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RICAM-Report 2022-07

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EXPONENTIAL TRACTABILITY OF L_2-APPROXIMATION WITH FUNCTION VALUES

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Abstract. We study the complexity of high-dimensional approximation in the L_2-norm when different classes of information are available; we compare the power of function evaluations with the power of arbtirary continuous linear measurements. Here, we discuss the situation when the number of linear measurements required to achieve an error $\varepsilon \in (0,1)$ in dimension $d \in \mathbb{N}$ depends only poly-logarithmically on ε^{-1}. This corresponds to an exponential order of convergence of the approximation error, which often happens in applications. However, it does not mean that the high-dimensional approximation problem is easy, the main difficulty usually lies within the dependence on the dimension d. We determine to which extent the required amount of information changes, if we allow only function evaluation instead of arbitrary linear information. It turns out that in this case we only lose very little, and we can even restrict to linear algorithms. In particular, several notions of tractability hold simultaneously for both types of available information.

1. Exposition

We want to approximate real- or complex-valued functions defined on some (nonempty) set $\mathcal{D}$, and belonging to a space F. We assume that F is a separable Banach space of functions defined on $\mathcal{D}$, such that function evaluation $f \mapsto f(x)$ is continuous on F for each $x \in \mathcal{D}$ and F is continuosly embedded in $L_2 = L_2(\mathcal{D}, \mu)$

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Date: May 6, 2022.
DK is supported by the Austrian Science Fund (FWF) Project F5506, which is part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

for some measure μ. Formally, the approximation problem is given as

$$\mathrm{APP}: F \to L_2, \quad \mathrm{APP}(f) := f,$$

which might be understood as a continuous embedding into L_2. The class of all spaces F satisfying the assumptions above will be denoted by $\mathcal{A}$. In particular, for each $F \in \mathcal{A}$ we have some associated nonempty set $\mathcal{D}$, measure μ on $\mathcal{D}$ and continuous embedding APP.

We approximate APP by using functionals from the class Λ^{std} consisting of all function evaluations, or from the class $\Lambda^{\mathrm{all}} = F^*$ of all continuous linear functionals.

Below B_F denotes the closed unit ball in F. Let us define, for $n \in \mathbb{N}$, the

- *n-th linear sampling width* as

$$e_n(F, L_2) := \inf_{\substack{x_1,\dots,x_n \in D \\ \varphi_1,\dots,\varphi_n \in L_2}} \sup_{f \in B_F} \Big\| f - \sum_{i=1}^n f(x_i)\,\varphi_i \Big\|_{L_2},$$

- *n-th sampling width* as

$$g_n(F, L_2) := \inf_{\substack{x_1,\dots,x_n \in D \\ \phi\colon \mathbb{R}^n \to L_2}} \sup_{f \in B_F} \Big\| f - \phi(f(x_1),\dots,f(x_n)) \Big\|_{L_2},$$

- *n-th linear width* as

$$a_n(F, L_2) := \inf_{\substack{T\colon L_2 \to L_2 \\ \mathrm{rank}(T) \le n}} \sup_{f \in B_F} \Big\| f - Tf \Big\|_{L_2},$$

- *n-th Gelfand width* as

$$c_n(F, L_2) := \inf_{\substack{\phi\colon \mathbb{R}^n \to L_2 \\ N \in (F^*)^n}} \sup_{f \in B_F} \Big\| f - \phi \circ N(f) \Big\|_{L_2}.$$

These quantities represent the *minimal worst case errors* that can be achieved with linear or nonlinear algorithms using at most n function values or linear measurements, respectively.

We also define the information-based complexity of the problem APP for the classes Λ^{std} and Λ^{all}, respectively, as the minimal number of evaluations from Λ^{std} or Λ^{all} necessary to obtain the absolute precision of approximation at most ε, i.e., as

$$n^{\mathrm{std}}(\varepsilon, F) := \min\Big\{ n\colon g_n(F, L_2) \le \varepsilon \Big\}$$

and

$$n^{\mathrm{all}}(\varepsilon, F) := \min\Big\{ n\colon c_n(F, L_2) \le \varepsilon \Big\}.$$

Note that, since $g_n(F, L_2) \le e_n(F, L_2)$, we have

$$n^{\mathrm{std}}(\varepsilon, F) \le \min\Big\{ n\colon e_n(F, L_2) \le \varepsilon \Big\},$$

and all our upper bounds are proven for the latter. There is a lot of literature on the size of these quantities for specific classes F. We refer to the monographs [3, 21, 22, 23, 27, 28] for more details and literature on the subject.

