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A SHARP UPPER BOUND FOR SAMPLING NUMBERS IN L_2

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Abstract. For a class F of complex-valued functions on a set D, we denote by $g_n(F)$ its sampling numbers, i.e., the minimal worst-case error on F, measured in L_2, that can be achieved with a recovery algorithm based on n function evaluations. We prove that there is a universal constant $c \in \mathbb{N}$ such that, if F is the unit ball of a separable reproducing kernel Hilbert space, then

$$g_{cn}(F)^2 \leq \frac{1}{n} \sum_{k \geq n} d_k(F)^2,$$

where $d_k(F)$ are the Kolmogorov widths (or approximation numbers) of F in L_2. We also obtain similar upper bounds for more general classes F, including all compact subsets of the space of continuous functions on a bounded domain $D \subset \mathbb{R}^d$, and show that these bounds are sharp by providing examples where the converse inequality holds up to a constant. The results rely on the solution to the Kadison-Singer problem, which we extend to the subsampling of a sum of infinite rank-one matrices.

1. Introduction and main results

The general question of how well point-wise evaluations perform for approximating a function, which is often called *sampling recovery* or approximation using *standard information*, is a classical question in theoretical and applied mathematics. A historical treatment and various basics may be found in the monographs [6, 31, 32, 33, 38, 39]. It is of particular interest to compare the *power of function evaluations* with the power of optimal linear measurements (which could be Fourier coefficients or derivatives), since the latter are well understood in many cases and easier to handle from a theoretical point of view.

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The quest for a systematic comparison has attracted much attention recently, see [4, 5, 8, 9, 11, 12, 13, 14, 17, 18, 22, 27, 28, 40, 42, 43, 44], which all appeared in the past five years. It is often expressed in terms of sampling numbers and Kolmogorov (or approximation) numbers, as we summarize below.

Let $(D, \mathcal{A}, \mu)$ be a measure space and $L_2 := L_2(D, \mathcal{A}, \mu)$ be the space of square-integrable complex-valued functions on D. Let F be a set of functions contained in L_2. The *Kolmogorov widths* of F in L_2 are defined by

$$d_k(F) := \inf_{\substack{\ell_1,\dots,\ell_k\colon F\to\mathbb{C}\\ \varphi_1,\dots,\varphi_k\in L_2}} \sup_{f\in F} \Big\| f - \sum_{i=1}^{k} \ell_i(f)\,\varphi_i \Big\|_{L_2}.$$

This is the worst-case error of an optimal approximation within a linear space of dimension k. It coincides with the kth approximation number (or linear width) of F, which is the worst-case error of an optimal linear algorithm that uses at most k linear functionals as information, see Remark 5. On the other hand, the *sampling numbers* are given by

$$g_n(F) := \inf_{\substack{x_1,\dots,x_n\in D\\ \varphi_1,\dots,\varphi_n\in L_2}} \sup_{f\in F} \Big\| f - \sum_{i=1}^{n} f(x_i)\,\varphi_i \Big\|_{L_2},$$

i.e., $g_n(F)$ is the minimal worst-case error of linear algorithms based on n function evaluations. Therefore, the task is to compare the numbers $d_k(F)$ and $g_n(F)$.

It is clear that we have $g_n(F) \geq d_n(F)$. Here, we aim for an upper bound of $g_n(F)$ in terms of the numbers $d_k(F)$. We first describe the situation where F is the unit ball of a separable reproducing kernel Hilbert space (RKHS). A priori, it is not clear whether such a bound is even possible. And indeed, there can be no such bound in the case that $(d_k(F)) \notin \ell_2$. More precisely, it is shown in [7] that for any non-negative and non-increasing sequence $(\sigma_k) \notin \ell_2$ and any sequence (τ_n) tending to infinity, e.g. $\tau_n = \log\log n$, there exists a RKHS with unit ball F such that $d_k(F) = \sigma_k$ for all k but $\limsup_{n\to\infty} \tau_n \cdot g_n(F) > 0$.

The situation is completely different when $(d_k(F)) \in \ell_2$, which is equivalent to the finite trace assumption on the kernel K of the Hilbert space, i.e.,

$$\int_D K(x,x)\, d\mu(x) < \infty. \tag{1}$$

Under this assumption, first upper bounds on $g_n(F)$ in terms of the numbers $d_k(F)$ were obtained more than 20 years ago in [47]. These upper bounds were later improved in [17, 20, 28]. On the other hand, a lower bound from [11, Theorem 2] tells us how far these improvements might go: for every non-negative and non-increasing $(\sigma_k) \in \ell_2$, there exists a separable RKHS with unit ball F such that $d_k(F) = \sigma_k$ for all $k \in \mathbb{N}$ and

$$g_{\lfloor m/8 \rfloor}(F) \;\geq\; \sqrt{\frac{1}{m}\sum_{k\geq m} d_k(F)^2} \tag{2}$$

for infinitely many values of $m \in \mathbb{N}$. And indeed, it turns out that this is already the worst possible scenario. The main result of this paper is an upper bound, which matches the above lower bound (2) up to a universal constant, and which is true for any separable reproducing kernel Hilbert space.

Theorem 1. *Let F be the unit ball of a separable RKHS satisfying the finite trace assumption* (1)*. Then, there is a universal constant $c \in \mathbb{N}$ such that, for all $m \geq 1$,*

$$g_{cm}(F) \;\leq\; \sqrt{\frac{1}{m}\sum_{k\geq m} d_k(F)^2}.$$

This settles the question on the power of standard information compared to general linear information for the problem of L_2-approximation on Hilbert spaces, and solves the open problems from [11, 17], Open Problem 140 in [33], as well as Outstanding Open Problem 1.4 in [6] for L_2-approximation. The latter is discussed in Example 24, where we consider tensor product spaces. We note that the case of L_p-approximation ($p \neq 2$) is widely open.

Theorem 1 is a direct continuation of the series of works initiated in [17], in which the sampling numbers were bounded by

$$g_{\lfloor c\, m \log m \rfloor}(F) \;\leq\; \sqrt{\frac{1}{m}\sum_{k\geq m} d_k(F)^2},$$

see also [12, 44], and an improvement from [28], where the logarithmic oversampling was removed in exchange for an additional factor $\sqrt{\log m}$ on the right hand side.

The ingredients for the proof are still the existence of good point sets with $\mathcal{O}(m \log m)$ points from [17], and a subsampling of $\mathcal{O}(m)$ points based on the solution to the Kadison-Singer problem from [24], that was introduced into the study of this problem in [28], see also [22]. In [28], the subsampling was, roughly speaking, only performed for a finite-dimensional sub-problem, and this was not enough to ensure that the subsampled point set satisfies the same upper bound as the original point set. The new ingredient here is an infinite-dimensional version of the subsampling theorem that might be of independent interest, see Proposition 13.

If we apply Theorem 1 and the lower bound from [11] to sequences with polynomial decay, we obtain the following characterization.

Corollary 2. *Let F be the unit ball of a separable RKHS with*

$$d_n(F) \lesssim n^{-\alpha} \log^{-\beta} n$$

for some $\alpha \geq 1/2$ and $\beta \in \mathbb{R}$. Then

$$g_n(F) \lesssim \begin{cases} d_n(F) & \text{if } \alpha > 1/2, \\ d_n(F) \cdot \sqrt{\log n} & \text{if } \alpha = 1/2 \text{ and } \beta > 1/2. \end{cases}$$

Moreover, there exist F such that these bounds are sharp.

