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RICAM-Report 2021-41

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AN INVERSE SOURCE PROBLEM FOR VECTOR FIELD IN ABSORBING AND SCATTERING MEDIUM

DAVID OMOGBHE AND KAMRAN SADIQ

ABSTRACT. We consider in a two dimensional absorbing and scattering medium, an inverse source problem in the stationary radiative transport, where the source is generated by a vector field. The medium has an anisotropic scattering property that is neither negligible nor large enough for the diffusion approximation to hold. The attenuating and scattering properties of the medium are assumed known and the unknown vector field source is isotropic. For scattering kernels of finite Fourier content in the angular variable, we show how to recover the isotropic vector field sources from boundary measurements. The approach is based on the Cauchy problem for a Beltrami-like equation associated with A-analytic maps in the sense of Bukhgeim. As an application, we determine necessary and sufficient conditions for the attenuated and scattering 1-tensor field data and 0-tensor field data to be mistaken for each other.

1. INTRODUCTION

In this work, we consider an inverse source problem for stationary radiative transfer (transport) [8, 9], in a two-dimensional bounded, strictly convex domain $\Omega \subset \mathbb{R}^2$, with boundary Γ. The stationary radiative transport models the linear transport of particles through a medium and includes absorption and scattering phenomena. In the steady state case, when generated solely by a source $f = \boldsymbol{\theta} \cdot \mathbf{F}$ inside Ω, the density $u(z, \boldsymbol{\theta})$ of particles at z traveling in the direction $\boldsymbol{\theta}$ solves the stationary radiative transport boundary value problem

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) + a(z, \boldsymbol{\theta}) u(z, \boldsymbol{\theta}) - \int_{\mathbf{S}^1} k(z, \boldsymbol{\theta}, \boldsymbol{\theta}') u(z, \boldsymbol{\theta}') d\boldsymbol{\theta}' = \underbrace{\boldsymbol{\theta} \cdot \mathbf{F}(z)}_{f(z,\boldsymbol{\theta})}, \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1, \\ &u|_{\Gamma_-} = 0, \end{aligned} \tag{1}$$

where the function $a(z, \boldsymbol{\theta})$ is the medium capability of absorption per unit path-length at z moving in the direction $\boldsymbol{\theta}$ called the attenuation coefficient, function $k(z, \boldsymbol{\theta}, \boldsymbol{\theta}')$ is the scattering coefficient which accounts for particles from an arbitrary direction $\boldsymbol{\theta}'$ which scatter in the direction $\boldsymbol{\theta}$ at a point z, and $\varGamma_- := \{(\zeta, \boldsymbol{\theta}) \in \varGamma \times \mathbf{S}^1 : \nu(\zeta) \cdot \boldsymbol{\theta} < 0\}$ is the incoming unit tangent sub-bundle of the boundary, with $\nu(\zeta)$ being the outer unit normal at $\zeta \in \varGamma$. The attenuation and scattering coefficients are assumed known real valued functions. The boundary condition in (1) indicates that no radiation is coming from outside the domain. Throughout, the measure $d\boldsymbol{\theta}$ on the unit sphere $\mathbf{S}^1$ is normalized to $\int_{\mathbf{S}^1} d\boldsymbol{\theta} = 1$.

Under various "subcritical" assumptions, e.g., [12, 10, 11, 1, 30], the (forward) boundary value problem (1) is known to have a unique solution, with a general result in [49] showing that, for an open and dense set of coefficients $a \in C^2(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, the boundary value problem (1) has a unique solution $u \in L^2(\Omega \times \mathbf{S}^1)$ for any $f \in L^2(\Omega \times \mathbf{S}^1)$. In [21], it is shown that without any subcritical condition, for attenuation merely *once* differentiable, $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$ and

Date: December 1, 2021.

2010 *Mathematics Subject Classification.* Primary 35J56, 30E20; Secondary 35R30, 45E05.

Key words and phrases. Radiative transport, source reconstruction, scattering, A-analytic maps, Hilbert transform, Bukhgeim-Beltrami equation, Dopplar tomography, vector tomography.

$k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, the boundary value problem (1) has a unique solution $u \in L^p(\Omega \times \mathbf{S}^1)$ for any $f \in L^p(\Omega \times \mathbf{S}^1)$, $p > 1$. Moreover, uniqueness result of the forward problem (1) are also establish in weighted L^p spaces in [16]. In our reconstruction method here, some of our arguments require solutions $u \in C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$, $\frac{1}{2} < \mu < 1$. We revisit the arguments in [49, 21] and show that such a regularity can be achieved for sources $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, $p > 4$; see Theorem 2.2 (iii) below.

For a given medium, i.e., a and k both known, we consider the inverse problem of determining the vector field $\mathbf{F}$ from measurements g of exiting radiation on Γ,

$$u|_{\Gamma_+} = g, \tag{2}$$

where $\Gamma_+ := \{(z, \boldsymbol{\theta}) \in \Gamma \times \mathbf{S}^1 : \nu(z) \cdot \boldsymbol{\theta} > 0\}$ is the outgoing unit tangent sub-bundle of the boundary, with $\nu(\zeta)$ being the outer unit normal at $\zeta \in \Gamma$.

For non-absorbing non-scattering media, the inversion problem has been investigated in [45] in the general framework of Riemannian geometry, and in [48] in the Euclidean setting. The problem does not have a unique solution: the superposition of any compactly supported gradient field is indistinguishable from the data. However, the solenoidal part of the field is uniquely determined by the traces of the field on the boundary.

For the absorbing non-scattering media in planar domains, interesting enough, the full vector field can be recovered in the regions of positive absorption as shown in [25, 3, 50, 41], without using some additional data information [44, 13, 29, 24]. It is due to a surprising fact that the two-dimensional attenuated Doppler transform with positive attenuation is injective while the non-attenuated Doppler transform is not. Moreover, for the absorbing and scattering media in planar domains, we show in here that the full vector field can be recovered in the subcritical regions, see (47) below and [32].

In the absorbing non-scattering medium, this is the mathematical formulation of Doppler Tomography [36, 34]. The problem has been studied in various settings by multiple authors, e.g. see [48, 26, 3, 35, 41, 23, 43, 39, 14, 45, 31, 27, 32, 29] and the references therein. Such a problem appears in ultrasound tomography [48, 43], occurs in the investigation to detect tumors using blood flow measurements [51], in the investigation of velocity distribution in a flow [6], and also in non-invasive industrial measurements for reconstructing the velocity of a moving fluid [37, 38].

Similar inverse source problems in radiative transport for 0-tensor sources has been well studied, e.g., [28, 5, 49, 16, 18, 47, 20] in Euclidean domains, and [46, 32] in refractive media (Riemannian domains). The numerical reconstruction results for the inverse source problems for 0-tensor has recently been provided in [47, 18, 21]. When full boundary data is available, the numerical source reconstruction has been given in [18, 19], and in the partial data case, the numerical source reconstruction result are provided in [47, 21].

In Section 2, we remark on the existence and regularity of the forward boundary value problem. The results in Section 2 consider both attenuation coefficient and scattering kernel in general setting.

In this work, except for the results in Section 2, the attenuation coefficient are assumed isotropic $a = a(z)$, and that the scattering kernel $k(z, \boldsymbol{\theta}, \boldsymbol{\theta}') = k(z, \boldsymbol{\theta} \cdot \boldsymbol{\theta}')$ depends polynomially on the angle between the directions. Moreover, the functions a, k and vector field $\mathbf{F}$ are assumed real valued.

In Section 3, we recall some basic properties of A-analytic theory, and in Section 4 we provide the reconstruction method for the vector field. Our approach is based on the Cauchy problem for a Beltrami-like equation associated with A-analytic maps in the sense of Bukhgeim [7]. The A-analytic approach developed in [7] treats the non-attenuating case, and the absorbing but non-scattering case is treated in [2]. The original idea of Bukhgeim from the absorbing non-scattering media [7, 2] to the absorbing and scattering media has been extended in [18, 21]. In here we extend the results in [18, 21] for 0-tensor to 1-tensor. Our main result, Theorem 4.1, shows that $u|_{\Gamma_+}$ determines the vector field

$\mathbf{F}$, and provides a method of reconstruction. One of the main crux in our reconstruction method is the observation that any finite Fourier content in the angular variable of the scattering kernel splits the problem into an infinite system of non-scattering case and a boundary value problem for a finite elliptic system. The role of the finite Fourier content has been independently recognized in [18] and [32].

