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Abstract

It is well known that the presence, in a homogeneous acoustic medium, of a small inhomogeneity (of size ε), enjoying a high contrast of both its mass density and bulk modulus, amplifies the generated total fields. This amplification is more pronounced when the incident frequency is close to the Minnaert frequency ω_M. Here we explain the origin of such a phenomenon: at first we show that the scattering of an incident wave of frequency ω is described by a self-adjoint ω-dependent Schrödinger operator with a singular δ-like potential supported at the inhomogeneity interface. Then we show that, in the low energy regime (corresponding in our setting to $\varepsilon \ll 1$) such an operator has a non-trivial limit (i.e., it asymptotically differs from the Laplacian) if and only if $\omega = \omega_M$. The limit operator describing the non-trivial scattering process is explicitly determined and belongs to the class of point perturbations of the Laplacian. When the frequency of the incident wave approaches ω_M, the scattering process undergoes a transition between an asymptotically trivial behaviour and a non-trivial one. As a by-product, we get the existence of a local maximum of the scattering amplitude occurring at the Minnaert frequency. It is this last property that is experimentally observed and used in the applications.

1 Introduction

Models related to the wave propagation in the presence of small scaled but highly contrasted inhomogeneities appear in different areas of applied sciences as in acoustics, electromagnetism and elasticity where these inhomogeneities model micro-bubbles, nano-particles and micro- or nano-inclusions, respectively. There is a critical ratio between the size and the contrast of the inhomogeneities under which the generated fields can be drastically enhanced. This enhancement has tremendous applications in imaging, in the broad sense, and material sciences, to cite a few. It has been observed and quantified in the stationary regimes for certain values of the incident frequencies (see, e.g., [14]) which are referred to as resonances. The purpose of our work is understanding the origin of such particular frequencies, enlightening the mechanism which leads to their emergence.

To study this question, we consider the stationary acoustic wave propagation in the presence of micro-bubbles. We deal with a linear model described by the mass density and the bulk modulus see [10]-[11]. When the background medium is homogeneous, with constant mass density ρ_0 and bulk modulus k_0, and the micro-bubble has shape Ω^ε, with diameter ε of about few tens of micrometers, mass density ρ_ε and bulk modulus k_ε, then the resonant frequencies are expected to appear in the following asymptotic regimes:

- *Low density / low bulk bubble*, characterized by the small-ε behaviour: $\rho_\varepsilon/\rho_0 \sim k_\varepsilon/k_0 \sim \varepsilon^{-r}$ with $r > 0$. In this regime the relative speed of propagation: $c_\varepsilon^2/c_0^2 := \frac{\rho_\varepsilon}{k_\varepsilon}\frac{\rho_0}{k_0} \sim 1$ is moderate, but the contrast of the transmission coefficient is large as $\varepsilon \ll 1$.

- *Moderate density / low bulk bubble,* defined by $\rho_\varepsilon/\rho_0 \sim 1$ and $k_\varepsilon/k_0 \sim \varepsilon^{-r}$, $r > 0$, as $\varepsilon \ll 1$. These properties mean that the relative speed of propagation is small. But the contrast of the transmission coefficient is moderate. Such bubbles are not known to exist in nature but they might be designed, see [26].

These configurations give rise to different types of resonant frequencies. We classify them as follows:

- The *Minnaert resonance*, which corresponds to a surface-mode for the low density / low bulk bubbles.
- The *body resonances*, which correspond to body-modes for the moderate density / low bulk bubbles.

The size of such resonances depends on the value of r. In particular, they are very large when $r < 2$ and very small when $r > 2$, in terms of the relative diameter ε, $\varepsilon \ll 1$. However, when $r = 2$, they are moderate and their dominant parts are independent of ε. In what follows we present the general setting of our problem and provide a qualitative argument showing how these resonances indeed appear.

1.1 Resonant frequencies generated by a micro-bubble

Let $\Omega \subset \mathbb{R}^3$ be an open bounded and connected domain with a smooth boundary[1] $\Gamma := \partial\Omega$. We define the contracted domain

$$\Omega^\varepsilon := \{x : x = y_0 + \varepsilon\,(y - y_0)\,,\; y \in \Omega\} \tag{1.1}$$

and denote with $\Gamma^\varepsilon := \partial\Omega^\varepsilon$ its boundary. The acoustic medium is defined by the density ρ and the bulk k both discontinuous across Γ^ε

$$\rho := \begin{cases} \rho_\varepsilon & \text{inside } \Omega^\varepsilon\,, \\ \rho_0 & \text{outside } \Omega^\varepsilon\,, \end{cases} \qquad \text{and} \qquad k := \begin{cases} k_\varepsilon & \text{inside } \Omega^\varepsilon\,, \\ k_0 & \text{outside } \Omega^\varepsilon\,. \end{cases} \tag{1.2}$$

Let $u^{\text{in}}(x,\omega,\theta) := e^{i\omega\sqrt{\rho_0/k_0}\;x\cdot\theta}$ be the incident plane wave, propagating in the direction θ, and ν be the exterior unit normal to Γ^ε. The scattering of u^{in} by the medium perturbation introduced in (1.2) is described by the boundary value problem (see [3], [7])

$$\begin{cases} \left(\nabla\cdot\frac{1}{\rho}\nabla + \omega^2\frac{1}{k}\right)u = 0\,, \quad \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\[2ex] u\big|_{\text{in}} = u\big|_{\text{ex}}\,, \quad \frac{1}{\rho_\varepsilon}\nu\cdot\nabla u\Big|_{\text{in}} = \frac{1}{\rho_0}\nu\cdot\nabla u\Big|_{\text{ex}}\,, \quad \text{at } \Gamma^\varepsilon\,, \\[2ex] u^{\text{sc}} := u - u^{\text{in}}\,, \quad \frac{\partial u^{\text{sc}}}{\partial|x|} - i\omega\sqrt{\frac{\rho_0}{k_0}}\,u^{\text{sc}} = o\left(|x|^{-1}\right)\,, \quad \text{as } |x|\to+\infty\,, \end{cases} \tag{1.3}$$

where $f\big|_{\text{in/ex}}$ denote the lateral traces on Γ^ε. It is worth noticing that these interface conditions provide with the regularity of the total field $u = u^{\text{sc}} + u^{\text{in}}$ across the boundary: we further refer to [10]-[11] for the physical setting justifying (1.3). Denoting with $\mathcal{K}^0_\omega$ the Green's function of the background medium (ρ_0, k_0) satisfying the outgoing Sommerfeld radiation conditions and with

$$\alpha := \frac{1}{\rho_\varepsilon} - \frac{1}{\rho_0}\,, \qquad \beta := \frac{1}{k_\varepsilon} - \frac{1}{k_0}\,, \tag{1.4}$$

the contrasts between the inner and the outer acoustic coefficients, by the Lippmann-Schwinger representation of the total acoustic field u we have

$$u(x) - \alpha\,\nabla_x\cdot\int_{\Omega^\varepsilon}\mathcal{G}^0_\omega(x-y)\nabla u(y)dy - \beta\omega^2\int_{\Omega^\varepsilon}\mathcal{G}^0_\omega(x-y)u(y)dy = u^{\text{in}}(x)\,. \tag{1.5}$$

An integration by parts allows to transform this integro-differential equation into a solely integral equation (see detailed computations in [12, Sec. 3])

$$u(x) - (\beta - \alpha\rho_\varepsilon/k_\varepsilon)\,\omega^2\int_{\Omega^\varepsilon}\mathcal{G}^0_\omega(x-y)u(y)dy + \alpha\int_{\Gamma^\varepsilon}\mathcal{G}^0_\omega(x-y)\frac{\partial u}{\partial\nu}(y)dy = u^{\text{in}}(x)\,. \tag{1.6}$$

We next rephrase this problem using the Newtonian (volume-type) operator

$$N_\omega(\varepsilon) : L^2(\Omega^\varepsilon) \to L^2(\Omega^\varepsilon), \qquad (N_\omega(\varepsilon)u)(x) := \int_{\Omega^\varepsilon}\mathcal{G}^0_\omega(x-y)u(y)dy\,, \tag{1.7}$$

with image in $H^2(\Omega^\varepsilon)$, and the surface-type operator[2]

$$K^*_\omega(\varepsilon) : H^{-1/2}(\Gamma^\varepsilon) \to H^{-1/2}(\Gamma^\varepsilon), \qquad (K^*_\omega(\varepsilon)\varphi)(x) := p.v.\int_{\Gamma^\varepsilon}\frac{\partial}{\partial\nu_x}\mathcal{G}^0_\omega(x-y)\varphi(y)dy\,. \tag{1.8}$$

[1]In most of the computations, the Lipschitz regularity is enough, and all the results here presented hold with a boundary of class $\mathcal{C}^{1,1}$. However, to avoid too many technicalities, we prefer to work with a smooth boundary.

[2]Here $p.v$ refers to the Cauchy principal value.

