,\left|\langle u_2; e^{(2)}_{n_0}\rangle\right|^2 + \mathcal{O}\left(|\log(a)|^{\max(h-1-2s,2h-2,-h-s)}\right). \tag{1.24}$$

(b) *When the distance is estimated as $d=\mathcal{O}\left(a^{|\log(a)|^{-(\ell+h)}}\right)$, where $\ell>0$, we obtain*[9]

$$\tilde{p}(t,x) = \frac{\omega\,\beta_0}{c_p}\,\frac{-4\,t\,Im\,(\epsilon_p)}{(t^2-|x-z_2|^2)^{\frac{3}{2}}}\,\left|\langle u_2; e^{(2)}_{n_0}\rangle\right|^2 + \mathcal{O}\left(|\log(a)|^{\max(h-1-s-\ell,h-1-2s,-h-s,2h-2)}\right). \tag{1.25}$$

Here also, p^+ and p^- correspond to the pressure after injecting the couples of nano-particle and exciting with frequencies of incidence (1.22), with the sign $+$ and $-$ respectively, while p_0 is the pressure in the absence of the nano-particles.

Let discuss the implications of these two theorems on how we can solve the inverse problem related to the photo-acoustic modality using injected single or double (closely spaced) dielectric nano-particles.

(1). The formula (1.23) means that if we measure before and after injecting one nano-particle, then we can reconstruct the phaseless data $|u_0|(x)$, $x\in\Omega$. Hence, we transform the photoacoustic inverse problem to the inverse scattering using phaseless internal data. Taking h close to 1, i.e. the used frequency very close to the resonance, we see that the dominant term is more pronounced which might avoid division by a small quantity.

(2). The formula (1.24) can be expressed using u_0 instead of u_2 under the condition $0<s<1-h$ as for (1.23). The formula (1.24) means that if we measure before and after injecting two closely spaced

[8]Since z_1 and z_2 are sufficiently close, we make in (1.24) an arbitrary choice of one of them, i.e. (1.24) does not distinguish between z_1 and z_2. In (1.24) and (1.25) the frequency ω can be taken as ω_+ or ω_- and u_2 is evaluated at either frequencies $\omega_\pm$ as well. Similar observations apply to (1.23).

[9]Comparing (1.25) to (1.23), we remark that they differ by a multiplicative constant equals to two and this is justified by the fact that in (1.25) we deal with two nano-particles.

nano-particles, then we can reconstruct $\int_D u_2(x) e_{n_0}(x) dx$ and hence $\int_D |u_2(x)|^2 dx$. In addition, a slightly different form of formula (1.23), see (3.19),

$$\left(p^+ + p^- - 2p_0\right)(t,x) = \frac{-2\, t \,\mathrm{Im}\,(\epsilon_p)\, \left|\int_D u_1(x) e_{n_0}(x)\, dx\right|^2}{\left(t^2 - |x-z|^2\right)^{3/2}} + \mathcal{O}\left(|\log(a)|^{2h-2}\right),$$

shows that if we measure before and after injecting one nano-particle, we can reconstruct $\int_D |u_1(x)|^2 dx$.

Using these two last data, i.e. $\int_D |u_1(x)|^2 dx$ and $\int_D |u_2(x)|^2 dx$, we apply Theorem 1.1 to reconstruct, via (1.21), $|\epsilon_0(\cdot)|$. Hence, using two different resonances, we reconstruct both the permittivity ϵ_0 and the conductivity σ_Ω.

(3). Finally, let us show how we can use (1.23) to localize the position z of the injected nano-particle and estimate $|u_0(z)|$. The corresponding results can also be shown using (1.24). For this, we use the notations

$$\tilde{p}\,(t,x) := \left(p^+ + p^- - 2p_0\right)(t,x), \quad A := \frac{\omega\, \beta_0}{c_p} \frac{-2\ \mathrm{Im}\,(\epsilon_p)\ |u_0(z)|^2}{|1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2} \left(\int_D e_{n_0}(x)\, dx\right)^2$$

and

$$Err := \mathcal{O}\left(|\log(a)|^{\max(-h-s,2h-2,h-1-2s)}\right).$$

Let $t_1 \neq t_2$ then we have

$$\frac{\tilde{p}(t_1,x)}{\tilde{p}(t_2,x)} = \frac{A\dfrac{t_1}{(t_1^2-|x-z|^2)^{3/2}} + Err}{A\dfrac{t_2}{(t_2^2-|x-z|^2)^{3/2}} + Err} = \frac{t_1}{t_2}\left(\frac{t_2^2-|x-z|^2}{t_1^2-|x-z|^2}\right)^{3/2} + \mathcal{O}\left(|\log(a)|^{\max(1-2h,s+h-1,-s)}\right), \tag{1.26}$$

where

$$0 < s < 1-h \quad \text{and} \quad h > 1/2. \tag{1.27}$$

From (1.26) we derive the formula

$$|x-z| = \left[\frac{t_1^2\,(t_2\tilde{p}(t_1,x))^{2/3} - t_2^2\,(t_1\tilde{p}(t_2,x))^{2/3}}{(t_2\tilde{p}(t_1,x))^{2/3} - (t_1\tilde{p}(t_2,x))^{2/3}}\right]^{\frac{1}{2}} + \mathcal{O}\left(|\log(a)|^{\max(1-2h,s+h-1,-s)}\right). \tag{1.28}$$

The expression (1.28) tells that, for $x \in \partial\Omega$, the point z is in the arc given by the intersection of Ω and the circle S with center x and radius computable as

$$\left[\frac{t_1^2\,(t_2\tilde{p}(t_1,x))^{2/3} - t_2^2\,(t_1\tilde{p}(t_2,x))^{2/3}}{(t_2\tilde{p}(t_1,x))^{2/3} - (t_1\tilde{p}(t_2,x))^{2/3}}\right]^{\frac{1}{2}}. \tag{1.29}$$

Then in order to localise z, we repeat the same experience with another point $x_\star \neq x$, and take the intersection of two arcs, see Figure 1.

Assume that z is localized, then from the equation (1.23), we get

$$|u_0(z)|^2 = -\frac{c_p\,|1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2 (t^2-|x-z|^2)^{3/2}\ \tilde{p}(t,x)}{\omega\,\beta_0\, 2\, t\,\mathrm{Im}\,(\epsilon_p)\ \left(\int_D e_{n_0}(x)\, dx\right)^2} + \mathcal{O}\left(|\log(a)|^{\max(1-2h,s+h-1,-s)}\right).$$

with

$$0 < s < 1-h \quad \text{and} \quad h > 1/2. \tag{1.30}$$

As a conclusion of the points (2) and (3) above, if we measure the pressure, created before and after injecting one and then two closely spaced dielectric nano-particles, on two single points, x_1 and x_2, located on the boundary of $\partial\Omega$ and at two single different times t_1 and t_2, with $t_1, t_2 \geq \mathrm{diam}(\Omega)$, then we can localize these injected nano-particles and reconstruct, with explicit reconstruction formulas, the

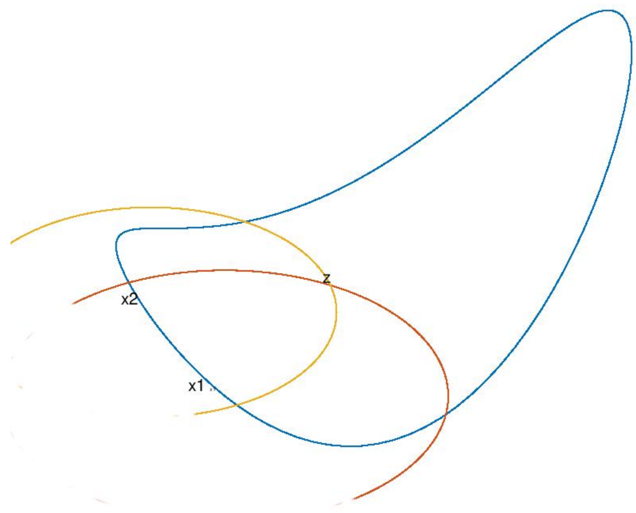


FIGURE 1. Localization of the particles. The blue curve represents $\partial\Omega$ while the red and yellow ones the circles of center $x_1 := x$ and $x_2 := x^*$ and radius (1.29), with x and $x^\star$ respectively.

modulus of the index of refraction. These reconstruction are possible using incident frequencies close to the dielectric resonances of the used nano-particles. Using two distinct resonances, we can then estimate both the permittivity and the conductivity of the imaged region.

Let us finish this introduction by comparing our findings with the previous results. To our knowledge, the only work published to analyze the photo-acoustic imaging modality using contrast agents is the recent work [28]. The authors propose to use plasmonic resonances instead of dielectric ones. Assuming the acoustic inversion to be known and done, as described in section 1.2, they perform the electromagnetic inversion. They state the 2D-electromagnetic model where the magnetic fields satisfy a divergence form equation. Performing asymptotic expansions, close to these resonances they derive the dominant part of the magnetic field and reconstruct the permittivity by an optimization step applied on this dominating term. This result could be compared to the first part of Theorem 1.1, i.e formula (1.20).

The rest of the paper is organized as follows. In section 2 and section 3, we prove Theorem 1.1 and Theorem 1.2 respectively. In section 4, we derive the needed estimates on the electric fields, used in section 2 and section 3 in terms of the contrast of the permittivity, conductivity and for frequencies close to the dielectric resonances. Finally, in section 5.1, we discuss the validity of the conditions in **Hypotheses1**.

Notations: Only $\mathbb{L}^2$-norms on domains are involved in the text. Therefore, unless indicated, we use $\|\cdot\|$ without specifying the domain. In addition, we use $\langle\cdot,\cdot\rangle$ for the corresponding scalar product. For a given function f defined on $\overset{M}{\underset{j=1}{\cup}} D_j$, we denote by $f_j := f_{|_{D_j}}$, $j = 1, \cdots, M$.

The eigenfunctions $\left(e_n^{(i)}\right)_{n\in\mathbb{N}}$ of the Newtonian operator stated on D_i depend, of course, on D_i. Nevertheless, unless specified, we use the notation (e_n) even when dealing with multiple particles located in different positions.

Remark 1.3. To avoid making the text more cumbersome, we warn the reader that we omit, except for the Appendix B (5.2), to note the dependency of $u_0, u_1, u_2, v_1, \cdots, v_M, k$ and p with respect to the frequency variable ω.

2. Proof of Theorem 1.1

We split the proof into two subsections. In the first one, we derive the Foldy-Lax algebraic system, see (2.12) in Proposition 2.4, as an approximation of the continuous L.S.E satisfied by the electric field. In the second subsection, we invert the algebraic system and extract the needed formulas, see (2.34).

2.1. Approximation of the L.S.E.

In the following, we denote by G_k the Green kernel for Helmholtz equation in dimension two. This means that G_k is a solution of:

$$(\Delta + \omega^2 \, \boldsymbol{n_0}^2(\cdot)) G_{k(\cdot)}(\cdot, \cdot) = -\delta.(\cdot), \qquad in \qquad \mathbb{R}^2 \quad \text{satisfying the S.R.C.} \tag{2.1}$$

Lemma 2.1. *The Green kernel G_k admits the following asymptotic expansion*

$$G_k(|x-y|) = \frac{-1}{2\pi} \log(|x-y|) + \varrho(y) + \mathcal{O}\left(|x-y|\right), \quad x \text{ near } y. \tag{2.2}$$

Proof. We set $\Phi_{k(z)}$ to be the solution of 2D-Helmholtz equation with constant coefficients, i.e:

$$(\Delta + \omega^2 \, \boldsymbol{n_0}^2(z)) \Phi_{k(z)}(\cdot, \cdot) = -\delta.(\cdot), \tag{2.3}$$

which is, see for instance [10], pages 72 to 74, given by:[10]

$$\Phi_{k(z)}(x,z) := \frac{-1}{2\pi} \log\left(|x-z|\right) + \frac{i}{4} - \frac{1}{2\pi} \log\left(\frac{\omega n_0(z)}{2}\right) - \frac{C^{te}_{=}}{2\pi} + \mathcal{O}\left(|x-z|^2 \log\left(|x-z|\right)\right), \tag{2.4}$$

where $C^{te}_{=}$ denote the Euler's constant and it is equals to:

$$C^{te}_{=} := \lim_{p \to +\infty} \left\{ \sum_{m=1}^{p} \frac{1}{m} - \log(p) \right\}.$$

Now, by subtracting (2.1) from (2.3), we obtain:

$$(\Delta + \omega^2 \, \boldsymbol{n_0}^2(\cdot)) \left(G_{k(\cdot)} - \Phi_{k(z)} \right) = k^2 \left(\boldsymbol{n_0}^2(z) - \boldsymbol{n_0}^2(\cdot) \right) \Phi_{k(z)}.$$

Remark that the right hand side is an element in $\mathbb{L}^p$ for every $p \geq 1$. Then, using the regularity of Helmholtz equation we deduce that:

$$\left(G_{k(\cdot)} - \Phi_{k(z)} \right) \in \mathbb{W}^{2,p}, \quad \forall \, p \geq 1,$$

and, by embedding results, then $\left(G_{k(\cdot)} - \Phi_{k(z)} \right) \in \mathcal{C}^1$. Next, for shortness raison, we set:

$$f(x,z) := \left(G_{k(\cdot)} - \Phi_{k(z)} \right)(x,z),$$

and we use the Taylor expansion for $f(\cdot, z)$ to get:

$$f(x,z) = f(z,z) + \int_0^1 \underset{x}{\nabla} f(z + t(x-z), z) \cdot (x-z) \, dt = f(z,z) + \mathcal{O}\left(|x-z|\right).$$

Then, we deduce that:

$$\begin{aligned} G_{k(\cdot)}(x,z) &= \Phi_{k(z)}(|x-z|) + f(z,z) + \mathcal{O}\left(|x-z|\right) \\ &\overset{(2.4)}{=} \frac{-1}{2\pi} \log\left(|x-z|\right) + \frac{i}{4} - \frac{1}{2\pi} \log\left(\frac{\omega n_0(z)}{2}\right) - \frac{C^{te}_{=}}{2\pi} + f(z,z) + \mathcal{O}\left(|x-z|\right). \end{aligned}$$

[10] We define the logarithm of a complex number z as follows $\log(z) = \log(|z|) + iArg(z)$.

To end the proof of the lemma we set $\varrho(z)$, constant in x, to be:

$$\varrho(z) := \frac{i}{4} - \frac{1}{2\pi} \log\left(\frac{\omega n_0(z)}{2}\right) - \frac{C^{te}_{=}}{2\pi} + f(z,z). \tag{2.5}$$

Finally,

$$G_{k(\cdot)}(x,z) = \frac{-1}{2\pi} \log\left(|x-z|\right) + \varrho(z) + \mathcal{O}\left(|x-z|\right).$$

□

Definition 2.2. We define

$$a := \frac{1}{2} \underset{1\leq m\leq M}{diam}(D_m), \quad d_{mj} := dist(D_m, D_j), \quad d := \min_{\substack{m\neq j \\ 1\leq m,j\leq M}} d_{mj},$$

where $D_m = z_m + a\,B$ with B is a bounded domain containing the origin.

The unique solution of the problem (1.7), with $D := \overset{M}{\underset{j=1}{\cup}} D_j$, satisfies the L.S.E

$$u(x) - \omega^2 \mu_\infty \int_D G_k(x,y)\,(\epsilon_p - \epsilon_0)(y)\,u(y)\,dy = u_0(x) \quad in\ D. \tag{2.6}$$

We set[11] $v_m := u_{|_{D_m}}, m = 1, \cdots, M$. Then (2.6), for $x \in D_m$, rewrites as

$$v_m(x) - \omega^2\mu_\infty \int_{D_m} G_k(x,y)(\epsilon_p - \epsilon_0(y))v_m(y)dy - \omega^2\mu_\infty \sum_{j\neq m} \int_{D_j} G_k(x,y)(\epsilon_p - \epsilon_0(y))v_j(y)dy = u_0(x).$$

We set: $\tau_j := (\epsilon_p - \epsilon_0(z_j))$. Assuming $\epsilon_{0|_\Omega}$ to be $W^{1,\infty}(\Omega)$, the solution u of the scattering problem

$$\begin{cases} \left(\Delta + \omega^2\, \boldsymbol{n_0^2}(x)\right) u = 0 \quad \text{in } \mathbb{R}^2 \\ u := u^i + u^s \text{ and } u^s\ S.R.C, \end{cases}$$

has a $W^{1,\infty}(\Omega)$ regularity. Set

$$\Phi_0(x,y) := \frac{-1}{2\pi} \log(|x-y|) \tag{2.7}$$

Expanding $(\epsilon_p - \epsilon_0(\cdot))$, $u_0(\cdot)$ and $G_{k(\cdot)}(\cdot,\cdot)$ near the center z_m, we obtain

$$\begin{aligned} v_m(x) \quad &- \quad \omega^2\,\mu_\infty\,\tau_m \int_{D_m} \left(\Phi_0(x,y) + \varrho(z_m)\right)\, v_m(y)\,dy - \omega^2\,\mu_\infty\,\tau_m\,\mathcal{O}\left(\int_{D_m} |x-y|\,v_m(y)\,dy\right) \\ &- \quad \omega^2\,\mu_\infty\,\tau_m \int_{D_m}\int_0^1 \nabla\varrho(z_m + t(y-z_m))\cdot(y-z_m)dt\,v_m(y)dy \\ &+ \quad \omega^2\,\mu_\infty \int_{D_m} G_k(x,y) \int_0^1 (y-z_m)\cdot\nabla\epsilon_0(z_m + t(y-z_m))\,dt\,v_m(y)\,dy \\ &- \quad \omega^2\,\mu_\infty \sum_{j\neq m} \tau_j \int_{D_j} \Bigg[G_k(z_m;z_j) + \int_0^1 \underset{x}{\nabla} G_k(z_m + t(x-z_m);z_j)\cdot(x-z_m)\,dt \\ &+ \quad \int_0^1 \underset{y}{\nabla} G_k(z_m; z_j + t(y-z_j))\cdot(y-z_j)\,dt \\ &+ \quad \int_0^1 \underset{x}{\nabla} \int_0^1 \underset{y}{\nabla} G_k(z_m + t(x-z_m); z_j + t(y-z_j))\cdot(y-z_j)\,dt\cdot\,(x-z_m)\,dt\Bigg]\,v_j(y)\,dy \end{aligned}$$

[11]We use the notation $v_m := u_{|_{D_m}}$ instead of $u_m := u_{|_{D_m}}$ to avoid confusion with u_0, u_1 and u_2 we defined before concerning the electric fields in the absence or the presence of one or two particles.

$$
\begin{aligned}
&+ \quad \omega^2 \, \mu_\infty \sum_{j \neq m} \int_{D_j} G_k(x,y) \int_0^1 (y - z_j) \cdot \nabla \epsilon_0(z_j + t(y-z_j)) \, dt \, v_j(y) \, dy \\
&= \quad u_0(z_m) + \int_0^1 \nabla u_0(z_m + t(x - z_m)) \cdot (x - z_m) \, dt.
\end{aligned}
$$

Now, define $w_m(\cdot)$ to be

$$
w_m(\cdot) := \omega^2 \, \mu_\infty \, \tau_m \left[I - \omega^2 \, \mu_\infty \, \tau_m \, A_0 \right]^{-1} (1)(\cdot) = \left[\frac{1}{\omega^2 \, \mu_\infty \, \tau_m} I - A_0 \right]^{-1} (1)(\cdot), \tag{2.8}
$$

and set the following notations

$$
\boldsymbol{C}_m = \int_{D_m} w_m dx \quad \& \quad \boldsymbol{C}_m^\star = \boldsymbol{C}_m \left[1 - \varrho(z_m) \, \boldsymbol{C}_m \right]^{-1} \quad \& \quad Q_m = \omega^2 \, \mu_\infty \tau_m (\boldsymbol{C}_m^\star)^{-1} \int_{D_m} v_m dx. \tag{2.9}
$$

Using the definition of w_m, and integrate y over D_m, the self adjointness of the operator $(\lambda I - A_0)$ and we multiplying both sides of this equation by $\omega^2 \, \mu_\infty \, \tau_m \, C_m^{-1}$, we obtain

$$
\begin{aligned}
Q_m \quad &- \quad \sum_{j \neq m} G_k(z_m; z_j) \, \mathbf{C}_j^\star \, Q_j = u_0(z_m) \\
&- \quad \omega^2 \, \mu_\infty \, \tau_m \, \mathbf{C}_m^{-1} \Bigg[\mathbf{C}_m \int_{D_m} \int_0^1 \nabla \varrho(z_m + t(y - z_m)) \cdot (y - z_m) \, dt \, v_m(y) dy \\
&+ \quad \int_{D_m} w_m(x) \int_{D_m} |x - y| v_m(y) dy dx \\
&+ \quad \tau_m^{-1} \int_{D_m} w_m(x) \int_{D_m} G_k(x,y) \int_0^1 (y - z_m) \cdot \nabla \epsilon_0(z_m + t(y - z_m)) \, dt \; v_m(y) \, dy \, dx \\
&- \quad \tau_m^{-1} \sum_{j \neq m} \tau_j \int_{D_m} w_m(x) \int_0^1 \underset{x}{\nabla} G_k(z_m + t(x - z_m); z_j) \cdot (x - z_m) \, dt \, dx \int_{D_j} v_j(y) \, dy \\
&- \quad \tau_m^{-1} \boldsymbol{C}_m \sum_{j \neq m} \tau_j \int_{D_j} \int_0^1 \underset{y}{\nabla} G_k(z_m; z_j + t(y - z_j)) \cdot (y - z_j) dt \, v_j(y) \, dy \\
&- \quad \tau_m^{-1} \sum_{j \neq m} \tau_j \int_{D_m} w_m \int_{D_j} \int_0^1 \underset{x}{\nabla} \int_0^1 \underset{y}{\nabla} G_k(z_m + t(x - z_m); z_j + t(y - z_j)) \cdot (y - z_j) dt \cdot (x - z_m) dt v_j dy dx \\
&+ \quad \tau_m^{-1} \sum_{j \neq m} \int_{D_m} w_m(x) \int_{D_j} G_k(x,y) \int_0^1 (y - z_j) \cdot \nabla \epsilon_0(z_j + t(y - z_j)) \, dt v_j(y) \, dy \, dx \\
&+ \quad \left(\omega^2 \, \mu_\infty \, \tau_m \right)^{-1} \int_{D_m} w_m(x) \int_0^1 (x - z_m) \cdot \nabla u_0(z_m + t(x - z_m)) dt \, dx \Bigg].
\end{aligned}
$$

For the right side, we keep $u_0(z_m)$ as a dominant term and estimate the other terms as an error. To achieve this goal, we need the following proposition.