As an application, the method used in the reconstruction will explain when (and only then) the attenuated and scattering 0-tensor field data and attenuated and scattering 1-tensor field data, can be confounded for each other, see Section 5.

2. Some remarks on the existence and regularity of the forward problem

In this section, we revisit the arguments in [49, 21], and remark on the well posedness in $L^p(\Omega \times \mathbf{S}^1)$ of the boundary value problem (1). Adopting the notation in [49, 21], we consider the operators

$$[T_1^{-1}g](x,\boldsymbol{\theta}) = \int_{-\infty}^{0} e^{-\int_s^0 a(x+t\boldsymbol{\theta},\boldsymbol{\theta})dt} g(x+s\boldsymbol{\theta},\boldsymbol{\theta})ds, \text{ and } [Kg](x,\boldsymbol{\theta}) = \int_{\mathbf{S}^1} k(x,\boldsymbol{\theta},\boldsymbol{\theta}')g(x,\boldsymbol{\theta}')d\boldsymbol{\theta}',$$

where the intervening functions are extended by 0 outside Ω.

Using the above operators, the problem (1) can be rewritten as

$$(I - T_1^{-1}K)u = T_1^{-1}f, \qquad u|_{\Gamma_-} = 0. \tag{3}$$

If the operator $I - T_1^{-1}K$ is invertible, then the problem (3) is uniquely solvable, and has the form $u = (I - T_1^{-1}K)^{-1}T_1^{-1}f$. By using the formal expansion

$$u = T_1^{-1}f + T_1^{-1}KT_1^{-1}f + T_1^{-1}(KT_1^{-1}K)[I - T_1^{-1}K]^{-1}T_1^{-1}f, \tag{4}$$

the well posed-ness in $L^p(\Omega \times \mathbf{S}^1)$ of the (forward) boundary value problem (1) is reduced to showing that the operator $I - T_1^{-1}K$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$.

We recall some results in [21].

Proposition 2.1. [21, Proposition 2.1] *Let $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$. Then the operator*

$$KT_1^{-1}K : L^p(\Omega \times \mathbf{S}^1) \longrightarrow W^{1,p}(\Omega \times \mathbf{S}^1) \textit{ is bounded}, \ 1 < p < \infty. \tag{5}$$

The following simple result is useful.

Lemma 2.1. [21, Lemma 2.2] *Let X be a Banach space and $A : X \to X$ be bounded. Then $I \pm A$ have bounded inverses in X, if and only if $I - A^2$ has a bounded inverse in X.*

For $\lambda \in \mathbb{C}$, we note that $(T_1^{-1}(\lambda K))^2 = \lambda^2 T_1^{-1}(KT_1^{-1}K)$. By Proposition 2.1, the operator $(T_1^{-1}(\lambda K))^2$ is compact for any $\lambda \in \mathbb{C}$. By Lemma 2.1, if the operator $I - (T_1^{-1}(\lambda K))^2$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$, then the operator $I - T_1^{-1}(\lambda K)$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$. Since $I - (T_1^{-1}(\lambda K))^2$ is invertible for λ in a neighborhood of 0, an application of the analytic Fredholm alternative in Banach spaces, e.g., [15, Theorem VII.4.5], yields the following result.

Theorem 2.1. [21, Theorem 2.1] *Let $p > 1$, $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$, and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$. At least one of the following statements is true.*
(i) $I - T_1^{-1}K$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$.
(ii) there exists $\epsilon > 0$ such that $I - T_1^{-1}(\lambda K)$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$, for any $0 < |\lambda - 1| < \epsilon$.

The regularity of the solution u of (1) increases with the regularity of f as follows.

Theorem 2.2. *Consider the boundary value problem* (1) *with $a \in C^3(\overline{\Omega} \times \mathbf{S}^1)$. For $p > 1$, let $k \in C^3(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$ be such that $I - T_1^{-1}K$ is invertible in $L^p(\Omega \times \mathbf{S}^1)$, and let $u \in L^p(\Omega \times \mathbf{S}^1)$ in* (4) *be the solution of* (1).

(i) If $f \in W^{1,p}(\Omega \times \mathbf{S}^1)$*, then* $u \in W^{1,p}(\Omega \times \mathbf{S}^1)$*.*
(ii) If $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$*, then* $u \in W^{2,p}(\Omega \times \mathbf{S}^1)$*.*
(iii) If $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$*, then* $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$*.*

Proof. (i) We consider the regularity of the solution u of (1) term by term as in (4):

$$u = T_1^{-1}f + T_1^{-1}KT_1^{-1}f + T_1^{-1}[KT_1^{-1}K](I - T_1^{-1}K)^{-1}T_1^{-1}f.$$

It is easy to see that the operator T_1^{-1} preserve the space $W^{1,p}(\Omega \times \mathbf{S}^1)$, and also the operator K preserve the space $W^{1,p}(\Omega \times \mathbf{S}^1)$, so that the first two terms, $T_1^{-1}f$ and $T_1^{-1}KT_1^{-1}f$, both belong to $W^{1,p}(\Omega \times \mathbf{S}^1)$. Moreover, $(I - T_1^{-1}K)^{-1}T_1^{-1}f \in L^p(\Omega \times \mathbf{S}^1)$, and now, by using Proposition 2.1, the last term is also belong in $W^{1,p}(\Omega \times \mathbf{S}^1)$.

(ii) We define the following operators

$$(6)\qquad \begin{aligned} T_0^{-1}u(x,\boldsymbol{\theta}) &:= \int_{-\infty}^{0} u(x+t\boldsymbol{\theta},\boldsymbol{\theta})dt, & K_{\xi_j}u(x,\boldsymbol{\theta}) &:= \int_{\mathbf{S}^1} \frac{\partial k}{\partial \xi_j}(x,\boldsymbol{\theta},\boldsymbol{\theta}')u(x,\boldsymbol{\theta}')d\boldsymbol{\theta}', \\ \widetilde{T}_{0,j}^{-1}u(x,\boldsymbol{\theta}) &:= \int_{-\infty}^{0} u(x+t\boldsymbol{\theta},\boldsymbol{\theta})t^j dt, & K_{\eta_i\xi_j}u(x,\boldsymbol{\theta}) &:= \int_{\mathbf{S}^1} \frac{\partial^2 k}{\partial \eta_i \partial \xi_j}(x,\boldsymbol{\theta},\boldsymbol{\theta}')u(x,\boldsymbol{\theta}')d\boldsymbol{\theta}', \end{aligned}$$

where $\eta_i = \{x_i, \theta_i\}$ and $\xi_j = \{x_j, \theta_j\}$ for $i, j = 1, 2$.
It is easy to see that $T_0^{-1}, \widetilde{T}_{0,j}^{-1}, K_{\xi_j}$ and $K_{\eta_i\xi_j}$ preserve $W^{1,p}(\Omega \times \mathbf{S}^1)$.