Here, we are specifically interested in the comparison of these quantities for general classes F. That is, since $n^{\rm all}(\varepsilon, F) \le n^{\rm std}(\varepsilon, F)$ is obvious for all $F \in \mathcal{A}$, we ask for an upper bound on $n^{\rm std}(\varepsilon, F)$ based on knowledge of the function $n^{\rm all}(\varepsilon, F)$. However, it is known that such a bound cannot hold without certain assumptions on F, see [23, Chapter 26] and references therein, and even then, the involved "constants" depend in a non-trivial way on F. One approach to obtain qualitative statements on the relation of the complexities, is to consider a whole sequence of spaces $(F_d)_{d\in\mathbb{N}}$, where d can be interpreted as the dimension of the underlying domain. We then assume a certain bound on $n^{\rm all}(\varepsilon, F_d)$, depending only on $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$, and ask for an upper bound on $n^{\rm std}(\varepsilon, F_d)$, hopefully not much worse than the bound on $n^{\rm all}(\varepsilon, F_d)$.

A typical, but restricting, assumption is that the classes F are Hilbert spaces. In this case, many computations rely on a singular value decomposition that is not available for arbitrary classes F. However, under the assumption that F is Hilbert, there is often a complete knowledge about the linear/Gelfand width, and hence about $n^{\rm all}(\varepsilon, F_d)$, see, e.g., [12, 13, 17, 18, 21]. Moreover, there are also some general results on the power of $\Lambda^{\rm std}$ in this case, see [23, Chapter 26], and even more results in specific settings, see, e.g., [2, 4, 5, 30].

Our results are based on the following theorem from [16], see also [7, 15, 20, 29], which allows us to treat more general classes of functions.

Theorem 1. *Assume that $(a_n(F, L_2)) \in \ell_r$ for some $0 < r < 2$. There is a universal constant $b \in \mathbb{N}$ and a constant $c_r > 0$, depending only on r, such that, for all $n \ge 2$, we have*

$$e_{bn}(F, L_2) \;\le\; c_r \sqrt{\ln n}\, \left(\frac{1}{n} \sum_{k\ge n} a_k(F, L_2)^r\right)^{1/r}.$$

We are also going to use a known fact that for all $n \ge 1$

$$a_n(F, L_2) \;\le\; \left(1 + \sqrt{n}\right) c_n(F, L_2) \;\le\; 2\sqrt{n}\, c_n(F, L_2), \tag{1}$$

see [1, 19, 25] for more details.

In the present paper, we allow for arbitrary Banach spaces of functions F_d, but we assume that $n^{\rm all}(\varepsilon, F_d)$ depends only poly-logarithmically on ε^{-1}. That is, we assume that

$$n^{\rm all}(\varepsilon, F_d) \le C_d \left(1 + \ln \varepsilon^{-1}\right)^{\alpha_d}$$

for some $C_d, \alpha_d > 0$, and study how this translates into bounds on $n^{\mathrm{std}}(\varepsilon, F_d)$. We will show that in this case, we do not lose much when we only allow function evaluations as information.

One of our main results states that if $(F_d)_{d\in\mathbb{N}}$ satisfies

$$n^{\mathrm{all}}(\varepsilon, F_d) \le C\, d^q\, (1+\ln\varepsilon^{-1})^p$$

for some $p, q > 0$ and all $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$, then

$$n^{\mathrm{std}}(\varepsilon, F_d) \le D\, d^q\, (1+\ln d)^p\, (1+\ln\varepsilon^{-1})^p,$$

see Corollary 3. In particular, this improves upon Theorem 26.21 from [23] and solves Open Problem 128 therein. However, let us add that we do not know if the additional $(1+\ln d)^p$ is necessary, although we conjecture that it is.

2. Exponential tractability of approximation

The notions of tractability are defined as follows. Let us fix, for every $d \in \mathbb{N}$, some space $F_d \in \mathcal{A}$. For each $F_d \in \mathcal{A}$ we have some associated set $\mathcal{D}_d$ equipped with a measure μ_d, and a continuous embedding $\mathrm{APP}_d : F_d \to L_2(\mathcal{D}_d, \mu_d)$. The index $d \in \mathbb{N}$ is an arbitrary parameter, but it usually stands for the dimension of the domain $\mathcal{D}_d$. *A multivariate approximation problem* is simply a sequence of embeddings

$$\widetilde{\mathrm{APP}} = \Big(\mathrm{APP}_d : F_d \to L_2(\mathcal{D}_d, \mu_d)\Big)_{d\in\mathbb{N}}.$$