Proposition 2.3. *We have:*

$$
\|u\|_{\mathbb{L}^2(D)} \leq |\log(a)|^h \, \|u_0\|_{\mathbb{L}^2(D)}, \tag{2.10}
$$

and

$$
\boldsymbol{C}_m = \mathcal{O}(|\log(a)|^{h-1}).
$$

Proof. See Section 4. ☐

As the incident wave is smooth and independent on a, thanks to (2.10), we get

$$
\|w\|_{\mathbb{L}^2(D)} \leq a^{-1} \, |\log(a)|^{h-1}. \tag{2.11}
$$

We recall that
$$\tau_m \sim a^{-2}\,|\log(a)|^{-1}, \quad m=1,\cdots,M.$$
The error part contains eight terms. Next we define and estimate every term, then we sum them up. More precisely, we have

- <u>Estimation of</u> $S_0 := \tau_m \int_{D_m}\int_0^1 \nabla\varrho(z_m+t(y-z_m))\cdot(y-z_m)\,dt\,v_m(y)dy$

$$|S_0| \lesssim a^{-2}\,|\log(a)|^{-1}\,\|v_m\|\,\left\|\int_0^1 \nabla\varrho(z_m+t(\cdot-z_m))\cdot(\cdot-z_m)dt\right\| = \mathcal{O}\left(|\log(a)|^{-1}\,\|v_m\|\right),$$

and then
$$S_0 = \mathcal{O}\left(a\,|\log(a)|^{h-1}\,M^{\frac{1}{2}}\right).$$

- <u>Estimation of</u> $S_1 := \tau_m\,\mathbf{C}_m^{-1}\int_{D_m} w_m(x)\int_{D_m}|x-y|\,v_m(y)dy\,dx$

$$\begin{aligned}
|S_1| &\lesssim a^{-2}\,|\log(a)|^{-1}\,|\log(a)|^{1-h}\,\|w\|\left[\int_{D_m}\left|\int_{D_m}|x-y|\,v_m(y)dy\right|^2 dx\right]^{\frac{1}{2}}\\
&\lesssim a^{-2}\,|\log(a)|^{-h}\,a^{-1}\,|\log(a)|^{h-1}\left[\int_{D_m}\int_{D_m}|x-y|^2dy\,dx\right]^{\frac{1}{2}}\|v_m\|\\
&= \mathcal{O}\Big(|\log(a)|^{-1}\,\|v_m\|\Big),
\end{aligned}$$

and then
$$S_1 = \mathcal{O}\Big(a\,|\log(a)|^{h-1}\,M^{\frac{1}{2}}\Big).$$

- <u>Estimation of</u> $S_2 := \mathbf{C}_m^{-1}\int_{D_m} w_m(x)\int_{D_m} G_k(x,y)\int_0^1(y-z_m)\cdot\nabla\epsilon_0(z_m+t(y-z_m))dt\,v_m(y)\,dy\,dx$

$$|S_2| \lesssim a^{-1}\left[\int_{D_m}\left(\int_{D_m}|G_k|(x;y)\left|\int_0^1(y-z_m)\cdot\nabla\epsilon_0(z_m+t(y-z_m))dt\right|\,|v_m|(y)dy\right)^2dx\right]^{\frac{1}{2}}.$$

The smoothness of ϵ_0 implies $\left|\int_0^1(y-z_m)\cdot\nabla\epsilon_0(z_m+t(y-z_m))dt\right| \lesssim \mathcal{O}(a)$, hence

$$|S_2| \quad\lesssim\quad \|v_m\|\left[\int_{D_m}\int_{D_m}|G_k|^2(x;y)\,dy\,dx\right]^{\frac{1}{2}} \lesssim \|u\|\,a^2\,|\log(a)|,$$

and then
$$S_2 = \mathcal{O}\Big(a^3\,|\log(a)|^{1+h}\,M^{\frac{1}{2}}\Big).$$

- <u>Estimation of</u> $S_3 := \boldsymbol{C}_m^{-1}\sum\limits_{j\neq m}\tau_j\int_{D_m} w_m(x)\int_0^1 \underset{x}{\nabla} G_k(z_m+t(x-z_m);z_j)\cdot(x-z_m)\,dt\,dx\int_{D_j}v_j(y)\,dy$

$$|S_3| \lesssim \frac{1}{a\,|\log(a)|^h}\sum_{j\neq m}\|w\|\,\|v_j\|\left[\int_{D_m}\left|\int_0^1\underset{x}{\nabla} G_k(z_m+t(x-z_m);z_j)\cdot(x-z_m)\,dt\right|^2dx\right]^{\frac{1}{2}}.$$

Without difficulties, we can check that

$$\left[\int_{D_m}\left|\int_0^1\underset{x}{\nabla} G_k(z_m+t(x-z_m);z_j)\cdot(x-z_m)\,dt\right|^2dx\right]^{\frac{1}{2}} \lesssim \frac{a^2}{d_{mj}},$$

then we plug this on the previous equation and use Cauchy-Schwartz inequality, to get

$$|S_3| \lesssim |\log(a)|^{-1}\,\|v\|\left(\sum_{j\neq m}\frac{1}{d_{mj}^2}\right)^{\frac{1}{2}} \lesssim |\log(a)|^{h-1}\,a\,M\,d^{-1}.$$

Set

$$S_4 := \sum_{j \neq m} \tau_j \int_{D_j} \int_0^1 \underset{y}{\nabla} G_k(z_m; z_j + t(y - z_j)) \cdot (y - z_j) dt \, v_j(y) \, dy,$$

and remark that S_4 has a similar expression as S_3, then we obtain:

$$S_3 = \mathcal{O}(|\log(a)|^{h-1} \; a \; M \; d^{-1}) \quad \text{and} \quad S_4 = \mathcal{O}(|\log(a)|^{h-1} \; a \; M \; d^{-1}).$$

- Estimation of

$$S_5 := \boldsymbol{C}_m^{-1} \sum_{j \neq m} \tau_j \int_{D_m} w_m \int_{D_j} \int_0^1 \underset{x}{\nabla} \int_0^1 \underset{y}{\nabla} G_k(z_m + t(x - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (x - z_m) dt v_j(y) dy dx$$

$$\begin{aligned} |S_5| &\lesssim \frac{\|w\|}{a^2 \, |\log(a)|^h} \sum_{j \neq m} \left\| \int_{D_j} \int_0^1 \underset{\cdot}{\nabla} \int_0^1 \underset{y}{\nabla} G_k(z_m + t(\cdot - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (\cdot - z_m) dt \, v_j(y) \, dy \right\| \\ &\lesssim \frac{a^{-3}}{|\log(a)|} \sum_{j \neq m} \|v_j\| \left[\int_{D_m} \left| \int_{D_j} \int_0^1 \underset{x}{\nabla} \int_0^1 \underset{y}{\nabla} G_k(z_m + t(x - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (x - z_m) dt dy \right|^2 dx \right]^{\frac{1}{2}}. \end{aligned}$$

we have

$$\int_{D_m} \left| \int_{D_j} \int_0^1 \underset{x}{\nabla} \int_0^1 \underset{y}{\nabla} G_k(z_m + t(x - z_m); z_j + l(y - z_j)) \cdot (y - z_j) dl \cdot (x - z_m) dt dy \right|^2 dx \lesssim \mathcal{O}\Big(\frac{a^8}{d_{mj}^4} \Big),$$

hence

$$|S_5| \quad \lesssim \quad a \, |\log(a)|^{-1} \, \|u\| \left(\sum_{j \neq m} \frac{1}{d_{mj}^4} \right)^{\frac{1}{2}} \lesssim a^2 \; |\log(a)|^{h-1} \; M \; d^{-2},$$

then

$$S_5 = \mathcal{O}(a^2 \; |\log(a)|^{h-1} \; M \, d^{-2}).$$

- Estimation of $S_6 := \boldsymbol{C}_m^{-1} \sum\limits_{j \neq m} \int_{D_m} w_m(x) \int_{D_j} G_k(x, y) \int_0^1 (y - z_j) \cdot \nabla \epsilon_0(z_j + t(y - z_j)) \, dt \, v_j(y) \, dy \, dx$

$$|S_6| \quad \lesssim \quad \sum_{j \neq m} \|v_j\| \left[\int_{D_m} \int_{D_j} |G_k|^2(x; y) \, dy \, dx \right]^{\frac{1}{2}} \lesssim a^2 \; \|v\| \; M^{\frac{1}{2}} \leq a^3 \; |\log(a)|^h \; M.$$

Then

$$S_6 = \mathcal{O}\Big(a^3 \; |\log(a)|^h \; M\Big).$$

- Estimation of $S_7 := \boldsymbol{C}_m^{-1} \; \int_{D_m} w_m(x) \int_0^1 (x - z_m) \cdot \nabla u_0(z_m + t(x - z_m)) \, dt \, dx$

$$|S_7| \quad \lesssim \quad |\log(a)|^{1-h} \; \|w\| \; \left[\int_{D_m} \left| \int_0^1 (x - z_m) \cdot \nabla u_0(z_m + t(x - z_m)) dt \right|^2 dx \right]^{\frac{1}{2}}.$$

As u_0 is smooth, we have

$$\left[\int_{D_m} \left| \int_0^1 (x - z_m) \cdot \nabla u_0(z_m + t(x - z_m)) dt \right|^2 dx \right]^{\frac{1}{2}} = \mathcal{O}(a^2).$$

Hence

$$S_7 = \mathcal{O}(a).$$

Finally, the error part is

$$Error = S_0 + \cdots + S_7 = \mathcal{O}(a \; d^{-1} \; |\log(a)|^{h-1} \; M).$$

Proposition 2.4. *The vector* $\left(Q_m\right)_{m=1}^{M}$ *satisfie the following algebraic system*

$$Q_m - \sum_{j \neq m} G_k(z_m; z_j)\ \boldsymbol{C}_j^{\star}\ Q_j = u_0(z_m) + \mathcal{O}\left(a\ d^{-1}\ |\log(a)|^{h-1}\ M\right). \tag{2.12}$$

The algebraic system 2.12 can be written, in a matrix form, as

$$(I - B)\, Q = V + Error \tag{2.13}$$

with $B = \left(B_{mj}\right)_{m,j=1}^{M}$ such that $B_{mj} := G_k(z_m; z_j)\, \mathbf{C}_j^{\star}$ and $V := (u_0(z_1), \cdots, u_0(z_M))^{\top}$.
In the next proposition, we give conditions under which the linear system (2.13) is invertible.

Lemma 2.5. *The algebraic system* (2.13) *is invertible if*

$$d > exp\Big(-|\log(a)|^{1-h}\Big), \tag{2.14}$$

where d is the minimal distance between the particles.

Proof. of **Lemma**(2.5). Let us evaluate the norm of B. For this we have:

$$\|B\| = \max_m \sum_{j \neq m} |B_{mj}| \overset{\text{def}}{=} \max_m \sum_{j \neq m} \left| G_k(z_m; z_j)\left[\mathbf{C}_j^{-1} - \varrho(z_j)\right]^{-1} \right| \leq |\log(a)|^{h-1} \sum_{j \neq m} \log\left(\frac{1}{d_{mj}}\right).$$

We need the following lemma

Lemma 2.6. *We have*

$$\sum_{j \neq m} \log(1/d_{mj}) = \log(1/d). \tag{2.15}$$

Proof. of **Lemma 2.6**
We set $\log(1/d_{mj}) = 1/l_{mj}$ and $l = \min\limits_{j \neq m} l_{mj}$. Then

$$\sum_{j \neq m} \log\left(\frac{1}{d_{mj}}\right) = \sum_{j \neq m} \frac{1}{l_{mj}} \overset{(\star)}{=} \frac{1}{l}. \tag{2.16}$$

At first we assume that $(\star)$ is checked. Then we have

$$l = \min_{j \neq m} \frac{1}{\log(1/d_{mj})} = \frac{1}{\log(\max\limits_{j \neq m}(1/d_{mj}))} = \frac{1}{\log(1/(\min\limits_{j \neq m} d_{mj}))} = \frac{1}{\log(1/d)}. \tag{2.17}$$

Then (2.16) combined with (2.17) give a justification of (2.15).
Now, in order to prove $(\star)$ we modify to the two dimensional case the proof, done for three dimensional case, given in ([3], page 13). We get

$$\sum_{\substack{i=1 \\ i \neq j}}^{M} \frac{1}{l_{ij}^{k}} = \left\{ \begin{array}{lll} \mathcal{O}(l^{-k}) + \mathcal{O}(l^{-2\alpha}) & if & k < 2 \\ \mathcal{O}(l^{-2}) + \mathcal{O}(l^{-2\alpha}|\log(l)|) & if & k = 2 \\ \mathcal{O}(l^{-k}) + \mathcal{O}(l^{-\alpha k}) & if & k > 2. \end{array} \right.$$

□

Based on lemma 2.6 the condition $\|B\| < 1$, is fulfilled if

$$\log(1/d) < |\log(a)|^{1-h} \quad \text{or} \quad d > \exp\Big(-|\log(a)|^{1-h}\Big). \tag{2.18}$$

□

2.2. **Inversion of the derived Foldy-Lax algebraic system (2.12).** Here, we deal with the case of two particles, i.e $M=2$. In the equation (2.12) we use the condition $d \backsim a^{|\log(a)|^{-h}}$, then we get

$$\left\{\begin{array}{rcl} Q_1 - G_k(z_1;z_2)\,\mathbf{C}_2^{\star} Q_2 &=& u_0(z_1) + \mathcal{O}\left(a^{1-|\log(a)|^{-h}}\,|\log(a)|^{h-1}\right), \\ Q_2 - G_k(z_2;z_1)\,\mathbf{C}_1^{\star} Q_1 &=& u_0(z_2) + \mathcal{O}\left(a^{1-|\log(a)|^{-h}}\,|\log(a)|^{h-1}\right). \end{array}\right.$$

We check that the condition $d \backsim a^{|\log(a)|^{-h}}$ is sufficient for the invertibility of the last system. For this, we have from (2.18)

$$d > \exp\left(-\left|\log(a)\right|^{1-h}\right) = \left(e^{-|\log(a)|}\right)^{|\log(a)|^{-h}} = a^{|\log(a)|^{-h}}.$$

Now, we use the expansion of $G_k(z_m;z_j)$, see (2.2), to obtain

$$\left\{\begin{array}{rcl} Q_1 - \left[\Phi_0(z_1;z_2) + \varrho(z_2)\right]\,\mathbf{C}_2^{\star} Q_2 &=& u_0(z_1) + \mathcal{O}\left(a^{1-|\log(a)|^{-h}}\,|\log(a)|^{h-1}\right) + \mathcal{O}\left(d\,\mathbf{C}_2^{\star} Q_2\right), \\ Q_2 - \left[\Phi_0(z_2;z_1) + \varrho(z_1)\right]\,\mathbf{C}_1^{\star} Q_1 &=& u_0(z_2) + \mathcal{O}\left(a^{1-|\log(a)|^{-h}}\,|\log(a)|^{h-1}\right) + \mathcal{O}\left(d\,\mathbf{C}_1^{\star} Q_1\right). \end{array}\right.$$

We can estimate

$$d\,\mathbf{C}_i^{\star} Q_i = \mathcal{O}\left(a^{|\log(a)|^{-h}}\,\left|\log(a)\right|^{h-1}\right), \quad for \quad i=1,2, \tag{2.19}$$

because, by the definition of Q_i, see (2.9), we have

$$d\;\mathbf{C}_i^{\star} Q_i = d\,\mathbf{C}_i^{\star}\,\omega^2\,\mu_\infty\,\tau_i\left(\mathbf{C}_i^{\star}\right)^{-1}\int_{D_i} v\,dx \lesssim d\,|\tau_i|\,\|1\|\,\|u\|.$$

The value of d and $|\tau_i|$ are known, and we have an a priori estimate about $\|u\|$ given by (2.10), then

$$d\;\mathbf{C}_i^{\star}\;Q_i \lesssim a^{|\log(a)|^{-h}}\,a^{-2}\,|\log(a)|^{-1}\,a\,|\log(a)|^{h}\,a = a^{|\log(a)|^{-h}}\,\left|\log(a)\right|^{h-1}.$$

This proves (2.19).
With these estimations the last system can be written as

$$\left\{\begin{array}{rcl} Q_1 - \left[\Phi_0(z_1;z_2) + \varrho(z_2)\right]\,\mathbf{C}_2^{\star} Q_2 &=& u_0(z_1) + \mathcal{O}\left(d\,\left|\log(a)\right|^{h-1}\right), \\ Q_2 - \left[\Phi_0(z_2;z_1) + \varrho(z_1)\right]\,\mathbf{C}_1^{\star} Q_1 &=& u_0(z_2) + \mathcal{O}\left(d\,\left|\log(a)\right|^{h-1}\right). \end{array}\right. \tag{2.20}$$

We need the following lemmas to simplify the last system.

Lemma 2.7. *For $j=1,2$, we have*

$$\boldsymbol{C}_j = \boldsymbol{C} + \mathcal{O}\left(a^2\,|\log(a)|^{2h}\right), \tag{2.21}$$

where

$$\boldsymbol{C} := \int_D \overline{w}(x)\,dx \quad and \quad \overline{w}(\cdot) := \omega^2\,\mu_\infty\,\epsilon_p\,\left(I - \omega^2\,\mu_\infty\,\epsilon_p\,A_0\right)^{-1}(1)(\cdot).$$

Proof. See Section 4.2. □

Lemma 2.8. *Since $\varrho(\cdot)$ is $W^{1,\infty}$-smooth and z_1 is close to z_2 at a distance d, we obtain*

$$\varrho(z_2) = \varrho(z_1) + \mathcal{O}\left(d\right) \quad and \quad \boldsymbol{C}_2^{\star} = \boldsymbol{C}_1^{\star} + \mathcal{O}\big(d\;\boldsymbol{C}^2\big).$$

Proof. Use Taylor expansion of the function ϱ to get the first equality. Now the first one is proved, we use the definition of $\boldsymbol{C}_{1,2}^{\star}$ and the fact that $\boldsymbol{C}_j = \boldsymbol{C} + \mathcal{O}\left(a^2\;\left|\log(a)\right|^{2h}\right), j=1,2$, to obtain the second equality. □

We use the last lemma to write the system (2.20) as

$$\left\{ \begin{array}{lll} Q_1 - [\Phi_0(z_1;z_2) + \varrho(z_1)] \; \mathbf{C}^\star_1 Q_2 & = & u_0(z_1) + \mathcal{O}\left(d \; |\log(a)|^{h-1}\right), \\ Q_2 - [\Phi_0(z_2;z_1) + \varrho(z_1)] \; \mathbf{C}^\star_1 Q_1 & = & u_0(z_2) + \mathcal{O}\left(d \; |\log(a)|^{h-1}\right). \end{array} \right. \tag{2.22}$$

Remark 2.9. To simplify notations, we write Φ_0 (respectively ϱ, $\mathbf{C}^\star$) instead of $\Phi_0(z_1;z_2)$ (respectively $\varrho(z_1)$, $\mathbf{C}^\star_1$).

After resolution of this algebraic system, we obtain

$$\left\{ \begin{array}{lll} Q_1 & = & \dfrac{u_0(z_1)}{1 - [\Phi_0 + \varrho] \; \mathbf{C}^\star} + \mathcal{O}(d), \\ Q_2 & = & \dfrac{u_0(z_2)}{1 - [\Phi_0 + \varrho] \; \mathbf{C}^\star} + \mathcal{O}(d). \end{array} \right.$$

We use the definition of $Q_{1,2}$, see (2.9), to get

$$\int_{D_1} v \; dx = \frac{u_0(z_1)}{\omega^2 \, \mu_\infty \, \tau_1 \left[(\boldsymbol{C}^\star)^{-1} - (\Phi_0 + \varrho) \right]} + \mathcal{O}(d \, a^2 \, |\log(a)|^h).$$

Recall that $\tau_1 = \epsilon_p - \epsilon_0(z_1)$ and use the fact that $\epsilon_p \gg \epsilon_0$ to obtain

$$\int_{D_1} v \; dx = \frac{u_0(z_1)}{\omega^2 \, \mu_\infty \, \epsilon_p \left[(\boldsymbol{C}^\star)^{-1} - (\Phi_0 + \varrho) \right]} + \mathcal{O}(d \, a^2 \, |\log(a)|^h).$$

Then

$$\int_{D_1} v \; dx = \frac{u_0(z_2)}{\omega^2 \, \mu_\infty \, \epsilon_p \, [(\mathbf{C}^\star)^{-1} - (\Phi_0 + \varrho)]} + \frac{u_0(z_1) - u_0(z_2)}{\omega^2 \, \mu_\infty \, \epsilon_p \big[(\mathbf{C}^\star)^{-1} - (\Phi_0 + \varrho)\big]} + \mathcal{O}(d \, a^2 \, |\log(a)|^h),$$

we estimate the term

$$\frac{u_0(z_1) - u_0(z_2)}{\omega^2 \, \mu_\infty \, \epsilon_p \, [(\mathbf{C}^\star)^{-1} - (\Phi_0 + \varrho)]}$$

as $\mathcal{O}(d \, a^2 \, |\log(a)|^h)$, and use this to obtain

$$\begin{array}{lll} \displaystyle\int_{D_1} v \; dx & = & \dfrac{u_0(z_2)}{\omega^2 \, \mu_\infty \, \epsilon_p \big[(\mathbf{C}^\star)^{-1} - (\Phi_0 + \varrho)\big]} + \mathcal{O}(d \, a^2 \, |\log(a)|^h) \\ & = & \displaystyle\int_{D_2} v \, dx + \mathcal{O}(d \, a^2 \, |\log(a)|^h), \end{array}$$

and finally

$$\int_{D_1} v \, dx = \int_{D_2} v \, dx + \mathcal{O}(d \, a^2 \, |\log(a)|^h). \tag{2.23}$$

By adding the two equations of system (2.22), we get

$$\boldsymbol{C}^{-1} - (\Phi_0 + 2\varrho) = \frac{u_0(z_1) + u_0(z_2)}{\omega^2 \, \mu_\infty \, \epsilon_p \, \left[\displaystyle\int_{D_1} v \, dy + \int_{D_2} v \, dy \right]} + \mathcal{O}\left(d\right). \tag{2.24}$$

We use equation (2.23) to rewrite the denominator as

$$\begin{array}{lll} \omega^2 \, \mu_\infty \, \epsilon_p \left[\displaystyle\int_{D_1} v \, dy + \int_{D_2} v \, dy \right] & = & \omega^2 \, \mu_\infty \, \epsilon_p \left[2 \displaystyle\int_{D_2} v \, dy + \mathcal{O}\left(d \, a^2 \, |\log(a)|^h\right) \right] \\ & = & 2 \, \omega^2 \, \mu_\infty \, \epsilon_p \displaystyle\int_{D_2} v \, dy \left[1 + \dfrac{\mathcal{O}\left(d \, a^2 \, |\log(a)|^h\right)}{\int_{D_2} v \, dy} \right] \end{array}$$

$$= \quad 2\,\omega^2\,\mu_\infty\,\epsilon_p \int_{D_2} v\,dy\,\left[1+\mathcal{O}\left(d\right)\right],$$

then equation (2.24) takes the form

$$\boldsymbol{C}^{-1} - (\Phi_0 + 2\,\varrho) = \frac{u_0(z_1)+u_0(z_2)}{2\,\omega^2\,\mu_\infty\,\epsilon_p \displaystyle\int_{D_2} v\,dy}\left[1+\mathcal{O}\left(d\right)\right] + \mathcal{O}\left(d\right),$$

We manage the errors

$$\begin{aligned}
\mathbf{C}^{-1} - (\Phi_0 + 2\varrho) &= \frac{u_0(z_1)+u_0(z_2)}{2\,\omega^2\,\mu_\infty\,\epsilon_p \displaystyle\int_{D_2} v\,dy} + \mathcal{O}(d\,|\log(a)|^{1-h}) \\
&= \frac{2\,u_0(z_2)}{2\,\omega^2\,\mu_\infty\,\epsilon_p \displaystyle\int_{D_2} v\,dy} + \frac{\int_0^1 (z_1-z_2)\cdot\nabla u_0(z_2+t(z_1-z_2))dt}{2\,\omega^2\,\mu_\infty\,\epsilon_p \displaystyle\int_{D_2} v\,dy} + \mathcal{O}(d\,|\log(a)|^{1-h}) \\
&= \frac{u_0(z_2)}{\omega^2\,\mu_\infty\,\epsilon_p \displaystyle\int_{D_2} v\,dy} + \mathcal{O}(d\,|\log(a)|^{1-h}),
\end{aligned}$$

and take the modulus, we derive the identity:

$$\left|\mathbf{C}^{-1} - (\Phi_0 + 2\varrho)\right|^2 = \frac{|u_0(z_2)|^2}{|\omega^2\,\mu_\infty\,\epsilon_p|^2\,\left|\displaystyle\int_{D_2} v\,dy\right|^2} + \mathcal{O}(d\,|\log(a)|^{2(1-h)}). \tag{2.25}$$

Unfortunately, from the acoustic inversion, we get only data of the form $\int_{D_{1,2}} |v|^2 dx$ and in the last equation we deal with $|\int_{D_{1,2}} v\,dx|^2$. The next lemma makes a link between these two quantities.

Lemma 2.10. *We have*

$$\left|\int_{D_i} v\,dy\right|^2 = a^2\left(\int_B \overline{e}_{n_0}\,dy\right)^2 \int_{D_i} |v|^2\,dy + \mathcal{O}(a^4\,|\log(a)|^h), \quad i=1,2. \tag{2.26}$$

Proof. We split the proof into two steps.
Step 1: Estimation of $|\int_{D_1} v\,dy|^2$.
We use the same techniques as in the proof of the a priori estimation i.e proposition 2.3. We have

$$\begin{aligned}
\int_{D_1} v\,dy &= \langle v; e^{(1)}_{n_0}\rangle \int_D e_{n_0}\,dx + a^2 \sum_{n\neq n_0} \langle \tilde{v}; \overline{e}_n\rangle \int_B \overline{e}_n\,d\eta \\
&\overset{(4.20)}{=} \langle v; e^{(1)}_{n_0}\rangle \int_D e_{n_0} dx + \mathcal{O}\left(a^2\right) \sum_{n\neq n_0}\left[\frac{\langle \tilde{u}_0; \overline{e}_n\rangle}{(1-\omega^2\mu_\infty\tau\lambda_n)} + \mathcal{O}(|\log(a)|^{-h})\langle 1, \overline{e}_n\rangle\right] \int_B \overline{e}_n d\eta.
\end{aligned}$$

When the used frequency is not close to the resonance the following estimation holds

$$\sum_{n\neq n_0}\left[\frac{\langle \tilde{u}_0; \overline{e}_n\rangle}{(1-\omega^2\,\mu_\infty\,\tau\,\lambda_n)} + \mathcal{O}(|\log(a)|^{-h})\langle 1, \overline{e}_n\rangle\right] \int_B \overline{e}_n\,d\eta \sim \mathcal{O}(1),$$

and plug this in the previous equation to obtain

$$\int_{D_1} v\,dy \overset{(4.24)}{=} a^2\left[\frac{\langle \tilde{u}_0; \overline{e}^{(1)}_{n_0}\rangle}{(1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\tau\,a^2\,\Phi_0\left(\int_B \overline{e}_{n_0}\right)^2} + \mathcal{O}(1)\right] \int_B \overline{e}_{n_0}\,d\eta + \mathcal{O}(a^2).$$

Then

$$\left|\int_{D_1} v\,dy\right|^2 = a^4 \frac{|\langle \tilde{u}_0; \overline{e}^{(1)}_{n_0}\rangle|^2}{\left|(1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\tau\,a^2\,\Phi_0\Big(\int_B \overline{e}_{n_0}\Big)^2\right|^2} \left(\int_B \overline{e}_{n_0}\,d\eta\right)^2 + \mathcal{O}(a^4\,|\log(a)|^h). \tag{2.27}$$

Step 2: Estimation of $\int_{D_1} |v|^2\,dy$.
We have

$$\begin{aligned}
\int_{D_1} |v|^2\,dx &= \sum_n |\langle v, e^{(1)}_n\rangle|^2 = a^2 \left(|\langle \tilde{v_1}, \overline{e}_{n_0}\rangle|^2 + \sum_{n\neq n_0} |\langle \tilde{v_1}, \overline{e}_n\rangle|^2\right) \\
&= a^2\,|\langle \tilde{v_1}, \overline{e}_{n_0}\rangle|^2 + \mathcal{O}(a^2) \\
&\overset{(4.24)}{=} a^2\left[\frac{|\langle \tilde{u_0}; \overline{e}^{(1)}_{n_0}\rangle|^2}{\left|(1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\tau\,a^2\,\Phi_0(z_1,z_2)\Big(\int_B \overline{e}_{n_0}\Big)^2\right|^2} + \mathcal{O}(|\log(a)|^h)\right] + \mathcal{O}\left(a^2\right).
\end{aligned}$$

Then

$$\int_{D_1} |v|^2\,dx = a^2 \frac{|\langle \tilde{u_0}; \overline{e}^{(1)}_{n_0}\rangle|^2}{\left|(1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\tau\,a^2\,\Phi_0(z_1,z_2)\Big(\int_B \overline{e}_{n_0}\Big)^2\right|^2} + \mathcal{O}(a^2\,|\log(a)|^h). \tag{2.28}$$

Combining (2.27) and (2.28), we obtain

$$\begin{aligned}
\left|\int_{D_1} v\,dy\right|^2 &= a^4 \frac{|\langle \tilde{u_0}; \overline{e}^{(1)}_{n_0}\rangle|^2}{\left|(1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\tau\,a^2\,\Phi_0(z_1,z_2)\Big(\int_B \overline{e}_{n_0}\Big)^2\right|^2} \left(\int_B \overline{e}_{n_0}\,d\eta\right)^2 + \mathcal{O}(a^4\,|\log(a)|^h) \\
&= a^2\left[\int_{D_1} |v|^2\,dx + \mathcal{O}(a^2\,|\log(a)|^h)\right] \left(\int_B \overline{e}_{n_0}\,d\eta\right)^2 + \mathcal{O}(a^4\,|\log(a)|^h) \\
&= a^2 \int_{D_1} |v|^2\,dx \left(\int_B \overline{e}_{n_0}\,d\eta\right)^2 + \mathcal{O}(a^4\,|\log(a)|^h),
\end{aligned}$$

which proves the formula (2.26). □

We continue with equation (2.25), then

$$\begin{aligned}
\left|\boldsymbol{C}^{-1} - (\Phi_0 + 2\varrho)\right|^2 &= \frac{|u_0(z_2)|^2}{|\omega^2\,\mu_\infty\,\epsilon_p|^2\left|\int_{D_2} v\,dy\right|^2} + \mathcal{O}\left(d\,|\log(a)|^{2(1-h)}\right) \\
&\overset{(2.26)}{=} \frac{|u_0(z_2)|^2}{|\omega^2\,\mu_\infty\,\epsilon_p|^2\left[a^2\left(\int_B \overline{e}_{n_0} dy\right)^2 \int_{D_2} |v|^2\,dy + \mathcal{O}(a^4|\log(a)|^h)\right]} + \mathcal{O}(d\,|\log(a)|^{2(1-h)}) \\
&= \frac{|u_0(z_2)|^2}{|\omega^2\,\mu_\infty\,\epsilon_p|^2\,a^2\left(\int_B \overline{e}_{n_0} dy\right)^2 \int_{D_2} |v|^2\,dy\left[1 + \mathcal{O}(|\log(a)|^{-h})\right]} + \mathcal{O}(d\,|\log(a)|^{2(1-h)}) \\
&= \frac{|u_0(z_2)|^2}{|\omega^2\,\mu_\infty\,\epsilon_p|^2\,a^2\left(\int_B \overline{e}_{n_0} dy\right)^2 \int_{D_2} |v|^2\,dy} + \mathcal{O}(|\log(a)|^{2-3h}).
\end{aligned}$$

In the following proposition, we write an estimation of $|u_0(z_2)|$ in the case of one particle inside the domain.