The notation adopted is justified by the fact that $K_\omega^*(\varepsilon)$ identifies with the $L^2(\Gamma^\varepsilon)$-adjoint of the well known Neumann-Poincaré operator (see Subsection A.5). Taking the normal derivative (here simply denoted with ∂_ν) and trace on Γ^ε, from (1.6) we obtain the surface integral equation

$$\left(1+\frac{\alpha}{2}\right)\partial_\nu u-(\beta-\alpha\rho_\varepsilon/k_\varepsilon)\,\omega^2\partial_\nu N_\omega(\varepsilon)u+\alpha\,K_\omega^*(\varepsilon)\,(\partial_\nu u)=\partial_\nu u^{\mathrm{in}}\,,\quad \text{at } \Gamma^\varepsilon\,. \tag{1.9}$$

Hence, the total acoustic field in the exterior of the bubble $\mathbb{R}^3\backslash\overline{\Omega^\varepsilon}$ is fully computable from the values $u|\Omega^\varepsilon$ and $\partial_\nu u|_{\mathrm{in}}$ which are solutions of the following closed-form system of integral equations

$$\left[I-(\beta-\alpha\rho_\varepsilon/k_\varepsilon)\,\omega^2N_\omega(\varepsilon)\right]u+\alpha\int_{\Gamma^\varepsilon}\mathcal{G}_\omega^0(x-y)\frac{\partial u}{\partial\nu}(y)\,d\sigma(y)=u^{\mathrm{in}}(x)\,,\qquad \text{in } \Omega^\varepsilon\,, \tag{1.10}$$

$$\left[\frac{1}{\alpha}+\frac{1}{2}+K_\omega^*\left(\varepsilon\right)\right]\partial_\nu u-\frac{(\beta-\alpha\rho_\varepsilon/k_\varepsilon)}{\alpha}\,\omega^2\partial_\nu N_\omega(\varepsilon)u=\frac{1}{\alpha}\,\partial_\nu u^{\mathrm{in}}\,,\qquad \text{at } \Gamma^\varepsilon\,. \tag{1.11}$$

The Newtonian and the Neumann-Poincaré operators appearing above can be similarly defined on the dilated domain Ω; let us simply denote them as N_ω and K_ω^* in this case. When $\omega=0$, each of the operators N_0 and K_0^* generates discrete sequences of eigenvalues

$$\sigma_p\left(N_0\right)=\left\{\lambda_m\right\}_{m\in\mathbb{N}}\subset\mathbb{R}\,,\quad \text{and}\quad \sigma_p\left(K_0^*\right)\subset\left[-\frac{1}{2},\frac{1}{2}\right)\,. \tag{1.12}$$

These are the key properties in estimating the resonances. Since $N_\omega(\varepsilon)$ and $K_\omega^*(\varepsilon)$ scale as: $N_\omega(\varepsilon)\sim\varepsilon^2N_0$ and $K_\omega^*(\varepsilon)\sim-1/2+\varepsilon^2\left(\partial_\omega K_\omega^*\right)(0)$ as $\varepsilon\to0_+$, these operators may excite the eigenvalues of N_0 or K_0^* and create a singularity in (1.10)-(1.11) depending on the scales defining our micro-bubbles.

- For low density / low bulk bubbles, we have $(\beta-\alpha\rho_\varepsilon/k_\varepsilon)\sim1$ and then $(\beta-\alpha\rho_\varepsilon/k_\varepsilon)\,\omega^2N_\omega(\varepsilon)\ll1$ as $\varepsilon\ll1$. Hence, there is no singularity coming from (1.10). Nevertheless, if $\alpha\sim\varepsilon^{-2}$ as $\varepsilon\ll1$ (which corresponds to the assumption of a low density and bulk regime with $r=2$) then a suitable choice of ω allows to excite the eigenvalue $-1/2$ of K_0^* and create a singularity in (1.11). In this case, we have the Minnaert resonance with surface-modes. This resonance was first observed in [7] based on indirect integral equation methods. This result was extended to more general families of micro-bubbles in [3] and [4].
- For moderate density / low bulk bubbles, we have $\alpha\sim1$ and then we keep away from the full spectrum of K_0^*. Hence there is no singularity coming from (1.11). But as $(\beta-\alpha\rho_\varepsilon/k_\varepsilon)\sim\varepsilon^{-2}\gg1$, suitable choices of ω allow to excite the eigenvalues of the Newtonian operators N_0 and create singularities in (1.10). This gives us a sequence of resonances with volumetric-modes which were observed in [5] and [21].
- Observe that if α is negative (i.e. *negative mass densities*, similar to the Drude model for electromagnetism for instance) then we could excite the other sequence of eigenvalues of K_0^*. This gives us another sequence of resonances (i.e. corresponding to the sequence of plasmonics in electromagnetics).

When the incident frequency ω is close to the ones generating the singularities, the total field inside the bubble, solution of the system (1.10)-(1.11) becomes large. This implies an enhancement of the scattered and far-fields, and motivates the definition of *resonant frequency*, which became widely used in a somehow generic sense. The limit behaviour of the scale-dependent scattering problem (1.6) has been described in [7],[12], where the asymptotic analysis is developed using layer potential techniques. The expansions provided in these works are carried out under specific constraints allowing to consider the scattering problem when the frequency is only close to the resonant value, but does not coincide with. Hence, while suggesting a critical scattering enhancement, these results do not clarify if the resonant frequencies correspond to – or are related with – singular values of a scattering amplitude, neither if – conversely – the corresponding scattering problem is well posed.

1.2 An equivalent frequency-dependent Schrödinger operator

The asymptotic framework is next realized by contrasting an homogeneous acoustic background with a small homogeneous inclusion, supported on Ω^ε, whose acoustic density and bulk are both defined by the piecewise constant field $1_{\mathbb{R}^3\backslash\Omega_\varepsilon}+\varepsilon^{-2}1_{\Omega^\varepsilon}$. Following the notation introduced in (1.1)-(1.2), we assume

$$\rho=k:=\begin{cases}1/\varepsilon^2 & \text{inside } \Omega^\varepsilon\,,\\ 1 & \text{outside } \Omega^\varepsilon\,.\end{cases} \tag{1.13}$$

Since both contrasts are of size ε^{-2}, this regime defines a low density / low bulk bubble and – according to our previous discussion – generates an asymptotically bounded Minnaert resonance with dominant part independent of ε. Furthermore, in this particular scaling we have $(\beta - \alpha\rho_\varepsilon/k_\varepsilon) = 0$ (compare with (1.2)) which cancels the body-potential contribution in (1.6). In what follows we incorporate the assumption (1.13) in our scattering problem and provide with a frequency-dependent auxiliary operator allowing to rephrase (1.3) in terms of a generalized eigenfunction problem. This approach requires a large use of layer mappings, potentials and integral operators which naturally appear in the modeling of scattering from interfaces and obstacles. The precise definitions, the related mapping properties and the common notation are recalled in the Appendix. When these operators refer to the contracted boundary Γ^ε we adopt appropriate notation which are next introduced. Let $\gamma_0^{\mathrm{in/ex}}(\varepsilon)$ and $\gamma_1^{\mathrm{in/ex}}(\varepsilon)$ denote the lateral traces and normal-traces operators on Γ^ε; the corresponding mean-traces and jumps are $\gamma_0(\varepsilon)$, $\gamma_1(\varepsilon)$ and $[\gamma_0(\varepsilon)]$, $[\gamma_1(\varepsilon)]$ respectively. The acoustic scattering equation (1.3) writes as

$$\begin{cases} \left(\nabla\left(1_{\mathbb{R}^3\backslash\Omega^\varepsilon} + \varepsilon^{-2}1_{\Omega^\varepsilon}\right)\nabla + \omega^2\left(1_{\mathbb{R}^3\backslash\Omega^\varepsilon} + \varepsilon^{-2}1_{\Omega^\varepsilon}\right)\right)u = 0\,, & \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\ \\ [\gamma_0(\varepsilon)]\,u = 0\,, \qquad \left(\gamma_1^{\mathrm{ex}}(\varepsilon) - \varepsilon^{-2}\gamma_1^{\mathrm{in}}(\varepsilon)\right)u = 0\,, & \text{on } \Gamma^\varepsilon\,, \end{cases} \tag{1.14}$$

Since the co-normal jump condition implies

$$[\gamma_1(\varepsilon)]u = (\varepsilon^{-2} - 1)\gamma_1^{\mathrm{in}}(\varepsilon)u\,, \tag{1.15}$$

this problem rephrases as

$$\begin{cases} \left(\Delta + \omega^2\right)u = 0\,, & \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\ \\ [\gamma_0(\varepsilon)]\,u = 0\,, \qquad [\gamma_1(\varepsilon)]\,u = \left(\varepsilon^{-2} - 1\right)\gamma_1^{\mathrm{in}}(\varepsilon)u\,, & \text{on } \Gamma^\varepsilon\,. \end{cases} \tag{1.16}$$

Let us recall that the Dirichlet-to-Neumann operator for the domain Ω^ε, next denoted with $DN_z(\varepsilon)$, is defined by

$$DN_z(\varepsilon)\varphi := \gamma_1^{\mathrm{in}}(\varepsilon)\,u\,, \qquad \begin{cases} \left(\Delta + z^2\right)u = 0\,, & \text{in } \Omega^\varepsilon\,, \\ \\ \gamma_0^{\mathrm{in}}(\varepsilon)u = \varphi & \text{on } \Gamma^\varepsilon\,. \end{cases} \tag{1.17}$$

Such a definition is well-posed whenever $z^2 \notin \sigma(-\Delta^D_{\Omega^\varepsilon})\}$, where $\Delta^D_{\Omega^\varepsilon}$ is the Dirichlet Laplacian in $L^2(\Omega^\varepsilon)$. Since $\lambda_\Omega := \inf\sigma(-\Delta^D_\Omega) > 0$, by

$$z^2 \in \sigma\left(-\Delta^D_{\Omega^\varepsilon}\right) \quad \Longleftrightarrow \quad \varepsilon^2 z^2 \in \sigma\left(-\Delta^D_\Omega\right)\,, \tag{1.18}$$

there follows that for each $z \in \mathbb{C}$ there exists $\varepsilon_0 > 0$ small enough (depending on z) such that $DN_z(\varepsilon)$ exists for all $0 < \varepsilon < \varepsilon_0$. Assuming $\varepsilon^2\omega^2 \notin \sigma(\Delta^D_\Omega)$, the solution of (1.16) solves the homogeneous problem in (1.17) with $z^2 = \omega^2$ and with boundary datum $\gamma_0^{\mathrm{in}}(\varepsilon)u$. By definition, we have

$$\gamma_1^{\mathrm{in}}(\varepsilon)\,u = DN_\omega(\varepsilon)\,\gamma_0^{\mathrm{in}}(\varepsilon)\,u \tag{1.19}$$

and (1.16) recasts to

$$\begin{cases} \left(\Delta + \omega^2\right)u = 0\,, & \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\ \\ [\gamma_0(\varepsilon)]\,u = 0\,, \qquad [\gamma_1(\varepsilon)]\,u = \left(\varepsilon^{-2} - 1\right)DN_\omega(\varepsilon)\,\gamma_0(\varepsilon)u\,, & \text{on } \Gamma^\varepsilon\,. \end{cases} \tag{1.20}$$