By evaluating the radiative transport equation in (1) at $x + t\boldsymbol{\theta}$ and integrating in t from $-\infty$ to 0, the boundary value problem (1) with zero incoming fluxes is equivalent to the integral equation:

$$(7)\qquad u + T_0^{-1}(au) - T_0^{-1}Ku = T_0^{-1}f.$$

According to part (i), for $f \in W^{1,p}(\Omega \times \mathbf{S}^1)$, the solution $u \in W^{1,p}(\Omega \times \mathbf{S}^1)$, and so $u_{x_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular u_{x_j} solves the integral equation:

$$(8)\qquad u_{x_j} + T_0^{-1}(au_{x_j}) - T_0^{-1}Ku_{x_j} = -T_0^{-1}(a_{x_j}u) + T_0^{-1}K_{x_j}u + T_0^{-1}f_{x_j}.$$

Moreover, since $a \in C^2(\overline{\Omega} \times \mathbf{S}^1)$, $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, the right-hand-side of (8) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. By applying part (i) above, we get that the unique solution to (8)

$$(9)\qquad u_{x_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1, 2.$$

For $f \in W^{1,p}(\Omega \times \mathbf{S}^1)$, also according to part (i), $u_{\theta_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular u_{θ_j} is the unique solution of the integral equation

(10)
$$u_{\theta_j} + T_0^{-1}(au_{\theta_j}) = -\widetilde{T}_{0,1}^{-1}(au_{x_j}) - T_0^{-1}(a_{\theta_j}u) - \widetilde{T}_{0,1}^{-1}(a_{x_j}u) + T_0^{-1}K_{\theta_j}u + \widetilde{T}_{0,1}^{-1}K_{x_j}u + T_0^{-1}f_{\theta_j},$$

which is of the type (7) with $K = 0$. Moreover, since $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, and, according to (9), $u_{x_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1, 2$, the right-hand-side of (10) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. Again, by applying part (i), we get

$$u_{\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1, 2.$$

Thus, $u \in W^{2,p}(\Omega \times \mathbf{S}^1)$.

(iii) For $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, according to part (ii), $u_{x_j}, u_{\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1)$, and $u_{x_ix_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular $u_{x_ix_j}$ is the unique solution of the integral equation

(11)
$$\begin{aligned} u_{x_ix_j} + T_0^{-1}(au_{x_ix_j}) - T_0^{-1}(Ku_{x_ix_j}) &= T_0^{-1}f_{x_ix_j} - T_0^{-1}(a_{x_j}u_{x_i}) - T_0^{-1}(a_{x_ix_j}u) + T_0^{-1}(K_{x_j}u_{x_i}) \\ &\quad + T_0^{-1}(K_{x_ix_j}u) - T_0^{-1}(a_{x_i}u_{x_j}) - T_0^{-1}(K_{x_i}u_{x_j}). \end{aligned}$$

Moreover, since $a \in C^3(\overline{\Omega} \times \mathbf{S}^1)$, $k \in C^3(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, the right-hand-side of (11) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. By applying part (i) above, we get that the unique solution to (11)

$$u_{x_i x_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ i,j = 1,2. \tag{12}$$

For $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, also according to part (ii), $u_{x_j}, u_{\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1)$, and $u_{\theta_i\theta_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular $u_{\theta_i\theta_j}$ is the unique solution of the integral equation

$$\begin{aligned} u_{\theta_i\theta_j} + T_0^{-1}(a u_{\theta_i\theta_j}) = {} & T_0^{-1}(f_{\theta_i\theta_j}) - \widetilde{T}_{0,2}^{-1}(a_{x_i} u_{x_j}) - \widetilde{T}_{0,1}^{-1}(a_{x_i} u_{\theta_j}) - \widetilde{T}_{0,1}^{-1}(a_{\theta_i} u_{x_j}) \\ & - T_0^{-1}(a_{\theta_i} u_{\theta_j}) - \widetilde{T}_{0,2}^{-1}(a_{x_j} u_{x_i}) - \widetilde{T}_{0,1}^{-1}(a_{x_j} u_{\theta_i}) - \widetilde{T}_{0,2}^{-1}(a_{x_i x_j} u) \\ & - \widetilde{T}_{0,1}^{-1}(a_{x_j\theta_i} u) - \widetilde{T}_{0,1}^{-1}(a_{\theta_j} u_{x_i}) - T_0^{-1}(a_{\theta_j} u_{\theta_i}) - \widetilde{T}_{0,1}^{-1}(a_{\theta_i\theta_j} u) \\ & - \widetilde{T}_{0,1}^{-1}(K_{\theta_j} u_{x_i}) - T_0^{-1}(K_{\theta_i\theta_j} u) - \widetilde{T}_{0,2}^{-1}(K u_{x_i x_j}) - \widetilde{T}_{0,1}^{-1}(K_{\theta_i} u_{x_j}), \end{aligned} \tag{13}$$

which is of the type (7) with $K = 0$.

Moreover, since $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, and, according to (12), $u_{x_i x_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ j = 1,2$, the right-hand-side of (13) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. Again, by applying part (i), we get

$$u_{\theta_i\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ i,j = 1,2. \tag{14}$$

For $f \in W^{2,p}(\Omega \times \mathbf{S}^1)$, also according to part (ii), $u_{x_i} u_{\theta_j} \in L^p(\Omega \times \mathbf{S}^1)$. In particular $u_{x_i\theta_j}$ is the unique solution of the integral equation

$$\begin{aligned} u_{x_i\theta_j} + T_0^{-1}(a u_{x_i\theta_j}) - T_0^{-1}(K u_{x_j\theta_i}) = {} & T_0^{-1}(f_{x_j\theta_i}) - \widetilde{T}_{0,1}^{-1}(a_{x_i} u_{x_j}) - T_0^{-1}(a_{\theta_i} u_{x_j}) \\ & - T_0^{-1}(a_{x_j\theta_i} u) - \widetilde{T}_{0,1}^{-1}(a_{x_j} u_{x_i}) - T_0^{-1}(u_{\theta_i} a_{x_j}) \\ & + \widetilde{T}_{0,1}^{-1}(K_{x_j} u_{x_i}) + T_0^{-1}(K_{\theta_i} u_{x_j}) + T_0^{-1}(K_{x_j\theta_i} u), \end{aligned} \tag{15}$$

which is of the type (7). Moreover, since $a \in C^3(\overline{\Omega} \times \mathbf{S}^1)$, $k \in C^3(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and $f \in W^{3,p}(\Omega \times \mathbf{S}^1)$, the right-hand-side of (15) lies in $W^{1,p}(\Omega \times \mathbf{S}^1)$. Again, by applying part (i), we get

$$u_{x_i\theta_j} \in W^{1,p}(\Omega \times \mathbf{S}^1),\ i,j = 1,2. \tag{16}$$

From (12), (14), and (16), we get $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$.

□

We remark that for Theorem 2.2 part (i) we only need $a \in C^1(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$, and we only require $a \in C^2(\overline{\Omega} \times \mathbf{S}^1)$ and $k \in C^2(\overline{\Omega} \times \mathbf{S}^1 \times \mathbf{S}^1)$ for Theorem 2.2 part (ii). We also refer to [21, Theorem 2.2] for part (i) and (ii) of Theorem 2.2. Moreover, in a similar fashion, one can show that under sufficiently increased regularity of a and k, the solution u of (1) belong to $u \in W^{m,p}(\Omega \times \mathbf{S}^1)$ for $\mathbb{Z} \ni m \geqslant 1$, provided $f \in W^{m,p}(\Omega \times \mathbf{S}^1)$.

3. Ingredients from A-analytic theory

In this section we briefly introduce the properties of A-analytic maps needed later, and introduce notation. We recall some of the existing results and concepts used in our reconstruction method.

For $z = x_1 + ix_2$, we consider the Cauchy-Riemann operators

$$\overline{\partial} = (\partial_{x_1} + i\partial_{x_2})/2, \quad \partial = (\partial_{x_1} - i\partial_{x_2})/2. \tag{17}$$

Let l_∞ be the space of bounded (, respectively summable) sequences, $L : l_\infty \to l_\infty$ be the left shift

$$L\langle v_0, v_{-1}, v_{-2}, \cdots \rangle = \langle v_{-1}, v_{-2}, \cdots \rangle,$$

and $L^m = \underbrace{L \circ \cdots \circ L}_{m}$ be its m-th composition.