We now turn to general function classes F that are assumed to satisfy the following assumption.

Assumption A. Let F be a class of complex-valued functions on a set D. We say that F satisfies Assumption A, if there is a metric on F such that F is continuously embedded into L_2, separable, and function evaluation $f \mapsto f(x)$ is, for each $x \in D$, continuous on F.

Note that Assumption A is satisfied, for example, if

- F is a separable subset of the space of bounded functions equipped with the maximum distance and the measure μ is finite, **or**
- F is the unit ball of a separable normed space that is continuously embedded in L_2 and on which function evaluation at each point is a continuous functional, **or**
- F is a countable set of square-integrable functions, equipped with the discrete metric.

In this setting, we prove the following bound.

Theorem 3. *Let F satisfy Assumption A. Then, for every $0 < p < 2$, there is a constant $c \in \mathbb{N}$, depending only on p, such that, for all $m \geq 1$,*

$$g_{cm}(F) \leq \left(\frac{1}{m} \sum_{k \geq m} d_k(F)^p \right)^{1/p}.$$

We will also show that the result is not true for $p = 2$, see Example 26. However, we provide a variant of Theorem 3 under the weaker condition $\big((\log k)^s d_k(F)\big) \in \ell_2$ for some $s > 1/2$ in Section 6.2. This leads to the following corollary.

Corollary 4. *Let F satisfy Assumption A and*

$$d_n(F) \lesssim n^{-\alpha} \log^{-\beta} n$$

for some $\alpha > 0$ and $\beta \in \mathbb{R}$. Then

$$g_n(F) \lesssim \begin{cases} d_n(F) & \text{if } \alpha > 1/2, \\ d_n(F) \cdot \log n & \text{if } \alpha = 1/2 \text{ and } \beta > 1, \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, there exist F such that these bounds are sharp.

The difference compared to unit balls of RKHSs is the case $\alpha = 1/2$, where we need $\beta > 1$ instead of $\beta > 1/2$ and lose a factor $\log n$ instead of $\sqrt{\log n}$, see Example 25. In addition, if $(d_k) \notin \ell_2$, then (g_n) might be bounded below by a constant, opposite to the RKHS setting where g_n tends to zero as soon as d_k does, see [7]. However, for $\alpha > 1/2$, the results for general function classes are just as strong as before.

The rest of the paper is devoted to the proof of the above results. Sections 2–5 form the proof of Theorem 1. In Section 2, we collect some basics on the RKHS setting. In Section 3, we obtain our initial sample of $\mathcal{O}(m \log m)$ points based on a concentration inequality for infinite matrices. The subsampling is performed in Section 4, which applies the solution to the Kadison-Singer problem in a slightly original way, leading to the core of the proof in Section 5. In Section 6, we prove our results for general function classes by constructing a suitable RKHS, on which a local version of Theorem 1 can be applied. Finally, in Section 7, we present examples showing that our results are sharp.

Remark 5 (Equivalent widths)**.** There are several quantities to measure the "width" of a set F. Although we work here with the Kolmogorov numbers $d_k(F)$ as benchmark, let us add that these quantities coincide in L_2 with the *approximation numbers* of F, i.e.

$$d_k(F) \;=\; a_k(F) \;:=\; \inf_{\substack{\ell_1,\dots,\ell_k\colon F\to\mathbb{C} \text{ linear} \\ \varphi_1,\dots,\varphi_k \in L_2}} \; \sup_{f\in F} \Big\| f - \sum_{i=1}^k \ell_i(f)\,\varphi_i \Big\|_{L_2},$$

as the infimum in the definition of $d_k(F)$ for given $\varphi_1, \dots, \varphi_k$ is attained by the L_2-orthogonal projection onto their span, which is linear in any case. The approximation numbers of a class represent the worst-case error of an optimal linear algorithm that uses at most k linear functionals as information. If F is the unit ball of some Hilbert space H, then the approximation numbers agree with the *singular values* of the identity $\mathrm{Id}\colon H \to L_2$. In this case, the $d_k(F)$ also coincide with the *Gelfand k-widths* $c_k(F)$, which represent the minimal worst-case error of (possibly non-linear) algorithms based on k arbitrary linear functionals, see, e.g., Chapter 4 in [31].

Remark 6 (Assumptions on F)**.** Contrarily to the ℓ_2 summability of the Kolmogorov widths, it should be possible to remove the separability assumption on the RKHS H in Theorem 1, by adding a term $\mathrm{tr}_0(K)/m$ inside the square root in the right-hand side, as done in [27]. We also note that our bounds make sense even if $d_k(F)$ is infinite for small k.

Remark 7 (Extreme classes F)**.** It is interesting to note that the lower bound (2) from [11] is attained already for univariate Sobolev spaces of periodic functions. By Theorem 1, this means that these basic classes already represent the most difficult RKHSs for sampling recovery when the numbers $d_k(F)$ are fixed.

Remark 8 (Expected error)**.** A different approach to L_2-approximation is by using randomized algorithms and taking the worst case expected error instead of a worst case deterministic error. The results in this randomized setting are quite different; the error of optimal algorithms does not depend on the tail of the sequence $(d_k(F))$. We refer to [4, 5, 14, 23, 33, 48].

Remark 9 (Upper bounds for infinite trace)**.** An alternative approach is to bound the numbers $g_n(F)$ by the Kolmogorov widths $d_k(F, L_\infty)$ in L_∞: it is shown in [40] that there is a universal constant $c \in \mathbb{N}$ such that $g_{cm}(F) \leq c\, d_m(F, L_\infty)$ for probability spaces $(D, \mathcal{A}, \mu)$. Although this bound is sometimes weaker than Theorem 3 (see Example 1 in [18]), it has the great advantage that it may also be applied in situations where the Kolmogorov widths in L_2 are not square summable, see, e.g., [42, 43]. It would be very interesting to see whether it is possible to unify the two approaches.

Remark 10 (Tractability)**.** Assume now that a whole sequence of classes F_d is given, where d could be the dimension of the underlying domain. For some classes we know that the curse of dimensionality is present, if only standard information (function values) is allowed, while the problem is tractable for general linear information, see, e.g., [10, 34, 45]. However, since the constants from Theorems 1 and 3 are independent of the dimension, it is possible to transfer certain tractability properties from linear information to standard information, see, e.g., [12, 16, 33].

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2. Hilbert space setting

We first consider the case where F is the unit ball of a separable Hilbert space H with reproducing kernel $K \in \mathbb{C}^{D\times D}$. We refer to [27] and references therein for theoretical background on RKHSs.

Thanks to the finite trace assumption (1), we know that the identity map $\mathrm{Id}\colon H \to L_2$ is Hilbert-Schmidt, thus its left and right singular vectors $(b_k)_{k\in\mathbb{I}}$ and $(\sigma_k b_k)_{k\in\mathbb{I}}$ are orthonormal families in L_2 and H, respectively. Here, we only list the singular vectors with respect to the nonzero singular values $\sigma_k > 0$, and the index set is of the form $\mathbb{I} = \{k \in \mathbb{N}_0 \colon k < M\}$ with $M \in \mathbb{N} \cup \{\infty\}$. The singular vectors satisfy

$$\langle f, b_k\rangle_{L_2} = \langle f, \sigma_k^2 b_k\rangle_H \quad \text{for all } f \in H \text{ and } k \in \mathbb{I}.$$

We use the convention that $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and the singular values are arranged in a non-increasing order. In particular, $\sum_{k\in\mathbb{I}} \sigma_k^2 < \infty$ and the Kolmogorov width $d_m(F) = \sigma_m$ is attained by the L_2-orthogonal projection P_m onto $V_m = \mathrm{Span}\{b_k \colon k < m\}$. Moreover, the separability of H ensures that the equality

$$K(x, y) = \sum_{k\in\mathbb{I}} \sigma_k^2 \, b_k(x)\overline{b_k(y)}$$

holds for all $x, y \in D_0$ with some set $D_0 \subset D$ satisfying $\mu(D \setminus D_0) = 0$. We therefore have the identity

$$(3) \qquad f(x) = \sum_{k\in\mathbb{I}} \langle f, b_k\rangle_{L_2} \, b_k(x) \quad \text{for all } f \in H \text{ and } x \in D_0.$$

Our sampling points will be contained in the set D_0.