Let us recall from [17] and [19] that, given: $s \in (0, 1/2)$ and $\Theta \in \mathsf{B}\left(H^s(\Gamma^\varepsilon), H^{-s}(\Gamma^\varepsilon)\right)$ selfadjoint (in the sense of the duality) and defining $\Theta\delta_{\partial\Omega_\varepsilon} \in \mathcal{D}'\left(\mathbb{R}^3\right)$ as

$$\Theta\delta_{\partial\Omega_\varepsilon}u := \int_{\partial\Omega_\varepsilon} d\sigma\,\Theta\left(\gamma(\varepsilon)\,u\right)\,, \tag{1.21}$$

a selfadjoint (in $L^2(\mathbb{R}^3)$) realization of $u \mapsto -\Delta u + \alpha\delta_{\partial\Omega_\varepsilon}u|$ is provided by the restriction of $\left(\Delta|\ker\left(\gamma_0(\varepsilon)\right)\right)^*$ to functions fulfilling interface conditions of the kind

$$[\gamma_0(\varepsilon)]\,u = 0\,, \qquad [\gamma_1(\varepsilon)] = \Theta\gamma_0(\varepsilon)\,u\,. \tag{1.22}$$

This suggests a formal analogy between our problem and the generalized eigenfunction equation for singular perturbations of the Laplacian with δ-type transmission conditions (see results in [18]). As a further support to this remark, we also notice that the integral form of (1.20) simply reads as (compare with (1.6))

$$u = u^{\mathrm{in}} - \left(\varepsilon^{-2} - 1\right)SL_\omega(\varepsilon)\,\gamma_1^{\mathrm{in}}(\varepsilon)\,u\,, \tag{1.23}$$

where u^{in} is an incoming wave (in this case a solution of $\left(\Delta+\omega^2\right)u^{\mathrm{in}}=0$ in $\mathbb{R}^3$) and $SL_\omega(\varepsilon)$ is the single-layer operator related to Γ^ε. Hence, the scattered field is represented in terms of a single-layer potential, which, as it has been shown in [18], corresponds to the solution form of the scattering problem for δ-type singular perturbations of the free Laplacian.

A specific feature of classical scattering problems consists in the fact that the total field identifies with a generalized eigenfunction of an auxiliary Schrödinger-type operator which usually depends on the frequency. This is quite evident when one considers the simpler stationary problem for classical waves propagating in a medium with a local perturbation of the bulk. Assume for instance to have a piecewise constant bulk described by $b_0 1_\Omega + 1_{\mathbb{R}^3\backslash\Omega}$; then, a stationary wave with frequency $\omega>0$ solves the equation

$$\left(\Delta-\omega^2\left(1-b_0\right)1_\Omega+\omega^2\right)u=0\,,$$

corresponding to the generalized eigenfunction problem at energy ω^2 for the Schrödinger operator $-\Delta+\omega^2\left(1-b_0\right)1_\Omega$. In the attempt of adapting this construction to the more complex framework considered in (1.20), which involves a discontinuity on Γ^ε both for the acoustic bulk and density, we push further the analogy and consider $H_\omega(\varepsilon)$ of the form

$$\Delta+\left(\varepsilon^{-2}-1\right)DN_\omega\left(\varepsilon\right)\delta_{\Gamma^\varepsilon}\,, \tag{1.24}$$

as a candidate for the frequency-dependent operator to identify the solutions of (1.20) in terms of generalized eigenfunctions of $H_\omega(\varepsilon)$ at energy ω^2.

1.3 The main results

The precise definition of $H_\omega(\varepsilon)$ is

$$H_\omega\left(\varepsilon\right)u:=\Delta u\,,\qquad \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \tag{1.25}$$

for any u in the domain

$$\operatorname{dom}\left(H_\omega\left(\varepsilon\right)\right):=\left\{u\in H^0_\Delta(\Omega^\varepsilon\backslash\Gamma^\varepsilon)\cap H^1(\mathbb{R}^3):[\gamma_1\left(\varepsilon\right)]u=\left(\varepsilon^{-2}-1\right)DN_\omega\left(\varepsilon\right)\gamma_0\left(\varepsilon\right)u\right\}\,, \tag{1.26}$$

where $H^0_\Delta(\Omega^\varepsilon\backslash\Gamma^\varepsilon)$ is the set of the functions $u\in L^2(\mathbb{R}^3)$ such that

$$\Delta_{\mathbb{R}^3\backslash\Gamma^\varepsilon}u:=1_{\Omega^\varepsilon}\Delta u+1_{\mathbb{R}^3\backslash\overline{\Omega^\varepsilon}}\Delta u\in L^2(\mathbb{R}^3)\,. \tag{1.27}$$

Notice that the jump condition $[\gamma_0\left(\varepsilon\right)]u=0$ is incorporated into $\operatorname{dom}(H_\omega(\varepsilon))\subseteq H^1(\mathbb{R}^3)$. The properties of $H_\omega(\varepsilon)$ are investigated in Section 3. We next resume the main features of this model. Let S_0 denote the single layer operator of the Laplacian in the whole space (see the definition in Subsection A.4). The *capacitance* of Ω is defined by

$$c_\Omega:=\int_\Gamma(S_0^{-1}1)(x)\,d\sigma(x)\,, \tag{1.28}$$

and the related *Minnaert frequency* is

$$\omega_M:=\sqrt{\frac{c_\Omega}{|\Omega|}}\,, \tag{1.29}$$

where $|\Omega|$ denotes the volume of Ω. It is worth recalling that the positiveness of S_0 implies $c_\Omega>0$. According to Theorem 3.1 and definitions in Subsection 3.3, for each $\omega>0$ and $\varepsilon>0$ sufficiently small, (1.26) and (1.25) define a self-adjoint operator in $L^2\left(\mathbb{R}^3\right)$. The corresponding resolvent equation

$$(H_\omega\left(\varepsilon\right)+z^2)u=f\,, \tag{1.30}$$

is nothing but

$$\begin{cases}\left(\Delta+z^2\right)u=f\,, & \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,,\\[2mm] [\gamma_0\left(\varepsilon\right)]\,u=0\,, & [\gamma_1\left(\varepsilon\right)]\,u=\left(\varepsilon^{-2}-1\right)DN_\omega\left(\varepsilon\right)\gamma_0\left(\varepsilon\right)u\,.\end{cases} \tag{1.31}$$

Theorem 4.1 – whose statement is here reproduced – provides the asymptotic expansion of $R^\omega_z(\varepsilon):=\left(-H_\omega(\varepsilon)-z^2\right)^{-1}$.

Theorem 1.1 *For any $z\in\mathbb{C}_+\backslash i\mathbb{R}_+$ and for any $\varepsilon>0$ sufficiently small, one has*

$$\omega\neq\omega_M\quad\Longrightarrow\quad\|R^\omega_z(\varepsilon)-R_z\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)}\leq c\,\varepsilon\,, \tag{1.32}$$

$$\omega=\omega_M\quad\Longrightarrow\quad\left\|R^\omega_z(\varepsilon)-\widehat{R}_z\right\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)}\leq c\,\varepsilon^{1/2}\,. \tag{1.33}$$

Here R_z denotes the resolvent of the free Laplacian with kernel $\mathcal{G}_z(x-y)=\frac{e^{iz|x-y|}}{4\pi|x-y|}$, while the operator $\widehat{R}_z$ has kernel

$$\widehat{\mathcal{R}}_z(x,y):=\mathcal{G}_z(x-y)+4\pi\,\frac{i}{z}\,\mathcal{G}_z(x-y_0)\,\mathcal{G}_z(y-y_0)\,. \tag{1.34}$$

As this result shows, the singular frequency ω_M identifies with the value corresponding to a non-trivial resolvent limit of $H_\omega(\varepsilon)$.

According to the resolvent equation (1.30), the acoustic scattering problem – in the equivalent form (1.20) – identifies with the generalized eigenfunctions problem for $H_\omega(\varepsilon)$. This important feature is discussed in details in Section 5. The interest in establishing such relation is not merely formal. Indeed, it allows to represent the scattered field in terms of a Limiting Absorption Principle. Hence, the asymptotic behavior of the solutions of (1.20) is determined by similar computation to the ones leading to the norm-resolvent asymptotic expansions of $H_\omega(\varepsilon)$. In particular, we consider two regimes in terms of ω and ε. In the first one, discussed in Corollary 6.1, we fix ω and provide the expansion in term of ε only.