Moreover, tractability notions are defined relative to the considered class of information operations, i.e., we can consider tractability for Λ^{std} or Λ^{all}. Therefore, for $\mathrm{x} \in \{\mathrm{std}, \mathrm{all}\}$, we say that $\widetilde{\mathrm{APP}}$ is

- *exponentially strongly polynomially tractable* (EXP-SPT) for the class Λ^{x} if and only if

$$n^{\mathrm{x}}(\varepsilon, F_d) \le C\, (1+\ln\varepsilon^{-1})^p$$

  for some $C, p > 0$ and for all $d \in \mathbb{N}$ and $\varepsilon \in (0,1)$,
- *exponentially polynomially tractable* (EXP-PT) for the class Λ^{x} if and only if

$$n^{\mathrm{x}}(\varepsilon, F_d) \le C\, d^q\, (1+\ln\varepsilon^{-1})^p$$

  for some $C, p, q > 0$ and for all $d \in \mathbb{N}$ and $\varepsilon \in (0,1)$,
- *exponentially quasi-polynomially tractable* (EXP-QPT) for the class Λ^{x} if and only if

$$n^{\mathrm{x}}(\varepsilon, F_d) \le C\, \exp(t(1+\ln d)(1+\ln(1+\ln\varepsilon^{-1})))$$

  for some $C, t > 0$ and for all $d \in \mathbb{N}$ and $\varepsilon \in (0,1)$,

- *exponentially uniformly weakly tractable* (EXP-UWT) for the class Λ^{x} if and only if for all $\alpha, \beta > 0$ we have
$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n^{\mathrm{x}}(\varepsilon, F_d)}{d^{\alpha} + (1+\ln\varepsilon^{-1})^{\beta}} = 0,$$
- *exponentially weakly tractable (EXP-WT)* for the class Λ^{x} if and only if
$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n^{\mathrm{x}}(\varepsilon, F_d)}{d + (1+\ln\varepsilon^{-1})} = 0.$$

It is easy to see that we have the following logical relation between the tractability notions defined above

$$\text{EXP-SPT} \implies \text{EXP-PT} \implies \text{EXP-QPT} \implies \text{EXP-UWT} \implies \text{EXP-WT}.$$

For a multivariate approximation problem we will prove that *exponential strong polynomial tractability* (EXP-SPT), *exponential polynomial tractability* (EXP-PT), *exponential uniform weak tractability* (EXP-UWT) and *exponential weak tractability* (EXP-WT) for the classes Λ^{all} and Λ^{std} are equivalent. Moreover, *exponential quasi-polynomial tractability* (EXP-QPT) for the class Λ^{all} implies *exponential uniform weak tractability* (EXP-UWT) for the class Λ^{std}, i.e, the next tractability notion in the tractability hierarchy considered here. Whether the equivalence of *exponential quasi-polynomial tractability* (EXP-QPT) for the classes Λ^{all} and Λ^{std} holds remains an open problem.

These equivalences are in sharp contrast to the results for *algebraic tractability*, i.e., when the error depends polynomially on ε^{-1}, instead of $\ln\varepsilon^{-1}$. See, e.g., [8, 24, 26] for examples where the problem is algebraically tractable for Λ^{all} but the curse of dimensionality holds for Λ^{std}. In particular, [24, Example 5] shows that for the tensor product $W^s_{2,d}$ of certain univariate periodic Sobolev spaces, $s > 1/2$, we have QPT for Λ^{all}, but the curse of dimensionality for Λ^{std}.

Let us also add here that we do not know many non-Hilbert examples $(F_d)_{d\in\mathbb{N}}$ where we know the precise behavior of $n^{\mathrm{std}}(\varepsilon, F_d)$, while $n^{\mathrm{all}}(\varepsilon, F_d)$ is more often known. One exception are the classical smoothness spaces $\mathcal{C}^k_d := C^k([0,1]^d)$ of k-times differentiable functions on the d-dimensional unit cube. For these classes, the complexity of the integration problem and the L_∞-approximation problem using standard information is analyzed in [9, 10, 11, 14]. The complexity of L_2-approximation lies in between these two complexities.

3. Results

We now present our results. The first results are concerned with EXP-(S)PT and EXP-QPT. Both are direct corollaries of the following theorem.