As a consequence of the following lemma, we only have to show the validity of Theorem 1 for all $1 \leq m < M$.

Lemma 11. *Let $M < \infty$. Then we have $g_n(F) = 0$ for all $n \geq M$.*

Proof. For $x \in D_0$, we write $b(x) = (b_0(x), \dots, b_{M-1}(x))$. Then there are points $x_0, \dots, x_{M-1} \in D_0$ such that every $b(x)$ is contained in the span of the vectors $b(x_i)$. We write $b(x) = \sum \varphi_i(x) b(x_i)$ with coefficients $\varphi_i(x) \in \mathbb{C}$. By (3), we have

$$f(x) = \sum_{k<M} \langle f, b_k\rangle_{L_2} \sum_{i<M} \varphi_i(x) b_k(x_i) = \sum_{i<M} f(x_i)\varphi_i(x),$$

for all $x \in D_0$ and $f \in H$. Thus, the identity $f = \sum f(x_i)\varphi_i$ holds almost everywhere. Moreover, the functions $b_0, \dots, b_{M-1}$ restricted to D_0 form a basis of $\mathrm{Span}\{\varphi_i \colon i < M\}$, and thus $\varphi_i \in L_2$. □

We fix an integer $1 \leq m < M$ for the rest of the proof of Theorem 1.

3. Concentration inequality

As proposed in [17] and applied in [12, 18, 27, 28, 44], we define the probability density

$$\rho_m(x) = \frac{1}{2}\left(\frac{1}{m}\sum_{k<m} |b_k(x)|^2 + \frac{\sum_{k\geq m} \sigma_k^2 \, |b_k(x)|^2}{\sum_{k\geq m} \sigma_k^2}\right).$$

and draw i.i.d. random points $x_1, \dots, x_n \in D$ according to this density. We define the M-dimensional vectors $y_1, \dots, y_n$ by

$$(y_i)_k = \begin{cases} \rho_m(x_i)^{-1/2} \, b_k(x_i) & \text{if } 0 \leq k < m, \\ \rho_m(x_i)^{-1/2} \, \gamma_m^{-1} \, \sigma_k \, b_k(x_i) & \text{if } m \leq k < M, \end{cases}$$

where

$$\gamma_m := \max\left\{\sigma_m, \sqrt{\frac{1}{m}\sum_{k\geq m}\sigma_k^2}\right\} > 0.$$

Note that $\rho_m(x_i) > 0$ almost surely. It follows from these definitions that $y_i \in \ell_2(\mathbb{I})$ with

$$\|y_i\|_2^2 = \rho_m(x_i)^{-1}\left(\sum_{k<m}|b_k(x_i)|^2 + \gamma_m^{-2}\sum_{k\geq m}\sigma_k^2\,|b_k(x_i)|^2\right) \leq 2m,$$

and

$$\mathbb{E}(y_i y_i^*) = \mathrm{diag}(1,\dots,1,\sigma_m^2/\gamma_m^2,\sigma_{m+1}^2/\gamma_m^2,\dots) =: E,$$

with $\|E\|_{2\to 2} = 1$ since $\sigma_k^2/\gamma_m^2 \leq 1$ for $k \geq m$. Here, $\mathrm{diag}(v)$ denotes a diagonal matrix with diagonal v, and $\|\cdot\|_{2\to 2}$ denotes the spectral norm of a matrix.

We apply the following concentration inequality for infinite matrices, which was proved by Mendelson and Pajor in [25, Theorem 2.1]. We use a version of this result from [27, Theorem 1.1] and [28, Theorem 5.3].

Lemma 12. *Let $n \geq 3$ and $y_1,\dots,y_n$ be i.i.d. random sequences from $\ell_2(\mathbb{I})$ satisfying $\|y_i\|_2^2 \leq 2m$ almost surely and $\|E\|_{2\to 2} \leq 1$, with $E = \mathbb{E}(y_i y_i^*)$. Then, for $0 \leq t \leq 1$,*

$$\mathbb{P}\left(\Big\|\frac{1}{n}\sum_{i=1}^n y_i y_i^* - E\Big\|_{2\to 2} > t\right) \leq 2^{3/4}\, n\, \exp\left(-\frac{nt^2}{42\,m}\right).$$

For $t = 1/2$, this probability is less than $1/2$ as soon as $\frac{n}{\log(4n)} \geq 168\,m$. In the sequel we take

$$n = \lfloor C_0 m \log(m+1)\rfloor,$$

with C_0 large enough, so that the previous inequality holds true. (One can take $C_0 = 10^4$, for instance.) Thanks to Lemma 12, we know that there exists a deterministic sample $x_1,\dots,x_n \in D_0$ such that the corresponding family $y_1,\dots,y_n$ satisfies

$$\Big\|\frac{1}{n}\sum_{i=1}^n y_i y_i^* - E\Big\|_{2\to 2} \leq \frac{1}{2}. \tag{4}$$

We fix such a sequence for the rest of the proof of Theorem 1.

4. Subsampling of infinite vectors

We now want to apply the solution to the Kadison-Singer problem, or specifically to Weaver's conjecture, to the sum of rank-one matrices

$$\frac{1}{n}\sum_{i=1}^n y_i y_i^*,$$

in order to find a subsampling of order m preserving the spectral properties of the sum. The original result comes from the celebrated paper [24] by Marcus, Spielman and Srivastava, and has already been applied numerous times in approximation theory, see for instance [4, 14, 17, 18, 27, 28, 30, 40]. However, the original subsampling strategy only works for finite matrices. The main results of this section is the following infinite-dimensional variant, that might be of independent interest.

Proposition 13. *There are absolute constants $c_1, c_2, c_3 > 0$ with the following properties. Let $n, m \in \mathbb{N}$ and $y_1, \ldots, y_n$ be vectors from $\ell_2(\mathbb{N}_0)$ satisfying $\|y_i\|_2^2 \le 2m$ and*

$$\left\| \frac{1}{n} \sum_{i=1}^{n} y_i y_i^* - \begin{pmatrix} I_m & 0 \\ 0 & \Lambda \end{pmatrix} \right\|_{2\to 2} \le \frac{1}{2}, \tag{5}$$

for some Hermitian matrix Λ with $\|\Lambda\|_{2\to2} \le 1$, where $I_m \in \mathbb{R}^{m\times m}$ denotes the identity.
Then, there is a subset $J \subset \{1, \ldots, n\}$ with $|J| \le c_1 m$, such that

$$c_2 \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} \le \frac{1}{m} \sum_{i\in J} y_i y_i^* \le c_3 I.$$

We can choose $c_1 = 43200$, $c_2 = 50$ and $c_3 = 21600$.