Proposition 2.11. *We have*

$$|u_0(z_2)|^2 = \frac{\left|1-\omega^2\mu_\infty\tau\,\lambda_{n_0} - \omega^2\mu_\infty\tau\,\varrho(z)\left(\int_D e_{n_0}\right)^2\right|^2}{\left(\int_D e_{n_0}\right)^2} \int_D |u_1|^2\,dx + \mathcal{O}\left(|\log(a)|^{-1}\right). \tag{2.29}$$

Proof. To fix notations recall the L.S.E for one particle

$$u_1(x) - \omega^2\,\mu_\infty \int_D G_k(x,y)\,(\epsilon_p-\epsilon_0)(y)\,u_1(y)\,dy = u_0(x), \qquad x \in D.$$

With this notation the equation (4.13) takes the following form

$$\langle u_1; e_{n_0}\rangle = \frac{\langle u_0; e_{n_0}\rangle}{\left[1-\omega^2\mu_\infty\tau\,\lambda_{n_0} - \omega^2\mu_\infty\tau\,\varrho(z)\,\left(\int_D e_{n_0}\right)^2\right]} + \mathcal{O}\left(a\,|\log(a)|^{h-1}\right). \tag{2.30}$$

Next,

$$\begin{aligned}
\int_D |u_1|^2\,dx &= |\langle u_1; e_{n_0}\rangle|^2 + a^2 \sum_{n\neq n_0} |\langle \tilde{u}_1; \overline{e}_n\rangle|^2 \\
&\overset{(2.30)}{=} \frac{|\langle u_0; e_{n_0}\rangle|^2}{\left|1-\omega^2\mu_\infty\tau\,\lambda_{n_0} - \omega^2\mu_\infty\tau\,\varrho(z)\left(\int_D e_{n_0}\right)^2\right|^2} + \mathcal{O}\left(a^2\,|\log(a)|^{2h-1}\right).
\end{aligned} \tag{2.31}$$

We develop u_0 near the point z to obtain

$$\begin{aligned}
\int_D |u_1|^2\,dx &= \frac{\left[|u_0(z_2)|^2\left(\int_D e_{n_0}dx\right)^2 + \mathcal{O}(a^3)\right]}{\left|1-\omega^2\mu_\infty\tau\,\lambda_{n_0} - \omega^2\mu_\infty\tau\,\varrho(z)\left(\int_D e_{n_0}\right)^2\right|^2} + \mathcal{O}\left(a^2\,|\log(a)|^{2h-1}\right) \\
&= \frac{|u_0(z_2)|^2\left(\int_D e_{n_0}dx\right)^2}{\left|1-\omega^2\mu_\infty\tau\,\lambda_{n_0} - \omega^2\mu_\infty\tau\,\varrho(z)\left(\int_D e_{n_0}\right)^2\right|^2} + \mathcal{O}\left(a^2\,|\log(a)|^{2h-1}\right).
\end{aligned}$$

This proves (2.29).
We can, by keeping the dominant term of denominator, write the previous result as

$$\int_D |u_1|^2\,dx = \frac{|u_0(z_2)|^2\,\left(\int_D e_{n_0}dx\right)^2}{|1-\omega^2\mu_\infty\epsilon_p\,\lambda_{n_0}|^2} + \mathcal{O}\left(a^2\,|\log(a)|^{3h-1}\right).$$

This proves (1.20). □

In (2.29), we use the following notation

$$\Psi := \left|1-\omega^2\mu_\infty\tau\,\lambda_{n_0} - \omega^2\mu_\infty\tau\,\varrho\left(\int_D e_{n_0}\right)^2\right|^2.$$

With this, we get

$$\begin{aligned}
\left|\boldsymbol{C}^{-1} - (\Phi_0 + 2\varrho)\right|^2 &= \frac{|u_0(z_2)|^2}{|\omega^2\,\mu_\infty\,\epsilon_p|^2\left(\int_D e_{n_0}dy\right)^2\int_{D_2}|v|^2\,dy} + \mathcal{O}\left(|\log(a)|^{2-3h}\right) \\
&\overset{(2.29)}{=} \frac{\Psi\displaystyle\int_D |u_1|^2\,dx}{|\omega^2\,\mu_\infty\,\epsilon_p|^2\left(\int_D e_{n_0}dy\right)^4\int_{D_2}|v|^2\,dy} + \mathcal{O}(|\log(a)|^{2-3h}).
\end{aligned}$$

We set

$$B := \frac{\int_D |u_1|^2 \, dx}{|\omega^2 \, \mu_\infty \, \epsilon_p|^2 \, \Big(\int_D e_{n_0} dy \Big)^4 \int_{D_2} |v|^2 \, dy}. \tag{2.32}$$

We develop the left side of the last equation as

$$\big| \mathbf{C}^{-1} - (\Phi_0 + 2\varrho) \big|^2 = \Big(\mathbf{C}^{-1} - \Phi_0 \Big)^2 - 4 \Big(\mathbf{C}^{-1} - \Phi_0 \Big) \mathrm{Re}\,(\varrho) + 4 \, (\mathrm{Re}\,(\varrho))^2 + 4 \, (\mathrm{Im}\,(\varrho))^2 ,$$

then, we have

$$\big(\mathbf{C}^{-1} - \Phi_0 \big)^2 - 4 \, \big(\mathbf{C}^{-1} - \Phi_0 \big) \, \mathrm{Re}\,(\varrho) + 4 \, (\mathrm{Re}\,(\varrho))^2 + 4 \, (\mathrm{Im}\,(\varrho))^2 = \Psi \, B + \mathcal{O}(|\log(a)|^{2-3h}). \tag{2.33}$$

Remark that Ψ can be rewritten as

$$\begin{aligned} \Psi &= \Big| 1 - \omega^2 \mu_\infty \epsilon_p \, \lambda_{n_0} - \omega^2 \mu_\infty \epsilon_p \, \varrho \Big(\int_D e_{n_0} \Big)^2 \Big|^2 + \mathcal{O} \left(a^2 \, |\log(a)|^{1-h} \right) \\ &= \Big| 1 - \omega^2 \mu_\infty \epsilon_p \, \lambda_{n_0} \Big|^2 + (\omega^2 \mu_\infty)^2 \, |\epsilon_p|^2 \Big(\int_D e_{n_0} \Big)^4 \, \Big[(\mathrm{Re}\,(\varrho))^2 + (\mathrm{Im}\,(\varrho))^2 \Big] \\ &- \; 2\omega^2 \mu_\infty \Big(\int_D e_{n_0} \Big)^2 \, \mathrm{Re} \Big[\overline{\epsilon}_p \, \overline{\varrho} \, (1 - \omega^2 \mu_\infty \epsilon_p \lambda_{n_0}) \Big] + \mathcal{O} \left(a^2 \, |\log(a)|^{1-h} \right). \end{aligned}$$

Hence using (4.28), we have

$$\begin{aligned} \Psi &= \mathbf{C}^{-2} \, (\omega^2 \, \mu_\infty)^2 \, |\epsilon_p|^2 \Big(\int_D e_{n_0} \Big)^4 + (\omega^2 \mu_\infty)^2 \, |\epsilon_p|^2 \Big(\int_D e_{n_0} \Big)^4 \, \Big[(\mathrm{Re}\,(\varrho))^2 + (\mathrm{Im}\,(\varrho))^2 \Big] \\ &- \; 2 \, \mathbf{C}^{-1} \, (\omega^2 \mu_\infty)^2 \, |\epsilon_p|^2 \Big(\int_D e_{n_0} \Big)^4 \, \mathrm{Re}\,(\varrho) + \mathcal{O}\big(|\log(a)|^{-3h} \big). \end{aligned}$$

Replace Ψ in (2.33) and use the fact that $B = \mathcal{O}(|\log(a)|^2)$ to cancel all the terms of order $\mathcal{O}(1)$. The formula (2.33) will be

$$\begin{aligned} \big(\mathbf{C}^{-1} - \Phi_0 \big)^2 - 4 \, \big(\mathbf{C}^{-1} - \Phi_0 \big) \, \mathrm{Re}\,(\varrho) &= -2 \, \mathbf{C}^{-1} \, (\omega^2 \mu_\infty)^2 \, |\epsilon_p|^2 \Big(\int_D e_{n_0} \Big)^4 \, \mathrm{Re}\,(\varrho) \, B \\ &+ \; \mathbf{C}^{-2} \, (\omega^2 \, \mu_\infty)^2 \, |\epsilon_p|^2 \, \Big(\int_D e_{n_0} \Big)^4 B + \mathcal{O} \Big(|\log(a)|^{\max(0, 2-3h)} \Big). \end{aligned}$$

Then

$$\mathrm{Re}\,(\varrho) \, \Bigg[-4 \, \big(\mathbf{C}^{-1} - \Phi_0 \big) + 2 \, \boldsymbol{C}^{-1} \, (\omega^2 \mu_\infty)^2 \, |\epsilon_p|^2 \, \Big(\int_D e_{n_0} \Big)^4 \, B \Bigg] \; =$$

$$\boldsymbol{C}^{-2} \, (\omega^2 \, \mu_\infty)^2 \, |\epsilon_p|^2 \, \Big(\int_D e_{n_0} \Big)^4 B - \big(\mathbf{C}^{-1} - \Phi_0 \big)^2 + \mathcal{O} \Big(|\log(a)|^{\max(0, 2-3h)} \Big).$$

Using (2.32), we get an explicit expression

$$\mathrm{Re}\,(\varrho) = \frac{1}{2\,\mathbf{C}} \; \frac{\dfrac{\int_D |u_1|^2 \, dx}{\int_{D_2} |v|^2 \, dx} - (1 - \mathbf{C} \, \Phi_0)^2}{\dfrac{\int_D |u_1|^2 \, dx}{\int_{D_2} |v|^2 \, dx} - 2 \, (1 - \mathbf{C} \, \Phi_0)} + \mathcal{O}\Big(|\log(a)|^{\max(h-1; 1-2h)} \Big). \tag{2.34}$$

Remark 2.12. To justify that (2.34) is well defined, we use (2.31), (2.28) and (4.28) to obtain the following relation

$$\frac{\displaystyle\int_D |u_1|^2\,dx}{\displaystyle\int_{D_2} |v|^2\,dx} = \frac{(1-\mathbf{C}\,\Phi_0)^2}{|1-\mathbf{C}\,\varrho|^2} + \mathcal{O}\left(|\log(a)|^{h-1}\right).$$

Hence,

$$\frac{\dfrac{\displaystyle\int_D |u_1|^2\,dx}{\displaystyle\int_{D_2} |v|^2\,dx} - (1-\mathbf{C}\,\Phi_0)^2}{\dfrac{\displaystyle\int_D |u_1|^2\,dx}{\displaystyle\int_{D_2} |v|^2\,dx} - 2\,(1-\mathbf{C}\,\Phi_0)} = \frac{(1-\mathbf{C}\,\Phi_0)\;\mathbf{C}\,\left\{2\,\mathrm{Re}\,(\varrho) - \mathbf{C}\,|\varrho|^2\right\}}{-1-\mathbf{C}\,\Phi_0 + 2\,\mathbf{C}\,\left\{2\,\mathrm{Re}\,(\varrho) - \mathbf{C}\,|\varrho|^2\right\}} \sim \mathcal{O}\left(\boldsymbol{C}\right).$$

Therefore the error term in (2.34) is indeed negligible as soon as $\frac{1}{2} < h < 1$.

2.3. **Reconstruction of ϵ_0 from** $\mathrm{Re}\,G_k$**.** Here, we show how to use the formula (2.34) to reconstruct $\epsilon_0(\cdot)$ inside Ω. From (2.34) we can reconstruct $\mathrm{Re}\,(\varrho(z))$, z fixed, and then combined with (2.2) we get

$$\mathrm{Re}\,\left(G_k(|x-z|)\right) = \frac{-1}{2\pi}\log(|x-z|) + \mathrm{Re}\,\left(\varrho(z)\right) + \mathcal{O}\left(a\right), \quad x \in D(z,a), \tag{2.35}$$

and then we can reconstruct, via numerical differentiation methods,

$$\Delta\mathrm{Re}\,\left(G_k(|x-z|)\right). \tag{2.36}$$

Now, after taking the reel part on both sides of (2.1), we obtain

$$\left(-\omega^2\,\mu_\infty\right)^{-1}\,\underset{x}{\Delta}\,\mathrm{Re}\,\left(G_{k(z)}(x,z)\right) = \mathrm{Re}\,\left(\epsilon_0(z)\,G_{k(z)}(x,z)\right), \qquad x \neq z, \tag{2.37}$$

and this allows us to reconstruct

$$\mathrm{Re}\,\left(\epsilon_0(z)\,G_{k(z)}(x,z)\right). \tag{2.38}$$

Next, we need the following lemma.

Lemma 2.13. *For x near z, we have:*

$$\mathrm{Re}\,(G_k(|x-z|)) = \frac{-1}{2\pi}\log(|x-z|) + \mathcal{O}\left(1\right) \quad \textit{and} \quad \mathrm{Im}\,(G_k(|x-z|)) = \mathcal{O}\left(1\right). \tag{2.39}$$

Proof. Observe that (2.39) is nothing but (2.35). To prove that $\mathrm{Im}\,(G_k(|x-z|)) = \mathcal{O}\,(1)$, we start by recalling, from (2.1), that the imaginary part satisfies the following equation

$$\Delta\mathrm{Im}\,(G_k(|x-z|)) + \omega^2\,\mu_\infty\,\mathrm{Re}\,(\epsilon_0(x))\,\mathrm{Im}\,(G_k(|x-z|)) = -\omega^2\,\mu_\infty\,\mathrm{Im}\,(\epsilon_0(x))\,\mathrm{Re}\,(G_k(|x-z|)). \tag{2.40}$$

Remark that the right hand side term of (2.40) is locally in $\mathbb{L}^p$, $\forall p \geq 1$. Then, we deduce from the regularity of Helmholtz equation that $\mathrm{Im}\,(G_k)$ is locally in $\mathbb{W}^{2,p}$, $\forall p \geq 1$. Consequently, from Sobolev embedding results, we deduce that $\mathrm{Im}\,(G_k) \in \mathbb{L}^\infty$. This ends the proof. □

Moving back to the reconstructed formula, given by (2.38), and write it as

$$\begin{aligned}\mathrm{Re}\,\left(\epsilon_0(z)\,G_{k(z)}(x,z)\right) \quad &= \quad \mathrm{Re}\,\left(\epsilon_0(z)\right)\,\mathrm{Re}\,\left(G_{k(z)}(x,z)\right) - \mathrm{Im}\,\left(\epsilon_0(z)\right)\,\mathrm{Im}\,\left(G_{k(z)}(x,z)\right) \\ &\overset{(2.39)}{=} \quad \frac{-1}{2\pi}\,\mathrm{Re}\,\left(\epsilon_0(z)\right)\,\log(|x-z|) + \mathcal{O}(1).\end{aligned}$$

Then, we reconstruct $\mathrm{Re}\,\left(\epsilon_0(z)\right)$ by:

$$\mathrm{Re}\,\left(\epsilon_0(z)\right) = \frac{-2\pi\,\mathrm{Re}\,\left(\epsilon_0(z)\,G_{k(z)}(x,z)\right)}{\log(|x-z|)} + \mathcal{O}\left(|\log(a)|^{-1}\right). \tag{2.41}$$

Let us now show how to construct the imaginary part, $\mathrm{Im}\,(\epsilon_0(z))$. We already know $\mathrm{Re}\,(\epsilon_0(z)), \mathrm{Re}\,(G_k)$ and $\mathrm{Re}\,(\epsilon_0(z)G_k)$, see for instance (2.41), (2.35) and (2.38), respectively, and we know that

$$\mathrm{Im}\,(\epsilon_0(z))\,\mathrm{Im}\,(G_k) = \mathrm{Re}\,(\epsilon_0(z))\,\mathrm{Re}\,(G_k) - \mathrm{Re}\,(\epsilon_0(z)G_k), \tag{2.42}$$

and this formula allows us to construct the term:

$$\mathrm{Im}\,(\epsilon_0(z))\,\mathrm{Im}\,(G_k). \tag{2.43}$$

As the function $\mathrm{Im}\,(\epsilon_0(\cdot))$ has the $\mathcal{C}^1$-regularity, we use the Taylor expansion, for x near z, $\mathrm{Im}\,(\epsilon_0(x)) = \mathrm{Im}\,(\epsilon_0(z)) + \mathcal{O}\,(|x-z|)$, and we multiply the equation (2.40) by $\mathrm{Im}\,(\epsilon_0(x))$, for x near z, to obtain

$$\begin{aligned} \Delta\,(\mathrm{Im}\,(\epsilon_0(z))\mathrm{Im}\,(G_k(|x-z|))) &+ \omega^2\,\mu_\infty\,\mathrm{Re}\,(\epsilon_0(z))\,\mathrm{Im}\,(\epsilon_0(z))\,\mathrm{Im}\,(G_k(|x-z|)) \\ &+ \omega^2\,\mu_\infty\,\mathrm{Im}\,(\epsilon_0(z))\,\mathrm{Im}\,(\epsilon_0(z))\,\mathrm{Re}\,(G_k(|x-z|)) \\ &= \Delta\,(\mathrm{Im}\,(G_k(|x-z|)))\,\mathcal{O}\,(|x-z|) \\ &= \mathcal{O}\,(|x-z|\,\log\,(|x-z|))\,, \end{aligned}$$

where the last step is justified using the two equations (2.40) and (2.39). Using (2.42), we rewrite the last equation as

$$\begin{aligned} -\,(\mathrm{Im}\,(\epsilon_0(z)))^2\,\mathrm{Re}\,(G_k(|x-z|)) &= \left(\omega^2\,\mu_\infty\right)^{-1}\Delta\,(\mathrm{Im}\,(\epsilon_0(z))\mathrm{Im}\,(G_k(|x-z|))) \\ &+ (\mathrm{Re}\,(\epsilon_0(z)))^2\,\mathrm{Re}\,(G_k(|x-z|)) \\ &- \mathrm{Re}\,(\epsilon_0(z))\,\mathrm{Re}\,(\epsilon_0(z)\,G_k(|x-z|)) + \mathcal{O}\,(|x-z|\,\log\,(|x-z|))\,. \end{aligned} \tag{2.44}$$

We have seen from the last computation how to reconstruct the three first terms appearing on the right hand side of (2.44). Then, consequently, we are able to reconstruct $\mathrm{Im}\,(\epsilon_0(z))$. Finally, for z fixed, we reconstruct $\epsilon_0(z)$. Moving the point z in Ω, we reconstruct the function $\epsilon_0(\cdot)$.

3. Proof of Theorem 1.2

We recall the model problem for photo-acoustic imaging:

$$\begin{cases} \partial_t^2 p(x,t) - \underset{x}{\Delta} p(x,t) &= 0 \qquad in \quad \mathbb{R}^2 \times \mathbb{R}^+, \\ p(x,0) &= \dfrac{\omega\,\beta_0}{c_p}\mathrm{Im}\,(\boldsymbol{\varepsilon})(x)\,|u|^2(x)\,\chi_\Omega, \qquad in \quad \mathbb{R}^2 \\ \partial_t p(x,0) &= 0 \qquad in \quad \mathbb{R}^2. \end{cases} \tag{3.1}$$

Remark 3.1. Next, when we solve the equation (3.1), we omit the multiplicative term[12] $\dfrac{\omega\,\beta_0}{2\pi\,c_p}$.

3.1. Photo-acoustic imaging using one particle. Proof of (1.23).

The next lemma gives an estimation of the total field for $x \in \Omega \setminus D$.

Lemma 3.2. *The total field behaves as*

$$|u_1(x)|^2 = \mathcal{O}(1) + \mathcal{O}(|\log(a)|^{h-1}\,|\log(dist)|) \quad in \quad \Omega \setminus D, \tag{3.2}$$

where $dist = dist(x, D)$.