Theorem 1.2 *Let $u_\omega(\varepsilon) = u_\omega^{\rm in} + u_\omega^{\rm sc}(\varepsilon)$ be the unique radiating solution of the boundary value problem*

$$\begin{cases} \left(\nabla\cdot(1_{\mathbb{R}^3\backslash\Omega^\varepsilon}+\varepsilon^{-2}1_{\Omega^\varepsilon})\nabla+\omega^2(1_{\mathbb{R}^3\backslash\Omega^\varepsilon}+\varepsilon^{-2}1_{\Omega^\varepsilon})\right)u_\omega(\varepsilon)=0\,, \\ \gamma_0^{\rm in}(\varepsilon)u_\omega\,(\varepsilon)=\gamma_0^{\rm ex}(\varepsilon)u_\omega(\varepsilon)\,, \qquad \varepsilon^{-2}\gamma_1^{\rm in}(\varepsilon)u_\omega\,(\varepsilon)=\gamma_1^{\rm ex}(\varepsilon)u_\omega(\varepsilon)\,. \end{cases} \tag{1.35}$$

Then, for any $\varepsilon > 0$ sufficiently small, one has, uniformly with respect to the choice of the incoming wave $u_\omega^{\rm in}$,

$$\omega\neq\omega_M \quad\Longrightarrow\quad \begin{cases} \left(u_\omega^{\rm sc}(\varepsilon)\right)(x)=\varepsilon\,\frac{\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\;u_\omega^{\rm in}(y_0)\,\frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|}+\left(r_\omega(\varepsilon)\right)(x)\,, \\ \|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)}\leq c\,\varepsilon^{3/2}\,, \qquad \alpha>1/2\,, \end{cases} \tag{1.36}$$

$$\omega=\omega_M \quad\Longrightarrow\quad \begin{cases} \left(u_\omega^{\rm sc}(\varepsilon)\right)(x)=\frac{4\pi i}{\omega}\;u_\omega^{\rm in}(y_0)\,\frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|}+\left(r_\omega\,(\varepsilon)\right)(x)\,, \\ \|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)}\leq c\,\varepsilon^{1/2}\,, \qquad \alpha>1/2\,, \end{cases} \tag{1.37}$$

In the second one, discussed in Theorem 6.4, we provide the expansions by varying both ω and ε.

Theorem 1.3 *Let $\mathcal{I}_M\subset\mathbb{R}_+$ be a bounded interval containing ω_M. For any $\varepsilon>0$ sufficiently small, uniformly with respect to $\omega\in\mathcal{I}_M$ such that $|\omega-\omega_M|\geq c_M\,\varepsilon$, $c_M>0$, and with respect to the choice of the incoming wave $u_\omega^{\rm in}$, one has*

$$\left(u_\omega^{\rm sc}(\varepsilon)\right)(x)=\frac{\varepsilon\,\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\,\left(1-i\,\frac{\varepsilon\,\omega^3\,c_\Omega}{4\pi\,(\omega_M^2-\omega^2)}\right)^{-1}u_\omega^{\rm in}(y_0)\,\frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|}+\left(r_\omega(\varepsilon)\right)(x)\,, \tag{1.38}$$

$$\|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)}\leq c\;\frac{\varepsilon^{3/2}}{\omega_M^2-\omega^2}\,, \qquad \alpha>1/2\,. \tag{1.39}$$

When ω approaches this singularity, the scattering system undergoes a transition between an asymptotically trivial scattering and a non-trivial one. The transition is enlighten in (1.38) with the lower-bound condition $|\omega-\omega_M|>c_M\,\varepsilon$. Since the remainders fulfill an outgoing radiation conditions, the previous expansions rephrase in terms of the far-fields pattern (a.k.a. scattering amplitude) $u_\omega^\infty(\varepsilon)$ as follows (see Lemmata 6.2 and 6.5)

$$u_\omega^\infty(\varepsilon)=\frac{\varepsilon\,\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\;u_\omega^{\rm in}(y_0)\,e^{-i\omega\,(\cdot)\cdot y_0}+O(\varepsilon^{3/2})\,, \qquad \omega\neq\omega_M\,, \tag{1.40}$$

$$u_\omega^\infty(\varepsilon)=\frac{4\pi i}{\omega}\;u_\omega^{\rm in}(y_0)\,e^{-i\omega_M\,(\cdot)\cdot y_0}+O(\varepsilon^{1/2})\,, \qquad \omega=\omega_M\,, \tag{1.41}$$

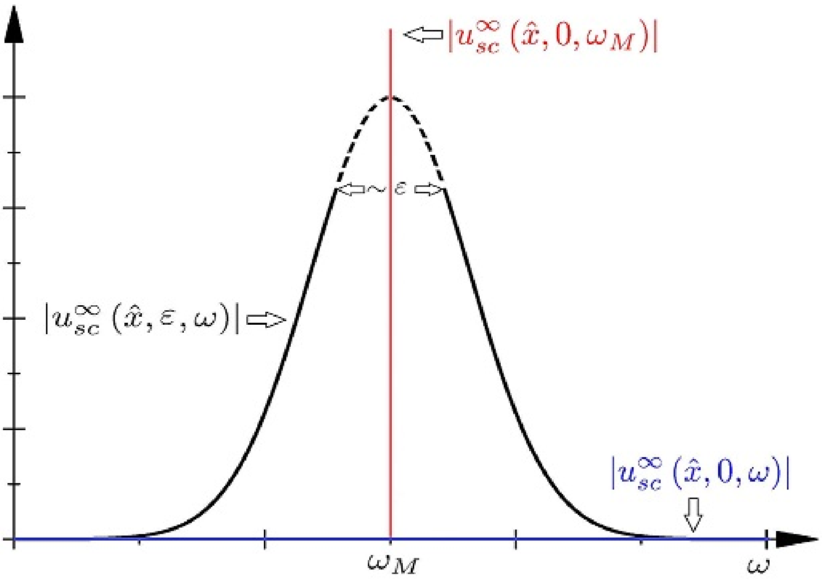
$$u_\omega^\infty(\varepsilon)=\frac{\varepsilon\,\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\,\left(1-i\,\frac{\omega}{4\pi}\,\frac{\varepsilon\,\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\right)^{-1}u_\omega^{\rm in}(y_0)\,e^{-i\omega\,(\cdot)\cdot y_0}+O\left(\frac{\varepsilon^{3/2}}{\omega_M^2-\omega^2}\right), \quad |\omega-\omega_M|\geq c_M\,\varepsilon\,, \tag{1.42}$$

where the $O\,(\cdot)$'s are in the sense of the $L^2(\mathbb{S}^2)$-norm, $\mathbb{S}^2$ denoting the unit sphere in $\mathbb{R}^3$. Let us assume $\left|u_\omega^{\rm in}(y_0)\right|\neq 0$; taking the limits $\varepsilon\to 0_+$ in the above expansions we get

$$|u_\omega^\infty(0_+)|=0\,, \qquad \omega\neq\omega_M\,, \tag{1.43}$$

$$\left|u_{\omega_M}^\infty\,(0_+)\right|=\frac{4\pi}{\omega_M}\left|u_{\omega_M}^{\rm in}(y_0)\right|\,. \tag{1.44}$$

This provides the following qualitative picture:



Here the dash line represents the expected behaviour of $\omega \mapsto |(u_{\omega}^{\infty}(\varepsilon))(\hat{x})|$ close to ω_M. It is worth noticing that our quasi-resonant expansion has a qualitative agreement with experimental observations, where one observes a scattering peak in correspondence with the Minnaert frequency.

Although the scattering enhancement described so far and referred to as a Minnaert resonance involves the generalized eigenfunctions at frequency ω_M of the frequency-dependent operator $H_{\omega_M}(\varepsilon)$, it is worth to underline that this phenomenon does not originates from the appearance of spectral poles of our equivalent model. As the resolvent analysis shows indeed (see Remark 3.7), the map $z \to \left(-H_{\omega}(\varepsilon) - z^2\right)^{-1}$ is analytic in the neighbourhood of any $\omega > 0$: this corresponds to the absence of eigenvalues or resonances of $H_{\omega}(\varepsilon)$ around any point $z^2 = \omega^2$ of the positive continuous spectrum. There is, however, a strong indication that in the resonant regime $H_{\omega_M}(\varepsilon)$ may have an eigenvalue/resonance localized in a neighbourhood of size $\sim \varepsilon$ of the origin.

The mechanism behind the appearance of such Minnaert's frequencies can be explained in terms of the discontinuous behavior in the limit $\varepsilon \to 0_+$ of the operator $H_{\omega}(\varepsilon)$, whose generalized eigenfunction at frequency ω identifies with the solution of the acoustic scattering problem. According to Theorem 1.1, $H_{\omega}(\varepsilon)$ converges to the unperturbed Laplacian in the norm-resolvent sense for all $\omega \neq \omega_M$, while for $\omega = \omega_M$, we have a well-defined non-trivial limit: the limit resolvent operator $\widehat{R}_z$ appearing in Theorem 1.1, is the resolvent of a point interaction Hamiltonian supported in y_0 with infinite scattering length, which is also known to have a zero energy resonance (we refer to Subsection 4 and to [2, Chap. 1] for details). Thus, ω_M is the only value of ω for which $H_{\omega}(\varepsilon)$ converges toward the generator of a non-trivial dynamics, giving rise therefore to a non-trivial asymptotic scattering. As a by-product, our computations allow to recover the asymptotic formulae provided in [7] and [12].