Note that the constants, and hence also the final sampling size, are independent of n, the original sampling size. The rest of this section is devoted to the proof of this proposition.

4.1. **Reduction to finite dimension.** Let U_0 be a matrix whose columns form an orthonormal basis of

$$\operatorname{span}\{(y_i)_{\ge m} \colon i = 1, \ldots, n\} \subset \ell_2,$$

where $(y_i)_{\ge m} = ((y_i)_k)_{k\ge m}$. Clearly, U_0 has at most n columns. Then we have that $U_0^* U_0$ is the identity matrix and in particular the spectral norm of U_0 and U_0^* equals one. We set

$$U = \begin{pmatrix} I_m & 0 \\ 0 & U_0 \end{pmatrix},$$

which is a matrix that satisfies $U^*U = I_p$, where $p \le m + n$, and therefore also U and U^* have unit norm. We choose vectors $z_i \in \mathbb{C}^p$ that satisfy $Uz_i = y_i$ for all $i \le n$. Such vectors exist since y_i is contained in the span of the columns of U. Then we also have $z_i = U^*Uz_i = U^*y_i$.

Let $E = \begin{pmatrix} I_m & 0 \\ 0 & \Lambda \end{pmatrix}$ be the matrix from Proposition 13. We define

$$\hat{E} = U^* E\, U = \begin{pmatrix} I_m & 0 \\ 0 & E' \end{pmatrix} \qquad \text{where} \qquad \|E'\|_{2\to2} \le \|E\|_{2\to2} \le 1.$$

With the norm bounds on U and U^*, equation (5) gives

$$\Big\|\frac{1}{n}\sum_{i=1}^{n} z_i z_i^* - \hat{E}\Big\|_{2\to 2} = \Big\|U^*\left(\frac{1}{n}\sum_{i=1}^{n} y_i y_i^* - E\right)U\Big\|_{2\to 2} \le \frac{1}{2}.$$

4.2. **Approximating the identity.** In addition to finite dimension, the result from [24] requires the matrix $\frac{1}{n}\sum_{i=1}^{n} z_i z_i^*$ to be close to the identity in spectral norm, and this is not ensured here. To mitigate this defect, we artificially add rank-one matrices $z_i z_i^* \in \mathbb{C}^{p\times p}$ for $i = n+1,\dots,q$ in the following way.

As $I_p - \hat{E}$ is positive semi-definite, we can decompose it as a sum of rank-one matrices

$$I_p - \hat{E} = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-m} - E' \end{pmatrix} = \sum_{j=1}^{p-m} t_j t_j^*,$$

where $t_j \in \mathbb{C}^p$. We now choose

$$z_i = \sqrt{\frac{n}{n_{j(i)}}}\, t_{j(i)}, \quad n_j = \left\lceil \frac{n}{2m}\, \|t_j\|_2^2 \right\rceil,$$

with $j(i) \in \{1,\dots,p-m\}$ such that $\{z_i,\, i = n+1,\dots,q\}$ contains exactly n_j copies of each $\sqrt{n/n_j}\, t_j$. In this way, for $i > n$, the first m entries of z_i are zero since this is true for the t_j,

$$\|z_i\|_2^2 \le \frac{n}{n_{j(i)}}\, \|t_{j(i)}\|_2^2 \le 2m,$$

and

$$\begin{aligned}\Big\|\frac{1}{n}\sum_{i=1}^{q} z_i z_i^* - I_p\Big\|_{2\to 2} &= \Big\|\frac{1}{n}\sum_{i=1}^{n} z_i z_i^* + \sum_{j=1}^{p-m} t_j t_j^* - I_p\Big\|_{2\to 2} \\ &= \Big\|\frac{1}{n}\sum_{i=1}^{n} z_i z_i^* - \hat{E}\Big\|_{2\to 2} \le \frac{1}{2}.\end{aligned}$$

Remark 14. As $\|t_j\|_2^2 \le \|I_p - \hat{E}\|_{2\to 2} \le 1$, we count

$$q = n + \sum_{j=1}^{p-m} n_j \le n + \sum_{j=1}^{p-m}\left(1 + \frac{n}{2m}\right) \le n + (p-m)\frac{n}{m} = \frac{np}{m}.$$

Conversely, taking traces in $\mathbb{C}^{p\times p}$, we find

$$\frac{p}{2} = \operatorname{Tr}\left(\frac{1}{2}I_p\right) \le \operatorname{Tr}\left(\frac{1}{n}\sum_{i=1}^{q} z_i z_i^*\right) = \frac{1}{n}\sum_{i=1}^{q}\|z_i\|_2^2 \le \frac{2mq}{n}.$$

So, we obtain $n/m \ge q/p \ge n/4m$. Recall that, given m the dimension of the approximation space V_m, we took $n = \mathcal{O}(m\log m)$ initial sample points, and vectors z_i of size $p = \mathcal{O}(m\log m)$. Hence, the number of such vectors is $q = \mathcal{O}(m \log^2 m)$. Surprisingly, we do not use estimates on p and q in the rest of the argument.

Remark 15. In fact, we did not need an exponential speed of convergence in the concentration inequality. The reduction of the sample size to $\mathcal{O}(m)$ points works for any initial set of sampling points satisfying (4). If the cardinality of the initial sample is $n = m\,\ell(m)$, where $\ell(m)$ is any positive function of m, we get $p = \mathcal{O}(m\,\ell(m))$ and $q = \mathcal{O}(m\,\ell(m)^2)$.

4.3. **Reduction of the sample size.** We can now use the Kadison-Singer solution from [24] in an iterated way, as proposed in Lemma 3 of [30], and later used in [4, 14, 17, 18, 27, 28, 40]. The following lemma is obtained from Corollary B and Lemma 1 in [30].

Lemma 16. *Let $z_1, \ldots, z_q \in \mathbb{C}^p$ with $\|z_i\|_2^2 \le \delta$ and*

$$\alpha I_p \le \sum_{i=1}^{q} z_i z_i^* \le \beta I_p$$

for some $\beta \ge \alpha > 100\,\delta > 0$. Then there is a partition of $\{1, \ldots, q\}$ into sets $J_1, \ldots, J_t$ such that, for all $s \le t$, we have

$$25\,\delta I_p \;\le\; \sum_{i \in J_s} z_i z_i^* \;\le\; 3600\,\frac{\beta}{\alpha}\,\delta I_p.$$

Proof. Since the matrix $M = \sum_{i=1}^{q} z_i z_i^*$ is positive, we may define $\tilde{z}_i = M^{-1/2} z_i$. Then we have $\sum_{i=1}^{q} \tilde{z}_i \tilde{z}_i^* = I_p$ and $\|\tilde{z}_i\|_2^2 \le \delta/\alpha =: \delta' < 1/100$. By Corollary B and Lemma 1 in [30], noting that the constant C from Lemma 1 is at most 36, we get a partition of $\{1, \ldots, q\}$ into sets $J_1, \ldots, J_t$ such that, for all $s \le t$, we have

$$25\,\delta'\, I_p \;\le\; \sum_{i \in J_s} \tilde{z}_i \tilde{z}_i^* \;\le\; 3600\,\delta'\, I_p.$$