Theorem 2. *Assume that $F \in \mathcal{A}$ satisfies*

$$n^{\text{all}}(\varepsilon, F) \;\leq\; A\left(1+\ln\varepsilon^{-1}\right)^{B}$$

for some $B > 0$ and $A \geq 1$ and all $\varepsilon \in (0,1)$. Then

$$n^{\text{std}}(\varepsilon, F) \;\leq\; C\left(1+\ln\varepsilon^{-1}\right)^{B}$$

for all $\varepsilon \in (0,1)$, where

$$C = c\,A\,(c + c\,B^3 + \ln A)^B$$

and $c > 0$ is an absolute constant.

Proof. Observe that the inequality

$$n^{\text{all}}(\varepsilon, F) \leq A\Big(1+\ln\varepsilon^{-1}\Big)^{B}$$

implies that

$$c_n(F, L_2) \leq e\;\exp(-(n/A)^{1/B}),$$

hence from inequalities (1) it follows that for all $n \geq 1$

$$a_n(F, L_2) \leq 2\,e\,n^{1/2}\;\exp(-(n/A)^{1/B}).$$

Applying first Lemma 6 and then Lemma 7 from the Appendix, we deduce that

$$\sum_{k\geq n} a_k(F, L_2) \leq 2\,e \sum_{k\geq n} k^{1/2}\exp(-(k/A)^{1/B})$$
$$\leq 6\,A^{1/B}\,B\;\max(3B/2,1)\,(n-1)^{3/2-1/B}\;\exp(-((n-1)/A)^{1/B})$$

for all $n \geq A\;\max(3B/2,1)^B + 1$. In particular, $(a_n(F, L_2)) \in \ell_1$. It follows from Theorem 1 that there exist absolute constants $b \in \mathbb{N}$ and $c_1 \geq 1$ such that

$$e_{bn}(F, L_2) \leq c_1\,\frac{\sqrt{\ln n}}{n} \sum_{k\geq n} a_k(F, L_2) \leq c_1\,(n-1)^{-1/2} \sum_{k\geq n} a_k(F, L_2)$$
$$\leq 6\,c_1\,A^{1/B}\,B\;\max(3B/2,1)\,(n-1)^{1-1/B}\;\exp(-((n-1)/A)^{1/B})$$

for all $n \geq n_0(A,B) := A\;\max(3B/2,1)^B + 1$.

In the case $B \leq 1$ we have $(n-1)^{1-1/B} \leq A^{1-1/B}$, and thus

$$e_{bn}(F, L_2) \;\leq\; 9\,c_1 A\;\exp(-((n-1)/A)^{1/B}).$$

If $B > 1$, then Lemma 8 with $u = 1 - 1/B$ yields for any $\delta \in (0, \frac{1}{B-1})$ that

$$e_{bn}(F, L_2) \leq 9\,c_1\,A^{1/B}\,B^2\,\delta^{1-B}\;\exp\Big(((B-1)\delta - 1)((n-1)/A)^{1/B}\Big)$$

and taking $\delta = 1/B$ yields

$$e_{bn}(F_d, L_2) \;\leq\; 9\,c_1 A^{1/B} B^{B+1} \exp\left(-\frac{1}{B}\left(\frac{n-1}{A}\right)^{1/B}\right).$$

If we put $B_0 := \max\{B,1\}$, we have for all $B>0$ and $n \ge n_0(A,B)$ the bound

$$e_{bn}(F_d, L_2) \;\le\; 9\, c_1 A^{1/B_0} B_0^{B+1} \exp\left(-\frac{1}{B_0}\left(\frac{n-1}{A}\right)^{1/B}\right)$$

which is smaller than ε if

$$n \;\ge\; A\, B_0^B \left(\ln\left(9\, c_1 A^{1/B_0} B_0^{B+1} \varepsilon^{-1}\right)\right)^B + 1.$$

Thus

$$\begin{aligned} n^{\rm std}(\varepsilon, F_d) &\le b \max\left\{A\, B_0^B \left(\ln\left(9\, c_1 A^{1/B_0} B_0^{B+1} \varepsilon^{-1}\right)\right)^B + 2,\; n_0(A,B)\right\}, \\ &\le 3b \max\left\{A\, B_0^B R^B \left(1+\ln(\varepsilon^{-1})\right)^B,\; A\,(2B_0)^B\right\} \end{aligned}$$

with

$$R \;:=\; \ln(9\, c_1) + \frac{\ln A}{B_0} + (B+1)\ln B_0 \;\le\; \frac{\ln A}{B_0} + cB_0^2$$

which gives the desired estimate.