For $0 < \mu < 1$, $p = 1, 2$, we consider the Banach spaces:

$$l^{1,p}_{\infty}(\Gamma) := \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ...\rangle \; : \; \|\mathbf{g}\|_{l^{1,p}_{\infty}(\Gamma)} := \sup_{\xi \in \Gamma} \sum_{j=0}^{\infty} \langle j \rangle^p |g_{-j}(\xi)| < \infty \right\},$$

$$C^{\mu}(\Gamma; l_1) := \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ...\rangle : \sup_{\xi \in \Gamma} \|\mathbf{g}(\xi)\|_{l_1} + \sup_{\substack{\xi,\eta \in \Gamma \\ \xi \neq \eta}} \frac{\|\mathbf{g}(\xi) - \mathbf{g}(\eta)\|_{l_1}}{|\xi - \eta|^{\mu}} < \infty \right\}, \tag{18}$$

$$Y_{\mu}(\Gamma) := \left\{ \mathbf{g} : \mathbf{g} \in l^{1,2}_{\infty}(\Gamma) \text{ and } \sup_{\substack{\xi,\eta \in \Gamma \\ \xi \neq \eta}} \sum_{j=0}^{\infty} \langle j \rangle \frac{|g_{-j}(\xi) - g_{-j}(\eta)|}{|\xi - \eta|^{\mu}} < \infty \right\},$$

where, for brevity, we use the notation $\langle j \rangle = (1 + |j|^2)^{1/2}$. Similarly, we consider $C^{\mu}(\overline{\Omega}; l_1)$, and $C^{\mu}(\overline{\Omega}; l_{\infty})$.

A sequence valued map $\Omega \ni z \mapsto \mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), ...\rangle$ in $C(\overline{\Omega}; l_{\infty}) \cap C^1(\Omega; l_{\infty})$ is called L^2*-analytic* (in the sense of Bukhgeim), if

$$\overline{\partial} \mathbf{v}(z) + L^2 \partial \mathbf{v}(z) = 0, \quad z \in \Omega. \tag{19}$$

Bukhgeim's original theory [7] shows that solutions of (19), satisfy a Cauchy-like integral formula,

$$\mathbf{v}(z) = \mathcal{B}[\mathbf{v}|_{\Gamma}](z), \quad z \in \Omega, \tag{20}$$

where $\mathcal{B}$ is the Bukhgeim-Cauchy operator acting on $\mathbf{v}|_{\Gamma}$. We use the formula in [17], where $\mathcal{B}$ is defined component-wise for $n \geqslant 0$ by

$$(\mathcal{B}\mathbf{v})_{-n}(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{v_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\Gamma} \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \right\} \sum_{j=1}^{\infty} v_{-n-2j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^j, \; z \in \Omega. \tag{21}$$

The theorems below comprise some results in [40, 41]. For the proof of the theorem below we refer to [41, Proposition 2.3].

Theorem 3.1. *Let* $0 < \mu < 1$*, and let* $\mathcal{B}$ *be the Bukhgeim-Cauchy operator in* (21).
If $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, ...\rangle \in Y_{\mu}(\Gamma)$ *for* $\mu > 1/2$*, then* $\mathbf{v} := \mathcal{B}\mathbf{g} \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_{\infty})$ *is* L^2*-analytic in* Ω*.*

Similar to the analytic maps, the traces of L^2-analytic maps on the boundary must satisfy some constraints, which can be expressed in terms of a corresponding Hilbert-like transform introduced in [40]. More precisely, the Bukhgeim-Hilbert transform $\mathcal{H}$ acting on g,

$$\Gamma \ni z \mapsto (\mathcal{H}\mathbf{g})(z) = \langle (\mathcal{H}\mathbf{g})_0(z), (\mathcal{H}\mathbf{g})_{-1}(z), (\mathcal{H}\mathbf{g})_{-2}(z), ...\rangle \tag{22}$$

is defined component-wise for $n \geqslant 0$ by

$$(\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi} \int_{\Gamma} \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi} \int_{\Gamma} \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - \overline{z}} \right\} \sum_{j=1}^{\infty} g_{-n-2j}(\zeta) \left(\frac{\overline{\zeta} - \overline{z}}{\zeta - z} \right)^j, \; z \in \Gamma, \tag{23}$$

and we refer to [40] for its mapping properties.

Key to the proof of the reconstruction is the following characterization of traces of L^2-analytic maps.

Theorem 3.2. *Let $0 < \mu < 1$. Let $\mathbf{g} \in Y_\mu(\varGamma)$ for $\mu > 1/2$ be defined on the boundary Γ, and let $\mathcal{H}$ be the Bukhgeim-Hilbert transform acting on $\mathbf{g}$ as in* (23).

(i) If $\mathbf{g}$ is the boundary value of an L^2-analytic function, then $\mathcal{H}\mathbf{g} \in C^\mu(\varGamma; l_1)$ and satisfies

$$(I + i\mathcal{H})\mathbf{g} = \mathbf{0}. \tag{24}$$

(ii) If $\mathbf{g}$ satisfies (24)*, then there exists an L^2-analytic $\mathbf{v} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty)$, such that*

$$\mathbf{v}|_{\varGamma} = \mathbf{g}. \tag{25}$$

For the proof of Theorem 3.2 we refer to [40, Theorem 3.2, Corollary 4.1, and Proposition 4.2] and [41, Proposition 2.3].

Another ingredient, in addition to L^2-analytic maps, consists in the one-to-one relation between solutions $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, ...\rangle$ satisfying

$$\overline{\partial} u_{-n}(z) + \partial u_{-n-2}(z) + a(z)u_{-n-1}(z) = 0, \quad z \in \Omega,\ n \geqslant 0. \tag{26}$$

and the L^2-analytic map $\mathbf{v} = \langle v_0, v_{-1}, v_{-2}, ...\rangle$ satisfying

$$\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z) = 0, \quad z \in \Omega,\ n \geqslant 0; \tag{27}$$

via a special function h, see (38), and [42, Lemma 4.2] for details. The function h is defined as

$$h(z, \boldsymbol{\theta}) := Da(z, \boldsymbol{\theta}) - \frac{1}{2}\,(I - iH)\, Ra(z \cdot \boldsymbol{\theta}^\perp, \boldsymbol{\theta}^\perp), \tag{28}$$

where $\boldsymbol{\theta}^\perp$ is orthogonal to $\boldsymbol{\theta}$, $Ra(s, \boldsymbol{\theta}^\perp) = \displaystyle\int_{-\infty}^{\infty} a\left(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}\right) dt$ is the Radon transform of the attenuation a, $Da(z, \boldsymbol{\theta}) = \displaystyle\int_0^\infty a(z + t\boldsymbol{\theta})dt$ is the divergent beam transform of the attenuation a, and $Hf(x) = \displaystyle\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(t)}{x-t}dt$ is the classical Hilbert transform [33], taken in the first variable and evaluated at $s = z\cdot \boldsymbol{\theta}^\perp$. The function h appeared first in [34] and enjoys the crucial property of having vanishing negative Fourier modes. We refer the following Lemma 3.1 for the properties of h used in here.

Lemma 3.1. [42, Lemma 4.1] *Assume $\Omega \subset \mathbb{R}^2$ is $C^{2,\mu}, \mu > 1/2$, convex domain. For $p = 1, 2$, let $a \in C^{p,\mu}(\overline{\Omega})$, $\mu > 1/2$, and h defined in* (28)*. Then $h \in C^{p,\mu}(\overline{\Omega} \times \mathbf{S}^1)$ and the following hold*
(i) h satisfies

$$\boldsymbol{\theta} \cdot \nabla h(z, \boldsymbol{\theta}) = -a(z), \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1. \tag{29}$$

(ii) h has vanishing negative Fourier modes yielding the expansions

$$e^{-h(z,\boldsymbol{\theta})} := \sum_{m=0}^{\infty} \alpha_m(z) e^{\mathrm{i}m\theta}, \quad e^{h(z,\boldsymbol{\theta})} := \sum_{m=0}^{\infty} \beta_m(z) e^{\mathrm{i}m\theta}, \quad (z, \theta) \in \overline{\Omega} \times \mathbf{S}^1. \tag{30}$$

with (iii)

$$z \mapsto \boldsymbol{\alpha}(z) := \langle \alpha_0(z), \alpha_1(z), \alpha_2(z), \alpha_3(z), ...,\rangle \in C^{p,\mu}(\Omega; l_1) \cap C(\overline{\Omega}; l_1), \tag{31}$$

$$z \mapsto \boldsymbol{\beta}(z) := \langle \beta_0(z), \beta_1(z), \beta_2(z), \beta_3(z), ...,\rangle \in C^{p,\mu}(\Omega; l_1) \cap C(\overline{\Omega}; l_1). \tag{32}$$