Now, using

$$\sum_{i \in J_s} z_i z_i^* = M^{1/2} \sum_{i \in J_s} \tilde{z}_i \tilde{z}_i^*\, M^{1/2},$$

we get the statement. □

Note that one could obtain better constants by adapting the proof of Theorem 2.3 from [28]. In our case, we have $\delta = 2m$, $\alpha = n/2$ and $\beta = 3n/2$. The relation $\alpha > 100\,\delta$ is satisfied. We thus obtain

$$50\, m I_p \;\le\; \sum_{i \in J_s} z_i z_i^* \;\le\; 21600\, m I_p.$$

for every J_s from the partition. Moreover, the inequality

$$\frac{n}{2} I_p \leq \sum_{i=1}^{q} z_i z_i^* = \sum_{s=1}^{t} \sum_{i \in J_s} z_i z_i^* \leq 21600\, tm I_p$$

implies that one of the sets $J' = J_s$ from the partition must satisfy

$$|J' \cap \{1, \dots, n\}| \leq \frac{n}{t} \leq 43200\, m.$$

After applying Lemma 16 and removing the indices from $J' \cap \{n + 1, \dots, q\}$ corresponding to artificially added vectors, we are left with a set $J := J' \cap \{1, \dots, n\}$ of cardinality

$$|J| \leq 43200\, m.$$

It remains to show that the artificial vectors do not interfere with our desired properties. For this, recall that $(z_i)_k = (y_i)_k$ for $k < m$ and $i \leq n$, whereas the first m entries of $z_i \in \mathbb{C}^p$ are zero for $i > n$. Hence,

$$\left(\sum_{i \in J} y_i y_i^* \right)_{<m} = \left(\sum_{i \in J'} z_i z_i^* \right)_{<m} \geq 50\, m\, I_m$$

with $A_{<m} = (A_{k,l})_{k,l<m}$, where we use a simple linear algebra fact on self-adjoint matrices A:

$$\lambda_{\min}(A_{<m}) = \inf_{\substack{z \in \mathbb{C}^p,\, \|z\|_2 = 1 \\ z_k = 0 \text{ for } k \geq m}} z^* A z \geq \inf_{z \in \mathbb{C}^p,\, \|z\|_2 = 1} z^* A z = \lambda_{\min}(A).$$

Similarly, and using positive definiteness, we have

$$\sum_{i \in J} z_i z_i^* \leq \sum_{i \in J'} z_i z_i^* \leq 21600\, m\, I_p.$$

With the orthogonal transformation U from Section 4.1, we get

$$\Big\| \sum_{i \in J} y_i y_i^* \Big\|_{2 \to 2} = \Big\| U \left(\sum_{i \in J} z_i z_i^* \right) U^* \Big\|_{2 \to 2} \leq \Big\| \sum_{i \in J} z_i z_i^* \Big\|_{2 \to 2} \leq 21600\, m.$$

This proves Proposition 13.

□

Remark 17. It would be an interesting improvement to use the result [2] of Batson, Spielman and Srivastava instead of [24] for the subsampling. This earlier paper is applied to approximation theory in e.g. [22, 29, 41] and more recently in [3]. It presents a slightly less powerful method, requiring additional weights, but comes with an almost linear algorithmic complexity, see [21], and much smaller constants, which could make the bound presented here sharp also in terms of numerical values.

5. Proof of the main theorem

We now have all the tools for proving Theorem 1.

To obtain our sampling points, we combine (4) for our initial vectors $y_i \in \ell_2(\mathbb{I})$ with Proposition 13. Clearly, Proposition 13 stays true if we replace $\mathbb{N}_0$ by the possibly finite index set $\mathbb{I}$. We obtain points $x_1, \ldots, x_n \in D_0$ with $n \leq 43200\, m$ such that the vectors

$$(y_i)_k = \begin{cases} \rho_m(x_i)^{-1/2}\, b_k(x_i) & \text{if } 0 \leq k < m, \\ \rho_m(x_i)^{-1/2}\, \gamma_m^{-1}\, \sigma_k\, b_k(x_i) & \text{if } m \leq k < M, \end{cases}$$

satisfy

$$\left(\sum_{i=1}^n y_i y_i^*\right)_{<m} \geq\ 50\, m\, I,$$

and

$$\left(\sum_{i=1}^n y_i y_i^*\right)_{\geq m} \leq\ 21600\, m\, I,$$

where we use the notation $A_{\geq m} = (A_{k,l})_{k,l \geq m}$ for a matrix A.

As in earlier papers, we use the *weighted least squares estimator*

$$A_n(f) \,:=\, \underset{g \in V_m}{\operatorname{argmin}} \sum_{i=1}^n \frac{|g(x_i) - f(x_i)|^2}{\rho_m(x_i)}$$

with V_m and ρ_m as defined in Sections 2 and 3, respectively, see [17]. This algorithm may be written as

$$A_n(f) \,=\, \sum_{k=1}^m (G^+ N f)_k\, b_k$$

where $N\colon F \to \mathbb{C}^n$ with $N(f) := \big(\rho_m(x_i)^{-1/2} f(x_i)\big)_{i \leq n}$ is the *information mapping* and $G^+ \in \mathbb{C}^{m \times n}$ is the Moore-Penrose inverse of the matrix

$$G := \big(\rho_m(x_i)^{-1/2}\, b_k(x_i)\big)_{i \leq n,\, k \leq m} \in \mathbb{C}^{n \times m}.$$

Since we have the identity $\overline{G^* G} = (\sum_{i=1}^n y_i y_i^*)_{<m}$, the matrix G has full rank and the spectral norm of G^+ is bounded by $(50m)^{-1/2}$. In particular, the argmin in the definition of A_n is uniquely defined and A_n satisfies $A_n(f) = f$ for all $f \in V_m$.
Denoting with Q_m the L_2-orthogonal projection onto $\operatorname{Span}\{b_k\colon k \geq m\}$, we obtain for any $f \in H$ that

$$\begin{aligned} \|f - A_n(f)\|_{L_2}^2 \,&=\, \|f - P_m f\|_{L_2}^2 + \|P_m f - A_n(f)\|_{L_2}^2 \\ &=\, \|Q_m f\|_{L_2}^2 + \|A_n(f - P_m f)\|_{L_2}^2 \\ &=\, \|Q_m f\|_{L_2}^2 + \big\|G^+ N(f - P_m f)\big\|_{\ell_2^n}^2 \\ &\leq\, \sigma_m^2\, \|Q_m f\|_H^2 + \big\|G^+\big\|_{2 \to 2}^2 \cdot \|N(f - P_m f)\|_{\ell_2^m}^2 . \end{aligned}$$

By (3) we have $N(f - P_m f) = \Phi\xi_f$, where

$$\Phi = \big(\rho_m(x_i)^{-1/2}\sigma_k\, b_k(x_i)\big)_{i\le n,\, k\ge m} \quad \text{and} \quad \xi_f = (\langle f, \sigma_k\, b_k\rangle_H)_{k\ge m}.$$

The matrix Φ satisfies

$$\overline{\Phi^*\Phi} \;=\; \gamma_m^2 \left(\sum_{i=1}^n y_i y_i^*\right)_{\ge m}$$

and therefore its spectral norm is bounded by $(21600\, m\, \gamma_m^2)^{1/2}$. Thus,

$$\|N(f - P_m f)\|_{\ell_2^m}^2 \;\le\; 21600\, m\, \gamma_m^2 \|\xi_f\|_2^2 \;=\; 21600\, m\, \gamma_m^2\, \|Q_m f\|_H^2.$$

In summary, we obtain for all $1 \le m < M$ the bound

$$\text{(6)} \qquad \|f - A_n(f)\|_{L_2}^2 \;\le\; 433\, \max\left\{\sigma_m^2,\, \frac{1}{m}\sum_{k\ge m}\sigma_k^2\right\} \|Q_m f\|_H^2\,.$$

for all $f \in H$ and some $n \le 43200\, m$. Taking the supremum over $f \in F$ and using that

$$\max\left\{\sigma_m^2,\, \frac{1}{m}\sum_{k\ge m}\sigma_k^2\right\} \;\le\; \frac{2}{m}\sum_{k\ge\lceil m/2\rceil}\sigma_k^2,$$

we obtain

$$g_{43200\, m}(F)^2 \;\le\; \frac{866}{m}\sum_{k\ge\lceil m/2\rceil}\sigma_k^2.$$

This finishes the proof of Theorem 1 with $c = 43200 \cdot 866$.