1.4 Approach and perspectives

A common approach to the analysis of perturbations with small support and high contrast consists in introducing an equivalent dilated system. Following this strategy, in Theorem 3.1, we build a family of self-adjoint operators $H_{\varepsilon,\omega}$, depending both on the scale parameter ε and the frequency ω, which encode the dilated interface conditions at Γ

$$[\gamma_0]u = 0\,, \quad [\gamma_1]u = (\varepsilon^{-2} - 1) DN_{\omega} \gamma u\,. \tag{1.45}$$

The definition of $H_{\varepsilon,\omega}$ requires to overcome the loss of regularity entailed by the use of the Dirichlet-to-Neumann map (which is a pseudodifferential operator of order one). This problem is solved using the regime of small ε and the estimates provided in Section 2. The physical operators $H_{\omega}(\varepsilon)$ are defined in Subsection 3.3 by the resolvent identity

$$R_z^{\omega}(\varepsilon) := (-H_{\omega}(\varepsilon) - z^2)^{-1} = \varepsilon^2 U_{\varepsilon} \left(H_{\varepsilon,\omega} - z^2\right)^{-1} U_{\varepsilon}^{-1}\,, \tag{1.46}$$

where U_{ε} is the unitary dilation mapping $L^2(\Omega^{\varepsilon})$ (and $L^2(\mathbb{R}^3 \backslash \overline{\Omega^{\varepsilon}})$) onto $L^2(\Omega)$ (and $L^2(\mathbb{R}^3 \backslash \overline{\Omega})$). This allows to work with integral operators defined on the fixed boundary Γ and to use the ε-expansion, provided in Sections 2, of the operator $\Lambda_z^{\omega}(\varepsilon)$ which appears in the resolvent and in the scattering representations. In particular, one has the the Kreĭn-like resolvent formula

$$(R_z^\omega(\varepsilon) - R_z)\, f = -\varepsilon U_\varepsilon S L_{\varepsilon z} \Lambda_z^\omega(\varepsilon) \gamma_0 R_{\varepsilon z} U_\varepsilon^{-1} f\,. \tag{1.47}$$

Building on the spectral decomposition of the Neumann-Poincaré operator K_z, we prove that (see Theorem 2.6)

$$\Lambda_z^\omega(\varepsilon) = \frac{1}{\varepsilon}\, S_{\varepsilon\omega}^{-1} \left(\frac{\omega_M^2}{\omega_M^2 - \omega^2}\, P_0 + P_0 O(\varepsilon) P_0 + O(\varepsilon^2) \right) S_{\varepsilon\omega} DN_{\varepsilon\omega}\,, \quad \omega \neq \omega_M\,, \tag{1.48}$$

$$\Lambda_z^\omega(\varepsilon) = \frac{1}{\varepsilon^2}\, S_{\varepsilon\omega}^{-1} \left(\frac{4\pi}{c_\Omega}\, \frac{i}{z}\, P_0 + P_0 O(\varepsilon) P_0 + O(\varepsilon^2) \right) S_{\varepsilon\omega} DN_{\varepsilon\omega}\,, \qquad \omega = \omega_M\,, \tag{1.49}$$

where P_0 is the rank-one projector on the first eigenstate of K_0 and S_0 is the single layer operator. In Theorem 1.1 these expansions and the ones provided in Lemma A.4 and Corollary A.5 are implemented in the resolvent formula (1.47) to get the resolvent limits (see the proof in Theorem 3.1). In Theorem 2, a similar formula for the scattering solution

$$u_\omega^{\mathrm{sc}}(\varepsilon) := -\varepsilon U_\varepsilon S L_{\varepsilon\omega} \Lambda_\omega^\omega(\varepsilon) \gamma_0 U_\varepsilon^{-1} u_\omega^{\mathrm{in}}\,, \tag{1.50}$$

combined with the above expansions leads to the asymptotic representations of the scattered field.

Most of the computations can be carried out in the case of multiple low density / low bulk bubbles, each having a specific Minnaert frequency. When multiple bubbles share the same excitation frequency, the expected limit scattering in the resonant regime will be described by a multiple point-scattering system.

Expansions similar to (1.38) have been recently provided in the regime of highly contrasted small inclusions with non-homogeneous acoustic backgrounds and different bulk/density ratios (see [12]). These results suggest that the mechanism leading to the scattering enhancement, enlighten in this work, actually characterizes a much larger class of models. It is worth noticing that, while our simplified setting allows to implement purely singular perturbation methods in the resolvent analysis of the problem, in more general frameworks, the auxiliary Schrödinger operator associated to the scattering problem would exhibit both singular and regular (potential-like) perturbation terms whose resolvent analysis requires some generalization with respect to the one here employed.

Perturbations with small support and high contrast have been considered in connection with the low energy behaviour of Schrödinger operators. In the potential case, the asymptotic problem has been discussed in [1], where the Schrödinger operator $-\Delta + \lambda(\varepsilon)V_\varepsilon$, $V_\varepsilon(x) := \varepsilon^{-2}V(x/\varepsilon)$, $\lambda(0_+) = 1$, was considered. Making use of an unitary dilation, the authors recast this problem in terms of the dilated operator $H_\varepsilon := -\Delta + \lambda(\varepsilon)V$ and use this setting to study the resolvent limit of the original Schrödinger operator as $\varepsilon \to 0_+$. It turns out that it converges either to the unperturbed Laplacian $-\Delta$ or to a point perturbation of $-\Delta$ depending on the spectral profile of H_0. In particular it has a non-trivial limit *if and only if* $H_0 = -\Delta + V$ has a zero-energy resonance. In [23], the case of a δ-shell interaction supported on the sphere $\mathbb{S}_\varepsilon^2 \subset \mathbb{R}^3$ of radius ε was considered in a similar setting. The author shows that $-\Delta + \varepsilon^{-1}\lambda(\varepsilon)\alpha\,\delta_{\mathbb{S}_\varepsilon^2}$, $\alpha \in \mathcal{C}^\infty(\mathbb{S}^2)$, $\lambda(0_+) = 1$, converges in the resolvent sense toward a point perturbation of $-\Delta$ as $\varepsilon \to 0_+$ whenever $-\Delta + \alpha\,\delta_{\mathbb{S}_1^2}$ has a zero-energy resonance.

When the equivalent setting involving the operator $H_\omega(\varepsilon)$ is considered, the physical behaviour of our model has an evident analogy with the ones described in [1] and [23]. Although $H_\omega(\varepsilon)$ does not directly fit in the framework considered in these works, the conclusions of [1] and [23] suggest that a zero-energy resonance should appear at $(\varepsilon, \omega) = (0, \omega_M)$ for the dileted operator $H_{\varepsilon,\omega}$. Despite its theoretical relevance, the analysis of this point is outside the main scope of our work and it is postponed to a further development.

1.5 Notation

· $\mathbb{R}_\pm := \{x \in \mathbb{R} : \pm x > 0\}$; $\mathbb{C}_\pm := \{z \in \mathbb{C} : \mathrm{Im}(z) \in \mathbb{R}_\pm\}$;

· $c \in \mathbb{R}_+$ denotes a generic constant which may vary from line to line;

· $\|\cdot\|_X$ denotes the norm in the Banach space X;

· $\langle \cdot, \cdot \rangle_H$ denotes the (conjugate-linear w.r.t. the first variable) inner product in the Hilbert space H;

· $\langle \cdot, \cdot \rangle_{X^*,X}$ denotes the duality, assumed to be conjugate-linear w.r.t. the first variable, between the dual pair (X^*, X);

· $\mathrm{dom}(L)$, $\ker(L)$ and $\mathrm{ran}(L)$ denote the domain, kernel and range of the linear operator L;

· $\sigma(L)$ and $\varrho(L)$ denote the spectrum and the resolvent set of the closed operator L;

· $L^* : \mathrm{dom}(L^*) \subseteq Y^* \to X^*$ denotes the adjoint of the densely defined linear operator $L : \mathrm{dom}(L) \subseteq X \to Y$;

· $\mathcal{L}(X, Y)$ denotes the space of continuous linear maps on X to Y, where X and Y are topological vector space; $\mathcal{L}(X) \equiv \mathcal{L}(X, X)$;

· $\|\cdot\|_{X,Y}$ denotes the norm on the Banach space $\mathcal{L}(X,Y)$, X and Y Banach spaces;

· $\mathcal{L}_{HS}(X,Y)$ denotes the set of Hilbert-Schmidt operators on X to Y;

· Given $\varepsilon \mapsto L(\varepsilon) \in \mathcal{L}(X,Y)$, $L(\varepsilon) = O(\varepsilon^\lambda)$ means $\|L(\varepsilon)\|_{X,Y} \le c\,\varepsilon^\lambda$;

· $\mathcal{C}^\infty_{\text{comp}}(\mathcal{O})$ and $\mathcal{D}(\mathcal{O})$ both denote the set of smooth, compactly supported, test functions on the open set $\mathcal{O} \subseteq \mathbb{R}^3$; $\mathcal{D}'(\mathcal{O})$ denotes the space of Schwartz's distributions and $\mathcal{E}'(\mathcal{O})$ denotes the spaces of compactly supported Schwartz's distributions;

· $\Delta_{\mathcal{O}}$ denotes the distributional Laplacian in $\mathcal{D}'(\mathcal{O})$; $\Delta_{\mathbb{R}^3}$ is simply denoted by Δ;

· $u * v$ denotes convolution;

· $\delta_y \in \mathcal{E}'(\mathbb{R}^3)$ denotes Dirac's delta distribution supported at the point y;

· Given the bounded open set $\Omega \subset \mathbb{R}^3$ with a regular boundary Γ, $|\Omega|$ and $|\Gamma|$ denote the volume of Ω and the area of Γ respectively; d_Ω denotes the diameter of Ω and $d\sigma$ denotes the surface measure on Γ;

· $\Omega_{\text{in}} \equiv \Omega$ and $\Omega_{\text{ex}} := \mathbb{R}^3 \backslash \overline{\Omega}$;

· $H^s(\mathbb{R}^3)$, $H^s(\Omega_{\text{in/ex}})$, $s \in \mathbb{R}$, denote the usual scales of Sobolev spaces on $\mathbb{R}^3$ and $\Omega_{\text{in/ex}}$ respectively;

· $H^s(\Gamma)$, $s \in \mathbb{R}$, denotes the usual scales of Sobolev spaces on Γ;