Proof. We use L.S.E

$$u_1(x) = u_0(x) + \omega^2\,\mu_\infty \int_D (\epsilon_p - \epsilon_0)(y) G_k(y,x) u_1(y) dy, \qquad x \in \mathbb{R}^2. \tag{3.3}$$

Now, for x away from D

$$|u_1(x)| \;\leq\; |u_0(x)| + \mathcal{O}\left(\frac{1}{a^2\,|\log(a)|}\int_D |G_0|(y,x)\,|u_1(y)|\,dy\right)$$

[12] The constant 2π in the denominator comes from the Poisson formula.

$$= \mathcal{O}(1) + \mathcal{O}\left(\frac{1}{a^2\,|\log(a)|}\,\|u_1\| \Big[\int_D |G_0|^2(y,x)\,dy\Big]^{\frac{1}{2}}\right) = \mathcal{O}(1) + \mathcal{O}\Big(|\log(a)|^{h-1}\,|\log(dist)|\Big)$$

This proves (3.2). □

Let us recall from proposition (4.1), the following relation

$$\langle u_1, e_{n_0}\rangle = \frac{1}{[1-\omega^2\,\mu_\infty\,\epsilon_p\,\lambda_{n_0}]}\,\langle u_0; e_{n_0}\rangle + \mathcal{O}\left(a\,|\log(a)|^{2h-1}\right). \tag{3.4}$$

We use Poisson's formula to solve the system (3.1), see ([23], Chapter 9), to represent the pressure as follows

$$\begin{aligned} p(t,x) &= \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_p)|u_1|^2)(y)}{\sqrt{t^2-|x-y|^2}}\chi_D dy + \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_0)|u_1|^2)(y)}{\sqrt{t^2-|x-y|^2}}\chi_{\Omega\setminus D} dy \\ &= \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_p-\epsilon_0)|u_1|^2)(y)}{\sqrt{t^2-|x-y|^2}}\chi_D dy + \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_0)|u_1|^2)(y)}{\sqrt{t^2-|x-y|^2}}\chi_\Omega dy. \end{aligned}$$

Let $t > diam(\Omega)$. For $x \in \partial\Omega$, the representation above reduces to:

$$p(t,x) = \int_{D(z,a)} \partial_t \frac{(\mathrm{Im}\,(\epsilon_p-\epsilon_0)|u_1|^2)(y)}{\sqrt{t^2-|x-y|^2}} dy + \int_\Omega \partial_t \frac{(\mathrm{Im}\,(\epsilon_0)|u_1|^2)(y)}{\sqrt{t^2-|x-y|^2}} dy.$$

Set T_4 to be

$$T_4 := \int_\Omega \partial_t \frac{(\mathrm{Im}\,(\epsilon_0)|u_1|^2)(y)}{\sqrt{t^2-|x-y|^2}} dy.$$

Recalling that $\tau := \epsilon_p - \epsilon_0(z)$, we have

$$p(t,x) = -t\,\mathrm{Im}\,(\tau)\int_{D(z,a)} \frac{|u_1|^2(y)}{(t^2-|x-y|^2)^{3/2}} dy + T_4 + \int_{D(z,a)} |u_1|^2(y)\;\partial_t \frac{\mathrm{Im}\,\Big(\int_0^1 (y-z)\cdot\nabla\epsilon_0(z+s(y-z))ds\Big)}{\sqrt{t^2-|x-y|^2}} dy$$

We estimate the remainder term as follows

$$\left|\int_{D(z,a)} |u_1|^2(y)\;\partial_t \frac{\mathrm{Im}\,\Big(\int_0^1 (y-z)\cdot\nabla\epsilon_0(z+s(y-z))ds\Big)}{\sqrt{t^2-|x-y|^2}} dy\right| \leq a\,\|u_1\|^2_{\mathbb{L}^2(D)} = \mathcal{O}(a^3\;|\log(a)|^{2h}), \tag{3.5}$$

then

$$p(t,x) = -t\,\mathrm{Im}\,(\tau)\int_{D(z,a)} \frac{|u_1|^2(y)}{(t^2-|x-y|^2)^{3/2}} dy + T_4 + \mathcal{O}(a^3\;|\log(a)|^{2h}).$$

By Taylor expansion of the function $\left(t^2-|x-\cdot|^2\right)^{-3/2}$ near z, we have

$$\begin{aligned} p(t,x) &= \frac{-t\,\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\int_{D(z,a)} |u_1|^2(y) dy + T_4 \\ &+ \mathcal{O}\left(a^3\;|\log(a)|^{2h}\right) + \mathcal{O}\left[\mathrm{Im}\,(\tau)\int_{D(z,a)} \left(|y-z|^2 + 2\langle x-z; z-y\rangle\right)|u_1|^2(y)dy\right]. \end{aligned}$$

We estimate the remainder term as

$$\left|\mathrm{Im}\,(\tau)\int_D \left(|y-z|^2 + 2\langle x-z; z-y\rangle\right)|u_1|^2(y)dy\right| \leq \mathrm{Im}\,(\tau)\,a\,\|u_1\|^2 = \mathcal{O}(\mathrm{Im}\,(\tau)a^3|\log(a)|^{2h}), \tag{3.6}$$

and then

$$p(t,x) = \frac{-t\,\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\int_{D(z,a)} |u_1|^2(y)dy + T_4 + \mathcal{O}\big(a^3\;\mathrm{Im}\,(\tau)\;|\log(a)|^{2h}\big).$$

Writing u_1 as a Fourier series over the basis $\left(e_n\right)_{n\in\mathbb{N}}$, we obtain

$$p(t,x) = \frac{-t\,\mathrm{Im}\,(\tau)\,|\langle u_1; e_{n_0}\rangle|^2}{(t^2-|x-z|^2)^{3/2}} - \frac{t\,\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\sum_{n\neq n_0}|\langle u_1; e_n\rangle|^2 + T_4 + \mathcal{O}\left(a^3\,\mathrm{Im}\,(\tau)\,|\log(a)|^{2h}\right), \tag{3.7}$$

since $n \neq n_0$ we estimate the series as

$$\mathcal{O}\left(\mathrm{Im}\,(\tau)\sum_{n\neq n_0}|\langle u_1; e_n\rangle|^2\right) \sim \mathcal{O}\left(\mathrm{Im}\,(\tau)\;\|u_0\|^2_{\mathbb{L}^2(D)}\right) = \mathcal{O}\left(\mathrm{Im}\,(\tau)\,a^2\right). \tag{3.8}$$

Next,

$$p(t,x) \overset{(3.4)}{=} \frac{-t\,\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\left[\frac{|\langle u_0; e_{n_0}\rangle|^2}{|1-\omega^2\mu_\infty\lambda_{n_0}\tau|^2} + \mathcal{O}(a^2\,|\log(a)|^{3h-1})\right] + T_4 + \mathcal{O}\left(\mathrm{Im}\,(\tau)\,a^2\right),$$

hence

$$p(t,x) = \frac{-t\,\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\,\frac{|\langle u_0; e_{n_0}\rangle|^2}{|1-\omega^2\mu_\infty\lambda_{n_0}\tau|^2} + T_4 + \mathcal{O}(\mathrm{Im}\,(\tau)\,a^2) + \mathcal{O}(\mathrm{Im}\,(\tau)\,a^2\,|\log(a)|^{3h-1}). \tag{3.9}$$

In order to calculate the term T_4, we use L.S.E

$$u_1(x) - \omega^2\,\mu_\infty\int_D(\epsilon_p-\epsilon_0(\eta))G_k(x,\eta)\,u_1(\eta)d\eta = u_0(x) \quad in \quad \Omega,$$

and define

$$p_0(t,x) := \int_\Omega \partial_t\,\frac{1}{\sqrt{t^2-|x-y|^2}}\,\mathrm{Im}\,(\epsilon_0)(y)\;|u_0|^2(y)\,dy. \tag{3.10}$$

Observe that $p_0(t,x)$ is the measured pressure at point $x\in\partial\Omega$ and time t when no particle is inside Ω. We set

$$f := \partial_t\,\frac{1}{\sqrt{t^2-|x-y|^2}}\,\mathrm{Im}\,(\epsilon_0)(y).$$

With this, we get

$$T_4 = \int_\Omega \partial_t\,\frac{1}{\sqrt{t^2-|x-y|^2}}\,\mathrm{Im}\,(\epsilon_0)(y)\;|u_1|^2(y)\,dy = \int_{\Omega\setminus D} f\,|u_1|^2(y)\,dy + \int_D f\,|u_1|^2(y)\,dy.$$

If we compare (4.12) to (3.2) we deduce that the term $(\star) := \int_D f\,|u_1|^2(y)\,dy$ is less dominant than the one given on $\Omega\setminus D$. Now, since f is smooth we can estimate $(\star)$, with help of a priori estimation, as

$$|(\star)| = \left|\int_D f\,|u_1|^2(y)\,dy\right| \leq \|u_1\|^2 = \mathcal{O}\big(a^2\,|\log(a)|^{2h}\big),$$

and, from L.S.E, see for instance (3.3), we can rewrite T_4 as

$$\begin{aligned} T_4 &= p_0(t,x) - \int_D f|u_0|^2(y)dy + (\omega^2\mu_\infty)^2\int_{\Omega\setminus D} f\left|\int_D(\epsilon_p-\epsilon_0(\eta))G_k(\eta,y)u_1(\eta)d\eta\right|^2 dy \\ &+ 2\,\omega^2\mu_\infty\mathrm{Re}\left[\int_{\Omega\setminus D} f\overline{u}_0(y)\int_D(\epsilon_p-\epsilon_0(\eta))G_k(\eta,y)u_1(\eta)d\eta dy\right] + \mathcal{O}\big(a^2\,|\log(a)|^{2h}\big). \end{aligned}$$

The smoothness of u_0 is enough to justify the following estimation $\left|\int_D f|u_0|^2(y)dy\right| \sim \mathcal{O}\left(a^2\right)$.
To finish the estimation of T_4 we still have to deal with two terms. More exactly, we set

$$S_3 := \int_{\Omega\setminus D} f\left|\int_D(\epsilon_p-\epsilon_0(\eta))G_k(\eta,y)\,u_1(\eta)d\eta\right|^2 dy. \tag{3.11}$$

Expanding $(\epsilon_p-\epsilon_0(\cdot))$ near z, we obtain

$$|S_3| \leq |\tau|^2\int_{\Omega\setminus D}|f|\left(\int_D\left|G_k(\eta,y)\,u_1(\eta)\right|d\eta\right)^2 dy$$

$$
\begin{aligned}
&+ \int_{\Omega\setminus D} |f| \left(\int_D \left| \int_0^1 (z-\eta)\cdot\nabla\epsilon_0(z+s(\eta-z))ds G_k(\eta,y)\, u_1(\eta)\right| d\eta \right)^2 dy \\
&+ 2\int_{\Omega\setminus D} |f\,| \left| \mathrm{Re}\left[\overline{\tau} \int_D \overline{G}_k(\eta,y)\,\overline{u}_1(\eta)\,d\eta \int_D \int_0^1 (z-\eta)\cdot\nabla\epsilon_0(z+s(\eta-z))ds G_k(\eta,y)\, u_1(\eta) d\eta \right] \right| dy,
\end{aligned}
$$

then apply Cauchy Schwartz inequality and exchange the integration variables to obtain

$$
|S_3| \leq |\tau|^2 \, \|u_1\|^2 \int_D J(\eta)\, d\eta + \mathcal{O}(a^2)\, \|u_1\|^2 \int_D J(\eta)\, d\eta + \mathcal{O}(a\,\tau)\, \|u_1\|^2 \int_D J(\eta)\, d\eta \lesssim |\tau|^2 \, \|u_1\|^2 \int_D J(\eta)\, d\eta,
$$

where J is the function given by $\quad J(\eta) := \int_{\Omega\setminus D} |f| \, \left|G_k(\eta,y)\right|^2 \, dy$. Remark that J is a smooth function because f is a smooth and η and y are in two disjoint domains. Then

$$
S_3 = \mathcal{O}(|\log(a)|^{2h-2}).
$$

The last term to estimate, that we set as S_4, is more delicate. We split it as:

$$
\begin{aligned}
S_4 &:= 2\,\omega^2 \mu_\infty \,\mathrm{Re}\left[\int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D (\epsilon_p - \epsilon_0(\eta)) G_k(\eta,y)\, u_1(\eta)\, d\eta\, dy \right] \\
&= 2\,\omega^2 \mu_\infty \sum_n \mathrm{Re}\left[\langle u_1; e_n\rangle\, \tau \int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D G_k(\eta,y)\, e_n(\eta)\, d\eta\, dy \right] \\
&- 2\,\omega^2 \mu_\infty \,\mathrm{Re}\left[\int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D \int_0^1 (z-\eta)\cdot\nabla\epsilon_0(z+s(\eta-z))ds\, G_k(\eta,y)\, u_1(\eta)\, d\eta\, dy \right].
\end{aligned}
$$

The same techniques, as previously, allows to estimate the second term of S_4 as $\mathcal{O}\left(a^4\, |\log(a)|^h\right)$. Then

$$
\begin{aligned}
S_4 &= 2\,\omega^2 \mu_\infty \,\mathrm{Re}\left[\langle u_1; e_{n_0}\rangle\, \tau \int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D G_k(\eta,y)\, e_{n_0}(\eta)\, d\eta\, dy \right] \\
&+ \mathcal{O}\left(\sum_{n\neq n_0} \mathrm{Re}\left[\langle u_1; e_n\rangle\, \tau \int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D G_k(\eta,y)\, e_n(\eta)\, d\eta\, dy \right] \right) + \mathcal{O}\left(a^4\, |\log(a)|^h\right).
\end{aligned}
$$

We keep the term with index n_0 and estimate the series as

$$
\begin{aligned}
|\mathcal{O}(\cdots)| &\lesssim |\tau| \int_{\Omega\setminus D} |f| \, |\overline{u}_0(y)| \sum_{n\neq n_0} \left| \langle u_1; e_n\rangle \right| \left| \int_D G_k(\eta,y)\, e_n(\eta)\, d\eta \right| dy \\
&\lesssim |\tau| \, \|u_1\| \, \|f\,\overline{u}_0\|_{\mathbb{L}^2(\Omega\setminus D)} \left(\int_D \int_{\Omega\setminus D} \left|G_k(\eta,y)\,\right|^2 dy d\eta \right)^{\frac{1}{2}} = \mathcal{O}\Big(|\log(a)|^{-1}\Big). \qquad (3.12)
\end{aligned}
$$

Plug this in the last equation to obtain

$$
\begin{aligned}
S_4 &\overset{(3.4)}{=} 2\,\omega^2 \mu_\infty \,\mathrm{Re}\left[\frac{\langle u_0; e_{n_0}\rangle}{[1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}]}\, \tau \int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D G_k(\eta,y)\, e_{n_0}(\eta)\, d\eta\, dy \right] \\
&+ \mathcal{O}\left(a\, |\log(a)|^{2h-1} \,\mathrm{Re}\left[\tau \int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D G_k(\eta,y)\, e_{n_0}(\eta)\, d\eta\, dy \right] \right) + \mathcal{O}\big(|\log(a)|^{-1}\big),
\end{aligned}
$$

the same technique, as previously again, see (3.12), allows us to deduce that

$$
a\, |\log(a)|^{2h-1} \,\mathrm{Re}\left[\tau \int_{\Omega\setminus D} f \;\, \overline{u}_0(y) \int_D G_k(\eta,y)\, e_{n_0}(\eta)\, d\eta\, dy \right] = \mathcal{O}\big(|\log(a)|^{2h-2}\big).
$$

The last step is to use Taylor expansion to write $\langle u_0; e_{n_0}\rangle$ in function of the center z. We have

$$\begin{aligned} S_4 &= 2\,\omega^2\mu_\infty \int_D e_{n_0}\,dx\,\mathrm{Re}\left[\frac{u_0(z)}{[1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}]}\,\tau\int_{\Omega\setminus D} f\ \overline{u}_0(y)\int_D G_k(\eta,y)\,e_{n_0}(\eta)\,d\eta\,dy\right] \\ &+ \mathcal{O}\left(\frac{\int_D\int_0^1(z-\eta)\cdot\nabla u_0(z+s(\eta-z))ds\,e_{n_0}(\eta)d\eta}{[1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}]}\,\tau\int_{\Omega\setminus D} f\,\overline{u}_0(y)\int_D G_k(\eta,y)\,e_{n_0}(\eta)\,d\eta\,dy\right) \\ &+ \mathcal{O}\left(|\log(a)|^{2h-2}\right), \end{aligned}$$

then we compute an estimation of the remainder term from Taylor expansion. More precisely, we have

$$\begin{aligned} |\mathcal{O}(\cdots)| &\lesssim a\,|\log(a)|^h\left|\int_D e_{n_0}\,dx\right||\tau|\left|\int_{\Omega\setminus D} f\,\overline{u}_0(y)\int_D G_k(\eta,y)\,e_{n_0}(\eta)\,d\eta\,dy\right| \\ &\leq |\log(a)|^{h-1}\int_D\int_{\Omega\setminus D}\left|f\,\overline{u}_0(y)\,G_k(\eta,y)\right|dy\left|e_{n_0}(\eta)\right|d\eta = \mathcal{O}\big(a\,|\log(a)|^{h-1}\big). \end{aligned}$$

Finally,

$$S_4 = 2\omega^2\mu_\infty\int_D e_{n_0}\,dx\,\mathrm{Re}\left[\frac{u_0(z)\,\tau}{[1-\omega^2\mu_\infty\tau\lambda_{n_0}]}\int_{\Omega\setminus D} f\,\overline{u}_0(y)\int_D G_k(\eta,y)e_{n_0}(\eta)d\eta dy\right] + \mathcal{O}\left(|\log(a)|^{2h-2}\right).$$

Hence

$$\begin{aligned} T_4 &= p_0(t,x)+S_3+S_4 \\ &= p_0(t,x)+2\,\omega^2\mu_\infty\int_D e_{n_0}\,dx\,\mathrm{Re}\left[\frac{u_0(z)}{[1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}]}\,\tau\int_{\Omega\setminus D}\partial_t\frac{\mathrm{Im}\,(\epsilon_0)(y)\,\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_k(\eta,y)\,e_{n_0}(\eta)\,d\eta\,dy\right] \\ &+ \mathcal{O}\left(|\log(a)|^{2h-2}\right). \end{aligned}$$

The equation (3.9) takes the form

$$\begin{aligned} (p-p_0)(t,x) &= 2\omega^2\mu_\infty\int_D e_{n_0}dx\mathrm{Re}\left[\frac{u_0(z)\tau}{[1-\omega^2\mu_\infty\,\tau\lambda_{n_0}]}\int_{\Omega\setminus D}\partial_t\frac{\mathrm{Im}\,(\epsilon_0)(y)\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_k(\eta,y)e_{n_0}(\eta)d\eta dy\right] \\ &+ \frac{-t\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\frac{|\langle u_0;e_{n_0}\rangle|^2}{|1-\omega^2\mu_\infty\lambda_{n_0}\tau|^2}+\mathcal{O}\left(|\log(a)|^{2h-2}\right)+\mathcal{O}\left(\mathrm{Im}\,(\tau)a^2|\log(a)|^{3h-1}\right). \end{aligned} \tag{3.13}$$

Recall that we take,

$$\mathrm{Im}\,(\tau) = \frac{1}{a^2\,|\log(a)|^{1+h+s}}$$

with

$$0 < s < 1-h. \tag{3.14}$$

With this choice, the error part of (3.13) will be of order $\mathcal{O}\left(|\log(a)|^{2h-2}\right)$. Hence

$$\begin{aligned} (p-p_0)(t,x) &= 2\omega^2\mu_\infty\int_D e_{n_0}dx\mathrm{Re}\left[\frac{u_0(z)}{[1-\omega^2\mu_\infty\tau\lambda_{n_0}]}\tau\int_{\Omega\setminus D}\partial_t\frac{\mathrm{Im}\,(\epsilon_0)(y)\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_k(\eta,y)e_{n_0}(\eta)d\eta dy\right] \\ &+ \frac{-t\,\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\,\frac{1}{|1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\tau|^2}\,|\langle u_0,e_{n_0}\rangle|^2+\mathcal{O}\left(|\log(a)|^{2h-2}\right). \end{aligned}$$

Using again the estimate

$$|\langle u_0,e_{n_0}\rangle|^2 = |u_0(z)|^2\left(\int_D e_{n_0}(x)dx\right)^2+\mathcal{O}\left(a^3\right),$$

we get

$$\begin{aligned}\big(p-p_0\big)(t,x) &= 2\omega^2\mu_\infty \int_D e_{n_0} dx \mathrm{Re}\left[\frac{u_0(z)\tau}{[1-\omega^2\mu_\infty\tau\lambda_{n_0}]}\int_{\Omega\setminus D}\partial_t \frac{\mathrm{Im}\,(\epsilon_0)(y)\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_k(\eta,y)e_{n_0}(\eta)d\eta dy\right] \\ &+ \frac{-t\,\mathrm{Im}\,(\tau)}{(t^2-|x-z|^2)^{3/2}}\,\frac{|u_0(z)|^2}{|1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\tau|^2}\,\left(\int_D e_{n_0}(x)dx\right)^2 + \mathcal{O}\left(|\log(a)|^{2h-2}\right),\end{aligned}$$

or, with recalling that $\tau := \epsilon_p - \epsilon_0(z)$, we obtain

$$\begin{aligned}\big(p-p_0\big)(t,x) &= 2\omega^2\mu_\infty \int_D e_{n_0} dx \mathrm{Re}\left[\frac{u_0(z)\epsilon_p}{[1-\omega^2\mu_\infty\epsilon_p\lambda_{n_0}]}\int_{\Omega\setminus D}\partial_t \frac{\mathrm{Im}\,(\epsilon_0)(y)\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_k(\eta,y)e_{n_0}(\eta)d\eta dy\right] \\ &+ \frac{-t\,\mathrm{Im}\,(\epsilon_p)}{(t^2-|x-z|^2)^{3/2}}\,\frac{|u_0(z)|^2}{|1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2}\,\left(\int_D e_{n_0} dx\right)^2 + \mathcal{O}\left(|\log(a)|^{2h-2}\right).\end{aligned} \tag{3.15}$$

Now, if we take two frequencies $\omega^2_\pm$, such that[13]

$$\omega^2_\pm = \omega^2_{n_0} \pm |\log(a)|^{-h}, \tag{3.16}$$

we obtain

$$\begin{aligned}\left(p^\pm-p_0\right)(t,x) &= 2\,\omega^2_\pm\mu_\infty \int_D e_{n_0}\,dx\,\mathrm{Re}\left[\frac{u_0(z)\,\epsilon_p}{[1-\omega^2_\pm\,\mu_\infty\,\epsilon_p\,\lambda_{n_0}]}\int_{\Omega\setminus D}\partial_t\,\frac{\mathrm{Im}\,(\epsilon_0)(y)\,\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_{k_\pm}(\eta,y)\,e_{n_0}(\eta)\,d\eta\,dy\right] \\ &+ \frac{-t\,\mathrm{Im}\,(\epsilon_p)}{(t^2-|x-z|^2)^{3/2}}\,\frac{1}{|1-\omega^2_\pm\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2}\,|u_0(z)|^2\left(\int_D e_{n_0} dx\right)^2 + \mathcal{O}\left(|\log(a)|^{2h-2}\right).\end{aligned}$$

We use the expansion of $G_k(\cdot,\cdot)$, see formula (2.2), and recall that $k^2_\pm(\cdot) := \boldsymbol{n_0}^2(\cdot)\,\omega^2_\pm$ to obtain

$$G_{k_\pm}(\cdot,\cdot) = G_\kappa(\cdot,\cdot) + \mathcal{O}\left(|\log(a)|^{-h}\right), \quad \text{where} \quad \kappa(\cdot) := \boldsymbol{n_0}(\cdot)\,\omega_{n_0}. \tag{3.17}$$

Also, we use $1-\omega^2_{n_0}\,\mu_\infty\lambda_{n_0}\,\epsilon_p = 0$ to deduce that $|1-\omega^2_\pm\,\mu_\infty\lambda_{n_0}\,\epsilon_p| = \mathcal{O}(|\log(a)|^{-h})$. Then, after some simplifications we get

$$\begin{aligned}\left(p^\pm-p_0\right)(t,x) &= 2\omega^2_{n_0}\mu_\infty \int_D e_{n_0} dx \mathrm{Re}\left[\frac{u_0(z)\epsilon_p}{[1-\omega^2_\pm\,\mu_\infty\,\epsilon_p\,\lambda_{n_0}]}\int_{\Omega\setminus D}\partial_t \frac{\mathrm{Im}\,(\epsilon_0)(y)\,\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_\kappa(\eta,y)e_{n_0}(\eta)d\eta dy\right] \\ &+ \frac{-t\,\mathrm{Im}\,(\epsilon_p)}{(t^2-|x-z|^2)^{\frac{3}{2}}}\,\frac{|u_0(z)|^2}{|1-\omega^2_\pm\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2}\left(\int_D e_{n_0} dx\right)^2 + \mathcal{O}\left(|\log(a)|^{2h-2}\right).\end{aligned}$$

Next, we remark that the first term can be canceled if we sum up the two previous equations. More precisely,

$$\begin{aligned}\left(p^+ + p^- - 2p_0\right)(t,x) &= 4\omega^2_{n_0}\mu_\infty \int_D e_{n_0}\,dx\,\mathrm{Re}\left[\frac{u_0(z)\ \epsilon_p\ \left(1-\omega^2_{n_0}\,\mu_\infty\lambda_{n_0}\,\epsilon_p\right)}{(1-\omega^2_+\,\mu_\infty\lambda_{n_0}\,\epsilon_p)\,(1-\omega^2_-\,\mu_\infty\lambda_{n_0}\,\epsilon_p)}\right. \\ &\left.\int_{\Omega\setminus D}\partial_t\,\frac{\mathrm{Im}\,(\epsilon_0)(y)\,\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_D G_\kappa(\eta,y)\,e_{n_0}(\eta)\,d\eta\,dy\right] \\ &+ \frac{-t\,2\,\mathrm{Im}\,(\epsilon_p)}{(t^2-|x-z|^2)^{\frac{3}{2}}}\,\frac{|u_0(z)|^2}{|1-\omega^2_\pm\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2}\,\left(\int_D e_{n_0} dx\right)^2 + \mathcal{O}\left(|\log(a)|^{2h-2}\right),\end{aligned}$$

[13] The complex number $\omega_\pm$ is defined as follows

$$|\omega_\pm| = \left|\omega^2_{n_0} \pm |\log(a)|^{-h}\right|^{\frac{1}{2}} \text{ and } Arg\,(\omega_\pm) = \frac{1}{2}\,Arg\left(\omega^2_{n_0} \pm |\log(a)|^{-h}\right).$$

and thanks to (4.8), we know that $(1-\omega_{n_0}^2\,\mu_\infty\lambda_{n_0}\,\epsilon_p)=0$, hence[14]

$$\left(p^+ + p^- - 2p_0\right)(t,x) = \frac{-2\,t}{(t^2-|x-z|^2)^{\frac{3}{2}}}\,\frac{\mathrm{Im}\,(\epsilon_p)\;|u_0(z)|^2}{|1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2}\Big(\int_D e_{n_0}dx\Big)^2 + \mathcal{O}\left(|\log(a)|^{2h-2}\right), \tag{3.18}$$

or, with the help of (3.4),

$$\left(p^+ + p^- - 2p_0\right)(t,x) = \frac{-2\,t\;\mathrm{Im}\,(\epsilon_p)\;|\langle u_1; e_{n_0}\rangle|^2}{(t^2-|x-z|^2)^{\frac{3}{2}}} + \mathcal{O}\left(|\log(a)|^{2h-2}\right). \tag{3.19}$$

Observe that, see (3.16), the used frequencies are complex, to allows them to be real we use the following lemma.

Lemma 3.3. *Let* $\omega_1^2 := \omega_{n_0}^2 \pm |\log(a)|^{-h}$ *and* $\omega_2^2 = Re\left(\omega_1^2\right) = Re\left(\omega_{n_0}^2\right) \pm |\log(a)|^{-h}$. *Then,*

$$\left|(p-p_0)(t,x,\omega_1) - (p-p_0)(t,x,\omega_2)\right| = \mathcal{O}\left(|\log(a)|^{\max(h-1-2s;-(h+s);2h-2)}\right). \tag{3.20}$$

Proof. See Appendix (5.2). □

We schematize, in the following graph, the construction of ω_1^2 and ω_2^2.