□

Remark 18. For the purpose of Theorem 1 it was enough to bound $\|Q_m f\|_H \le 1$. However, the “local version” (6) of Theorem 1 will be of advantage later for the study of the general classes, since it is able to see additional decay of the Fourier coefficients compared to the decay implied by $f \in H$. Note that faster decay of the Fourier coefficients often corresponds to higher smoothness of the target function. In a certain sense, this means that the algorithm A_n is universal. The error has the optimal rate of decay for any smoothness higher than the smoothness of H.

Remark 19. The condition on the point sets can also be given by finite matrices that are related to the kernel K of the Hilbert space. For this, let us define $K_m(x,y) := \sum_{k<m} b_k(x)b_k(y)$, and $R_m(x,y) := \sum_{k\ge m}\sigma_k^2\, b_k(x)b_k(y)$. The non-zero singular values of GG^* are the same as those of G^*G, and the non-zero singular values of $\Phi\Phi^*$ are the same as those of $\Phi^*\Phi$, where G and Φ are from above. Hence, the algorithm A_n based on points $x_1, \ldots, x_n$ satisfies the error bound above (up to a

constant) if

$$m \lesssim \lambda_m(GG^*) = \lambda_m\left(\left(\frac{K_m(x_i,x_j)}{\sqrt{\rho_m(x_i)\rho_m(x_j)}}\right)_{i,j=1}^n\right)$$

and

$$\left(\frac{R_m(x_i,x_j)}{\sqrt{\rho_m(x_i)\rho_m(x_j)}}\right)_{i,j=1}^n = \Phi\Phi^* \lesssim m\,\gamma_m^2\, I,$$

where λ_m denotes the m-th eigenvalue. It would be interesting to find a property that only involves the kernel K directly (instead of the truncated kernels above), or to verify that a similar property characterizes *good* point sets, in a way similar to Proposition 1 of [10] for integration.

5.1. **Proof of Corollary 2.** For the given bounds on the sampling numbers for sequences of polynomial decay, we only need to note that

$$\frac{1}{n}\sum_{k\geq n} k^{-a}\log^{-b} k \lesssim \begin{cases} n^{-a}\log^{-b} n & \text{if } a>1,\, b\in\mathbb{R}, \\ n^{-a}\log^{-b+1} n & \text{if } a=1,\, b>1. \end{cases}$$

Hence, Corollary 2 immediately follows from Theorem 1, and the existence of F where the bounds are attained comes from (2), see [11]. □

6. General function classes

We now prove all results related to general function classes.

6.1. **Proof of Theorem 3.** Fix some $0<p<2$, and let us assume that $d_k(F)$ is finite for $k\geq k_0$ and that $(d_k(F))_{k\geq k_0}\in \ell_p$. Otherwise, the statement is trivial.

The basic idea is to construct a suitable reproducing kernel Hilbert space H that contains a dense subset of F and apply Theorem 1 to this Hilbert space. It will be important to use the local bound from (6) for this Hilbert space instead of the global bound from Theorem 1.

We start with the following simple observation from [18]. Without loss of generality, we assume that L_2 is infinite-dimensional.

Lemma 20. *There is an orthonormal system $\{b_k\colon k\in\mathbb{N}_0\}$ in L_2 such that for all $m\geq 1$, the orthogonal projection P_m onto $V_m=$ Span$\{b_k\colon k<m\}$ satisfies*

$$\sup_{f\in F}\|f-P_m f\|_{L_2} \leq 2\, d_{\lfloor m/4\rfloor}(F). \tag{7}$$

Proof. Clearly it is enough to find an increasing sequence of subspaces of L_2,

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \dots, \qquad \dim(U_m) \le m,$$

such that the projection P_m onto U_m satisfies (7). By the definition of $d_k(F)$, $k \in \mathbb{N}_0$, there is a subspace $W_k \subset L_2$ of dimension k and a mapping $T_k \colon F \to W_k$ such that

$$\sup_{f \in F} \|f - T_k f\|_{L_2} \le 2\, d_k(F).$$

We let U_m be the space that is spanned by the union of the spaces W_{2^ℓ} over all $\ell \in \mathbb{N}_0$ such that $2^\ell \le m/2$. Note that U_m contains a subspace W_k with $k \ge \lfloor m/4 \rfloor$. Therefore, $P_m f$ is at least as close to f as $T_k f$ for some $k \ge \lfloor m/4 \rfloor$, which implies (7). □

In what follows $\{b_k \colon k \in \mathbb{N}_0\}$ will always be the orthonormal system from Lemma 20. Note that we will consider b_k as a function, where we fix an arbitrary representer from the equivalence class in L_2. We call $\hat{f}(k) := \langle f, b_k \rangle_{L_2}$ the kth Fourier coefficient of f.

We now choose some $\alpha \in (1/2, 1/p)$ and set $\sigma_k = \max\{1, k\}^{-\alpha}$ for $k \in \mathbb{N}_0$, so that we have $(\sigma_k) \in \ell_2$. Moreover, we fix a countable dense subset F_0 of F and use the following observation, see also [18].

Lemma 21. *There is a measurable subset D_0 of D with $\mu(D \setminus D_0) = 0$ such that, for all $x \in D_0$ and $f \in F_0$, we have*

$$K_x := \sum_{k \in \mathbb{N}_0} \sigma_k^2 |b_k(x)|^2 < \infty \quad \textit{and} \quad f(x) = \sum_{k \in \mathbb{N}_0} \hat{f}(k)\, b_k(x).$$

Proof. First, we note that the integral $\int_D K_x \, d\mu(x)$ is finite and thus K_x is finite for all $x \in D \setminus E$ with a null set $E \subset D$. For all $f \in F_0$, we have

$$\sum_{k \ge 1} k\, |\hat{f}(k)|^2 = \sum_{n \ge 0} \sum_{k > n} |\hat{f}(k)|^2 = \sum_{n \ge 0} \|f - P_n f\|_{L_2}^2 < \infty,$$

where we use (7) and the assumptions on F. The Rademacher-Menchov Theorem, see e.g. [35], now implies that the Fourier series of f at x converges to $f(x)$ for all $x \in D \setminus E_f$ with a null set $E_f \subset D$. The desired properties therefore hold on $D_0 := D \setminus E_0$, where $E_0 := E \cup \bigcup_{f \in F_0} E_f$ is a null set. □

Thus, the set H of all square-integrable functions $f \colon D_0 \to \mathbb{C}$ which are point-wise represented by their Fourier series $\sum_k \hat{f}(k)\, b_k$ and which satisfy

$$\|f\|_H^2 := \sum_{k \in \mathbb{N}_0} \frac{|\hat{f}(k)|^2}{\sigma_k^2} < \infty$$

is indeed a separable reproducing kernel Hilbert space on D_0 with orthonormal basis $(\sigma_k b_k)_{k \in \mathbb{N}_0}$ and trace $\sum_{k \in \mathbb{N}_0} \sigma_k^2 < \infty$. Moreover:

Lemma 22. *The countable family F_0 (with functions restricted to D_0) is a subset of H, and, for all $f \in F_0$ and $m \in 8\mathbb{N}$,*

$$\|f - P_m f\|_H \le C\, m^{\alpha} \Big(\frac{1}{m} \sum_{k \ge m/8} d_k(F)^p \Big)^{1/p}, \tag{8}$$

where $C < \infty$ only depends on α and p.