□

Corollary 3. *Assume that $F_d \in \mathcal{A}$ for every $d \in \mathbb{N}$ and*

$$n^{\rm all}(\varepsilon, F_d) \;\le\; c\, d^q \,(1+\ln\varepsilon^{-1})^p$$

for some $p, c > 0$, $q \ge 0$, and all $\varepsilon \in (0,1)$. Then

$$n^{\rm std}(\varepsilon, F_d) \;\le\; C\, d^q \,(1+\ln d)^p\,(1+\ln\varepsilon^{-1})^p$$

for all $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$, and some $C > 0$ that depends only on c, p and q.

In particular, if $\widetilde{\rm APP}$ is exponentially (strongly) polynomially tractable for the class $\Lambda^{\rm all}$ then it is exponentially (strongly) polynomially tractable for $\Lambda^{\rm std}$.

Proof. We use Theorem 2 with $A = c\, d^q + 1$ and $B = p$.

□

We now turn to the assumption that $\widetilde{\rm APP}$ is exponentially quasi-polynomially tractable for the class $\Lambda^{\rm all}$. This is the only case where we do not know if it implies the same property for $\Lambda^{\rm std}$.

For convenience, let us write $\ln_+(x) := 1 + \ln(x)$.

Corollary 4. *Assume that $F_d \in \mathcal{A}$ for every $d \in \mathbb{N}$ and*

$$n^{\rm all}(\varepsilon, F_d) \;\le\; c\, \exp\left(t \cdot \ln_+ d \cdot \ln_+ \ln_+ \varepsilon^{-1}\right)$$

for some $c, t > 0$ and all $\varepsilon \in (0,1)$. Then

$$n^{\rm std}(\varepsilon, F_d) \;\le\; c\, \exp\left(t \cdot \ln_+ d \cdot \left(\ln_+ \ln_+ \varepsilon^{-1} \;+\; 3\ln\big(t\, \ln_+ d\big) \;+\; C\right)\right)$$

for all $\varepsilon \in (0,1)$ and $d > (e + \frac{1}{c})^{1/t} e^{-1}$, and some $C > 0$ that depends only on c.

In particular, if $\widetilde{\mathrm{APP}}$ is exponentially quasi-polynomially tractable for the class Λ^{all}, then it is exponentially uniformly weakly tractable for the class Λ^{std}.

Proof. Note that

$$c\,\exp\Big(t\,(1+\ln d)\,(1+\ln(1+\ln\varepsilon^{-1}))\Big) \;=\; c\,e^t\,d^t\,\Big(1+\ln\varepsilon^{-1}\Big)^{t(1+\ln d)}.$$

Hence, we can apply Theorem 2 with $A = c\,e^t\,d^t$ and $B = t\,\ln_+ d$, i.e., $A = c\,e^B$. Note that $A, B \geq 1$ for $d > (e+\frac{1}{c})^{1/t}e^{-1}$. We obtain that there exists an absolute constant $c' > 0$ such that

$$n^{\mathrm{std}}(\varepsilon, F) \;\leq\; C\,\Big(1+\ln\varepsilon^{-1}\Big)^B$$

for all $\varepsilon \in (0,1)$, with

$$\begin{aligned} C &= c'\,A\,(c'+c'\,B^3+\ln A)^B = c'\,A\,(c'+c'\,B^3+B+\ln(c))^B \\ &\leq (c'')^B\,A\,B^{3B} = c\,\exp\Big(B\Big(\ln_+(c'')+3\ln(B)\Big)\Big), \end{aligned}$$

where $c'' > 0$ only depends on c. This proves the bound.

Now, since $\ln n^{\mathrm{std}}(\varepsilon, F_d)$ depends only logarithmically on d and double-logarithmically on ε^{-1}, we obtain

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n^{\mathrm{std}}(\varepsilon, F_d)}{d^{\,\alpha} + (1+\ln\varepsilon^{-1})^{\beta}} = 0$$

for all $\alpha, \beta > 0$, i.e., $\widetilde{\mathrm{APP}}$ is exponentially uniformly weakly tractable for the class Λ^{std}.

□

We finally discuss EXP-UWT and EXP-WT.