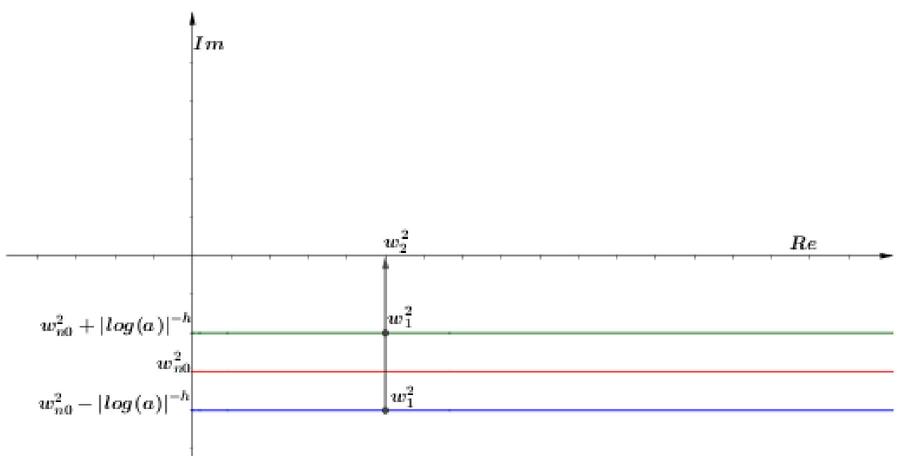


FIGURE 2. The red line contain the sequence $\left(\omega_n^2\right)_{n\in\mathbb{N}}$, we peak ω_1^2 on the band delimited by the blue line and the green one and by ω_2^2 we note its projection onto the real axis.

Combining (3.18) and Lemma 3.3, for real frequencies, we obtain

$$\left(p^+ + p^- - 2p_0\right)(t,x) = \frac{-2\,t}{(t^2-|x-z|^2)^{\frac{3}{2}}}\,\frac{\mathrm{Im}\,(\epsilon_p)\;|u_0(z)|^2}{|1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\epsilon_p|^2}\Big(\int_D e_{n_0}dx\Big)^2 + \mathcal{O}\left(|\log(a)|^{\max(-h-s,2h-2,h-1-2s)}\right).$$

This proves (1.23).

3.2. Photo-acoustic imaging using two close particles (Dimers). Proof of (1.24)

To avoid using, in the proof, more notations we keep the same ones as in the case of one particle whenever this is possible.

Lemma 3.4. *We have*

$$u_2(x) = \mathcal{O}(1) + \mathcal{O}(|\log(a)|^{h-1}\,|\log(dist(x, D_1\cup D_2))|), \qquad x\notin D_1\cup D_2. \tag{3.21}$$

Proof. We skip the proof since it is similar to that of one particle (see the proof of Lemma 3.2). □

[14]We take either $\omega^2=\omega_+^2$ or $\omega^2=\omega_-^2$ in the right hand side of the formulas (3.18) and (3.19), i.e. we do not distinguish between these frequencies.

Now, from Poisson's formula, the solution can be written as

$$\begin{aligned} p(t,x) &= \sum_{i=1}^{2} \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_p)|u_2|^2)(y)}{\sqrt{t^2-|x-y|^2}} \chi_{D_i}\, dy + \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_0)|u_2|^2)(y)}{\sqrt{t^2-|x-y|^2}} \chi_{\Omega\setminus D}\, dy \\ &= \sum_{i=1}^{2} \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_p-\epsilon_0)|u_2|^2)(y)}{\sqrt{t^2-|x-y|^2}} \chi_{D_i}\, dy + \partial_t \int_{|x-y|<t} \frac{(\mathrm{Im}\,(\epsilon_0)|u_2|^2)(y)}{\sqrt{t^2-|x-y|^2}} \chi_{\Omega}\, dy. \end{aligned}$$

For $t > diam(\Omega)$, we have

$$p(t,x) = \sum_{i=1}^{2} \partial_t \int_{D_i} \frac{(\mathrm{Im}\,(\epsilon_p-\epsilon_0)|u_2|^2)(y)}{\sqrt{t^2-|x-y|^2}}\, dy + \partial_t \int_{\Omega} \frac{(\mathrm{Im}\,(\epsilon_0)|u_2|^2)(y)}{\sqrt{t^2-|x-y|^2}}\, dy.$$

As before set

$$T_4^\star := \partial_t \int_{\Omega} \frac{(\mathrm{Im}\,(\epsilon_0)|u_2|^2)(y)}{\sqrt{t^2-|x-y|^2}}\, dy.$$

Next, we use Taylor expansion of $(\epsilon_p-\epsilon_0)(\cdot)$ and $\left(t^2-|x-\cdot|^2\right)^{-\frac{3}{2}}$ near z_1, z_2 to obtain

$$\begin{aligned} p(t,x) &= -t\,\mathrm{Im}\,(\epsilon_p) \sum_{i=1}^{2} \int_{D_i} \frac{|u_2|^2(y)}{\left(t^2-|x-y|^2\right)^{3/2}}\, dy + t \sum_{i=1}^{2} \mathrm{Im}\,(\epsilon_0(z_i)) \int_{D_i} \frac{|u_2|^2(y)}{\left(t^2-|x-y|^2\right)^{3/2}}\, dy \\ &+ T_4^\star + \mathcal{O}\left(\sum_{i=1}^{2} \int_{D_i} \frac{\int_0^1 (y-z_i)\cdot \nabla \epsilon_0(z_i + t(y-z_i))\, dt\, |u_2|^2(y)}{\left(t^2-|x-y|^2\right)^{3/2}}\, dy \right). \end{aligned}$$

The remainder term, as done in (3.5), is of order $\mathcal{O}\big(a^3\,|\log(a)|^{2h}\big)$ and using the a priori estimate, see (2.10), we have

$$t \sum_{i=1}^{2} \mathrm{Im}\,(\epsilon_0(z_i)) \int_{D_i} \frac{|u_2|^2(y)}{\left(t^2-|x-y|^2\right)^{3/2}}\, dy = \mathcal{O}\left(|\log(a)|^{2h}\ a^2 \right).$$

Then, as in the case of one particle, we have

$$\begin{aligned} p(t,x) &= \sum_{i=1}^{2} \frac{-t\,\mathrm{Im}\,(\epsilon_p)}{\left(t^2-|x-z_i|^2\right)^{3/2}} \int_{D_i} |u_2|^2 dy + T_4^\star \\ &+ \mathcal{O}\left(\sum_{i=1}^{2} \mathrm{Im}\,(\epsilon_p) \int_{D_i} \left(|y-z_i|^2 + 2\langle x-z_i; z_i-y\rangle\right) |u_2|^2(y) dy \right) + \mathcal{O}\big(a^2\,|\log(a)|^{2h}\big). \end{aligned}$$

We deduce as in (3.6) that the remainder term can be estimated as $\mathcal{O}\big(\mathrm{Im}\,(\epsilon_p)\, a^3\,|\log(a)|^{2h}\big)$. Next, we develop u_2 over the basis and we use (3.8) to estimate the remainder term to obtain

$$p(t,x) = -t\mathrm{Im}\,(\epsilon_p) \sum_{i=1}^{2} \frac{|\langle u_2; e_{n_0}^{(i)}\rangle|^2}{\left(t^2-|x-z_i|^2\right)^{3/2}} + T_4^\star + \mathcal{O}\left(\mathrm{Im}\,(\epsilon_p) \sum_{\substack{i=1 \\ n\neq n_0}}^{2} |\langle u_2; e_n^{(i)}\rangle|^2 \right) + \mathcal{O}\left(\mathrm{Im}\,(\epsilon_p)\, a^3\,|\log(a)|^{2h}\right).$$

Then we get

$$p(t,x) = -t\,\mathrm{Im}\,(\epsilon_p) \sum_{i=1}^{2} \frac{|\langle u_2; e_{n_0}^{(i)}\rangle|^2}{\left(t^2-|x-z_i|^2\right)^{3/2}} + T_4^\star + \mathcal{O}\big(\mathrm{Im}\,(\epsilon_p)\, a^2\big). \tag{3.22}$$

Set $\Omega_{1,2} := \Omega \setminus \big(D_1 \cup D_2\big)$ and write $T_4^\star$ as:

$$T_4^\star = \int_{\Omega} (\mathrm{Im}\,(\epsilon_0)|u_2|^2)(y)\ \partial_t \frac{1}{\sqrt{t^2-|x-y|^2}}\, dy = \int_{\Omega_{1,2}} |u_2|^2\ f\ dy + \int_{D_1\cup D_2} |u_2|^2\ f\ dy.$$

From the a priori estimate, see (4.19), and lemma (3.4) we deduce that the first integral dominates the second one. Now, since f is smooth, the a priori estimate allows to estimate the integral over $D_1 \cup D_2$ as follows

$$\Big| \int_{D_1 \cup D_2} |u_2|^2 \, f \, dy \Big| \lesssim \|u_2\|^2 = \mathcal{O}\big(a^2 |\log(a)|^{2h}\big).$$

Then we use L.S.E to obtain

$$\begin{aligned}
T_4^\star &= \int_\Omega f \, |u_0|^2 dy - \int_{D_1 \cup D_2} f \, |u_0|^2 dy + 2\, \omega^2 \, \mu_\infty \sum_{i=1}^{2} \mathrm{Re} \left[\int_{\Omega_{1,2}} f \, \overline{u}_0(y) \int_{D_i} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta \, dy \right] \\
&+ (\omega^2 \, \mu_\infty)^2 \sum_{i=1}^{2} \int_{\Omega_{1,2}} f \Big| \int_{D_i} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta \Big|^2 dy. \\
&+ 2 \, \mathrm{Re} \left[\int_{\Omega_{1,2}} f \int_{D_1} \overline{(\epsilon_p - \epsilon_0)}(\eta) \, \overline{G}_k(\eta, y) \, \overline{u}_2(\eta) \, d\eta \int_{D_2} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta dy \right] + \mathcal{O}\big(a^2 |\log(a)|^{2h}\big).
\end{aligned}$$

Clearly, by the smoothness of f and $|u_0|$, we have

$$\Big| \int_{D_1 \cup D_2} f \, |u_0|^2 dy \Big| = \mathcal{O}\big(a^2\big).$$

Then, we obtain[15]

$$\begin{aligned}
T_4^\star &= p_0(t,x) + 2\, \omega^2 \, \mu_\infty \sum_{i=1}^{2} \mathrm{Re} \left[\int_{\Omega_{1,2}} f \, \overline{u}_0(y) \int_{D_i} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta \, dy \right] \\
&+ 2 \, \mathrm{Re} \left[\int_{\Omega_{1,2}} f \int_{D_1} \overline{(\epsilon_p - \epsilon_0)}(\eta) \, \overline{G}_k(\eta, y) \, \overline{u}_2(\eta) \, d\eta \int_{D_2} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta dy \right] \\
&+ (\omega^2 \, \mu_\infty)^2 \sum_{i=1}^{2} \int_{\Omega_{1,2}} f \Big| \int_{D_i} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta \Big|^2 dy + \mathcal{O}\big(a^2 \, |\log(a)|^{2h}\big).
\end{aligned}$$

We remark that

$$(\omega^2 \, \mu_\infty)^2 \int_{\Omega_{1,2}} f \Big| \int_{D_i} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta \Big|^2 dy \quad \text{for } i = 1, 2$$

have the same expression as S_3 given in section (3.1) (more exactly see 3.11). Then we estimate it as $\mathcal{O}\big(|\log(a)|^{2h-2}\big)$. Similarly, regardless of whether the position of y is in D_1 or D_2, the same estimation holds for

$$2 \, \mathrm{Re} \left[\int_{\Omega_{1,2}} f \int_{D_1} \overline{(\epsilon_p - \epsilon_0)}(\eta) \, \overline{G}_k(\eta, y) \, \overline{u}_2(\eta) \, d\eta \int_{D_2} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta dy \right]$$

We synthesize the above to get

$$T_4^\star = p_0(t,x) + 2\, \omega^2 \, \mu_\infty \sum_{i=1}^{2} \mathrm{Re} \left[\int_{\Omega_{1,2}} f \, \overline{u}_0(y) \int_{D_i} (\epsilon_p - \epsilon_0)(\eta) \, G_k(\eta, y) \, u_2(\eta) \, d\eta \, dy \right] + \mathcal{O}\big(|\log(a)|^{2h-2}\big).$$

Next, we develop u_2 over the basis and use the Taylor expansion of $(\epsilon_p - \epsilon_0)(\cdot)$ to obtain

$$\begin{aligned}
T_4^\star &= p_0(t,x) + 2\, \omega^2 \, \mu_\infty \sum_{i=1}^{2} \mathrm{Re} \left[\tau_i \, \langle u_2; e_{n_0}^{(i)} \rangle \int_{\Omega_{1,2}} f \, \overline{u}_0(y) \int_{D_i} G_k(\eta, y) \, e_{n_0}^{(i)}(\eta) \, d\eta \, dy \right] \\
&- 2\, \omega^2 \, \mu_\infty \sum_{i=1}^{2} \mathrm{Re} \left[\int_{\Omega_{1,2}} f \, \overline{u}_0(y) \int_{D_i} \int_0^1 (z_i - \eta) \cdot \nabla \epsilon_0(z_i + s(\eta - z_i)) \, ds \, G_k(\eta, y) \, u_2(\eta) \, d\eta \, dy \right]
\end{aligned}$$

[15] For the definition of $p_0(t,x)$, see (3.10).

$$+ \quad 2\,\omega^2\,\mu_\infty \sum_{\substack{i=1,2\\ n\neq n_0}} \mathrm{Re}\left[\tau_i\,\langle u_2; e_n^{(i)}\rangle \int_{\Omega_{1,2}} f\,\overline{u}_0(y)\int_{D_i} G_k(\eta,y)\,e_n^{(i)}(\eta)\,d\eta\,dy\right] + \mathcal{O}\big(|\log(a)|^{2h-2}\big)$$

To precise the value of the error we need to estimate

$$\begin{aligned}
&\left|\sum_{i=1}^{2} \mathrm{Re}\left[\int_{\Omega_{1,2}} f\,\overline{u}_0(y)\int_{D_i}\int_0^1 (z_i-\eta)\cdot\nabla\epsilon_0(z_i+s(\eta-z_i))\,ds\,G_k(\eta,y)\,u_2(\eta)\,d\eta\,dy\right]\right|\\
\leq\;& a\sum_{i=1}^{2}\int_{\Omega_{1,2}}\Big|\,f\,\overline{u}_0(y)\Big|\left|\int_{D_i} G_k(\eta,y)\,u_2(\eta)\,d\eta\right|dy \lesssim a\sum_{i=1}^{2}\left(\int_{\Omega_{1,2}}\left|\int_{D_i} G_k(\eta,y)\,u_2(\eta)\,d\eta\right|^2 dy\right)^{\frac{1}{2}}\\
\lesssim\;& a\,\|u_2\|\sum_{i=1}^{2}\left(\int_{D_i}\int_{\Omega_{1,2}}|G_k|^2(\eta,y)\,dy\,d\eta\right)^{\frac{1}{2}} = \mathcal{O}\big(a^3\,|\log(a)|^h\big)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\sum_{\substack{i=1,2\\ n\neq n_0}} \mathrm{Re}\left[\tau_i\,\langle u_2; e_n^{(i)}\rangle\int_{\Omega_{1,2}} f\,\overline{u}_0(y)\int_{D_i} G_k(\eta,y)\,e_n^{(i)}(\eta)\,d\eta\,dy\right]\right|\\
\leq\;& \max(|\tau_1|,|\tau_2|)\Big(\sum_{\substack{i=1,2\\ n\neq n_0}}\big|\langle u_2; e_n^{(i)}\rangle\big|^2\Big)^{\frac{1}{2}}\left(\int_{\Omega_{1,2}}\Big|f\,\overline{u}_0(y)\Big|^2 dy\right)^{\frac{1}{2}}\left(\int_{\Omega_{1,2}}\sum_{\substack{i=1,2\\ n\neq n_0}}\Big|\int_{D_i} G_k(\eta,y)\,e_n^{(i)}(\eta)\,d\eta\Big|^2 dy\right)^{\frac{1}{2}}\\
\leq\;& \max(|\tau_1|,|\tau_2|)\,\|u_0\|\left(\int_D\int_{\Omega_{1,2}}|G_k|^2(\eta,y)dy\,d\eta\right)^{\frac{1}{2}} = \mathcal{O}\big(|\log(a)|^{-1}\big)
\end{aligned}$$

We keep the dominant term and sum the others as an error to obtain

$$T_4^\star = p_0(t,x) + 2\,\omega^2\,\mu_\infty\sum_{i=1}^{2}\mathrm{Re}\left[\tau_i\,\langle u_2; e_{n_0}^{(i)}\rangle\int_{\Omega_{1,2}} f\,\overline{u}_0(y)\int_{D_i} G_k(\eta,y)\,e_{n_0}^{(i)}(\eta)\,d\eta\,dy\right] + \mathcal{O}\left(|\log(a)|^{2h-2}\right).$$

Use (4.25) to obtain

$$\begin{aligned}
T_4^\star \;=\;& 2\,\omega^2\,\mu_\infty\sum_{i=1}^{2}\mathrm{Re}\left[\tau_i\,\frac{\langle u_0; e_{n_0}^{(i)}\rangle}{det^\star}\int_{\Omega_{1,2}} f\;\overline{u}_0(y)\int_{D_i} G_k(\eta;y)\,e_{n_0}^{(i)}(\eta)\,d\eta\,dy\right] + p_0(t,x)\\
+\;& \mathcal{O}\left(a\sum_{i=1}^{2}\tau_i\int_{\Omega_{1,2}} f\;\overline{u}_0(y)\int_{D_i} G_k(\eta,y)\,e_{n_0}^{(i)}(\eta)\,d\eta\,dy\right) + \mathcal{O}\left(|\log(a)|^{2h-2}\right),
\end{aligned} \tag{3.23}$$

where

$$det^\star := (1-\omega^2\,\mu_\infty\,\epsilon_p\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\epsilon_p\,a^2\,\Phi_0(z_1;z_2)\Big(\int_B \overline{e}_{n_0}\Big)^2,$$

and

$$a\sum_{i=1}^{2}\tau_i\int_{\Omega_{1,2}} f\overline{u}_0(y)\int_{D_i} G_k(\eta,y)e_{n_0}^{(i)}(\eta)d\eta dy = a\sum_{i=1}^{2}\tau_i\int_{D_i}\int_{\Omega_{1,2}} f\overline{u}_0(y)G_k(\eta,y)dy e_{n_0}^{(i)}(\eta)d\eta = \mathcal{O}\big(|\log(a)|^{-1}\big).$$

The last equality is justified by the fact that we integrate a smooth function over $\Omega_{1,2}$ and we know that the integral over D of an eigenfunction is of the order a.
Also we can write (3.23) as

$$T_4^\star \;=\; 2\,\omega^2\,\mu_\infty\int_D e_{n_0}dx\sum_{i=1}^{2}\mathrm{Re}\left[\tau_i\,\frac{u_0(z_i)}{det^\star}\int_{\Omega_{1,2}}\partial_t\frac{\mathrm{Im}\,(\epsilon_0)(y)\,\overline{u}_0(y)}{\sqrt{t^2-|x-y|^2}}\int_{D_i} G_k(\eta;y)\,e_{n_0}^{(i)}(\eta)\,d\eta\,dy\right] + p_0(t,x)$$

$$+ \quad \mathcal{O}\left(\sum_{i=1}^{2} \frac{\tau_i\, a^2}{det^\star} \int_{\Omega_{1,2}} f\, \overline{u}_0(y) \int_{D_i} G_k(\eta,y)\, e^{(i)}_{n_0}(\eta)\, d\eta\, dy \right) + \mathcal{O}\left(|\log(a)|^{2h-2} \right).$$

We have

$$\mathcal{O}\left(\sum_{i=1}^{2} \frac{\tau_i\, a^2}{det^\star} \int_{\Omega_{1,2}} f\, \overline{u}_0(y) \int_{D_i} G_k(\eta,y)\, e^{(i)}_{n_0}(\eta)\, d\eta\, dy \right) = \mathcal{O}(a\, |\log(a)|^{h-1})$$

since, if we compare it with the error term given in equation (3.23) we deduce that they are different by a term of order $a/det^\star$. Finally:

$$\begin{aligned} T_4^\star &= 2\,\omega^2\,\mu_\infty \int_D e_{n_0} dx \sum_{i=1}^{2} \mathrm{Re}\left[\frac{\tau_i\, u_0(z_i)}{det^\star} \int_{\Omega_{1,2}} \partial_t \frac{\mathrm{Im}\,(\epsilon_0)(y)\, \overline{u}_0(y)}{\sqrt{t^2 - |x-y|^2}} \int_{D_i} G_k(\eta,y)\, e^{(i)}_{n_0}(\eta)\, d\eta\, dy \right] + p_0(t,x) \\ &+ \mathcal{O}\left(|\log(a)|^{2h-2} \right). \end{aligned}$$

We set I_i to be

$$I_i := \int_{\Omega_{1,2}} \partial_t \frac{\mathrm{Im}\,(\epsilon_0)(y)\, \overline{u}_0(y)}{\sqrt{t^2 - |x-y|^2}} \int_{D_i} G_k(\eta,y)\, e^{(i)}_{n_0}(\eta)\, d\eta\, dy,$$

and use the estimation of $T_4^\star$ in the equation (3.22) to obtain:

$$\begin{aligned} (p - p_0)(t,x) &= 2\,\omega^2\,\mu_\infty \left(\int_D e_{n_0} dx \right) \sum_{i=1}^{2} \mathrm{Re}\left[\frac{\epsilon_p\, u_0(z_i)}{det^\star} I_i \right] - 2\,\omega^2\,\mu_\infty \left(\int_D e_{n_0} dx \right) \sum_{i=1}^{2} \mathrm{Re}\left[\frac{\epsilon_0(z_i)\, u_0(z_i)}{det^\star} I_i \right] \\ &\quad - t\, \mathrm{Im}\,(\epsilon_p) \sum_{i=1}^{2} \frac{|\langle u_2; e^{(i)}_{n_0} \rangle|^2}{(t^2 - |x - z_i|^2)^{3/2}} + \mathcal{O}\left(|\log(a)|^{2h-2} \right). \end{aligned}$$

We estimate

$$2\,\omega^2\,\mu_\infty \left(\int_D e_{n_0} dx \right) \sum_{i=1}^{2} \mathrm{Re}\left[\frac{\epsilon_0(z_i)\, u_0(z_i)}{det^\star} I_i \right] = \mathcal{O}\left(a^2\, |\log(a)|^h \right),$$

and we use the next lemma to simplify the expression of $p(t,x)$

Lemma 3.5. *We have*

$$\langle u_2; e^{(1)}_{n_0} \rangle = \langle u_2; e^{(2)}_{n_0} \rangle + \mathcal{O}\left(a \right). \tag{3.24}$$

Proof. Remember, from (2.23), that we have:

$$\int_{D_1} u_2\, dx = \int_{D_2} u_2\, dx + \mathcal{O}(d\, a^2\, |\log(a)|^h).$$

Write each integral over the basis:

$$\langle u_2, e^{(1)}_{n_0} \rangle \int_{D_1} e_{n_0}\, dx = \langle u_2, e^{(2)}_{n_0} \rangle \int_{D_1} e_{n_0}\, dx + \mathcal{O}\left(\sum_{\substack{i=1 \\ n \neq n_0}}^{2} \langle u_2, e^{(i)}_{n} \rangle \int_{D_i} e^{(i)}_{n}\, dx \right) + \mathcal{O}(d\, a^2\, |\log(a)|^h)$$

Clearly, by Holder inequality, we have

$$\left| \sum_{\substack{i=1 \\ n \neq n_0}}^{2} \langle u_2, e^{(i)}_{n} \rangle \int_{D_i} e^{(i)}_{n}\, dx \right| \lesssim \|u_0\|\, \|1\| = \mathcal{O}\left(a^2 \right)$$

and it follows that

$$\langle u_2; e^{(1)}_{n_0} \rangle = \langle u_2; e^{(2)}_{n_0} \rangle + \mathcal{O}\left(a \right).$$

From (3.24) we deduce:

$$|\langle u_2; e^{(1)}_{n_0} \rangle|^2 = |\langle u_2; e^{(2)}_{n_0} \rangle|^2 + \mathcal{O}\left(a^2\, |\log(a)|^h \right).$$

□

By lemma 3.5, we have

$$\begin{aligned}(p-p_0)(t,x) &= 2\,\omega^2\,\mu_\infty\left(\int_{D_1} e_{n_0}\right)\sum_{i=1}^{2}\text{Re}\left[\frac{\epsilon_p\,u_0(z_i)}{det^\star}I_i\right] - t\,\text{Im}\,(\epsilon_p)\,|\langle u_2; e^{(2)}_{n_0}\rangle|^2\sum_{i=1}^{2}\frac{1}{\left(t^2-|x-z_i|^2\right)^{3/2}}\\ &+ \mathcal{O}\left(|\log(a)|^{2h-2}\right).\end{aligned}$$

We have also:

$$\frac{1}{\left(t^2-|x-z_1|^2\right)^{3/2}} = \frac{1}{\left(t^2-|x-z_2|^2\right)^{3/2}}\Big(1+\mathcal{O}(d)\Big).$$

Then

$$(p-p_0)(t,x) = 2\,\omega^2\,\mu_\infty\left(\int_{D_1} e_{n_0}\right)\sum_{i=1}^{2}\text{Re}\left[\frac{\epsilon_p\,u_0(z_i)}{det^\star}I_i\right] - 2t\,\text{Im}\,(\epsilon_p)\,\frac{|\langle u_2; e^{(2)}_{n_0}\rangle|^2}{\left(t^2-|x-z_2|^2\right)^{3/2}} + \mathcal{O}\left(|\log(a)|^{2h-2}\right).$$