(iv) For any $z \in \Omega$,

$$\overline{\partial}\beta_0(z) = 0, \quad \overline{\partial}\beta_1(z) = -a(z)\beta_0(z), \quad \overline{\partial}\beta_{j+2}(z) + \partial\beta_j(z) + a(z)\beta_{j+1}(z) = 0, \quad j \geqslant 0. \tag{33}$$

(v) For any $z \in \Omega$,

$$\overline{\partial}\alpha_0(z) = 0, \quad \overline{\partial}\alpha_1(z) = a(z)\alpha_0(z), \quad \overline{\partial}\alpha_{j+2}(z) + \partial\alpha_j(z) - a(z)\alpha_{j+1}(z) = 0, \quad j \geqslant 0. \tag{34}$$

(vi) The Fourier modes $\alpha_j, \beta_j, j \geqslant 0$ satisfy

$$\alpha_0\beta_0 = 1, \quad \sum_{m=0}^{j} \alpha_m\beta_{j-m} = 0, \quad j \geqslant 1. \tag{35}$$

The one-to-one relation between (26) and (27):

$$\overline{\partial}\mathbf{u} + L^2\partial\mathbf{u} + a(z)L\mathbf{u} = \mathbf{0} \Longleftrightarrow \overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}, \tag{36}$$

can be expressed via the convolutions

$$\begin{aligned} v_{-n}(z) &= \sum_{j=0}^{\infty} \alpha_j(z)u_{-n-j}(z), \quad z \in \Omega,\ n \geqslant 0, \\ u_{-n}(z) &= \sum_{j=0}^{\infty} \beta_j(z)v_{-n-j}(z), \quad z \in \Omega,\ n \geqslant 0, \end{aligned} \tag{37}$$

where α_j's and β_j's are the Fourier modes of $e^{\mp h}$ in (30). Indeed, we note that v_{-n} for $n \geqslant 0$ satisfy

$$\begin{aligned} \overline{\partial}v_{-n} + \partial v_{-n-2} &= \sum_{j=0}^{\infty} \overline{\partial}\alpha_j u_{-n-j} + \sum_{j=0}^{\infty} \alpha_j\overline{\partial}u_{-n-j} + \sum_{j=0}^{\infty} \partial\alpha_j u_{-n-2-j} + \sum_{j=0}^{\infty} \alpha_j\partial u_{-n-2-j} \\ &= \overline{\partial}\alpha_0 u_{-n} + \overline{\partial}\alpha_1 u_{-n-1} + \sum_{j=0}^{\infty} \left(\overline{\partial}\alpha_{j+2} + \partial\alpha_j\right) u_{-n-2-j} + \sum_{j=0}^{\infty} \alpha_j \left(\overline{\partial}u_{-n-j} + \partial u_{-n-2-j}\right) \\ &= \overline{\partial}\alpha_0 u_{-n} + \overline{\partial}\alpha_1 u_{-n-1} + \sum_{j=0}^{\infty} \left(\overline{\partial}\alpha_{j+2} + \partial\alpha_j\right) u_{-n-2-j} + \sum_{j=0}^{\infty} \alpha_j \left(-au_{-n-1-j}\right) \\ &= \overline{\partial}\alpha_0 u_{-n} + \left(\overline{\partial}\alpha_1 - a\alpha_0\right) u_{-n-1} + \sum_{j=0}^{\infty} \left(\overline{\partial}\alpha_{j+2} + \partial\alpha_j - a\alpha_{j+1}\right) u_{-n-2-j} = 0, \end{aligned} \tag{38}$$

where in the third equality we use (26), and in the last equality we use (34). Thus $\mathbf{v} = \langle v_0, v_{-1}, ...\rangle$ solves $\overline{\partial}\mathbf{v} + L^2\partial\mathbf{v} = \mathbf{0}$.

An analogue calculation using the properties in Lemma 3.1 (iv) shows the converse of (36). Moreover, see [42, Lemma 4.2] for details.

4. Reconstruction of a sufficiently smooth isotropic vector field $\mathbf{F}$

For an isotropic real valued vector field $\mathbf{F} = \langle F_1, F_2\rangle$, recall the boundary value problem (1):

$$\begin{aligned} &\boldsymbol{\theta} \cdot \nabla u(z,\boldsymbol{\theta}) + a(z)u(z,\boldsymbol{\theta}) - \int_{\mathbf{S}^1} k(z,\boldsymbol{\theta}\cdot\boldsymbol{\theta}')u(z,\boldsymbol{\theta}')d\boldsymbol{\theta}' = \underbrace{\boldsymbol{\theta}\cdot\mathbf{F}(z)}_{f(z,\boldsymbol{\theta})}, \quad (z,\boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1, \\ &u|_{\Gamma_-} = 0, \end{aligned} \tag{39}$$

with an isotropic attenuation $a = a(z)$, and with the scattering kernel $k(z, \boldsymbol{\theta}, \boldsymbol{\theta}') = k(z, \boldsymbol{\theta} \cdot \boldsymbol{\theta}')$ depending polynomially on the angle between the directions,

$$k(z, \cos\theta) = k_0(z) + 2\sum_{n=1}^{M} k_{-n}(z)\cos(n\theta), \tag{40}$$

for some fixed integer $M \geqslant 1$. Note that, since $k(z, \cos\theta)$ is both real valued and even in θ, the coefficient k_{-n} is the $(-n)^{th}$ Fourier coefficient of $k(z, \cos(\cdot))$. Moreover k_{-n} is real valued, and $k_n(z) = k_{-n}(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} k(z, \cos\theta)e^{in\theta}d\theta$.

For vector field $\mathbf{F} = \langle F_1, F_2\rangle$ and $\boldsymbol{\theta} = (\cos\theta, \sin\theta) \in \mathbf{S}^1$, a calculation shows that

$$f(z, \boldsymbol{\theta}) = \boldsymbol{\theta} \cdot \mathbf{F}(z) = \overline{f_1(z)}e^{i\theta} + f_1(z)e^{-i\theta}, \quad \text{where } f_1 = (F_1 + \mathrm{i}F_2)/2. \tag{41}$$

We assume that the coefficients $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ are such that the forward problem (39) has a unique solution $u \in L^p(\Omega \times \mathbf{S}^1)$ for any $f \in L^p(\Omega \times \mathbf{S}^1)$,$p > 1$, see Theorem 2.1. Moreover, we assume also an *unknown* source of a priori regularity $\mathbf{F} \in W^{3,p}(\overline{\Omega}; \mathbb{R}^2)$, $p > 4$, and by Theorem 2.2 part (iii), the solution $u \in C^{2,\mu}(\Omega \times \mathbf{S}^1)$ with $\mu > 1/2$. Furthermore, the functions a, k and vector field $\mathbf{F}$ are assumed real valued, so that the solution u is also real valued.