Theorem 5. *Assume that $F_d \in \mathcal{A}$ for every $d \in \mathbb{N}$. If the problem $\widetilde{\mathrm{APP}}$ is exponentially (uniformly) weakly tractable for the class Λ^{all}, then it is exponentially (uniformly) weakly tractable for the class Λ^{std}.*

Proof. Assume that there are $0 < \alpha, \beta \leq 1$ such that

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n^{\mathrm{all}}(\varepsilon, F_d)}{d^{\alpha} + (1+\ln\varepsilon^{-1})^{\beta}} = 0.$$

It is enough to show that

$$\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln n^{\mathrm{std}}(\varepsilon, F_d)}{d^{\alpha} + (1+\ln\varepsilon^{-1})^{\beta}} = 0. \tag{2}$$

By assumption, for every $0 < h \leq 1/16$, there is some $v_0 \in \mathbb{N}$ such that

$$0 \leq \frac{\ln n^{\mathrm{all}}(\varepsilon, F_d)}{d^{\alpha} + (1+\ln\varepsilon^{-1})^{\beta}} \leq h$$

for all $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$ with $d^\alpha + (1+\ln \varepsilon^{-1})^\beta \geq v_0$. It follows that

$$c_n(F_d, L_2) \leq e \exp\left(- \left(\frac{\ln n}{h} - d^\alpha \right)^{1/\beta} \right)$$

for all $n \geq \exp(hv_0)$. From inequalities (1), we get

$$a_n(F_d, L_2) \leq 2e \exp\left(- \left(\left(\frac{\ln n}{h} - d^\alpha \right)^{1/\beta} - \frac{1}{2} \ln n \right) \right).$$

For all $n \geq \exp(2hd^\alpha)$ and $h \leq 1/16$, we have

$$\left(\frac{\ln n}{h} - d^\alpha \right)^{1/\beta} - \frac{1}{2} \ln n \geq \frac{1}{2} \left(\frac{\ln n}{h} - d^\alpha \right)^{1/\beta} + \frac{1}{8} \frac{\ln n}{h}$$

and hence we have for all $n \geq \max\{\exp(hv_0), \exp(2hd^\alpha)\}$ that

$$a_n(F_d, L_2) \leq 2e \exp\left(-\frac{1}{2} \left(\frac{\ln n}{h} - d^\alpha \right)^{1/\beta} \right) \cdot n^{-1/(8h)}.$$

It follows from Theorem 1 that for some absolute constants $b \in \mathbb{N}$ and $c_1 > 0$ and all $n \geq \max\{\exp(hv_0), \exp(2hd^\alpha)\}$, we have

$$\begin{aligned} e_{bn}(F_d, L_2) &\leq c_1 \frac{\sqrt{\ln n}}{n} \sum_{k \geq n} a_k(F_d, L_2) \leq c_1 \sum_{k \geq n} a_k(F_d, L_2) \\ &\leq 2e \exp\left(-\frac{1}{2} \left(\frac{\ln n}{h} - d^\alpha \right)^{1/\beta} \right) \sum_{k \geq n} k^{-1/(8h)} \\ &\leq 2e \exp\left(-\frac{1}{2} \left(\frac{\ln n}{h} - d^\alpha \right)^{1/\beta} \right), \end{aligned}$$

where we again used that $h \leq 1/16$. It follows that

$$n^{\text{std}}(\varepsilon, F_d) \leq D \exp\left(4h \left((1+\ln \varepsilon^{-1})^\beta + d^\alpha \right) \right)$$

for some absolute constant $D > 0$ and all $\varepsilon \in (0,1)$ and $d \in \mathbb{N}$ such that $d + \varepsilon^{-1}$ is sufficiently large. This implies

$$0 \leq \lim_{d+\varepsilon^{-1} \to \infty} \frac{\ln n^{\text{std}}(\varepsilon, F_d)}{d^\alpha + (1+\ln \varepsilon^{-1})^\beta} \leq 4h.$$

Since $h \in (0, 1/16)$ can be chosen arbitrarily close to 0, we obtain (2).
This allows us to conclude our thesis. Indeed, for uniform weak tractability we take arbitrary α and β from $(0,1)$, and for weak tractability we take $\alpha = \beta = 1$. $\square$

4. Appendix: technical lemmas

The following lemmas are used in the proofs of our results.