Proof. According to Lemma 21, every $f \in F_0$ is point-wise represented by its Fourier series and the norm of f in H is finite, as seen from the following computation. Clearly, it is enough to show (8). Note that the Kolmogorov widths of F_0 and F are the same. We use

$$d_{2m} = \big(d_{2m}^p\big)^{1/p} \le \Big(\frac{1}{m} \sum_{k \ge m} d_k^p \Big)^{1/p}$$

and obtain

$$\begin{aligned}
\|f - P_m f\|_H^2 &= \sum_{k \ge m} k^{2\alpha}\, |\hat{f}(k)|^2 \le \sum_{\ell \in \mathbb{N}_0} \big(m 2^{\ell+1}\big)^{2\alpha} \sum_{k = m2^{\ell}}^{m2^{\ell+1}-1} |\hat{f}(k)|^2 \\
&\le 4 \sum_{\ell \in \mathbb{N}_0} \big(m 2^{\ell+1}\big)^{2\alpha} d_{m2^{\ell-2}}^2 \le 4 \sum_{\ell \in \mathbb{N}_0} \big(m 2^{\ell+1}\big)^{2\alpha} \Big(\frac{1}{m 2^{\ell-3}} \sum_{k \ge m 2^{\ell-3}} d_k^p \Big)^{2/p} \\
&\le 2^{2+2\alpha+6/p} m^{2\alpha - 2/p} \sum_{\ell \in \mathbb{N}_0} 2^{(2\alpha - 2/p)\ell} \Big(\sum_{k \ge m/8} d_k^p \Big)^{2/p} < \infty
\end{aligned}$$

since $2\alpha - 2/p < 0$.

□

By Theorem 1 and bound (6), we find $n \le 43200\, m$ and a linear algorithm A_n of the form

$$A_n(f) = \sum_{i=1}^{n} f(x_i) g_i, \quad x_i \in D_0,\ g_i \in L_2,$$

such that

$$\|f - A_n f\|_{L_2(D_0,\mu)}^2 \le 433 \max\Big\{ \sigma_m^2, \frac{1}{m} \sum_{k \ge m} \sigma_k^2 \Big\} \|f - P_m f\|_H^2$$

for all $f \in H$ and thus, for all $f \in F_0$. Clearly, in the last inequality, D_0 can be replaced with D. If we now combine the estimate (8) with

$$\max\Big\{ \sigma_m^2, \frac{1}{m} \sum_{k \ge m} \sigma_k^2 \Big\} \lesssim m^{-2\alpha}, \tag{9}$$

we obtain that

$$\|f - A_n f\|_{L_2}^2 \le \Big(\frac{c_p}{m} \sum_{k \ge m/8} d_k(F)^p \Big)^{2/p}$$

for all $f \in F_0$ and some $c_p < \infty$ that only depends on p. Since F_0 is dense in F and both $\mathrm{id}\colon F \to L_2$ and $A_n\colon F \to L_2$ are continuous, the last bound is true for all $f \in F$. This finishes the proof of Theorem 3 with $c = 43200 \max(c_p, 8)$.

□

6.2. **The boundary case.** We provide a variant of Theorem 3 under a weaker condition than $(d_k(F)) \in \ell_p$ for $p < 2$. In fact, we show that the condition $\big((\log k)^s d_k(F)\big) \in \ell_2$ for some $s > 1/2$ is enough for a comparison of the sampling and the Kolmogorov widths, while the same assumption for $s = 1/2$ is not enough, see Example 26.

Theorem 23. *Let F satisfy Assumption A. Then, there is a universal constant $c \in \mathbb{N}$ and a constant $c_s > 0$, depending only on s, such that, for all $m \geq 2$, we have*

$$g_{cm}(F)^2 \;\leq\; c_s\, m^{-1} \log^{-2s+1} m \sum_{k \geq m} d_k(F)^2 \cdot \log^{2s} k.$$

Proof. The proof follows the same lines as the proof of Theorem 3. The only difference is that we now choose $\sigma_k = k^{-1/2} \log^{-s} k$ for $k \geq 2$. Then, inequality (8) becomes

$$\begin{aligned} \|f - P_m f\|_H^2 &= \sum_{k \geq m} |\hat{f}(k)|^2 k \log^{2s}(k) \leq \sum_{k \geq m} |\hat{f}(k)|^2 \sum_{m \leq r \leq 2k} \log^{2s}(r) \\ &\leq \sum_{r \geq m} \log^{2s}(r) \sum_{k \geq r/2} |\hat{f}(k)|^2 \leq 4 \sum_{r \geq m} \log^{2s}(r)\, d_{\lfloor r/8 \rfloor}^2 \\ &\leq 32 \sum_{k \geq \lfloor m/8 \rfloor} \log^{2s}(8k+7)\, d_k^2 \;< \infty. \end{aligned}$$

Likewise, inequality (9) becomes

$$\max\left\{ \sigma_m^2,\, \frac{1}{m} \sum_{k \geq m} \sigma_k^2 \right\} \;\lesssim\; m^{-1} \log^{-2s+1} m$$

and the stated inequality is obtained. □

6.3. **Proof of Corollary 4.** Using the same bound as in the proof of Corollary 2, the case $\alpha > 1/2$ immediately follows from Theorem 3 if we choose $1/\alpha < p < 2$, and the case $\alpha = 1/2$, $\beta > 1$ from Theorem 23 if we choose $1/2 < s < \beta - 1/2$.

All bounds are attained with the same classes F as in Corollary 2 for the first case, and with the constructions from the next section for the two other cases.

□

7. Examples

We first present an application of Theorem 1 to tensor product spaces.

Example 24. Let H be a RKHS on D that is compactly embedded into L_2 and let F be its unit ball. We consider L_2-approximation on the unit ball F_d of the d-fold tensor product H_d of H, which is a RKHS on the domain D^d. We assume that $g_n(F) \lesssim n^{-\alpha}$ for some $\alpha > 0$. The famous Smolyak algorithm, see [37], gives the estimate

$$g_n(F_d) \lesssim n^{-\alpha} \log^{(\alpha+1)(d-1)} n. \tag{10}$$

An example of such tensor product spaces are the spaces of dominating mixed smoothness $\alpha > 1/2$, see [6]. For these spaces, it is known that the error bound (10) for the Smolyak algorithm can be improved [36]; the exponent of the logarithm can be reduced to $(\alpha+1/2)(d-1)$. With Corollary 2 and known results on the approximation numbers of tensor product operators, see [1, 26], we now obtain

$$g_n(F_d) \lesssim n^{-\alpha} \log^{\alpha(d-1)} n \quad \text{if} \ \ \alpha > 1/2.$$

This bound is asymptotically optimal for the spaces of mixed smoothness. Let us note, however, that also preasymptotic estimates on the sampling numbers (say, for $n < d^d$) are of interest, especially if the dimension d is high, see [15, 19, 46].