Next, we use the same technique as before by taking two frequencies $\omega^2_\pm = \omega^2_{n_0} \pm |\log(a)|^{-h}$, we get

$$\begin{aligned}(p^\pm - p_0)(t,x) &= 2\,\omega^2_{n_0}\,\mu_\infty\int_{D_1} e_{n_0}(x)dx\sum_{i=1}^{2}\text{Re}\left[\frac{\epsilon_p\,u_0^\pm(z_i)\,I_i^\pm}{det^\star\left(\omega^2_\pm\right)}\right] - 2t\,\text{Im}\,(\epsilon_p)\,\frac{|\langle u_2^\pm; e^{(2)}_{n_0}\rangle|^2}{\left(t^2-|x-z_2|^2\right)^{3/2}}\\ &+ \mathcal{O}\left(|\log(a)|^{-h}\int_{D_1} e_{n_0}(x)dx\sum_{i=1}^{2}\text{Re}\left[\frac{\epsilon_p\,u_0^\pm(z_i)\,I_i^\pm}{det^\star\left(\omega^2_\pm\right)}\right]\right) + \mathcal{O}\left(|\log(a)|^{2h-2}\right).\end{aligned} \tag{3.25}$$

We estimate the error part as

$$|\log(a)|^{-h}\int_{D_1} e_{n_0}(x)dx\sum_{i=1}^{2}\text{Re}\left[\frac{\epsilon_p\,u_0^\pm(z_i)\;\,I_i^\pm}{det^\star\left(\omega^2_\pm\right)}\right] \sim \mathcal{O}(|\log(a)|^{-1}),$$

and we use (3.17) to write $I_i^\pm = I_i + \mathcal{O}\left(a\,|\log(a)|^{-h}\right)$. Also, from the smoothness of u_0, we deduce that

$$u_0^\pm(z_i) := u_0\left(\omega^2_\pm, z_i\right) = u_0\left(\omega^2_{n_0}, z_i\right) \pm \left|\omega^2_{n_0} - \omega^2_\pm\right| = u_0\left(z_i\right) + \mathcal{O}\left(|\log(a)|^{-h}\right).$$

Now, (3.25) takes the form

$$\left(p^\pm - p_0\right)(t,x) = 2\omega^2_{n_0}\mu_\infty\int_{D_1} e_{n_0}dx\sum_{i=1}^{2}\text{Re}\left[\frac{\epsilon_p\,u_0(z_i)\,I_i}{det^\star\left(\omega^2_\pm\right)}\right] - 2t\,\text{Im}\,(\epsilon_p)\frac{|\langle u_2^\pm; e^{(2)}_{n_0}\rangle|^2}{(t^2-|x-z_2|^2)^{\frac{3}{2}}} + \mathcal{O}\left(|\log(a)|^{2h-2}\right).$$

Define $\tilde{p}(t,x)$ as

$$\tilde{p}(t,x) := \left(p^+ - p_0\right)(t,x) + \Theta^*\,\left(p^- - p_0\right)(t,x),$$

where

$$\Theta^* := \frac{\left|\omega^2_\pm - \omega^2_{n_0}\right|\,\left[\lambda_{n_0} + a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\right] - a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\,Re\left(\omega^2_{n_0}\right)}{\left|\omega^2_\pm - \omega^2_{n_0}\right|\,\left[\lambda_{n_0} + a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\right] + a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\,Re\left(\omega^2_{n_0}\right)} \simeq \mathcal{O}\left(1\right),$$

hence

$$\begin{aligned}\tilde{p}(t,x) &= 2\,\omega^2_{n_0}\,\mu_\infty\int_{D_1} e_{n_0}(x)\,dx\sum_{i=1}^{2}\text{Re}\left[\epsilon_p\,u_0(z_i)\,I_i\left(\frac{1}{det^\star\left(\omega^2_+\right)} + \Theta^*\,\frac{1}{det^\star\left(\omega^2_-\right)}\right)\right]\\ &- \frac{2t\,\text{Im}\,(\epsilon_p)}{\left(t^2-|x-z_2|^2\right)^{3/2}}\left(|\langle u_2^+; e^{(2)}_{n_0}\rangle|^2 + \Theta^*\,|\langle u_2^-; e^{(2)}_{n_0}\rangle|^2\right) + \mathcal{O}\left(|\log(a)|^{2h-2}\right).\end{aligned} \tag{3.26}$$

We have

$$\frac{1}{det^\star\left(\omega^2_+\right)} + \Theta^*\,\frac{1}{det^\star\left(\omega^2_-\right)} =$$

$$\frac{-2\,\mu_{\infty}\,\epsilon_p\left[\lambda_{n_0}+a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\right]\left|\omega_{\pm}^2-\omega_{n_0}^2\right|\,a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\,i\,Im\left(\omega_{n_0}^2\right)}{\left(\left|\omega_{\pm}^2-\omega_{n_0}^2\right|\,\left[\lambda_{n_0}+a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\right]+a^2\,\Phi_0\left(z_1,z_2\right)\,\left(\int_B \overline{e}_{n_0}\right)^2\,Re\left(\omega_{n_0}^2\right)\right)\,det^{\star}\left(\omega_{+}^2\right)\,det^{\star}\left(\omega_{-}^2\right)}\simeq\left|\log(a)\right|^{-s},$$

then, we estimate the first term of (3.26) as

$$2\,\omega_{n_0}^2\,\mu_{\infty}\int_{D_1} e_{n_0}(x)\,dx\sum_{i=1}^{2}\mathrm{Re}\left[\epsilon_p\,u_0(z_i)\,I_i\left(\frac{1}{det^{\star}\left(\omega_{+}^2\right)}+\Theta^{*}\,\frac{1}{det^{\star}\left(\omega_{-}^2\right)}\right)\right]=\mathcal{O}\left(\left|\log(a)\right|^{-(1+s)}\right).$$

Hence

$$\tilde{p}(t,x)=-\frac{2t\,\mathrm{Im}\left(\epsilon_p\right)}{\left(t^2-|x-z_2|^2\right)^{3/2}}\left(|\langle u_2^{+};e_{n_0}^{(2)}\rangle|^2+\Theta^{*}\,|\langle u_2^{-};e_{n_0}^{(2)}\rangle|^2\right)+\mathcal{O}\left(|\log(a)|^{2h-2}\right).$$

We use the same technique as in the case of one particle by writing u_2 with respect to u_0 and using the smoothness of this last one to obtain

$$\tilde{p}(t,x)=-\frac{2t\,\mathrm{Im}\left(\epsilon_p\right)}{\left(t^2-|x-z_2|^2\right)^{3/2}}\left|\langle u_2;e_{n_0}^{(2)}\rangle\right|^2\,(1+\Theta^{*})+\mathcal{O}\left(|\log(a)|^{2h-2}\right). \tag{3.27}$$

We can write $(1+\Theta^{*})$ as $(1+\Theta^{*})=\Theta+\mathcal{O}\left(\left|\log(a)\right|^{-h}\right)$, where

$$\Theta:=\frac{2\,\left|\omega_{\pm}^2-\omega_{n_0}^2\right|\,\frac{1}{2\pi}\,\left|\log(a)\right|}{\Phi_0\left(z_1,z_2\right)\,\mathrm{Re}\,\left(\omega_{n_0}^2\right)+\left|\omega_{\pm}^2-\omega_{n_0}^2\right|\,\frac{1}{2\pi}\,\left|\log(a)\right|}=\frac{2\,\left|\log(a)\right|^{1-h}}{\Phi_0\left(z_1,z_2\right)\,\mathrm{Re}\,\left(\omega_{n_0}^2\right)+\left|\log(a)\right|^{1-h}}.$$

As in the case of one particle, when we use real frequencies instead of complex one's, and referring to the remark 5.1 and lemma 3.3, the formula (3.27) takes the form

$$\tilde{p}(t,x)=-\frac{2\,\Theta\,t\,\mathrm{Im}\left(\epsilon_p\right)}{\left(t^2-|x-z_2|^2\right)^{3/2}}\left|\langle u_2;e_{n_0}^{(2)}\rangle\right|^2+\mathcal{O}\left(|\log(a)|^{\max(h-1-2s,-h-s,2h-2)}\right). \tag{3.28}$$

This proves (1.24). The formula (1.25) is deduced from (1.24) as if $d=\mathcal{O}\left(a^{\left|\log(a)\right|^{-(\ell+h)}}\right)$, we have $\Theta=2+\mathcal{O}\left(\left|\log(a)\right|^{-\ell}\right)$ and (3.28) takes the following form

$$\tilde{p}(t,x)=-\frac{4\,t\,\mathrm{Im}\left(\epsilon_p\right)}{\left(t^2-|x-z_2|^2\right)^{3/2}}\left|\langle u_2;e_{n_0}^{(2)}\rangle\right|^2+\mathcal{O}\left(|\log(a)|^{\max(h-\ell-1-s,h-1-2s,-h-s,2h-2)}\right). \tag{3.29}$$

This proves (1.25).

4. A priori estimates

4.1. **A priori estimates on the electric field.**

Proof. of **Proposition 2.3**
In order to prove the a priori estimation (2.10), we proceed in two steps. First we do it for one single particle and then for multiple particles.

Step 1/ Case of one particle:
Remember that the eigenvalues and eigenfunctions of the logarithmic operator satisfy

$$\int_D \Phi_0(x,y)\,e_n(y)\,dy\;=\;\lambda_n\,e_n(x)\qquad in\quad D,$$

and after scaling we get, with $\tilde{e}_n(\cdot):=e_n\left(\frac{\cdot-z}{a}\right)$,

$$a^2\left[\int_B \Phi_0(\eta,\xi)\,\tilde{e}_n(\xi)\,d\xi-\frac{1}{2\pi}\,\log(a)\int_B \tilde{e}_n(\xi)d\xi\right]\;=\;\lambda_n\,\tilde{e}_n(\eta)\qquad in\quad B. \tag{4.1}$$

Integrating the equation (4.1) over B we obtain

$$\int_B \int_B \Phi_0(\eta,\xi)\,\tilde{e}_n(\xi)\,d\eta d\xi = \left[\frac{1}{2\pi}\log(a)|B| + \frac{\lambda_n}{a^2}\right]\int_B \tilde{e}_n d\eta.$$

Multiplying (4.1) by $\tilde{e}_m$ and integrating over B we get:

$$\int_B \int_B \Phi_0(\eta,\xi)\,\tilde{e}_n(\xi)\,\tilde{e}_m(\eta)\,d\eta d\xi - \frac{1}{2\pi}\,\log(a)\int_B \tilde{e}_n(\xi) d\xi \int_B \tilde{e}_m(\eta) d\eta = \frac{\lambda_n}{a^2}\int_B \tilde{e}_n\,\tilde{e}_m d\eta.$$

Remark that when $m \neq n$, thanks to the fact that $\left\{\tilde{e}_n\right\}_{n\in\mathbb{N}}$ forms an orthogonal basis in $\mathbb{L}^2(B)$, we obtain

$$\int_B \int_B \Phi_0(\eta,\xi)\,\tilde{e}_n(\xi)\,\tilde{e}_m(\eta)\,d\eta d\xi = \frac{1}{2\pi}\,\log(a)\int_B \tilde{e}_n d\xi \int_B \tilde{e}_m d\xi \tag{4.2}$$

and when $m = n$, we get

$$\int_B \int_B \Phi_0(\eta,\xi)\,\tilde{e}_n(\xi)\,\tilde{e}_n(\eta)\,d\eta d\xi - \frac{1}{2\pi}\,\log(a)\left(\int_B \tilde{e}_n d\xi\right)^2 = \frac{\lambda_n}{a^2}\,\|\tilde{e}_n\|^2. \tag{4.3}$$

After normalisation

$$\int_B \int_B \Phi_0(\eta,\xi)\,\frac{\tilde{e}_n(\xi)}{\|\tilde{e}_n\|}\,\frac{\tilde{e}_n(\eta)}{\|\tilde{e}_n\|}\,d\eta d\xi = \frac{1}{\|\tilde{e}_n\|^2}\left[\frac{1}{2\pi}\,\log(a)\left(\int_B \tilde{e}_n d\xi\right)^2 + \frac{\lambda_n}{a^2}\,\|\tilde{e}_n\|^2\right]. \tag{4.4}$$

We denote $\overline{e}_n := \tilde{e}_n/\|\tilde{e}_n\|$ the orthonormalized basis in $\mathbb{L}^2(B)$, and we set

$$\tilde{\lambda}_n := \int_B \int_B \Phi_0(\eta,\xi)\,\overline{e}_n(\eta)\,\overline{e}_n(\xi)\;d\eta\,d\xi, \tag{4.5}$$

from (4.4) and (4.5) we deduce that

$$\tilde{\lambda}_n = \frac{\lambda_n}{a^2} + \frac{1}{2\pi}\log(a)\left(\int_B \overline{e}_n d\xi\right)^2. \tag{4.6}$$

Thanks to L.S.E and Green kernel expansion (2.2), we have

$$\begin{aligned} u_1(x) \;&-\; \omega^2\mu_\infty\tau\int_D \left(\Phi_0(x,y)+\varrho(z)\right)\,u_1(y)\,dy = u_0(x) + \tau\,\mathcal{O}\left(\int_D |x-y|\,u_1(y)\,dy\right) \\ &+\; \omega^2\,\mu_\infty\,\tau\int_D\int_0^1 \nabla\varrho(z+t(y-z))\cdot(y-z)\,dt\,u_1(y)\,dy \quad in \quad D. \end{aligned}$$

Now scaling, we have

$$\begin{aligned} \tilde{u}_1(\eta) \;&-\; \omega^2\mu_\infty\tau\,a^2\int_B \Phi_0(\eta,\xi)\,\tilde{u}_1(\xi)\,d\xi + \omega^2\mu_\infty\tau\,a^2\left(\frac{1}{2\pi}\,\log(a)-\varrho(z)\right)\int_B \tilde{u}_1\,d\xi \\ &=\; \tilde{u}_0(\eta) + \mathcal{O}\left(a^3\,\tau\int_B |\eta-\xi|\,\tilde{u}_1(\xi)\,d\xi\right) \\ &+\; \omega^2\mu_\infty\tau a^3\int_B\int_0^1 \nabla\varrho(z+ta\xi)\cdot\xi\,dt\,\tilde{u}_1(\xi)\,d\xi. \end{aligned}$$

Using the basis, we obtain

$$\begin{aligned} \langle\tilde{u}_1;\overline{e}_{n_0}\rangle & \left[1-\omega^2\mu_\infty\tau\,a^2\int_B\int_B \Phi_0(\eta,\xi)\,\overline{e}_{n_0}(\xi)\,d\xi\overline{e}_{n_0}(\eta)\,d\eta + \omega^2\mu_\infty\tau\,a^2\left(\frac{\log(a)}{2\pi}-\varrho(z)\right)\left[\int_B \overline{e}_{n_0}d\xi\right]^2\right] \\ =\;& \langle\tilde{u_0};\overline{e}_{n_0}\rangle + \mathcal{O}\left(\tau\,a^3\int_B \overline{e}_{n_0}(\eta)\int_B |\eta-\xi|\,\tilde{u}_1(\xi)\,d\xi\,d\eta\right) \\ +\;& \omega^2\mu_\infty\tau a^3\int_B \overline{e}_{n_0}(\eta)d\eta\int_B\int_0^1 \nabla\varrho(z+ta\xi)\cdot\xi\,dt\,\tilde{u}_1(\xi)\,d\xi \end{aligned}$$

$$- \quad \omega^2 \mu_\infty \tau \, a^2 \left(\frac{1}{2\pi} \, \log(a) - \varrho(z) \right) \langle 1; \overline{e}_{n_0} \rangle \sum_{n \neq n_0} \langle \tilde{u}_1; \overline{e}_n \rangle \int_B \overline{e}_n d\xi$$

$$+ \quad \omega^2 \mu_\infty \tau \, a^2 \sum_{n \neq n_0} \langle \tilde{u}_1; \overline{e}_n \rangle \int_B \int_B \Phi_0(\eta, \xi) \, \overline{e}_n(\xi) \, d\xi \overline{e}_{n_0}(\eta) \, d\eta.$$

After simplifications and using (4.2) and (4.4) we get

$$\begin{aligned} \langle \tilde{u}_1; \overline{e}_{n_0} \rangle &= \frac{1}{\left[1 - \omega^2 \mu_\infty \tau \, \lambda_{n_0} - \omega^2 \mu_\infty \tau \, a^2 \, \varrho(z) \, \left(\int_B \overline{e}_{n_0}(\eta) d\eta \right)^2 \right]} \Bigg[\langle \tilde{u_0}; \overline{e}_{n_0} \rangle \\ &+ \quad \omega^2 \mu_\infty \tau \, a^2 \, \varrho(z) \int_B \overline{e}_{n_0} d\eta \sum_{n \neq n_0} \langle \tilde{u}_1; \overline{e}_n \rangle \langle 1; \overline{e}_n \rangle \\ &+ \quad \omega^2 \mu_\infty \tau a^3 \int_B \overline{e}_{n_0}(\eta) d\eta \int_B \int_0^1 \nabla \varrho(z + ta\xi) \cdot \xi \, dt \, \tilde{u}_1(\xi) \, d\xi \\ &+ \quad \mathcal{O}\left(\tau \, a^3 \int_B \overline{e}_{n_0}(\eta) \int_B |\eta - \xi| \, \tilde{u}_1(\xi) \, d\xi \, d\eta \right) \Bigg]. \end{aligned} \tag{4.7}$$

We take[16] τ and ω so that

$$\tau \simeq \frac{1}{a^2 \, |\log(a)|} \quad \text{and} \quad \omega^2 = \frac{\left(1 \pm |\log(a)|^{-h} \right)}{\mu_\infty \, \lambda_{n_0} \, a^{-2} \, |\log(a)|^{-1}}. \tag{4.9}$$

With this choice we have the estimation

$$\frac{1}{\left| 1 - \omega^2 \mu_\infty \tau \, \lambda_{n_0} - \omega^2 \mu_\infty \tau \, a^2 \, \varrho(z) \, \left(\int_B \overline{e}_{n_0}(\eta) d\eta \right)^2 \right|} = \mathcal{O}(|\log(a)|^h).$$

Then

$$\begin{aligned} |\langle \tilde{u}_1; \overline{e}_{n_0} \rangle| \quad &\lesssim \quad |\log(a)|^h \left[|\langle \tilde{u_0}; \overline{e}_{n_0} \rangle| + |\log(a)|^{-1} \left(\sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2 \right)^{\frac{1}{2}} \right. \\ &+ \quad a \, |\log(a)|^{-1} \left. \left| \int_B \overline{e}_{n_0}(\eta) \int_B |\eta - \xi| \, \tilde{u}_1(\xi) \, d\xi \, d\eta \right| \right]. \end{aligned}$$

We have:

$$\begin{aligned} \left| \int_B \overline{e}_{n_0}(\eta) \int_B |\eta - \xi| \, \tilde{u}_1(\xi) \, d\xi \, d\eta \right| \quad &\leq \quad |\langle \tilde{u}_1; \overline{e}_{n_0} \rangle| \left| \int_B \overline{e}_{n_0}(\eta) \int_B |\eta - \xi| \, \overline{e}_{n_0}(\xi) \, d\xi \, d\eta \right| \\ &+ \quad \sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle| \left| \int_B \overline{e}_{n_0}(\eta) \int_B |\eta - \xi| \, \overline{e}_n(\xi) \, d\xi \, d\eta \right| \\ &\lesssim \quad |\langle \tilde{u}_1; \overline{e}_{n_0} \rangle| + \left(\sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

[16]The dielectric-resonance that we want to excite is ω_{n_0} given by

$$\omega_{n_0}^2 = \frac{1}{\mu_\infty \, \lambda_{n_0} \, \epsilon_p} \simeq \frac{1}{\mu_\infty \, \lambda_{n_0} \, a^{-2} \, |\log(a)|^{-1}}. \tag{4.8}$$

Then:

$$|\langle \tilde{u}_1; \overline{e}_{n_0} \rangle|^2 \lesssim |\log(a)|^{2h} \left[|\langle \tilde{u_0}; \overline{e}_{n_0} \rangle|^2 + |\log(a)|^{-2} \sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2 \right]. \tag{4.10}$$

In what follows, we calculate an estimation of $\sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2$. We star with equation (4.7), since the other steps are the same, to obtain:

$$\begin{aligned} \langle \tilde{u}_1; \overline{e}_n \rangle &= \frac{1}{\left[1 - \omega^2 \mu_\infty \tau \, \lambda_n - \omega^2 \mu_\infty \tau \, a^2 \, \varrho(z) \, \left(\int_B \overline{e}_n(\eta) d\eta \right)^2 \right]} \Bigg[\langle \tilde{u_0}; \overline{e}_n \rangle \\ &+ \omega^2 \mu_\infty \tau \, a^2 \, \varrho(z) \int_B \overline{e}_n d\eta \sum_{j \neq n} \langle \tilde{u}_1; \overline{e}_j \rangle \, \langle 1; \overline{e}_j \rangle \\ &+ \omega^2 \, \mu_\infty \, \tau \, a^3 \int_B \overline{e}_n(\eta) \, d\eta \int_B \int_0^1 \nabla \varrho(z + ta\xi) \cdot \xi \, dt \tilde{u}_1(\xi) \, d\xi \\ &+ \mathcal{O}\left(\tau \, a^3 \int_B \overline{e}_n(\eta) \int_B |\eta - \xi| \, \tilde{u}_1(\xi) \, d\xi \, d\eta \right) \Bigg]. \end{aligned}$$

Then

$$\begin{aligned} \sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2 &\leq C^{te} \Bigg[\sum_{n \neq n_0} |\langle \tilde{u_0}; \overline{e}_n \rangle|^2 + |\log(a)|^{-2} \, \|\tilde{u}_1\|^2 \sum_{n \neq n_0} |\langle 1; \overline{e}_n \rangle|^2 \\ &+ a^2 \, |\log(a)|^{-2} \sum_{n \neq n_0} \left| \int_B \overline{e}_n(\eta) \int_B |\eta - \xi| \, \tilde{u}_1(\xi) \, d\xi \, d\eta \right|^2 \Bigg]. \end{aligned}$$

On the right side, except for the first term, we need to estimate the terms containing series. For this, we have

$$\sum_{n \neq n_0} \left| \int_B \overline{e}_n(\eta) \int_B |\eta - \xi| \, \tilde{u}_1(\xi) \, d\xi \, d\eta \right|^2 \leq \mathcal{O}(\|\tilde{u}_1\|^2).$$

Obviously we have also

$$\sum_{n \neq n_0} |\langle 1; \overline{e}_n \rangle|^2 \leq \|1\|^2.$$

Hence

$$\sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2 \leq C^{te} \left[\sum_{n \neq n_0} |\langle \tilde{u_0}; \overline{e}_n \rangle|^2 + |\log(a)|^{-2} \|\tilde{u}_1\|^2 \right],$$

and

$$\sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2 \leq C^{te} \left[\sum_{n \neq n_0} |\langle \tilde{u_0}; \overline{e}_n \rangle|^2 + |\log(a)|^{-2} |\langle \tilde{u_1}; \overline{e}_{n_0} \rangle|^2 \right]. \tag{4.11}$$

Combining (4.10) and (4.11), we get

$$\begin{aligned} \|\tilde{u}_1\|^2 = |\langle \tilde{u}_1; \overline{e}_{n_0} \rangle|^2 + \sum_{n \neq n_0} |\langle \tilde{u}_1; \overline{e}_n \rangle|^2 &\leq |\log(a)|^{2h} \left[|\langle \tilde{u_0}; \overline{e}_{n_0} \rangle|^2 + |\log(a)|^{-2} \sum_{n \neq n_0} |\langle \tilde{u_1}; \overline{e}_n \rangle|^2 \right] \\ &+ C^{te} \left[\sum_{n \neq n_0} |\langle \tilde{u_0}; \overline{e}_n \rangle|^2 + |\log(a)|^{-2} |\langle \tilde{u_1}; \overline{e}_{n_0} \rangle|^2 \right] \end{aligned}$$

hence

$$\|\tilde{u}_1\|^2 \leq |\log(a)|^{2h} \|\tilde{u_0}\|^2 + |\log(a)|^{2h-2} \|\tilde{u}_1\|^2,$$
$$\|\tilde{u}_1\|^2(1 - |\log(a)|^{2h-2}) \leq |\log(a)|^{2h} \|\tilde{u_0}\|^2$$

and, as $h < 1$,

$$\|\tilde{u}_1\|^2 \leq (1 - |\log(a)|^{2h-2})^{-1} |\log(a)|^{2h} \|\tilde{u_0}\|^2 \leq |\log(a)|^{2h} \|\tilde{u_0}\|^2,$$

or

$$\|u_1\|_{\mathbb{L}^2(D)} \leq |\log(a)|^h \|u_0\|_{\mathbb{L}^2(D)}. \tag{4.12}$$

The following proposition makes a link between the Fourier coefficient of the generated total field and that of the source field.