Let $u(z, \boldsymbol{\theta}) = \sum_{-\infty}^{\infty} u_n(z)e^{in\theta}$ be the formal Fourier series representation of the solution of (39) in the angular variable $\boldsymbol{\theta} = (\cos\theta, \sin\theta)$. Since u is real valued, the Fourier modes $\{u_n\}$ occurs in complex-conjugate pairs $u_{-n} = \overline{u_n}$, and the angular dependence is completely determined by the sequence of its nonpositive Fourier modes

$$\Omega \ni z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), ...\rangle. \tag{42}$$

For the derivatives $\partial, \overline{\partial}$ in the spatial variable as in (17), the advection operator $\boldsymbol{\theta} \cdot \nabla$ in (39) becomes $\boldsymbol{\theta} \cdot \nabla = e^{-\mathrm{i}\theta}\overline{\partial} + e^{\mathrm{i}\theta}\partial$. By identifying the Fourier coefficients of the same order, the equation (39) reduces to the system:

$$\overline{\partial}u_1(z) + \partial u_{-1}(z) + [a(z) - k_0(z)]u_0(z) = 0, \tag{43}$$

$$\overline{\partial}u_0(z) + \partial u_{-2}(z) + [a(z) - k_{-1}(z)]u_{-1}(z) = f_1(z), \tag{44}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-n-2}(z) + [a(z) - k_{-n-1}(z)]u_{-n-1}(z) = 0, \quad 1 \leqslant n \leqslant M - 1, \tag{45}$$

$$\overline{\partial}u_{-n}(z) + \partial u_{-n-2}(z) + a(z)u_{-n-1}(z) = 0, \qquad n \geqslant M, \tag{46}$$

where f_1 as in (41).

In the radiative transport literature, the attenuation coefficient $a = \sigma_a + \sigma_s$, where σ_a represents pure loss due to absorption and $\sigma_s(z) = \frac{1}{2\pi}\int_0^{2\pi} k(z, \theta)d\theta = k_0(z)$ is the isotropic part of scattering kernel. We consider the subcritical region:

$$\sigma_a := a - k_0 \geqslant \delta > 0, \quad \text{for some positive constant } \delta. \tag{47}$$

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ be a strictly convex bounded domain, and Γ be its boundary. Consider the boundary value problem* (39) *for some known real valued $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ such that* (39) *is well-posed. If the unknown vector field source $\mathbf{F}$ is real valued and $W^{3,p}(\Omega; \mathbb{R}^2)$-regular, with $p > 4$, and coefficients a, k_0 satisfying* (47)*, then $u|_{\Gamma_+}$ determines $\mathbf{F}$ in Ω.*

Proof. Let u be the solution of the boundary value problem (39) and let $\mathbf{u} = \langle u_0, u_{-1}, u_{-2}, ...\rangle$ be the sequence valued map of its non-positive Fourier modes, Since the isotropic vector field $\mathbf{F} \in W^{3,p}(\Omega; \mathbb{R}^2)$, $p > 4$, then the anistropic source $f = \boldsymbol{\theta} \cdot \mathbf{F} \in W^{3,p}(\Omega \times \mathbf{S}^1)$ and by applying Theorem 2.2 (iii), we have $u \in W^{3,p}(\Omega \times \mathbf{S}^1)$. By the Sobolev embedding, $W^{3,p}(\Omega \times \mathbf{S}^1) \subset C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$ with $\mu = 1 - \frac{2}{p} > \frac{1}{2}$, we have $u \in C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$, and thus, by [40, Proposition 4.1 (ii)], $\mathbf{u} \in Y_\mu(\Gamma)$.

We note from (46) that the shifted sequence valued map $L^M\mathbf{u} = \langle u_{-M}, u_{-M-1}, u_{-M-2}, \dots\rangle$ solves

$$\overline{\partial} L^M\mathbf{u}(z) + L^2\partial L^M\mathbf{u}(z) + a(z)L^{M+1}\mathbf{u}(z) = \mathbf{0}, \quad z \in \Omega, \tag{48}$$

and then the associated sequence valued map $L^M\mathbf{v} = \langle v_{-M}, v_{-M-1}, v_{-M-2}, \dots\rangle$ defined by the convolutions (37): $v_{-n}(z) = \sum_{j=0}^{\infty} \alpha_j(z)u_{-n-j}(z)$, for $n \geqslant M$, solves the system

$$\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z) = 0, \quad z \in \Omega,\ n \geqslant M. \tag{49}$$

In particular, the shifted sequence valued map $L^M\mathbf{v}$ is L^2-analytic.

By (2), the data $u|_{\Gamma_+} = g$ determines $L^M\mathbf{u}$ on Γ. By the convolution formula (37) for $n \geqslant M$, $L^M\mathbf{u}|_\Gamma$ determines the traces $L^M\mathbf{v} \in Y_\mu(\Gamma)$ on Γ.

Since $L^M\mathbf{v}|_\Gamma \in Y_\mu(\Gamma)$ is the boundary value of an L^2-analytic function in Ω, then Theorem 3.2 (i) yields

$$[I + \mathrm{i}\mathcal{H}]L^M\mathbf{v}|_\Gamma = \mathbf{0}, \tag{50}$$

where $\mathcal{H}$ is the Bukhgeim-Hilbert transform in (23).

From $L^M\mathbf{v}$ on Γ, we use the Bukhgeim-Cauchy Integral formula (21) to construct the sequence valued map $L^M\mathbf{v}$ inside Ω. By Theorem 3.1 and Theorem 3.2 (ii), the constructed sequence valued $L^M\mathbf{v} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty)$ is L^2-analytic in Ω. In particular, for each $n \geqslant M$, the modes v_{-n} in Ω is determined.

We use again the convolution formula (37): $u_{-n}(z) = \sum_{j=0}^{\infty} \beta_j(z)v_{-n-j}(z)$, and determine modes u_{-n} now inside Ω, for $n \geqslant M$. In particular, we recover modes $u_{-M-1}, u_{-M} \in C^2(\Omega)$.

Recall that the modes $u_{-1}, u_{-2}, \cdots, u_{-M}, u_{-M-1}$ satisfy

$$\overline{\partial} u_{-M+j} = -\partial u_{-M+j-2} - \left[(a - k_{-M+j-1})u_{-M+j-1}\right], \quad 1 \leqslant j \leqslant M-1, \tag{51a}$$

$$u_{-M+j}|_\Gamma = g_{-M+j}. \tag{51b}$$

By applying 4∂ to (51a), the mode u_{-M+1} (for $j = 1$) is then the solution to the Dirichlet problem for the Poisson equation

$$\Delta u_{-M+1} = -4\partial^2 u_{-M-1} - 4\partial\left[(a - k_{-M})u_{-M}\right], \tag{52a}$$

$$u_{-M+1}|_\Gamma = g_{-M+1}, \tag{52b}$$

where the right hand side of (52) is known.

We solve repeatedly (52) for $j = 2, \dots, M-1$ in (51), to recover $u_{-M+1}, u_{-M+2}, \cdots, u_{-1}$ in Ω.

Using the subcriticality condition (47):

$$\sigma_a(z) = a(z) - k_0(z) \geqslant \delta > 0, \quad \text{for some positive constant } \delta,$$

we define

$$u_0(z) := -\frac{2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}(z)}{a(z) - k_0(z)} = -\frac{2\,\mathbb{R}\mathrm{e}\,\partial u_{-1}(z)}{\sigma_a(z)}. \tag{53}$$

The real valued vector field $\mathbf{F} = \langle 2\,\mathbb{R}\mathrm{e}\, f_1, 2\,\mathbb{I}\mathrm{m}\, f_1\rangle$, is recovered pointwise by

$$f_1(z) = \overline{\partial} u_0(z) + \partial u_{-2}(z) + \left[a(z) - k_{-1}(z)\right] u_{-1}(z), \quad z \in \Omega. \tag{54}$$

□

We summarize below in a stepwise fashion the reconstruction of vector field $\mathbf{F}$.

Reconstruction procedure. Consider the data $\mathbf{u}|_{\Gamma}$.

(1) Using formula (37), and data $L^M\mathbf{u}$ on Γ, determine the traces $L^M\mathbf{v}|_{\Gamma}$ on Γ.
(2) By Bukhgeim-Cauchy formula (21), extend v_{-n} for $n \geqslant M$ from the boundary Γ to Ω.
(3) Using again formula (37), now in Ω, recover u_{-n} for $n \geqslant M$ inside Ω.
(4) Using u_{-M-1}, u_{-M}, recover the modes $u_{-M+1}, u_{-M+2}, \cdots, u_{-2}, u_{-1}$ recursively as follows:
    (a) Using $u_{-M+1}|_{\Gamma}$, recover u_{-M+1} inside Ω by solving (52).
    (b) Now iterate the steps (4 a) to find the modes $u_{-M+2}, \cdots, u_{-2}, u_{-1}$ in Ω.
(5) Using mode u_{-1}, define u_0 mode by (53).
(6) Recover $\mathbf{F}$ by formula (54).