□

We now present two examples that show that our upper bounds cannot be improved without further assumptions on the class F.

First, we show that the worst possible behavior of the sampling numbers in the case $d_n \lesssim n^{-1/2} \log^{-\beta} n$ with $\beta > 1$ is indeed $n^{-1/2} \log^{-\beta+1} n$.

Example 25. For $\ell \in \mathbb{N}_0$ and $k \in \{1, \ldots, 2^\ell\}$ define the interval $I_{\ell,k} = [(k-1)2^{-\ell}, k2^{-\ell})$ and denote with $\chi_{\ell,k}$ the indicator function of $I_{\ell,k}$. Let $\beta > 1$. We set

$$\mathcal{C}_\beta := \left\{ \mathbf{c} := (c_{\ell,k})_{\ell\in\mathbb{N}_0,\, 1\le k\le 2^\ell} \;\Big|\; \sum_{k=1}^{2^\ell} |c_{\ell,k}|^2 \le (\ell+1)^{-2\beta} \text{ for all } \ell \in \mathbb{N}_0 \right\}$$

and consider the class

$$F_\beta := \left\{ f_{\mathbf{c}} := \sum_{\ell\in\mathbb{N}_0} \sum_{k=1}^{2^\ell} c_{\ell,k}\chi_{\ell,k} \;\Big|\; \mathbf{c} \in \mathcal{C}_\beta \right\}.$$

Note that the series $f_{\mathbf{c}}$ converge uniformly, since the inner sum is bounded by $(\ell+1)^{-\beta}$. If F_β is equipped with the maximum distance on $[0, 1)$, it is a separable metric space, function evaluation is continuous, and the embedding in $L_2([0, 1))$ is continuous.

For every $L \in \mathbb{N}_0$, the span V_L of the functions $\chi_{\ell,k}$ with $\ell \le L$ has dimension 2^L. If P_L is the L_2-orthogonal projection onto V_L, we have for all $\mathbf{c} \in \mathcal{C}_\beta$ that

$$\begin{aligned}\big\|f_{\mathbf{c}} - P_L f_{\mathbf{c}}\big\|_2 &\le \Big\| \sum_{(\ell,k)\colon \ell>L} c_{\ell,k}\chi_{\ell,k}\Big\|_2 \le \sum_{\ell>L} \Big\| \sum_{k=1}^{2^\ell} c_{\ell,k}\chi_{\ell,k}\Big\|_2 \\ &= \sum_{\ell>L} \Big(\sum_{k=1}^{2^\ell} c_{\ell,k}^2 \|\chi_{\ell,k}\|_2^2 \Big)^{1/2} \le \sum_{\ell>L} 2^{-\ell/2}(\ell+1)^{-\beta} \lesssim 2^{-L/2}L^{-\beta},\end{aligned}$$

and thus

$$d_{2^L}(F_\beta) \lesssim 2^{-L/2}L^{-\beta},$$

or equivalently

$$d_n(F_\beta) \lesssim n^{-1/2}\log^{-\beta} n.$$

We now show a lower bound for the sampling numbers. Let $x_1, \dots, x_n \in [0,1)$. For all $\ell \in \mathbb{N}_0$, we let J_ℓ be the set of indices $1 \le k \le 2^\ell$ such that $I_{\ell,k}$ contains a point. Clearly, the cardinality of J_ℓ is at most n. We choose $L \in \mathbb{N}_0$ of order $\log n$ and define

$$f_L := \sum_{\ell>L} |J_\ell|^{-1/2}(\ell+1)^{-\beta} \sum_{k\in J_\ell} \chi_{\ell,k}.$$

This function is contained in F_β and for all $i \le n$, we have

$$h := f_L(x_i) = \sum_{\ell>L} |J_\ell|^{-1/2}(\ell+1)^{-\beta} \gtrsim n^{-1/2}\log^{-\beta+1} n,$$

where h is independent of i. On the other hand, as shown by our previous calculation,

$$\Big| \int_0^1 f_L(x)\,dx \Big| \le \big\|f_L\big\|_2 \lesssim 2^{-L/2}L^{-\beta} \lesssim n^{-1/2}\log^{-\beta} n.$$

Thus, if we set $f = h - f_L$, the function is contained in F_β, vanishes at all points $x_1, \dots, x_n$, and satisfies

$$\|f\|_2 \ge \int_0^1 f(x)\,dx \ge h - \Big| \int_0^1 f_L(x)\,dx \Big| \gtrsim n^{-1/2}\log^{-\beta+1} n.$$

This shows $g_n(F_\beta) \gtrsim n^{-1/2}\log^{-\beta+1} n$.

□

The next example shows that, in the case $d_n \lesssim n^{-1/2}\log^{-\beta} n$ with $\beta \le 1$, no general statement on the sampling numbers is possible.

Example 26. Similar to Example 25, we define

$$\mathcal{C} := \Big\{ \mathbf{c} \;\Big|\; \sum_{k=1}^{2^\ell} |c_{\ell,k}|^2 \le (\ell+1)^{-2}\log(\ell+e)^{-2} \text{ for all } \ell \in \mathbb{N}_0 \Big\}$$

and consider the class

$$F := \left\{ f_{\mathbf{c}} := \sum_{\ell\in\mathbb{N}_0} \sum_{k=1}^{2^\ell} c_{\ell,k}\chi_{\ell,k} \,\middle|\, \mathbf{c}\in\mathcal{C},\ \mathbf{c} \text{ finite} \right\}.$$

The finiteness of the sequences ensures that F, equipped with the maximum distance, is still a separable metric space, where function evaluation is continuous, and the embedding in $L_2([0,1))$ is continuous. As above, we obtain

$$d_n(F) \lesssim n^{-1/2}(\log n)^{-1}(\log\log n)^{-1}.$$

In particular, we have $(d_n(F)\log^{1/2} n) \in \ell_2$. On the other hand, given $x_1,\dots,x_n$ and $\varepsilon>0$, we choose $L\in\mathbb{N}_0$ with

$$\sum_{\ell>L} 2^{-\ell/2}(\ell+1)^{-1}(\log(\ell+e))^{-1} \le \varepsilon,$$

define the sets J_ℓ as above, and choose $N\in\mathbb{N}_0$ such that

$$h := \sum_{\ell=L+1}^{N} |J_\ell|^{-1/2}(\ell+1)^{-1}(\log(\ell+e))^{-1} \ge 1.$$

The function

$$f_L := \frac{1}{h}\sum_{\ell=L+1}^{N} |J_\ell|^{-1/2}(\ell+1)^{-1}(\log(\ell+e))^{-1} \sum_{k\in J_\ell}\chi_{\ell,k},$$

is contained in F, its integral is at most ε, and it satisfies $f_L(x_i)=1$ for all $i\le n$. Then $f=1-f_L$ is contained in F, vanishes at all points $x_1,\dots,x_n$, and satisfies

$$\|f\|_2 \ge \int_0^1 f(x)\,dx \ge 1-\left|\int_0^1 f_L(x)\,dx\right| \ge 1-\varepsilon.$$

This shows $g_n(F)\ge 1$ for all $n\in\mathbb{N}_0$.

□

We note that the lower bounds in Example 25 and 26 already hold for the easier problem of numerical integration on F_β. Thus, the upper bounds from Corollary 4 are also sharp for the minimal error of quadrature rules on probability spaces.

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