Proposition 4.1. *We have*

$$\langle u_1; e_{n_0}\rangle = \frac{\langle u_0; e_{n_0}\rangle}{(1 - \omega^2 \mu_\infty \epsilon_p \lambda_{n_0})} + \mathcal{O}\left(a\,|\log(a)|^{2h-1}\right).$$

Proof. From (4.7) we have:

$$\begin{aligned}
\langle \tilde{u}_1; \overline{e}_{n_0}\rangle &= \frac{1}{\left[1 - \omega^2 \mu_\infty \tau\, \lambda_{n_0} - \omega^2 \mu_\infty \tau\, a^2\, \varrho(z)\, \left(\int_B \overline{e}_{n_0} d\eta\right)^2\right]} \Bigg[\langle \tilde{u_0}; \overline{e}_{n_0}\rangle \\
&+ \omega^2 \mu_\infty \tau\, a^2\, \varrho\, \langle 1; \overline{e}_{n_0}\rangle \sum_{n \neq n_0} \langle \tilde{u}_1; \overline{e}_n\rangle \langle 1; \overline{e}_n\rangle \\
&+ \omega^2\, \mu_\infty\, \tau\, a^3\, \langle 1; \overline{e}_{n_0}\rangle \int_B \int_0^1 \nabla \varrho(z + ta\xi) \cdot \xi\, dt\, \tilde{u}_1(\xi)\, d\xi \\
&+ \mathcal{O}\left(\tau\, a^3 \int_B \overline{e}_{n_0}(\eta) \int_B |\eta - \xi|\, \tilde{u}_1(\xi)\, d\xi\, d\eta\right)\Bigg].
\end{aligned}$$

We need to estimate the two last terms between brackets. We have

$$\left|\omega^2 \mu_\infty \tau\, a^2\, \varrho\, \langle 1; \overline{e}_{n_0}\rangle \sum_{n \neq n_0} \langle \tilde{u}_1; \overline{e}_n\rangle \langle 1; \overline{e}_n\rangle\right| \lesssim \tau\, a^2\, \|\tilde{u_0}\|\, \|1\| = \mathcal{O}\left(|\log(a)|^{-1}\right),$$

and

$$\left|\omega^2\, \mu_\infty\, \tau\, a^3\, \langle 1; \overline{e}_{n_0}\rangle \int_B \int_0^1 \nabla \varrho(z + ta\xi) \cdot \xi\, dt\, \tilde{u}_1(\xi)\, d\xi\right| \lesssim a\, |\log(a)|^{-1}\, \|\tilde{u}_1\| = \mathcal{O}\left(a\, |\log(a)|^{h-1}\right).$$

Next, use Holder inequality and the a priori estimate to obtain

$$\left|\tau\, a^3 \int_B \overline{e}_{n_0}(\eta) \int_B |\eta - \xi|\, \tilde{u}_1(\xi)\, d\xi d\eta\right| \lesssim \mathcal{O}(a|\log(a)|^{h-1}).$$

Finally, we obtain

$$\langle u_1; e_{n_0}\rangle = \frac{\langle u_0; e_{n_0}\rangle}{\left[1 - \omega^2 \mu_\infty \tau\, \lambda_{n_0} - \omega^2 \mu_\infty \tau\, \varrho(z)\, \left(\int_D e_{n_0} d\eta\right)^2\right]} + \mathcal{O}(a\,|\log(a)|^{h-1}), \tag{4.13}$$

or in the following form

$$\begin{aligned}
\langle u_1; e_{n_0}\rangle &= \frac{\langle u_0; e_{n_0}\rangle}{(1 - \omega^2 \mu_\infty \tau\, \lambda_{n_0})\, \left[1 - \dfrac{\omega^2 \mu_\infty \tau\, \varrho(z)\, \left(\int_D e_{n_0} d\eta\right)^2}{(1 - \omega^2 \mu_\infty \tau\, \lambda_{n_0})}\right]} + \mathcal{O}\left(a\,|\log(a)|^{h-1}\right) \\
&= \frac{\langle u_0; e_{n_0}\rangle}{(1 - \omega^2 \mu_\infty \tau\, \lambda_{n_0})\, [1 + \mathcal{O}(|\log(a)|^{h-1})]} + \mathcal{O}(a\,|\log(a)|^{h-1})
\end{aligned}$$

$$= \frac{\langle u_0; e_{n_0}\rangle}{(1-\omega^2\mu_\infty\tau\,\lambda_{n_0})} + \mathcal{O}(a\,|\log(a)|^{2h-1})$$
$$= \frac{\langle u_0; e_{n_0}\rangle}{(1-\omega^2\mu_\infty\epsilon_p\,\lambda_{n_0})}\,\frac{1}{1+\frac{\omega^2\mu_\infty\epsilon_0\,\lambda_{n_0}}{(1-\omega^2\mu_\infty\epsilon_p\,\lambda_{n_0})}} + \mathcal{O}(a\,|\log(a)|^{2h-1})$$
$$= \frac{\langle u_0; e_{n_0}\rangle}{(1-\omega^2\mu_\infty\epsilon_p\,\lambda_{n_0})} + \mathcal{O}(a\,|\log(a)|^{2h-1})$$

which ends the proof. □

Step 2/ Case of multiples particles: Consider the L.S.E for multiple particles

$$v_i(x) - \omega^2\,\mu_\infty\,\tau_i\int_{D_i} G_k(x;y)v_i(y)\,dy - \omega^2\,\mu_\infty\sum_{m\neq i}^{M}\tau_m\int_{D_m} G_k(x;y)v_m(y)\,dy = u_0(x), \quad x\in D_i. \tag{4.14}$$

We use the expansion formula (2.2) of $G_k(x;y)$ to write

$$\begin{aligned} v_i(x) \;&-\; \omega^2\,\mu_\infty\,\tau_i\int_{D_i}[\Phi_0(x;y)+\varrho(z_i)]\,v_i(y)\,dy - \omega^2\,\mu_\infty\sum_{m\neq i}^{M}\tau_m\int_{D_m} G_k(x;y)v_m(y)\,dy \\ &= u_0(x) + \omega^2\mu_\infty\tau_i\int_{D_i}\int_0^1\nabla\varrho(z_i+t(y-z_i))\cdot(y-z_i)dtv_i(y)dy + \mathcal{O}\left(\tau_i\int_{D_i}|x-y|\,v_i(y)dy\right). \end{aligned}$$

Now expanding $G_k(\cdot,\cdot)$ near the center, we obtain:

$$\begin{aligned} v_i(x) \;&-\; \omega^2\,\mu_\infty\,\tau_i\int_{D_i}[\Phi_0(x;y)+\varrho(z_i)]\,v_i(y)\,dy \\ &= u_0(x) + \omega^2\,\mu_\infty\sum_{m\neq i}^{M}\tau_m\,G_k(z_i;z_m)\int_{D_m} v_m(y)\,dy + \mathcal{O}\left(\tau_i\int_{D_i}|x-y|\;v_i(y)dy\right) \\ &+ \omega^2\mu_\infty\tau_i\int_{D_i}\int_0^1\nabla\varrho(z_i+t(y-z_i))\cdot(y-z_i)dtv_i(y)dy \\ &+ \omega^2\,\mu_\infty\sum_{m\neq i}^{M}\tau_m\int_0^1\underset{x}{\nabla}G_k(z_i+t(x-z_i);z_m)\cdot(x-z_i)dt\int_{D_m} v_m(y)\,dy \\ &+ \omega^2\,\mu_\infty\sum_{m\neq i}^{M}\tau_m\int_{D_m}\int_0^1\underset{y}{\nabla}G_k(x;z_m+t(y-z_m))\cdot(y-z_m)\,dtv_m(y)\,dy. \end{aligned}$$

We recall that

$$A_0\,v(x) = \int_B \Phi_0(x,y)\;v(y)\,dy,$$

and denote

$$T\,v(x) := \int_B v(y)\,dy, \quad x\in B.$$

Scaling, the equations system, we obtain

$$\begin{aligned} &\left[I \;-\; \omega^2\,\mu_\infty\,\tau_i\,a^2\,A_0 + \omega^2\,\mu_\infty\,\tau_i\,a^2\left(\frac{\log(a)}{2\pi}-\varrho(z_i)\right)T\right](\tilde{v}_i)\,(\eta) = \tilde{u}_0(\eta) \\ &\quad+\; \mathcal{O}\left(\tau_i\,a^3\int_B|\eta-\xi|\,\tilde{v}_i(\xi)\,d\xi\right) + \omega^2\,\mu_\infty\sum_{m\neq i}^{M}\tau_m\,G_k(z_i;z_m)\int_{D_m} v_m(y)\,dy \\ &\quad+\; \omega^2\,\mu_\infty\,\tau_i\,a^3\int_B\int_0^1\nabla\varrho(z_i+t\,a\,\xi)\cdot\xi dt\tilde{v}_i(\xi)d\xi \end{aligned}$$

$$+ \quad \omega^2 \mu_\infty a^3 \sum_{m \neq i}^{M} \tau_m \int_0^1 \underset{x}{\nabla} G_k(z_i + t\, a\, \eta; z_m) \cdot \eta \, dt \int_B \tilde{v}_m(\xi)\, d\xi$$

$$+ \quad \omega^2 \mu_\infty a^3 \sum_{m \neq i}^{M} \tau_m \int_B \int_0^1 \underset{y}{\nabla} G_k(z_i + t\, a\, \eta; z_m + ta\xi) \cdot \xi \, dt \, \tilde{v}_m(\xi)\, d\xi. \tag{4.15}$$

Also, we note by

$$\mathfrak{Res}(A_0; T) := \left[I - \omega^2 \mu_\infty \tau_i a^2 A_0 + \omega^2 \mu_\infty \tau_i a^2 \left(\frac{\log(a)}{2\pi} - \varrho(z_i) \right) T \right]^{-1}.$$

Remark 4.2. In the definition of the operator $\mathfrak{Res}(A_0; T)$ we cannot neglect the operator T since it scales with the same order as A_0.

Then

$$\begin{aligned}
\tilde{v}_i \quad &= \quad \mathfrak{Res}(A_0; T)\,(\tilde{u}_0) + \mathcal{O}\left(\tau_i a^3 \, \mathfrak{Res}(A_0; T) \left(\int_B |\cdot - \xi| \, \tilde{v}_i(\xi) d\xi \right) \right) \\
&+ \quad \omega^2 \mu_\infty \tau_i a^3 \int_B \int_0^1 \nabla \varrho(z_i + ta\xi) \cdot \xi dt \, \tilde{v}_i(\xi)\, d\xi \, \mathfrak{Res}(A_0; T)(1) \\
&+ \quad \omega^2 \mu_\infty \sum_{m \neq i}^{M} \tau_m G_k(z_i; z_m) \int_{D_m} v_m(y) dy \, \mathfrak{Res}(A_0; T)(1) \\
&+ \quad \omega^2 \mu_\infty a^3 \sum_{m \neq i}^{M} \tau_m \int_B \tilde{v}_m(\xi)\, d\xi \;\; \mathfrak{Res}(A_0; T) \left(\int_0^1 \underset{x}{\nabla} G_k(z_i + t\, a \cdot; z_m) \cdot (\cdot) dt \right) \\
&+ \quad \omega^2 \mu_\infty a^3 \sum_{m \neq i}^{M} \tau_m \, \mathfrak{Res}(A_0; T) \left(\int_B \int_0^1 \underset{y}{\nabla} G_k(z_i + ta\cdot; z_m + at\xi) \cdot \xi \, dt \tilde{v}_m(\xi)\, d\xi \right).
\end{aligned}$$

Using the a priori estimate (4.12), we obtain

$$\begin{aligned}
\|\tilde{v}_i\| \quad &\lesssim \quad |\log(a)|^h \|\tilde{u}_0\| + a\, |\log(a)|^{h-1} \|\tilde{v}_i\| + a^{-2} \, |\log(a)|^{h-1} \sum_{m \neq i}^{M} |G_k(z_i; z_m)| \, \left| \int_{D_m} v_m(y) dy \right| \\
&+ \quad a\, |\log(a)|^{h-1} \sum_{m \neq i}^{M} \|\tilde{v}_m\| \, \left\| \int_0^1 \underset{x}{\nabla} G_k(z_i + ta\cdot; z_m) \cdot (\cdot)\, dt \right\| \\
&+ \quad a\, |\log(a)|^{h-1} \sum_{m \neq i}^{M} \left\| \int_B \int_0^1 \underset{y}{\nabla} G_k(z_i + ta\cdot; z_m + ta\xi) \cdot \xi dt \, \tilde{v}_m(\xi)\, d\xi \right\|.
\end{aligned}$$

We have

$$\left\| \int_0^1 \underset{x}{\nabla} G_k(z_i + ta\cdot; z_m) \cdot (\cdot)\, dt \right\| := \left[\int_B \left| \int_0^1 \underset{x}{\nabla} G_k(z_i + ta\eta; z_m) \cdot \eta \, dt \right|^2 d\eta \right]^{\frac{1}{2}} = \mathcal{O}\left(\frac{1}{|z_i - z_m|} \right),$$

and

$$\begin{aligned}
Term \quad &:= \quad \left\| \int_B \int_0^1 \underset{y}{\nabla} G_k(z_i + ta\cdot\,; z_m + ta\xi) \cdot \xi dt \, \tilde{v}_m(\xi)\, d\xi \right\| \\
&\leq \quad \left[\int_B \int_B \left| \int_0^1 \underset{y}{\nabla} G_k(z_i + ta\eta\,; z_m + ta\xi) \cdot \xi dt \right|^2 d\xi \, d\eta \right]^{\frac{1}{2}} \|\tilde{v}_m\| \\
&= \quad \mathcal{O}\left(\frac{\|\tilde{v}_m\|}{|z_i - z_m|} \right).
\end{aligned}$$

Gathering these estimates, we have

$$\begin{aligned} \|\tilde{v}_i\| &\lesssim |\log(a)|^h \|u_0(z_i + a\,\cdot)\| + a^{-2}\, |\log(a)|^{h-1} \sum_{m\neq i}^{M} |G_k(z_i; z_m)| \left| \int_{D_m} v_m(y)dy \right| \\ &+ a\,|\log(a)|^{h-1} \sum_{m\neq i}^{M} \frac{1}{|z_i - z_m|} \|\tilde{v}_m\|. \end{aligned} \tag{4.16}$$

We have

$$\sum_{m\neq i}^{M} \frac{1}{|z_i - z_m|} \|\tilde{v}_m\| \leq \left[\sum_{m\neq i}^{M} \frac{1}{|z_i - z_m|^2} \right]^{\frac{1}{2}} \left[\sum_{m\neq i}^{M} \|\tilde{v}_m\|^2 \right]^{\frac{1}{2}} = \mathcal{O}\left(d^{-1}\, \|\tilde{u}\|_{(\Pi\, \mathbb{L}^2(B))} \right), \tag{4.17}$$

and

$$\begin{aligned} \sum_{m\neq i}^{M} |G_k(z_i; z_m)| \left| \int_{D_m} v_m(y)dy \right| &\simeq \sum_{m\neq i}^{M} |\log|z_i - z_m|| \left| \int_{D_m} v_m(y)dy \right| \\ &\leq |\log(d)| \left[\sum_{m\neq i}^{M} \left| \int_{D_m} v_m(y)dy \right|^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Now, from the resolution of the system (2.12) we deduce that

$$\left[\sum_{m\neq i}^{M} \left| \int_{D_m} v_m(y)dy \right|^2 \right]^{\frac{1}{2}} \leq a^2\, |\log(a)|^{h-1}\, d^{-1}\, \|u\|_{(\Pi\, \mathbb{L}^2(D))} = a^3\, |\log(a)|^{h-1}\, d^{-1}\, \|\tilde{u}\|_{(\Pi\, \mathbb{L}^2(B))}.$$

Then, recalling that $d \sim a^{|\log(a)|^{-h}}$, we deduce

$$\sum_{m\neq i}^{M} |G_k(z_i; z_m)| \left| \int_{D_m} v_m(y)dy \right| \lesssim a^3\, d^{-1}\, \|\tilde{u}\|_{(\Pi\, \mathbb{L}^2(B))}. \tag{4.18}$$

Now, combining (4.17), (4.18) and (4.16) we get

$$\|\tilde{v}_i\| \lesssim |\log(a)|^h \|u_0(z_i + a\,\cdot)\| + a\, |\log(a)|^{h-1}\, d^{-1}\, \|\tilde{u}\|_{(\Pi\, \mathbb{L}^2(B))}$$

$$\|\tilde{v}_i\|^2_{\mathbb{L}^2(B)} \leq |\log(a)|^{2h} \|u_0(z_i + a\,\cdot)\|^2_{\mathbb{L}^2(B)} + a^2\, |\log(a)|^{2h-2}\, d^{-2}\, \|\tilde{u}\|^2_{(\Pi\, \mathbb{L}^2(B))},$$

we sum up to M, to obtain

$$\begin{aligned} \|\tilde{u}\|^2_{(\Pi\, \mathbb{L}^2(B))} &\leq |\log(a)|^{2h} \|\tilde{u}_0\|^2_{(\Pi\, \mathbb{L}^2(B))} + M\, a^2\, |\log(a)|^{2h-2}\, d^{-2}\, \|\tilde{u}\|^2_{(\Pi\, \mathbb{L}^2(B))} \\ (1 - M\, a^2\, |\log(a)|^{2h-2} d^{-2})\, \|\tilde{u}\|^2_{(\Pi\, \mathbb{L}^2(B))} &\leq |\log(a)|^{2h} \|\tilde{u}_0\|^2_{(\Pi\, \mathbb{L}^2(B))} \\ \|\tilde{u}\|^2_{(\Pi\, \mathbb{L}^2(B))} &\leq |\log(a)|^{2h} \|\tilde{u}_0\|^2_{(\Pi\, \mathbb{L}^2(B))}. \end{aligned}$$

We obtain after scaling back

$$\|u\|_{\mathbb{L}^2(D)} \leq |\log(a)|^h \|u_0\|_{\mathbb{L}^2(D)}. \tag{4.19}$$

□

In the next proposition, which is analogous to proposition (4.1), we estimate the Fourier coefficient of the total field for dimer particles when $n \neq n_0$.

Proposition 4.3. *For $n \neq n_0$, we have*

$$\langle u_2; e_n^{(i)} \rangle = \frac{1}{(1 - \omega^2\, \mu_\infty\, \tau_1\, \lambda_n)} \left[\langle u_0, e_n^{(i)} \rangle + \mathcal{O}(|\log(a)|^{-h})\, |\langle 1, e_n^{(i)} \rangle| \right], \; i = 1, 2. \tag{4.20}$$

Proof. First of all, recall that $v_m = u_{|_{D_m}}, m = 1,2$ and let $n \neq n_0$. Take the scalar product of (4.15) with respect to $\overline{e}_n^{(i)}$, $i = 1,2$, to obtain

$$\begin{aligned}
\langle \tilde{v}_1; \overline{e}_n \rangle &- \omega^2 \mu_\infty \tau_1 a^2 \int_B \overline{e}_n(\eta) \int_B \Phi_0(\eta,\xi) \tilde{v}_1(\xi)\, d\xi\, d\eta = \langle \tilde{u}_0, \overline{e}_n \rangle \\
&- \omega^2 \mu_\infty \tau_1 a^2 \frac{1}{2\pi} \log(a) \int_B \tilde{v}_1\, d\xi \int_B \overline{e}_n d\eta + \omega^2 \mu_\infty a^2 \Bigg[\tau_1\, \varrho(z_1) \int_B \tilde{v}_1\, d\xi \int_B \overline{e}_n d\eta \\
&+ \tau_1\, a \int_B \overline{e}_n(\eta) \int_B |\eta - \xi|\, \tilde{v}_1(\xi) d\xi\, d\eta + \tau_1\, a \int_B \overline{e}_n(\eta) d\eta \int_B \int_0^1 \nabla \varrho(z_1 + ta\xi)\cdot \xi dt \tilde{v}_1(\xi) d\xi \\
&+ \tau_2\, G_k(z_1; z_2) \int_B \tilde{v}_2\, d\xi \int_B \overline{e}_n d\eta + a\, \tau_2 \int_B \overline{e}_n(\eta) d\eta \int_0^1 \nabla G_k(z_1 + ta\eta; z_2)\cdot \eta dt \int_B \tilde{v}_2(\xi)\, d\xi \\
&+ a\, \tau_2 \int_B \overline{e}_n(\eta) \int_B \int_0^1 \nabla G_k(z_1 + at\eta; z_2 + at\xi)\cdot \xi\, dt\, \tilde{v}_2(\xi)\, d\xi\, d\eta \Bigg]_{:=error}.
\end{aligned}$$

The *error* part, with the help of Taylor's formula, behaves as $\mathcal{O}\left(|\log(a)|^{-h} |\langle 1, \overline{e}_n\rangle|\right)$ and we can write

$$\begin{aligned}
\int_B \overline{e}_n \int_B \Phi_0 \tilde{v}_1\, d\xi\, d\eta &= \langle \tilde{v}_1; \overline{e}_n \rangle \int_B \overline{e}_n \int_B \Phi_0 \overline{e}_n\, d\xi\, d\eta + \sum_{j\neq n} \langle \tilde{v}_1; \overline{e}_j \rangle \int_B \overline{e}_n \int_B \Phi_0 \overline{e}_j d\xi\, d\eta \\
&\overset{(4.4)}{=} \langle \tilde{v}_1; \overline{e}_n \rangle \left[\frac{\lambda_n}{a^2} + \boldsymbol{\frac{1}{2\pi} \log(a) \Big(\int_B \overline{e}_n d\eta \Big)^2} \right] \overset{(4.2)}{+} \frac{1}{2\pi} \log(a) \int_B \overline{e}_n d\eta \sum_{j\neq n} \langle \tilde{v}_1; \overline{e}_j \rangle \int_B \overline{e}_j d\eta.
\end{aligned}$$

we plug all this in the previous equation to obtain

$$\begin{aligned}
\langle \tilde{v}_1; \overline{e}_n \rangle &- \boldsymbol{\omega^2 \mu_\infty \tau_1 a^2} \Bigg[\boldsymbol{\langle \tilde{v}_1; \overline{e}_n \rangle} \int_B \overline{e}_n(\eta) \int_B \Phi_0(\eta,\xi) \overline{e}_n(\xi)\, d\xi\, d\eta \\
&\qquad + \sum_{j\neq n} \langle \tilde{v}_1; \overline{e}_j \rangle \int_B \overline{e}_n(\eta) \int_B \Phi_0(\eta,\xi) \overline{e}_j(\xi)\, d\xi\, d\eta \Bigg] = \langle \tilde{u}_0, \overline{e}_n \rangle \\
&- \boldsymbol{\omega^2 \mu_\infty \tau_1 a^2 \frac{1}{2\pi} \log(a)} \Bigg[\boldsymbol{\langle \tilde{v}_1; \overline{e}_n \rangle \int_B \overline{e}_n d\eta} + \sum_{j\neq n} \langle \tilde{v}_1; \overline{e}_j \rangle \int_B \overline{e}_j\, d\xi \Bigg] \boldsymbol{\int_B \overline{e}_n d\eta} \\
&+ \mathcal{O}\left(|\log(a)|^{-h}\right) |\langle 1, \overline{e}_n \rangle|.
\end{aligned}$$

Next, we cancel the two terms given by series and those written with **bold symbol** and scale back the obtained formula to get (4.20). ☐

The result in (4.20) also applies to the case $n = n_0$ with an error term of order $\mathcal{O}\big(|\log(a)|^{-h}\big)$. The next proposition improves the error term by improving the denominator term.

Proposition 4.4. *We have*

$$\langle u_2; e_{n_0}^{(i)} \rangle = \frac{\langle u_0; e_{n_0}^{(i)} \rangle}{(1 - \omega^2 \mu_\infty \tau \lambda_{n_0}) - \omega^2 \mu_\infty \tau a^2 \Phi_0(z_1; z_2) \Big(\int_B \overline{e}_{n_0} \Big)^2} + \mathcal{O}(a), \qquad i = 1,2. \tag{4.21}$$

Proof. In order to prove equality (4.21) we take a scalar product with respect to $\overline{e}_{n_0}$ at the equation (4.15), and after simplifications, we get[17]

(4.22)
$$\begin{bmatrix} (1-\omega^2\mu_\infty\lambda_{n_0}\tau) & -\omega^2\mu_\infty\tau a^2\Phi_0\Big(\int_B \overline{e}_{n_0}\Big)^2 \\ -\omega^2\mu_\infty\tau a^2\Phi_0\Big(\int_B \overline{e}_{n_0}\Big)^2 & (1-\omega^2\mu_\infty\lambda_{n_0}\tau) \end{bmatrix} \begin{bmatrix} \langle \tilde{u}_2;\overline{e}^{(1)}_{n_0}\rangle \\ \langle \tilde{u}_2;\overline{e}^{(2)}_{n_0}\rangle \end{bmatrix} = \begin{bmatrix} \langle \tilde{u}_0;\overline{e}^{(1)}_{n_0}\rangle + \mathcal{O}(|\log(a)|^{-h}) \\ \langle \tilde{u}_0;\overline{e}^{(2)}_{n_0}\rangle + \mathcal{O}(|\log(a)|^{-h}) \end{bmatrix}$$

We denote by *det* the determinant of the last matrix, i.e

$$(4.23)\qquad det = \big(1-\omega^2\,\mu_\infty\,\lambda_{n_0}\,\tau\big)^2 - \left(\omega^2\,\mu_\infty\,a^2\,\tau\,\Phi_0\,\left(\int_B \overline{e}_{n_0}\right)^2\right)^2, \text{ where } \Phi_0 = \Phi_0(z_1;z_2).$$

Next, we check that when we are close to the resonance the determinant $det \neq 0$. For this, and by construction of ω^2, we have

$$1-\omega^2\mu_\infty\tau\lambda_{n_0} = \mp|\log(a)|^{-h},$$

and the fact that $d \sim a^{|\log(a)|^{-h}}$ implies that $\tau a^2\Phi_0(z_1,z_2) \sim \frac{1}{2\pi}\,|\log(a)|^{-h}$. Plug this in (4.23) to obtain

$$det = |\log(a)|^{-2h}\left[1-\left(\omega^2\mu_\infty\,\frac{1}{2\pi}\,\left(\int_B \overline{e}_{n_0}\right)^2\right)^2\right] \overset{(4.9)}{=} |\log(a)|^{-2h}\left[1-\frac{\big(1\pm|\log(a)|^{-h}\big)}{\left(1+\frac{\tilde{\lambda}_{n_0}|\log(a)|^{-1}}{\frac{1}{2\pi}\left(\int_B \overline{e}_{n_0}\right)^2}\right)^2}\right]$$

from **Hypotheses1**, we deduce that

$$\left(\frac{\tilde{\lambda}_{n_0}|\log(a)|^{-1}}{\frac{1}{2\pi}\left(\int_B \overline{e}_{n_0}dx\right)^2}\right) \sim |\log(a)|^{-1},$$

then

$$det = |\log(a)|^{-2h}\left[1-\big(1\pm|\log(a)|^{-h}\big)\big(1+|\log(a)|^{-1}\big)\right] \sim |\log(a)|^{-3h}.$$

Since $det \neq 0$, the algebraic system (4.22) is invertible. We invert it and use the fact that

$$\langle \tilde{u}_0;\overline{e}^{(2)}_{n_0}\rangle = \langle \tilde{u}_0;\overline{e}^{(1)}_{n_0}\rangle + \mathcal{O}(d),$$

to obtain

$$(4.24)\qquad \langle \tilde{u}_2;\overline{e}^{(1)}_{n_0}\rangle = \frac{\langle \tilde{u}_0;\overline{e}^{(1)}_{n_0}\rangle}{(1-\omega^2\,\mu_\infty\,\tau\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\tau\,a^2\,\Phi_0(z_1;z_2)\left(\int_B \overline{e}_{n_0}\right)^2} + \mathcal{O}\big(1\big),$$

and, after scaling, we get (4.21). □

Remark 4.5. We use the definition of τ and the fact that $\epsilon_p \gg \epsilon_0$ to rewrite (4.21) as

$$(4.25)\qquad \langle u_2;e^{(i)}_{n_0}\rangle = \frac{\langle u_0;e^{(i)}_{n_0}\rangle}{(1-\omega^2\,\mu_\infty\,\epsilon_p\,\lambda_{n_0}) - \omega^2\,\mu_\infty\,\epsilon_p\,a^2\,\Phi_0(z_1;z_2)\Big(\int_B \overline{e}_{n_0}\Big)^2} + \mathcal{O}(a), \qquad i=1,2.$$

[17]To simplify the calculus we write the prove with τ, but of course we can do it with τ_1 and τ_2.