· $H^s_\alpha(\mathbb{R}^3)$, $H^s_\alpha(\Omega_{\text{ex}})$, $s \in \mathbb{R}$, $\alpha \in \mathbb{R}$, denote the scales of weighted Sobolev spaces on $\mathbb{R}^3$ and Ω_{ex} with weight $\langle x\rangle^\alpha \equiv (1+|x|^2)^{\alpha/2}$;

· γ_0 and γ_1 denote the Dirichlet and Neumann traces at Γ; $\gamma_0^{\text{in/ex}}$ and $\gamma_1^{\text{in/ex}}$ denote the analogous lateral traces at the boundary of $\Omega_{\text{in/ex}}$;

· SL_z and DL_z denote the single and double layer operators;

· S_z and K_z denote the boundary operators $\gamma_0 SL_z$ and $\gamma_0 DL_z$ respectively;

· DN_z denotes the Dirichlet-to-Neumann operator;

· R_z denotes the resolvent of the free Laplacian: $R_z := (-\Delta - z^2)^{-1}$;

· $\mathcal{G}_z(x)$ denotes 3-dimensional Green's function, i.e., $\mathcal{G}_z(x) := \frac{e^{iz|x|}}{4\pi|x|}$;

· c_Ω denotes the capacitance of Ω;

· $\omega_M > 0$ denotes the Minnaert frequency defined by $\omega_M^2 := c_\Omega/|\Omega|$;

· Δ^D_Ω and Δ^N_Ω denote the self-adjoint Dirichlet and Neumann Laplacians in $L^2(\Omega)$.

2 The main operator-expansions

The asymptotic expansion presented in this section provides the main technical tool of our work. The definitions and known properties regarding the functional spaces and the boundary integral operators involved in this construction are recalled in the Appendix.

In what follows, the convergence of the Neumann's series for $(1-L)^{-1}$ when $\|L\|<1$, is used in the following form.

Lemma 2.1 *Let $L(\varepsilon)$ be a bounded operator family such that $L(\varepsilon)=L_0+L_1O(\varepsilon^{\lambda_1})+O(\varepsilon^{\lambda_2})$, $0<\lambda_1<\lambda_2$. If L_0 has a bounded inverse and ε is sufficiently small, then $L(\varepsilon)$ has a bounded inverse as well and $L(\varepsilon)^{-1}=(1-L_1O(\varepsilon^{\lambda_1}))L_0^{-1}+O(\varepsilon^{\lambda})$. Here $\lambda=2\lambda_1$ whenever $L_1\neq 0$, $\lambda=\lambda_2$ otherwise.*

We next assume that $\Omega\subset\mathbb{R}^3$ is open and bounded with a smooth boundary Γ (a boundary of class $\mathcal{C}^{1,1}$ would suffice but, in order to simplify the exposition, here we do not strive for the maximum of generality).

Theorem 2.2 *Let P_0 and Q_0 be the spectral projectors (A.18)-(A.19) on $H^{1/2}(\Gamma)$ with the inner product* (A.15) *and $\widehat{\oplus}$ denotes the orthogonal sum. With respect to the decomposition $H^{1/2}(\Gamma)=\operatorname{ran}(P_0)\,\widehat{\oplus}\operatorname{ran}(Q_0)$, the operator*

$$\varepsilon^2+\left(1-\varepsilon^2\right)\left(1/2+K_{\varepsilon\omega}\right)S_{\varepsilon z}S_{\varepsilon\omega}^{-1}\in\mathcal{L}(H^{1/2}(\Gamma)) \tag{2.1}$$

writes as

$$\mathbb{M}(\varepsilon)\equiv\begin{bmatrix}M_{00}(\varepsilon) & M_{01}(\varepsilon)\\ M_{10}(\varepsilon) & M_{11}(\varepsilon)\end{bmatrix}:\operatorname{ran}(P_0)\,\widehat{\oplus}\operatorname{ran}(Q_0)\to\operatorname{ran}(P_0)\,\widehat{\oplus}\operatorname{ran}(Q_0)\,, \tag{2.2}$$

$$M_{00}(\varepsilon):=P_0\left(\left(1+\omega^2K_{(2)}\right)\varepsilon^2+\left((z-\omega)\,\omega^2K_{(2)}S_{(1)}S_0^{-1}+\omega^3K_{(3)}\right)\varepsilon^3+O(\varepsilon^4)\right)P_0\,, \tag{2.3}$$

$$M_{01}(\varepsilon):=P_0O(\varepsilon^2)Q_0\,,\qquad M_{10}(\varepsilon):=Q_0O(\varepsilon^2)P_0\,, \tag{2.4}$$

$$M_{11}(\varepsilon):=Q_0\left(1/2+K_0+O(\varepsilon^2)\right)Q_0\,, \tag{2.5}$$

Moreover, the Schur complement of $M_{11}(\varepsilon)$, defined by

$$C_{00}(\varepsilon):=M_{00}(\varepsilon)-M_{01}(\varepsilon)M_{11}(\varepsilon)^{-1}M_{10}(\varepsilon)\,, \tag{2.6}$$

writes as

$$C_{00}(\varepsilon)=P_0\left(\left(1+\omega^2K_{(2)}\right)\varepsilon^2+\left((z-\omega)\,\omega^2K_{(2)}S_{(1)}S_0^{-1}+\omega^3K_{(3)}\right)\varepsilon^3+O(\varepsilon^4)\right)P_0\,. \tag{2.7}$$

For each $\omega\in\mathbb{C}$, such expansions hold whenever ε is sufficiently small, uniformly with respect to z in any fixed ball of $\mathbb{C}$.

Proof. By Lemma A.1, when ε is sufficiently small, $\varepsilon\omega\in\mathbb{C}\backslash D_\Omega$ and the expansion

$$\begin{aligned}S_{\varepsilon z}S_{\varepsilon\omega}^{-1}=&1+\left(S_{\varepsilon z}-S_{\varepsilon\omega}\right)S_{\varepsilon\omega}^{-1}\\=&1+\left(\varepsilon(z-\omega)S_{(1)}+O((\varepsilon|z|)^2)+O((\varepsilon\omega)^2)\right)\left(S_0^{-1}+O(\varepsilon\omega)\right)\\=&1+\varepsilon(z-\omega)S_{(1)}S_0^{-1}+O(\varepsilon^2)\,.\end{aligned} \tag{2.8}$$

holds uniformly with respect to z in any fixed ball of $\mathbb{C}$. By the definition (A.17), $\operatorname{ran}(S_{(1)})=\mathbb{C}=\operatorname{ran}(P_0)$; it follows

$$S_{\varepsilon z}S_{\varepsilon\omega}^{-1}=1+\varepsilon\,(z-\omega)\,P_0S_{(1)}S_0^{-1}+O(\varepsilon^2)$$

and

$$Q_0S_{\varepsilon z}S_{\varepsilon\omega}^{-1}=Q_0(1+O(\varepsilon^2))\,.$$

Using (A.21)

$$\frac{1}{2}+K_{\varepsilon\omega}=\frac{1}{2}+K_0+(\varepsilon\omega)^2K_{(2)}+(\varepsilon\omega)^3K_{(3)}+O((\varepsilon\omega)^4)\,,$$

and taking into account the decomposition (A.20), one gets

$$\begin{aligned}\frac{1}{2}+K_{\varepsilon\omega}=&Q_0\left(\frac{1}{2}+K_0\right)Q_0+(\varepsilon\omega)^2K_{(2)}+(\varepsilon\omega)^3K_{(3)}+O((\varepsilon\omega)^4)\\=&P_0\left((\varepsilon\omega)^2K_{(2)}+(\varepsilon\omega)^3K_{(3)}+O((\varepsilon\omega)^4)\right)P_0+Q_0\left(\frac{1}{2}+K_0+O((\varepsilon\omega)^2)\right)Q_0\\&+P_0O((\varepsilon\omega)^2)Q_0+Q_0O((\varepsilon\omega)^2)P_0\,.\end{aligned} \tag{2.9}$$

Then, from (2.8) and (2.9) follows

$$
\begin{aligned}
&\varepsilon^2 + \left(1-\varepsilon^2\right)\left(\frac{1}{2}+K_{\varepsilon\omega}\right) S_{\varepsilon z} S_{\varepsilon\omega}^{-1} \\
=&P_0\left(\varepsilon^2+\left(1-\varepsilon^2\right)\left(\varepsilon^2\omega^2 K_{(2)}+\varepsilon^3\omega^3 K_{(3)}+O((\varepsilon\omega)^4)\right)\right) P_0\left(1+\varepsilon\,(z-\omega)\,P_0 S_{(1)} S_0^{-1}+O(\varepsilon^2)\right)(P_0+Q_0) \\
&+Q_0\left(\left(\frac{1}{2}+K_0\right)+O((\varepsilon\omega)^2)\right) Q_0(1+O(\varepsilon^2))(P_0+Q_0)+P_0 O(\varepsilon^2) Q_0+Q_0 O(\varepsilon^2) P_0 \\
=&P_0\left(\varepsilon^2\left(1+\omega^2 K_{(2)}\right)+\varepsilon^3\left(\omega^2 K_{(2)}\,(z-\omega)\,S_{(1)} S_0^{-1}+\omega^3 K_{(3)}\right)+O(\varepsilon^4)\right) P_0 \\
&+Q_0\left(\frac{1}{2}+K_0+O(\varepsilon^2)\right) Q_0+P_0 O(\varepsilon^2) Q_0+Q_0 O(\varepsilon^2) P_0\,.
\end{aligned}
$$

This yields (2.2).