5. When can the 0-tensor data and 1-tensor data be mistaken for each other ?

In this section we will address when the attenuated and scattering 0-tensor field data can be mistaken for the attenuated and scattering 1-tensor field data.

In the absorbing non-scattering medium ($a \neq 0, k = 0$), the 0-tensor data is the attenuated X-ray data of a function f, and the 1-tensor field data is the attenuated Doppler data of a vector field $\mathbf{F}$.

In the non-absorbing medium ($a = 0 = k$), by comparing the range conditions for the (non-attenuated) X-ray data for 0-tensors in [40, Theorem 4.1] and for 1-tensors in [41, Theorem 3.1], it is transparent that the two data cannot be mistaken for each other, unless they are both zero. However, in the attenuated and scattering case with $\sigma_a > 0$ the situation is different. To distinguish between the data coming from the 0- and 1-tensor fields we use the notations $g_{a,k,f}$, $g_{a,k,\mathbf{F}}$ for the attenuated and scattering data of the real valued function f, respectively, of the real valued vector field $\mathbf{F}$.

In the theorem below the 0- and 1-tensor fields data are assuming the same attenuation a and scattering coefficient k.

Theorem 5.1. *(i) Let $a \in C^3(\overline{\Omega})$, $k \in C^3(\overline{\Omega} \times \mathbf{S}^1)$ be real valued, with σ_a as in* (47)*, and $f \in W^{3,p}(\Omega)$, $p > 4$ be real valued with $f/\sigma_a \in C_0(\overline{\Omega})$. Then $\mathbf{F} := -\nabla\left(\dfrac{f}{\sigma_a}\right)$ is a real valued vector field whose attenuated and scattering 1-tensor field data $g_{a,k,\mathbf{F}}$ is the same as the attenuated and scattering 0-tensor field data $g_{a,k,f}$ of f.*

(ii) Let $a, k_0, k_{-1}, ..., k_{-M} \in C^3(\overline{\Omega})$ be real valued with σ_a as in (47)*. Assume that $\mathbf{F} \in W^{3,p}(\Omega; \mathbb{R}^2)$, $p > 4$ is a vector field whose attenuated and scattering 1-tensor field data $g_{a,k,\mathbf{F}}$ equals the attenuated and scattering 0-tensor field data $g_{a,k,f}$ of some real valued $f \in W^{3,p}(\Omega)$, $p > 4$. Then $\mathbf{F}$ must be a gradient field and $\mathbf{F} = -\nabla\left(\dfrac{f}{\sigma_a}\right)$.*

Proof. (i) Assume $g_{a,k,f}$ is the attenuated and scattering 0-tensor field data of some real valued function f, i.e., it is the trace on $\Gamma \times \mathbf{S}^1$ of solutions w to the stationary transport boundary value problem:

$$\begin{aligned} \boldsymbol{\theta} \cdot \nabla w + aw - Kw &= f, \\ w|_{\Gamma \times \mathbf{S}^1} &= g_{a,k,f}, \end{aligned} \tag{55}$$

where the operator $[Kw](z, \boldsymbol{\theta}) := \int_{\mathbf{S}^1} k(z, \boldsymbol{\theta} \cdot \boldsymbol{\theta}')w(z, \boldsymbol{\theta}')d\boldsymbol{\theta}'$, for $z \in \Omega$ and $\boldsymbol{\theta} \in \mathbf{S}^1$.

For $\sigma_a = a - k_0$ with $\sigma_a > 0$ (satisfying (47)), and isotropic real valued functions f and σ_a, we note:

$$\begin{aligned} \left[K\frac{f}{\sigma_a}\right](z, \boldsymbol{\theta}) &= \int_{\mathbf{S}^1} k(z, \boldsymbol{\theta} \cdot \boldsymbol{\theta}') \left[\frac{f}{\sigma_a}\right](z, \boldsymbol{\theta}')d\boldsymbol{\theta}' \\ &= \frac{f(z)}{\sigma_a(z)} \int_{\mathbf{S}^1} k(z, \boldsymbol{\theta} \cdot \boldsymbol{\theta}')d\boldsymbol{\theta}' = \frac{f(z)}{\sigma_a(z)} k_0(z), \end{aligned} \tag{56}$$

where second equality use the fact that both f and σ_a are angularly independent functions.

Let $u := w - f/\sigma_a$ and $\mathbf{F} := -\nabla\left(\dfrac{f}{\sigma_a}\right)$. Then

$$\begin{aligned}\boldsymbol{\theta}\cdot\nabla u + au - Ku &= \boldsymbol{\theta}\cdot\nabla\left(w - \frac{f}{\sigma_a}\right) + a\left(w - \frac{f}{\sigma_a}\right) - K\left(w - \frac{f}{\sigma_a}\right)\\ &= \boldsymbol{\theta}\cdot\nabla w + aw - Kw - f\left(\frac{a}{\sigma_a}\right) + K\left(\frac{f}{\sigma_a}\right) + \boldsymbol{\theta}\cdot\nabla\left(-\frac{f}{\sigma_a}\right)\\ &= \left(1 - \frac{a}{\sigma_a} + \frac{k_0}{\sigma_a}\right) f + \boldsymbol{\theta}\cdot\mathbf{F} = \boldsymbol{\theta}\cdot\mathbf{F},\end{aligned}$$

where the second equality uses the linearty of K, and the last equality uses (55), (56) and the definition of $\mathbf{F}$. Moreover, since f/σ_a vanishes on $\varGamma$, we get

$$g_{a,k,\mathbf{F}} = u|_{\varGamma\times\mathbf{S}^1} = w|_{\varGamma\times\mathbf{S}^1} - \left.\frac{f}{\sigma_a}\right|_{\varGamma} = w|_{\varGamma\times\mathbf{S}^1} = g_{a,k,f}.$$

(ii) Let $\mathbf{F} = \langle F_1, F_2\rangle \in W^{3,p}(\Omega;\mathbb{R}^2)$, $p > 4$ be a real valued vector field whose data $g_{a,k,\mathbf{F}}$ matches the 0-tensor field data $g_{a,k,f}$ of some real valued function $f \in W^{3,p}(\Omega)$, $p > 4$, i.e.

$$g_{a,k,f} = g = g_{a,k,\mathbf{F}}.$$

Then by Theorem 2.2 (iii), there exist $u, w \in W^{3,p}(\Omega\times\mathbf{S}^1)$ solutions to the corresponding transport equations

$$\boldsymbol{\theta}\cdot\nabla u + au - Ku = \boldsymbol{\theta}\cdot\mathbf{F}, \qquad \text{and} \quad \boldsymbol{\theta}\cdot\nabla w + aw - Kw = f \tag{57}$$

respectively, subject to

$$u|_{\varGamma\times\mathbf{S}^1} = g = w|_{\varGamma\times\mathbf{S}^1}.$$

Moreover, by the Sobolev embedding, $u, w \in C^{2,\mu}(\overline{\Omega}\times\mathbf{S}^1)$ with $\mu = 1 - \frac{2}{p} > \frac{1}{2}$, and the corresponding sequences of non-positive Fourier modes $\{u_{-n}\}_{n\geqslant 0}$ of u satisfy

$$\overline{\partial}\overline{u_{-1}}(z) + \partial u_{-1}(z) + [a(z) - k_0(z)]u_0(z) = 0, \tag{58}$$

$$\overline{\partial} u_0(z) + \partial u_{-2}(z) + [a(z) - k_{-1}(z)]u_{-1}(z) = (F_1(z) + iF_2(z))/2, \tag{59}$$

$$\overline{\partial} u_{-n}(z) + \partial u_{-n-2}(z) + [a(z) - k_{-n-1}(z)]u_{-n-1}(z) = 0, \quad 1 \leqslant n \leqslant M-1, \tag{60}$$

$$\overline{\partial} u_{-n}(z) + \partial u_{-n-2}(z) + a(z)u_{-n-1}(z) = 0, \qquad n \geqslant M, \tag{61}$$

whereas the non-positive Fourier modes $\{w_{-n}\}_{n\geqslant 0}$ of w satisfy

$$\overline{\partial}\overline{w_{-1}}(z) + \partial w_{-1}(z) + [a(z) - k_0(z)]w_0(z) = f(z), \tag{62}$$

$$\overline{\partial} w_0(z) + \partial w_{-2}(z) + [a(z) - k_{-1}(z)]w_{-1}(z) = 0, \tag{63}$$

$$\overline{\partial} w_{-n}(z) + \partial w_{-n-2}(z) + [a(z) - k_{-n-1}(z)]w_{-n-1}(z) = 0, \quad 1 \leqslant n \leqslant M-1, \tag{64}$$

$$\overline{\partial} w_{-n}(z) + \partial w_{-n-2}(z) + a(z)w_{-n-1}(z) = 0, \qquad n \geqslant M. \tag{65}$$