Lemma 6. *Let A and B be arbitrary positive real numbers. For $n \geq A(B/2)^B$ we have the following inequality*

$$\sum_{k\geq n+1} k^{1/2}\exp(-(k/A)^{1/B}) \leq \int_n^\infty t^{1/2}\exp(-(t/A)^{1/B})dt.$$

Proof. It is enough to show that the function $f:(0,\infty)\to\mathbb{R}$ given by $f(t) = t^{1/2}\exp(-(t/A)^{1/B})$ is decreasing on $(A(B/2)^B,\infty)$. Indeed, for $t > A(B/2)^B$ we have

$$f'(t) = \left(\exp\left(\frac{1}{2}\ln(t) - (t/A)^{1/B}\right)\right)' =$$
$$= \exp\left(\frac{1}{2}\ln(t) - (t/A)^{1/B}\right)\left(\frac{1}{2t} - (1/AB)(t/A)^{1/B-1}\right) < 0.$$

□

Lemma 7. *Let A and B be arbitrary positive real numbers. For every $n \geq A\max(3B/2,1)^B$ we have the following inequality*

$$\int_n^\infty t^{1/2}\exp(-(t/A)^{1/B})dt \;\leq\; A^{1/B}\,B\,\max(3B/2,1)\,n^{3/2-1/B}\,\exp(-(n/A)^{1/B}).$$

Proof. Using integration by substitution, with $u = (t/A)^{1/B}$, we obtain that

$$\int_n^\infty t^{1/2}\exp(-(t/A)^{1/B})dt \;=\; A^{3/2}\,B\,\Gamma(3B/2,(n/A)^{1/B})$$

where, for $a\in\mathbb{R}$ and $x>0$, $\Gamma(a,x) = \int_x^\infty v^{a-1}\exp(-v)dv$ is the incomplete gamma function.

It is known (see, e.g., Satz 4.4.3 in [6]) that for $a \geq 1$ and $x > a$ we have

$$\Gamma(a,x) \leq a\,x^{a-1}\,\exp(-x).$$

If, on the other hand, $0 < a < 1$ and $x > 1$ then since $v^{a-1} \leq x^{a-1}$ for $v \geq x$ we have

$$\Gamma(a,x) = \int_x^\infty v^{a-1}\exp(-v)dv \leq x^{a-1}\int_x^\infty \exp(-v)dv = x^{a-1}\exp(-x)$$

Therefore, for every $a > 0$ and $x > \max(a,1)$, the following bound holds

$$\Gamma(a,x) \;\leq\; \max(a,1)\,x^{a-1}\,\exp(-x).$$

Thus for $n > A\max(3B/2,1)^B$, and taking $a = 3B/2$ and $x = (n/A)^{1/B}$, we have

$$\int_n^\infty t^{1/2}\exp(-(t/A)^{1/B})dt \;\leq\; A^{1/B}\,B\,\max(3B/2,1)\,n^{3/2-1/B}\,\exp(-(n/A)^{1/B}).$$

□

Lemma 8. *For every $A, B, n, \delta, u > 0$ we have the following inequality*

$$n^u \exp\left(-(n/A)^{1/B}\right) \leq A^u \delta^{-uB} \exp\left((uB\delta - 1)(n/A)^{1/B}\right).$$

Proof. Let $x = \delta(n/A)^{1/B}$. Then $n^u = A^u \delta^{-uB} x^{uB}$. Using the fact that $\ln(x) \leq x$ for all $x > 0$ we obtain that

$$\begin{aligned} \ln(n^u) &= \ln(A^u\delta^{-uB}) + uB\ln(x) \leq \\ &\leq \ln(A^u\delta^{-uB}) + uBx = \ln(A^u\delta^{-uB}) + uB\delta(n/A)^{1/B} \end{aligned}$$

Hence, taking exponentials of both sides we derive that

$$n^u \leq A^u\delta^{-uB} \exp(uB\delta(n/A)^{1/B})$$

and thus

$$n^u \exp(-(n/A)^{1/B}) \leq A^u\delta^{-uB} \exp((uB\delta - 1)(n/A)^{1/B})$$

as claimed. □

References

[1] J. Creutzig and P. Wojtaszczyk. Linear vs. nonlinear algorithms for linear problems. *J. Complexity*, 20:807–820, 2004.

[2] J. Dick, P. Kritzer, F. Pillichshammer, and H. Woźniakowski. Approximation of analytic functions in Korobov spaces. *Journal of Complexity*, 30(2):2–28, 2014.

[3] D. Dũng, N. Temlyakov V. and T. Ullrich. *Hyperbolic Cross Approximation*. Advanced Courses in Mathematics - CRM Barcelona, Springer International Publishing, 2018.

[4] A. Ebert, P. Kritzer, and F. Pillichshammer. Tractability of approximation in the weighted Korobov space in the worst-case setting. *arXiv:2201.09940*, 2022.