By $-1/2 \in \varrho(Q_0K_0Q_0)$, one has $Q_0(1/2+K_0)^{-1}Q_0 \in\mathcal{L}(\mathrm{ran}(Q_0))$; hence, by (2.5) and Lemma 2.1, the inverse operator $M_{11}(\varepsilon)^{-1} \in\mathcal{L}(\mathrm{ran}(Q_0))$ exists whenever ε is sufficiently small and allows the expansion

$$
M_{11}(\varepsilon)^{-1} = Q_0(1/2+K_0)^{-1}Q_0 + Q_0O(\varepsilon^2)Q_0\,. \tag{2.10}
$$

holding for any $\omega \in \mathbb{C}$ uniformly w.r.t. z in any fixed ball of $\mathbb{C}$. This leads to (2.7). ∎

Lemma 2.3

$$
P_0\left(1+\omega^2K_{(2)}\right)P_0 = \left(1-\frac{\omega^2}{\omega_M^2}\right)P_0\,.
$$

Proof. By (A.22) and Green's identity,

$$
\begin{aligned}
(K_{(2)}1)(x) =&\frac{1}{8\pi}\int_\Gamma \nu(y)\cdot\frac{x-y}{|x-y|}\,d\sigma(y) = -\frac{1}{8\pi}\int_\Omega \Delta|x-y|\,dy \\
=&-\frac{1}{8\pi}\int_{\mathbb{R}^3}\Delta|y|\,1_\Omega(x-y)\,dy = -\frac{1}{4\pi}\int_{\mathbb{R}^3}1_\Omega(x-y)\,\frac{dy}{|y|}\,,
\end{aligned}
\tag{2.11}
$$

i.e.,

$$
K_{(2)}1 = -\gamma_0R_01_\Omega = -SL_0^*1_\Omega\,.
$$

This yields

$$
\left\langle 1,(1+\omega^2K_{(2)}1)\right\rangle_{S_0^{-1}} = \|1\|^2_{S_0^{-1}} - \omega^2\left\langle 1,\gamma_0SL_0^*1_\Omega\right\rangle_{S_0^{-1}} = c_\Omega - \omega^2\left\langle SL_0S_0^{-1}1,1\right\rangle_{L^2(\Omega)}\,. \tag{2.12}
$$

Since $u := (SL_0S_0^{-1}1)|\Omega$ solves the interior Dirichlet problem

$$
\begin{cases}\Delta_\Omega u = 0\,,\\ \gamma_0^{\mathrm{in}}u = 1\,,\end{cases}
$$

one gets $1_\Omega SL_0S_0^{-1}1 = 1_\Omega$ and (2.12) reduces to

$$
\left\langle 1,(1+\omega^2K_{(2)}1)\right\rangle_{S_0^{-1}} = c_\Omega - \omega^2|\Omega|\,.
$$

Therefore

$$
P_0\left(1+\omega^2K_{(2)}\right)P_0 = c_\Omega^{-1}\left\langle 1,\left(1+\omega^2K_{(2)}1\right)\right\rangle_{S_0^{-1}}1 = (1-\omega^2\,c_\Omega^{-1}|\Omega|)\,P_0 = \left(1-\frac{\omega^2}{\omega_M^2}\right)P_0\,. \tag{2.13}
$$

∎

Lemma 2.4

$$
P_0K_{(3)}P_0 = -i\,\frac{|\Omega|}{4\pi}\,P_0\,.
$$

Proof. By (A.22) and the divergence theorem,

$$
\begin{aligned}
\langle 1,K_{(3)}1\rangle_{S_0^{-1}} =&\frac{i}{12\pi}\int_\Gamma(S_0^{-1}1)(x)\left(\int_\Gamma\nu(y)\cdot(x-y)\,d\sigma(y)\right)d\sigma(x) \\
=&\frac{i}{12\pi}\int_\Gamma\nu(y)\cdot\left(\int_\Gamma(S_0^{-1}1)(x)(x-y)\,d\sigma(x)\right)d\sigma(y) \\
=&-i\,\frac{c_\Omega}{12\pi}\int_\Gamma\nu(y)\cdot y\,d\sigma(y) = -i\,\frac{c_\Omega}{12\pi}\int_\Omega\nabla\cdot y\,dy \\
=&-i\,c_\Omega\,\frac{|\Omega|}{4\pi}\,.
\end{aligned}
$$

Hence

$$P_0 K_{(3)} P_0 = c_\Omega^{-1} \langle 1, K_{(3)} 1 \rangle_{S_0^{-1}} 1 = -i \, \frac{|\Omega|}{4\pi} \, P_0 \,.$$

■

The capacitance of the set Ω and the Minnaert frequency, already introduced in (1.28)-(1.29), express as

$$c_\Omega = \|1\|^2_{S_0^{-1}} = \int_\Gamma (S_0^{-1} 1)(x) \, d\sigma(x) \,, \tag{2.14}$$

and

$$\omega_M = \sqrt{\frac{c_\Omega}{|\Omega|}} \,, \tag{2.15}$$

(see Appendix, Subsection A.4, for the definitions of S_0 of the associated inner product).

Theorem 2.5 *Let $\omega \in \mathbb{C}$, $r_0 > r_1 > 0$ and P_0 be the rank-one projector defined in* (A.18)*. There exists $\varepsilon_0 > 0$ such that, whenever $0 < \varepsilon < \varepsilon_0$ the following holds true:*

1) If $\omega \neq \omega_M$ and $|z| < r_0$ then the linear operator

$$\varepsilon^2 + \left(1 - \varepsilon^2\right) \left(1/2 + K_{\varepsilon\omega}\right) S_{\varepsilon z} S_{\varepsilon\omega}^{-1} \tag{2.16}$$

has a bounded inverse in $\mathcal{L}(H^{1/2}(\Gamma))$. Moreover, setting

$$E^0_\omega := 1 - \frac{\omega^2}{\omega_M^2} \,, \tag{2.17}$$

the expansion

$$\varepsilon^2 \left(\varepsilon^2 + \left(1 - \varepsilon^2\right) \left(1/2 + K_{\varepsilon\omega}\right) S_{\varepsilon z} S_{\varepsilon\omega}^{-1}\right)^{-1} = \frac{1}{E^0_\omega} \, P_0 + P_0 O(\varepsilon) P_0 + O(\varepsilon^2) \,, \tag{2.18}$$

holds uniformly w.r.t. z.

2) If $\omega = \omega_M$ and $r_1 < |z| < r_0$, then the linear operator (2.16) *has a bounded inverse in $\mathcal{L}(H^{1/2}(\Gamma))$ and the expansion*

$$\varepsilon^3 \left(\varepsilon^2 + \left(1 - \varepsilon^2\right) \left(1/2 + K_{\varepsilon\omega}\right) S_{\varepsilon z} S_{\varepsilon\omega}^{-1}\right)^{-1} = \frac{4\pi}{c_\Omega} \, \frac{i}{z} \, P_0 + P_0 O(\varepsilon) P_0 + O(\varepsilon^2) \,. \tag{2.19}$$

holds uniformly w.r.t. z.

Proof. By Lemma 2.2, (2.16) represents as a block operator matrix $\mathbb{M}(\varepsilon)$ acting on the decomposition $H^{1/2}(\Gamma) = \operatorname{ran}(P_0) \widehat{\oplus} \operatorname{ran}(Q_0)$, $Q_0 := 1 - P_0$. By the formula for the inversion of block operator matrices, one has

$$(\mathbb{M}(\varepsilon))^{-1} \equiv \begin{bmatrix} (C_{00}(\varepsilon))^{-1} & -\left(C_{00}(\varepsilon)\right)^{-1} M_{01}(\varepsilon) \left(M_{11}(\varepsilon)\right)^{-1} \\ -\left(M_{11}(\varepsilon)\right)^{-1} M_{10}(\varepsilon) \left(C_{00}(\varepsilon)\right)^{-1} & \left(M_{11}(\varepsilon)\right)^{-1} + \left(M_{11}(\varepsilon)\right)^{-1} M_{10}(\varepsilon) \left(C_{00}(\varepsilon)\right)^{-1} M_{01}(\varepsilon) M_{11}(\varepsilon) \end{bmatrix} \tag{2.20}$$

where, setting

$$E^1_\omega := -i \, \omega^3 \, \frac{|\Omega|}{4\pi} \,, \tag{2.21}$$

by (2.7) and Lemma 2.4 the expansion

$$C_{00}(\varepsilon) = P_0 \left(E^0_\omega \varepsilon^2 + \left(E^1_\omega + \omega^2 \,(z - \omega)\, K_{(2)} S_{(1)} S_0^{-1}\right) \varepsilon^3 + O(\varepsilon^4)\right) P_0 \,, \tag{2.22}$$

holds. In particular, for each $\omega \in \mathbb{C}$, the remainder $O(\varepsilon^4)$ has a uniform bound $\sim \varepsilon^4$ w.r.t. $z : |z| < r_0$, provided that ε_0 is small enough depending on ω and r_0 (see Lemma 2.2).