Furthermore, by [40, Proposition 4.1 (ii)], the corresponding sequence valued $\mathbf{u} = \langle u_0, u_{-1}, \cdots\rangle \in Y_\mu(\varGamma)$, and $\mathbf{w} = \langle w_0, w_{-1}, w_{-2}, \cdots\rangle \in Y_\mu(\varGamma)$ with $\mu > \frac{1}{2}$.

Since the boundary data g is the same $u|_{\varGamma\times\mathbf{S}^1} = w|_{\varGamma\times\mathbf{S}^1}$, we also have

$$u_{-n}|_{\varGamma} = w_{-n}|_{\varGamma}, \quad \forall n \geqslant 1. \tag{66}$$

We claim that the systems (61) and (65) subject to the identity (66) for $n \geqslant M$, yield

(67) $$u_{-n}(z) = w_{-n}(z), \quad z \in \Omega, \quad \forall n \geqslant M.$$

Recall the integrating factor $e^{\pm h}$ with h in (28). Since $a \in C^3(\overline{\Omega})$, then $e^h \in C^{2,\mu}(\overline{\Omega} \times \mathbf{S}^1)$ and its Fourier modes $\langle \beta_0, \beta_{-1}, ...\rangle \in C^{2,\mu}(\Omega; l_1) \cap C(\overline{\Omega}; l_1)$ by Lemma 3.1.

Recall the function $v = e^{-h}u$ and introduce $\rho := e^{-h}w$. The shifted sequence valued maps $L^M\mathbf{u} = (u_{-M}, u_{-M-1}, u_{-M-2}...)$, and $L^M\mathbf{w} = (w_{-M}, w_{-M-1}, w_{-M-2}...)$, respectively, solves systems (61), and (65). Then the associated sequence valued maps $L^M\mathbf{v} = (v_{-M}, v_{-M-1}, v_{-M-2}...)$, and $L^M\boldsymbol{\rho} = (\rho_{-M}, \rho_{-M-1}, \rho_{-M-2}...)$ defined by the convolutions (37), respectively, solves

$$\overline{\partial} v_{-n}(z) + \partial v_{-n-2}(z) = 0, \quad \text{for } z \in \Omega,\ \forall n \geqslant M, \quad \text{and}$$
$$\overline{\partial} \rho_{-n}(z) + \partial \rho_{-n-2}(z) = 0, \quad \text{for } z \in \Omega,\ \forall n \geqslant M.$$

In particular, $L^M\mathbf{v}$ and $L^M\boldsymbol{\rho}$ are L^2-analytic, and coincide at the boundary Γ. By uniqueness of L^2-analytic functions with a given trace, they coincide inside:

(68) $$v_{-n}(z) = \rho_{-n}(z), \quad \text{for } z \in \Omega,\ \forall n \geqslant M.$$

Using the convolutions (37) and equation (68), we conclude for all $n \geqslant M$ that

$$u_{-n}(z) = \sum_{j=0}^{\infty} \beta_j(z) v_{-n-j}(z) = \sum_{j=0}^{\infty} \beta_j(z) \rho_{-n-j}(z) = w_{-n}(z), \quad z \in \Omega.$$

Thus (67) holds.

Subjecting (60) and (64) to the boundary conditions (66), we claim that

(69) $$u_{-n}(z) = w_{-n}(z), \quad z \in \Omega, \quad \text{for all } 1 \leqslant n \leqslant M-1.$$

Define

(70) $$\psi_{-j} := u_{-j} - w_{-j}, \quad \text{for } j \geqslant 1,$$

and note that by (67), we have

(71) $$\psi_{-j} = 0, \quad \text{for } j \geqslant M.$$

By subtracting (64) from (60), and using (70), and (66), we have

(72a) $$\overline{\partial} \psi_{-M+j} = -\partial \psi_{-M+j-2} - \left[(a - k_{-M+j-1})\psi_{-M+j-1}\right], \quad 1 \leqslant j \leqslant M-1,$$

(72b) $$\psi_{-M+j}|_{\Gamma} = 0.$$

For the mode ψ_{-M+1} (when $j = 1$), the right hand side of (72a) contains modes ψ_{-M-1} and ψ_{-M} which are both zero by (71). Thus, the mode $\psi_{-M+1} \equiv 0$ is the unique solution to the Cauchy problem for the $\overline{\partial}$-equation,

(73a) $$\overline{\partial}\Psi = 0, \quad \text{in } \Omega,$$

(73b) $$\Psi = 0, \quad \text{on } \Gamma.$$

We then solve repeatedly (72) starting for $j = 2, ..., M-1$, where the right hand side of (72a) in each step is zero, yielding the Cauchy problem (73) for each subsequent modes, and thus, resulting in the recovering of the modes $\psi_{-M+1} = \psi_{-M+2} = \cdots \psi_{-2} = \psi_{-1} \equiv 0$ in Ω. Therefore, establishing (69).

By subtracting (63) from (59) and using (67) and (69), we obtain

$$\overline{\partial}(u_0 - w_0) = (F_1 + iF_2)/2.$$

Since both u_0 and w_0 are real valued we see that

$$\langle F_1, F_2 \rangle = \nabla(u_0 - w_0). \tag{74}$$

Moreover, by equation (62),

$$\begin{aligned} f &= \overline{\partial} \overline{w_{-1}} + \partial w_{-1} + (a - k_0) w_0 \\ &= \overline{\partial} \overline{u_{-1}} + \partial u_{-1} + (a - k_0) w_0 \\ &= \overline{\partial} \overline{u_{-1}} + \partial u_{-1} + (a - k_0) u_0 + (a - k_0)(w_0 - u_0) \\ &= (a - k_0)(w_0 - u_0), \end{aligned}$$

where the second equality uses (69) and the last equality uses (58). Therefore, using (47), we have

$$u_0 - w_0 = -\frac{f}{a - k_0} = -\frac{f}{\sigma_a},$$

and by (74),

$$\mathbf{F} := \langle F_1, F_2 \rangle = -\nabla \left(\frac{f}{\sigma_a} \right).$$

□

Remark 5.1. *Note that in Theorem 5.1(i), the assumption on scattering kernels of finite Fourier content in the angular variable is not assumed, and the result holds for a general scattering kernels which depends polynomially on the angle between the directions.*

Given the Helmholz decomposition of a vector field in gradient and a solenoidal field part, Theorem 5.1(ii) yields the following.

Corollary 5.1. *In an absorbing non-scattering media, the attenuated Doppler data of a solenoidal real valued smooth vector field cannot be mistaken by the attenuated X-ray data of a real valued smooth function.*

Corollary 5.2. *For a given attenuation and scattering kernel of finite Fourier content, the attenuated and scattering data of a solenoidal real valued smooth vector field cannot be mistaken by the attenuated and scattering data of a real valued smooth function.*

Acknowledgment

The work of D. Omogbhe and K. Sadiq were supported by the Austrian Science Fund (FWF), Project P31053-N32.

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