[5] A. Ebert and F. Pillichshammer. Tractability of approximation in the weighted Korobov space in the worst-case setting – a complete picture. *Journal of Complexity*, 67:101571, 2021.

[6] W. Gabcke. *Neue Herleitung und explizite Restabschätzung der Riemann-Siegel-Formel*. PhD thesis, University of Göttingen, 1979.

[7] A. Hinrichs, D. Krieg, E. Novak, J. Prochno, and M. Ullrich. Random sections of ellipsoids and the power of random information. *Transactions of the American Mathematical Society*, 374(12):8691 – 8713, 2021.

[8] A. Hinrichs, D. Krieg, E. Novak, and J. Vybíral. Lower bounds for the error of quadrature formulas for hilbert spaces. *Journal of Complexity*, 65:101544, 2021.

[9] A. Hinrichs, E. Novak, M. Ullrich, and H. Woźniakowski. The curse of dimensionality for numerical integration of smooth functions. *Math. Comp.*, 83(290):2853 – 2863, 2014.

[10] A. Hinrichs, E. Novak, M. Ullrich, and H. Woźniakowski. Product rules are optimal for numerical integration in classical smoothness spaces. *J. Complexity*, 38:39 – 49, 2017.

[11] A. Hinrichs, J. Prochno, and M. Ullrich. The curse of dimensionality for numerical integration on general domains. *Journal of Complexity*, 50:25 – 42, 2019.

[12] C. Irrgeher, P. Kritzer, F. Pillichshammer, and H. Woźniakowski. Approximation in Hermite spaces of smooth functions. *Journal of Approximation Theory*, 207:98 – 126, 2016.

[13] C. Irrgeher, P. Kritzer, F. Pillichshammer, and H. Woźniakowski. Tractability of multivariate approximation defined over hilbert spaces with exponential weights. *Journal of Approximation Theory*, 207:301–338, 2016.

[14] D. Krieg. Uniform recovery of high-dimensional C^r-functions. *Journal of Complexity*, 50:116–126, 2019.
[15] D. Krieg and M. Ullrich. Function values are enough for L_2-approximation. *Found. Comput. Math.*, 21(4):1141 – 1151, 2021.
[16] D. Krieg and M. Ullrich. Function values are enough for L_2-approximation: Part II. *J. Complexity*, 66:101569, 2021.
[17] P. Kritzer and H. Woźniakowski. Simple characterizations of exponential tractability for linear multivariate problems. *Journal of Complexity*, 51:110–128, 2019.
[18] T. Kühn, W. Sickel, and T. Ullrich. Approximation of mixed order Sobolev functions on the d-torus: Asymptotics, preasymptotics, and d-dependence. *Constr. Approx.*, 42:353–398, 2015.
[19] P. Mathé. s-numbers in information-based complexity. *J. Complexity*, 6(1):41–66, 1990.
[20] N. Nagel, M. Schäfer, and T. Ullrich. A new upper bound for sampling numbers. *CoRR*, abs/2010.00327, 2020.
[21] E. Novak and H. Woźniakowski. *Tractability of multivariate problems. Vol. 1: Linear information*, volume 6 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2008.
[22] E. Novak and H. Woźniakowski. *Tractability of multivariate problems. Volume II: Standard information for functionals*, volume 12 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2010.
[23] E. Novak and H. Woźniakowski. *Tractability of multivariate problems. Volume III: Standard information for operators*, volume 18 of *EMS Tracts in Mathematics*. European Mathematical Society (EMS), Zürich, 2012.
[24] E. Novak and H. Woźniakowski. Tractability of multivariate problems for standard and linear information in the worst case setting: part i. *J. Approx. Theory*, 207:177–192, 2016.
[25] A. Pietsch. *Operator ideals*. Deutscher Verlag, Wiss., Berlin, 1978.
[26] J. Vybíral. A variant of schur's product theorem and its applications. *Advances in Mathematics*, 368:107140, 2020.
[27] V. N. Temlyakov. *Approximation of periodic functions*. Computational Mathematics and Analysis Series, Nova Science Publishers, Inc., Commack, NY, 1993.
[28] V. N. Temlyakov. *Multivariate Approximation*, volume 32 of *Cambridge Monographs on Applied and Computational Mathematics*. Cambridge University Press, 2018.
[29] M. Ullrich. On the worst-case error of least squares algorithms for L_2-approximation with high probability. *J. Complexity*, 60, 2020.
[30] J. Zhang. A note on EC-tractability of multivariate approximation in weighted Korobov spaces for the standard information class. *Journal of Complexity*, 67, 2021.