4.2. **Estimation of the scattering coefficient C.** From (2.8) we have:

$$w = \omega^2 \, \mu_\infty \, \epsilon_p \Big[I - \omega^2 \, \mu_\infty \, \epsilon_p \, A_0 \Big]^{-1} (1) \quad \text{or} \quad \frac{1}{\omega^2 \, \mu_\infty \, \epsilon_p} \, \Big[I - \omega^2 \, \mu_\infty \, \epsilon_p \, A_0 \Big](w) = 1.$$

Hence

$$\langle 1, e_n \rangle = \frac{1}{\omega^2 \, \mu_\infty \, \epsilon_p} \langle e_n; \left[I - \omega^2 \, \mu_\infty \, \epsilon_p \, A_0 \right] (w) \rangle = \frac{1}{\omega^2 \, \mu_\infty \, \epsilon_p} \, \left[\langle e_n, w \rangle - \omega^2 \, \mu_\infty \, \epsilon_p \, \lambda_n \langle e_n, w \rangle \right]$$

and then

$$\langle w, e_n \rangle = \frac{\omega^2 \, \mu_\infty \, \epsilon_p}{1 - \omega^2 \, \mu_\infty \, \epsilon_p \, \lambda_n} \langle 1, e_n \rangle. \tag{4.26}$$

The next lemma uses (4.26) to gives a precision about the value of **C**.

Lemma 4.6. *The coefficient* $\boldsymbol{C}$ *can be approximated as*

$$\boldsymbol{C} = \frac{\omega^2 \, \mu_\infty \, \epsilon_p}{(1 - \omega^2 \, \mu_\infty \, \epsilon_p \, \lambda_{n_0})} \, \left(\int_D e_{n_0}(x) \, dx \right)^2 + \mathcal{O} \left(|\log(a)|^{-1} \right). \tag{4.27}$$

Proof. We use the definition of **C**, given by (2.9), to write

$$\mathbf{C} := \int_D w \, dx = \sum_n \langle w, e_n \rangle \, \langle 1, e_n \rangle,$$

apply (4.26) to obtain

$$\mathbf{C} = \omega^2 \, \mu_\infty \, \epsilon_p \, \left[\frac{1}{(1 - \omega^2 \, \mu_\infty \, \epsilon_p \, \lambda_{n_0})} \, \left(\int_D e_{n_0} \right)^2 + \sum_{n \neq n_0} \frac{1}{(1 - \omega^2 \, \mu_\infty \, \epsilon_p \, \lambda_n)} \, \left(\int_D e_n \right)^2 \right],$$

and, since the frequency ω is near ω_{n_0}, and hence away from the other resonances we have

$$\left| \sum_{n \neq n_0} \frac{1}{(1 - \omega^2 \, \mu_\infty \, \epsilon_p \, \lambda_n)} \, \left(\int_D e_n \right)^2 \right| \leq \sum_n |\langle 1, e_n \rangle|^2 = \|1\|_{\mathbb{L}^2(D)} = \mathcal{O}\big(a^2\big).$$

□

From (4.27), we see that

$$\mathbf{C} \sim |\log(a)|^{h-1}.$$

We deduce also the following formula:

$$(1 - \omega^2 \, \mu_\infty \, \epsilon_p \, \lambda_{n_0}) = \mathbf{C}^{-1} \, \omega^2 \, \mu_\infty \, \epsilon_p \, \left(\int_D e_{n_0} \right)^2 + \mathcal{O}(|\log(a)|^{-2h}). \tag{4.28}$$

Proof. of **Lemma** 2.7
We start by recalling the definition of w_j and $\overline{w}$,

$$\begin{cases} w_j - \omega^2 \, \mu_\infty \, \tau_j \, A_0(w_j) = \omega^2 \, \mu_\infty \, \tau_j \\ \overline{w} \; - \omega^2 \, \mu_\infty \, \epsilon_p \, A_0(\overline{w}) \; = \omega^2 \, \mu_\infty \, \epsilon_p, \end{cases}$$

and, by subtracting the two equations we obtain

$$(w_j - \overline{w}) - \omega^2 \, \mu_\infty \, \left[\tau_j \, A_0(w_j) - \epsilon_p \, A_0(\overline{w}) \right] = \omega^2 \, \mu_\infty \, (\tau_j - \epsilon_p) = -\omega^2 \, \mu_\infty \, \epsilon_0(z_j),$$

then, using the fact that $A_0(w_j) = -1 + \frac{1}{\omega^2 \, \mu_\infty \, \tau_j} \, w_j$, we get

$$\left[I - \omega^2 \, \mu_\infty \, \epsilon_p \, A_0 \right] (w_j - \overline{w}) = \frac{-\epsilon_0(z_j)}{\tau_j} \, w_j.$$

Taking the $\mathbb{L}^2(D_j)$-norm, we obtain

$$\begin{aligned}\|w_j - \overline{w}\|_{\mathbb{L}^2(D_j)} &= \frac{|\epsilon_0(z_j)|}{|\tau_j|} \left\| \left[I - \omega^2 \mu_\infty \epsilon_p A_0\right]^{-1} (w_j) \right\|_{\mathbb{L}^2(D_j)} \\ &\simeq a^2 \,|\log(a)|\,|\log(a)|^h \,\|w_j\|_{\mathbb{L}^2(D_j)} \overset{(2.11)}{=} \mathcal{O}\left(a\,|\log(a)|^{2h}\right),\end{aligned}$$

then

$$\int_{D_j} (w_j - \overline{w})(x)dx = \mathcal{O}\left(a^2\,|\log(a)|^{2h}\right), \tag{4.29}$$

and, for two particles $j = 1, 2$, we have

$$\int_{D_1} \overline{w}(x)dx = \int_{D_1} \overline{w}(x)dx + \mathcal{O}\left(a^2\, d\right). \tag{4.30}$$

Combining (4.29) and (4.30) we obtain

$$\int_{D_j} w_j(x)\,dx = \int_{D} \overline{w}(x)\,dx + \mathcal{O}\left(a^2\,|\log(a)|^{2h}\right), \tag{4.31}$$

where $j = 1, 2$ and D is any one of the two nano-particles D_1 or D_2. This ends the proof of the lemma. □

5. Appendices

5.1. **Appendix A.** To motivate the natural character of the hypotheses stated in **Hypotheses** 1, let us make the following observations:

a) We prove that the upper bound of λ_n is of order $a^2\,|\log(a)|$. For this, recalling and rescaling (1.15) we obtain, see section 4, in particular (4.6), for $a \ll 1$,

$$\lambda_n = a^2 \left(\tilde{\lambda}_n + \frac{1}{2}|\log(a)|(\int_B \overline{e}_n(\xi)d\xi)^2\right), \tag{5.1}$$

where

$$\tilde{\lambda}_n := \frac{1}{\|\tilde{e}_n\|^2_{\mathbb{L}^2(B)}} \int_B LP(\tilde{e}_n)(\eta)\; \tilde{e}_n(\eta)d\eta$$

and $\tilde{e}_n$ is the scaled of any eigenfunction e_n corresponding to λ_n. Take the absolute value in (5.1) to obtain

$$|\lambda_n| \leq a^2 \left(|\tilde{\lambda}_n| + \frac{1}{2}|\log(a)|\,| < 1; \bar{e}_n > |^2\right).$$

From the definition of $\tilde{\lambda}_n$, see (4.5), we have $|\tilde{\lambda}_n| \leq \|\Phi_0\|_{\mathbb{L}^2(B\times B)} < \infty$ and we use the Cauchy-Schwartz inequality to obtain

$$|\lambda_n| \leq a^2 \left(\|\Phi_0\|_{\mathbb{L}^2(B\times B)} + \frac{1}{2}|\log(a)|\,|B|^2\right) \lesssim a^2\,|\log(a)|.$$

b) For the lower bound, the situation is less clear. Nevertheless, we have the following results:

b.1) When the shape is a disc of radius a, we refer to (Theorem 4.1, [14]) for the existence of a sequence of eigenvalues given by

$$\lambda_{k,j} = a^2 \left[\mu_j^{(k)}\right]^{-2}, \quad k = 0, 1, 2, \cdots \; j = 1, 2, \cdots$$

and the corresponding eigenfunctions given by

$$u_{k,j}(r, \varphi) = \mathrm{J}_k\left(\mu_j^{(k)}\; r\; a^{-1}\right) e^{i\,k\,\varphi},$$

where J_k is the Bessel function of the first kind of order k and $\mu_j^{(k)}$ are the roots of the following transcendental equation

$$
(5.2) \qquad \begin{aligned}
k\mathrm{J}_k\left(\mu_j^{(k)}\right) + \frac{\mu_j^{(k)}}{2}\left(\mathrm{J}_{k-1}\left(\mu_j^{(k)}\right) - \mathrm{J}_{k+1}\left(\mu_j^{(k)}\right)\right) &= 0, \qquad k = 1, 2, \cdots \\
\mathrm{J}_0\left(\mu_j^{(0)}\right) - \mu_j^{(0)}\log(a)\left(\mathrm{J}_{-1}\left(\mu_j^{(0)}\right) - \mathrm{J}_1\left(\mu_j^{(0)}\right)\right) &= 0.
\end{aligned}
$$

We remark that (only) for $k = 0$, the associated eigenfunctions have a non zero average[18]. Next, in order to obtain a precision about the behaviour of $\{\lambda_{0,j}\}_{j\geq 1}$ with respect to a, we need to investigate the behaviour of $\mu_j^{(0)}$ solutions of (5.2). For this, we use the following properties of Bessel functions

$$
\mathrm{J}_{-1}(x) - \mathrm{J}_1(x) = 2\mathrm{J}_0^{'}(x) = -2\mathrm{J}_1(x),
$$

to write (5.2) as

$$
\mathrm{J}_0\Big(\mu_j^{(0)}\Big) + 2\,\log(a)\,\mu_j^{(0)}\mathrm{J}_1\Big(\mu_j^{(0)}\Big) = 0.
$$

Set $\Psi(x) := \mathrm{J}_0(x) + 2\,\log(a)\,x\,\mathrm{J}_1(x)$ and use *Dixon's* theorem, see [29] page 480, to deduce that the roots of Ψ are interlaced with those of J_0, noted by $\{x_{0,j}\}_{j\geq 1}$, and those of J_1, noted by $\{x_{1,j}\}_{j\geq 1}$. At this stage, we distinguish two cases

$\star$ The roots of Ψ exceeding $x_{0,1}$:

For this case, a direct application of *Dixon's* theorem, allows to deduce that

$$
\forall j \geq 2, \; x_{k,j-1} < \mu_j^{(0)} < x_{k,j}, \quad k = 0, 1
$$

and

$$
\forall j \geq 2, \; a^2\, x_{k,j}^{-2} < \lambda_{0,j} < a^2\, x_{k,j-1}^{-2}, \quad k = 0, 1,
$$

since $\left\{x_{k,j}\right\}_{\substack{j\geq 1\\ k=0,1}}$ are independent of a we deduce that $\lambda_{0,j}$ behaves as a^2.

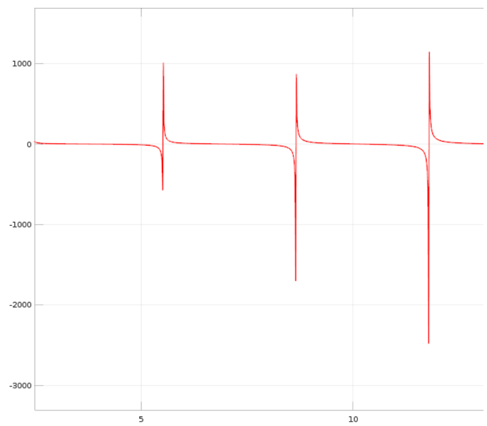
$\star$ The root of Ψ less than $x_{0,1}$:

The analysis of this case is more delicate. First, we observe that if, for a certain x, $\Psi(x) = 0$, then $\mathrm{J}_0(x) \neq 0$. Otherwise, we would have also $\mathrm{J}_1(x) = 0$ which is impossible as the zeros of J_0 and J_1 are disjoint, see *Bourget's Hypothesis*, page 484, section 15.28 in [29]. Hence the equation $\Psi(x) = 0$ can be rewritten as

$$
(5.3) \qquad \frac{1}{2\,\log(a)} = \frac{-x\,\mathrm{J}_1(x)}{\mathrm{J}_0(x)} := \mathbf{F}_0(x).
$$

Clearly, $\mathbf{F}_0$ is a smooth function on each interval not containing a zero of J_0 and from [17], see equation 27, we deduce that it is also a decreasing function, (see figure 3, for a schematic picture).

[18]We can compute $\int_D u_{0,j} = \int_0^{2\pi}\int_0^a u_{0,j}(r,\varphi)\, r\, dr\, d\varphi = 2\pi\, a^2\, \mathrm{J}_1\left(\mu_j^{(0)}\right)/\mu_j^{(0)}$.



.png

FIGURE 3. Graphic of $\mathbf{F}_0$.

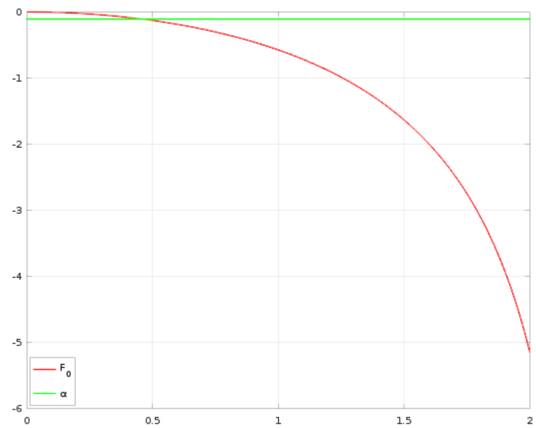


FIGURE 4. Solving, for $x \in (0, \nu)$, $\mathbf{F}_0(x) = 1/(-4\,\log(10))$.

So, if we restrict our study to $(0, \nu)$ with $\nu < x_{0,1}$ we deduce that $\mathbf{F}_0^{-1}$ exists and is continuous, then the equation (5.3) is solvable and the solution that we obtain is also small, (see figure (4), for numerical demonstration).
Now, since x is small we use the asymptotic behaviour of $\mathbf{F}_0$, see for instance (equation 25 in [17]), $\mathbf{F}_0(x) \sim -x^2/2$ to write (5.3) as

$$\frac{1}{2\,\log(a)} \sim \frac{-x^2}{2},$$

and this implies that $x \sim \Big(\log(1/a)\Big)^{\frac{-1}{2}}$. Finally

$$\lambda_{0,1} \sim a^2\,|\log(a)| \tag{5.4}$$

b.2) For the case of an arbitrary shape D, with $|D| = |B_a|$ where B_a is the disc of radius a, and referring to (Theorem 2.5, [24]) we have $\|LP_D\| \leq \|LP_{B_a}\|$. From the definition of $\|LP_D\|$, we write this inequality as a Faber-Krahn type inequality

$$\frac{1}{\lambda_{0,1}^2(D)} = \|LP_\Omega\| \leq \|LP_D\| = \frac{1}{\lambda_1^2(B_a)} \quad \text{or equivalently} \quad \lambda_1(B_a) \leq \lambda_{0,1}(D).$$

We deduce the lower bound, and hence the behavior, of the first eigenvalue

$$\lambda_{0,1}(D) \;\sim a^2|\log(a)|, \quad \forall\, a \ll 1. \tag{5.5}$$

In addition from (5.1), we see that

$$\left(\int_D e_1\right)^2 = \frac{\lambda_{0,1}}{a^2\,|\log(a)|} + \mathcal{O}\big(|\log(a)|^{-1}\big)$$

and hence

$$\left(\int_D e_1(x)\,dx\right)^2 \sim 1 \;\; for \;\; a \ll 1. \tag{5.6}$$

5.2. **Appendix B.** We prove in this appendix Lemma 3.3.
From the continuity of $u_0(\cdot,\omega)$ with respect to the frequency variable[19], we have

$$|u_0(\cdot,\omega_1) - u_0(\cdot,\omega_2)| \lesssim \left|\omega_1^2 - \omega_2^2\right|. \tag{5.7}$$

Now, we prove that $p_0(\cdot,\cdot,\omega)$ is also continuous with respect to the frequency variable. We start by recalling, see (3.10), the expression of p_0

$$p_0(t,x,\omega_j) := \int_\Omega \partial_t \frac{Im(\epsilon_0)(y)\; |u_0|^2(\omega_j,y)}{\sqrt{t^2-|x-y|^2}}\,dy, \quad j=1,2.$$

The smoothness of the function under the integral, for $x\in\partial\Omega$ and $t\geq \mathrm{diam}(\Omega)$, allows to deduce that

$$|p_0(\cdot,\cdot,\omega_1) - p_0(\cdot,\cdot,\omega_2)| \lesssim \left|\omega_1^2-\omega_2^2\right|. \tag{5.8}$$

Next, we prove a similar result when there is one particle inside the domain Ω. Taking two frequencies ω_j, $j=1,2$, and recalling that, see (3.7), the pressure in the presence of one particle is given by

$$\begin{aligned}
p(t,x,\omega_j) &= \frac{-t\,\mathrm{Im}(\epsilon_p)\,|\langle u_1(\cdot,\omega_j);e_{n_0}\rangle|^2}{(t^2-|x-z|^2)^{3/2}} + \int_{\Omega_j}\partial_t \frac{Im(\epsilon_0)(y)\;|u_1|^2(\omega_j,y)}{\sqrt{t^2-|x-y|^2}}\,dy + \mathcal{O}\left(|\log(a)|^{-(1+h+s)}\right),\\
p(t,x,\omega_j) &\overset{(4.13)}{=} p_0(t,x,\omega_j) - \frac{t\,\mathrm{Im}(\epsilon_p)}{(t^2-|x-z|^2)^{\frac{3}{2}}}\frac{\left(\int_D e_{n_0}(x)\,dx\right)^2 |u_0(z,\omega_j)|^2}{|\Psi(\omega_j)|^2}\\
&+ 2\,\omega_j^2\,\mu_\infty\left(\int_D e_{n_0}(x)\,dx\right) Re\left[\frac{u_0(z,\omega_j)\;\epsilon_p\;I(\omega_j)}{\Psi(\omega_j)}\right] + \mathcal{O}\left(|\log(a)|^{2h-2}\right),
\end{aligned}$$

where

$$\begin{aligned}
\Psi(\omega_j) &:= 1-\omega_j^2\mu_\infty\epsilon_p\,\lambda_{n_0} - \omega_j^2\mu_\infty\epsilon_p\left(-\frac{1}{2\pi}\log(k_j)(z)+\Gamma\right)\left(\int_D e_{n_0}d\eta\right)^2 = \mathcal{O}\left(|\log(a)|^{-h}\right)\\
&\overset{(1.19)}{=} \left(\omega_{n_0}^2-\omega_j^2\right)\,\mu_\infty\,\epsilon_p\,\lambda_{n_0} + \mathcal{O}\left(|\log(a)|^{-1}\right), \qquad (5.9)\\
I(\omega_j) &:= \int_{\Omega\setminus D}\partial_t\frac{Im(\epsilon_0(y))\,\overline{u}_0(\omega_j,y)}{\sqrt{t^2-|x-y|^2}}\int_D G_{k_j}(\eta,y)\,e_{n_0}(\eta)d\eta\,dy = \mathcal{O}(a).
\end{aligned}$$

Then

$$\begin{aligned}
|p(t,x,\omega_1)-p(t,x,\omega_2)| &\lesssim |p_0(t,x,\omega_1)-p_0(t,x,\omega_2)| + |Im(\epsilon_p)|\left(\int_D e_{n_0}\right)^2\left|\frac{|u_0(\omega_2,z)|^2}{|\Psi(\omega_2)|^2}-\frac{|u_0(\omega_1,z)|^2}{|\Psi(\omega_1)|^2}\right|\\
&+ \int_D e_{n_0}dx\left|\omega_2^2 Re\left[\frac{u_0(\omega_2,z)\,\epsilon_p\,I(\omega_2)}{\Psi(\omega_2)}\right] - \omega_1^2\,Re\left[\frac{u_0(\omega_1,z)\,\epsilon_p\,I(\omega_1)}{\Psi(\omega_1)}\right]\right|\\
&+ \mathcal{O}\left(|\log(a)|^{2h-2}\right).
\end{aligned}$$

A straightforward calculation, with help of (5.7), allows to obtain

$$I(\omega_2) = I(\omega_1) + \mathcal{O}\left(a\,\left|\omega_1^2-\omega_2^2\right|\right).$$

We use the previous equation, (5.8), (5.7) and (1.19) to obtain

$$\begin{aligned}
|p(t,x,\omega_1)-p(t,x,\omega_2)| &\lesssim \left|\omega_1^2-\omega_2^2\right| + |Im(\epsilon_p)|\left(\int_D e_{n_0}\right)^2\frac{|\Psi(\omega_1)\,-\,\Psi(\omega_2)|\,||\Psi(\omega_1)|\,+\,|\Psi(\omega_2)||}{|\Psi(\omega_1)|^2\,|\Psi(\omega_2)|^2}\\
&+ \left|\int_D e_{n_0}\,dx\right|\frac{|I(\omega_1)|\;|\epsilon_p|\,|\Psi(\omega_1)-\Psi(\omega_2)|}{|\Psi(\omega_1)|\;|\Psi(\omega_2)|} + \mathcal{O}\left(|\log(a)|^{h-1}\left|\omega_j^2-\omega_{n_0}^2\right|\right)\\
&+ \mathcal{O}\left(|\log(a)|^{h-1}\left|\omega_1^2-\omega_2^2\right|\right) + \mathcal{O}\left(|\log(a)|^{2h-2}\right).
\end{aligned}$$

[19] Consequence of the smoothness of the scattering problem for the Helmholtz equation in terms of the frequency.

From (5.9), we deduce that $\Psi(\omega_2) = \Psi(\omega_1) + \mathcal{O}\left(\left|\omega_1^2 - \omega_1^2\right|\right) + \mathcal{O}\left(\left|\log(a)\right|^{-1}\right)$, then

$$\begin{aligned} |p(t,x,\omega_1) - p(t,x,\omega_2)| &\lesssim \left|\omega_1^2 - \omega_2^2\right| + \left|\log(a)\right|^{2h-1-s} \left|\omega_1^2 - \omega_2^2\right| \\ &+ \mathcal{O}\left(\left|\log(a)\right|^{h-1} \left|\omega_j^2 - \omega_{n_0}^2\right|\right) + \mathcal{O}\left(\left|\log(a)\right|^{2h-2}\right). \end{aligned}$$

In the particular case, when we take $\omega_1^2 = \omega_{n_0}^2 \pm |\log(a)|^{-h}$ and $\omega_2^2 = Re\left(\omega_{n_0}^2\right) \pm |\log(a)|^{-h}$ (i.e. its projection onto the real axis), we obtain

$$\left|\omega_1^2 - \omega_2^2\right| = \left|Im(\omega_{n_0}^2)\right| = \mathcal{O}\left(\left|\log(a)\right|^{-(h+s)}\right) \quad \text{and} \quad \left|\omega_{1,2}^2 - \omega_{n_0}^2\right| = \mathcal{O}\left(\left|\log(a)\right|^{-h}\right),$$

hence

$$|p(t,x,\omega_1) - p(t,x,\omega_2)| \lesssim \left|\log(a)\right|^{\max(h-1-2s;-(h+s);2h-2)}.$$

This proves (3.20).

Remark 5.1. We obtain, with similar computations, the same result as in Lemma 3.3 for the case of a Dimer. We skip the details of the proof.

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