1) If $\omega \neq \omega_M$, by Lemma 2.3 $E^0_\omega \neq 0$; then the Schur complement writes as

$$C_{00}(\varepsilon) = E^0_\omega \varepsilon^2 P_0 \left(1 + \left(E^1_\omega + \omega^2 \,(z - \omega)\, K_{(2)} S_{(1)} S_0^{-1}\right) \frac{\varepsilon}{E^0_\omega} + O(\varepsilon^2)\right) P_0 \,.$$

Since $O(\varepsilon^2) \lesssim \varepsilon^2$ uniformly w.r.t. z s.t. $|z| < r_0$, Lemma 2.1 applies to the r.h.s. whenever both $\varepsilon\,|\omega|$ and $\varepsilon|z|$ are sufficiently small. Hence, for each ω and r_0, there exists $\varepsilon_0 > 0$ small enough that the expansion

$$\left(C_{00}(\varepsilon)\right)^{-1} = \frac{1}{\varepsilon^2} \, \frac{1}{E^0_\omega} \, P_0 \left(1 - \left(E^1_\omega + \omega^2 \,(z - \omega)\, K_{(2)} S_{(1)} S_0^{-1}\right) \frac{\varepsilon}{E^0_\omega} + O(\varepsilon^2)\right) P_0 \,, \tag{2.23}$$

holds uniformly whenever $|z| < r_0$. From (2.20), (2.4), (2.10) and (2.23) there follows

$$\varepsilon^2 (\mathbb{M}(\varepsilon))^{-1} = \begin{bmatrix} (E_\omega^0)^{-1} P_0 + P_0 O(\varepsilon) P_0 & P_0 O(\varepsilon^2) Q_0 \\ Q_0 O(\varepsilon^2) P_0 & Q_0 O(\varepsilon^2) Q_0 \end{bmatrix}.$$

2) Let us now assume that $\omega = \omega_M$, so that: $E^0_{\omega_M} = 0$. Since, by (A.17) and (2.14), $S_{(1)} S_0^{-1}(1) = i\, c_\Omega / 4\pi$, i.e.,

$$S_{(1)} S_0^{-1} P_0 = i\, \frac{c_\Omega}{4\pi}\, P_0\,,$$

one has

$$(z-\omega)\, \omega^2 P_0 K_{(2)} S_{(1)} S_0^{-1} P_0 = i\, \frac{c_\Omega}{4\pi}\, (z-\omega)\, \omega^2 P_0 K_{(2)} P_0 = -i\, \frac{c_\Omega}{4\pi}\, (z-\omega)\, P_0\,.$$

Moreover, from (2.21) and $\omega_M^2 := c_\Omega / |\Omega|$ there follows: $E^1_{\omega_M} = -i\, \omega_M \frac{c_\Omega}{4\pi}$. Then, (2.22) recasts as

$$C_{00}(\varepsilon) = \left(-i\, \omega_M\, \frac{c_\Omega}{4\pi} - i\, \frac{c_\Omega}{4\pi}\, (z - \omega_M)\right) \varepsilon^3 P_0 + P_0 O(\varepsilon^4) P_0 = \left(-i\, \frac{c_\Omega}{4\pi}\, z\, P_0 + P_0 O(\varepsilon) P_0\right) \varepsilon^3\,. \tag{2.24}$$

and, for $z \neq 0$, we obtain

$$C_{00}(\varepsilon) = -i\, \frac{c_\Omega}{4\pi}\, z\, P_0 \left(1 + P_0 \frac{1}{z} O(\varepsilon)\, P_0\right) \varepsilon^3\,.$$

Let us recall that $O(\varepsilon)$ has a uniform bound: $\sup_{|z|<r_0} \|O(\varepsilon)\| < C\varepsilon$, provided that ε_0 is small enough and $0 < \varepsilon < \varepsilon_0$. Choosing any $r_1 < r_0$ such that: $C\varepsilon_0 / r_1 < 1$, the Neumann series

$$\textstyle\sum_{j=0}^{+\infty} (-1)^j \left(P_0 \frac{1}{z} O(\varepsilon)\, P_0\right)^j = \left(1 + P_0 \frac{1}{z} O(\varepsilon)\, P_0\right)^{-1},$$

converges in $\mathcal{L}\big(H^{1/2}(\Gamma)\big)$ and the expansion

$$C_{00}(\varepsilon)^{-1} = \left(\frac{4\pi}{c_\Omega}\, \frac{i}{z}\, P_0 + P_0 O(\varepsilon)\, P_0\right) \frac{1}{\varepsilon^3}\,, \tag{2.25}$$

holds uniformly w.r.t. $z : r_1 < |z| < r_0$. By (2.20), (2.4), (2.10) and (2.25), one gets

$$\varepsilon^3 \mathbb{M}(\varepsilon)^{-1} = \begin{bmatrix} \frac{4\pi}{c_\Omega} \frac{i}{z} P_0 + P_0 O(\varepsilon) P_0 & P_0 O(\varepsilon^2) Q_0 \\ Q_0 O(\varepsilon^2) P_0 & Q_0 O(\varepsilon^2) Q_0 \end{bmatrix}.$$

■

Let us define

$$\Lambda_z^\omega(\varepsilon) := \varepsilon(1-\varepsilon^2)(\varepsilon^2 + (1-\varepsilon^2) DN_{\varepsilon\omega} S_{\varepsilon z})^{-1} DN_{\varepsilon\omega}\,. \tag{2.26}$$

By Theorem 2.5 this operator has the following asymptotic representation.

Theorem 2.6 *Let $\omega \in \mathbb{C}$, $r_0 > r_1 > 0$, P_0 be the rank-one projector defined in* (A.18) *and E^0_ω be given by (2.17). There exists $\varepsilon_0 > 0$ such that, whenever $0 < \varepsilon < \varepsilon_0$, the following holds true:*

1) If $\omega \neq \omega_M$, then: $z \mapsto \Lambda_z^\omega(\varepsilon)$ is a $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$-valued analytic map in the ball $\{z : |z| < r_0\}$ where it has the uniform-in-z expansion

$$\Lambda_z^\omega(\varepsilon) = \frac{1}{\varepsilon}\, S_{\varepsilon\omega}^{-1} \left(\frac{P_0}{E^0_\omega} + P_0 O(\varepsilon) P_0 + O(\varepsilon^2)\right) S_{\varepsilon\omega} DN_{\varepsilon\omega}\,. \tag{2.27}$$

2) If $\omega = \omega_M$, then: $z \mapsto \Lambda_z^\omega(\varepsilon)$ is a $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$-valued analytic map in $\{z : r_1 < |z| < r_0\} \setminus \{0\}$ where it has the uniform-in-z expansion

$$\Lambda_z^\omega(\varepsilon) = \frac{1}{\varepsilon^2}\, S_{\varepsilon\omega}^{-1} \left(\frac{4\pi}{c_\Omega}\, \frac{i}{z}\, P_0 + P_0 O(\varepsilon) P_0 + O(\varepsilon^2)\right) S_{\varepsilon\omega} DN_{\varepsilon\omega}\,. \tag{2.28}$$

Proof. According to (A.27),

$$DN_{\varepsilon\omega} = S_{\varepsilon\omega}^{-1} \left(\frac{1}{2} + K_{\varepsilon\omega}\right).$$

Hence,

$$\varepsilon^2 + \left(1-\varepsilon^2\right) DN_{\varepsilon\omega} S_{\varepsilon z} = S_{\varepsilon\omega}^{-1} \left(\varepsilon^2 + \left(1-\varepsilon^2\right) \left(\frac{1}{2} + K_{\varepsilon\omega}\right) S_{\varepsilon z} S_{\varepsilon\omega}^{-1}\right) S_{\varepsilon\omega} \tag{2.29}$$

and, by (2.26),

$$\Lambda_z^\omega(\varepsilon) = \varepsilon(1-\varepsilon^2)S_{\varepsilon\omega}^{-1}\left(\varepsilon^2 + \left(1-\varepsilon^2\right)\left(\frac{1}{2}+K_{\varepsilon\omega}\right)S_{\varepsilon z}S_{\varepsilon\omega}^{-1}\right)^{-1} S_{\varepsilon\omega}DN_{\varepsilon\omega}\,. \tag{2.30}$$

By Theorem 2.5 and the mapping properties of $S_{\varepsilon\omega}DN_{\varepsilon\omega}$ and $S_{\varepsilon\omega}^{-1}$ (see the Appendix), (2.30) defines an operator in $\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ for any $z : |z| < r_0$ and $\omega \neq \omega_M$ or for $z : r_1 < |z| < r_0$ and $\omega = \omega_M$, provided that ε_0 is small enough depending on ω, r_0 and r_1. From the analyticity of $z \mapsto S_z$ (see Lemma A.1) follows the analyticity of the operator (2.16); the analyticity of $z \mapsto \Lambda_z^\omega(\varepsilon)$ is then consequence of the existence the inverse, shown in Theorem 2.5, and of the analyticity of the inverse (see [24, Theorem 5.1]).

The ε-expansions (2.27) and (2.28) follow from the ones provided in Theorem 2.5. ■

3 The operator model for acoustic interface conditions

Here we introduce the Schrödinger-type operators modeling acoustic interface conditions. Their construction involves the boundary operators whose existence and mapping properties have been discussed in the small-scale limit in Section 2.

3.1 The dilated operator

In this subsection we provide, together with its resolvent, a self-adjoint realization of the Laplacian with boundary conditions at the interface Γ separating $\Omega_{\text{in}} = \Omega$ from $\Omega_{ex} = \mathbb{R}^3\backslash\overline{\Omega}$ given by

$$[\gamma_0]u = 0\,, \qquad [\gamma_1]u = (\varepsilon^{-2}-1)DN_{\varepsilon\omega}\gamma_0 u\,. \tag{3.1}$$

The vector space $H^0_\Delta(\mathbb{R}^3\backslash\Gamma)$ appearing in the next theorem is defined as the set of the functions $u \in L^2(\mathbb{R}^3)$ such that the distributional Laplacian $\Delta_{\mathbb{R}^3\backslash\Gamma}u$ is in $L^2(\mathbb{R}^3)$ (see (1.27) and (A.10) for more details).

Theorem 3.1 *Let $\omega > 0$ and $H_{\varepsilon,\omega}$ be the restriction of*

$$\Delta : H^0_\Delta(\mathbb{R}^3\backslash\Gamma) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \tag{3.2}$$

to the domain

$$\text{dom}(H_{\varepsilon,\omega}) = \left\{u \in H^0_\Delta(\mathbb{R}^3\backslash\Gamma) \cap H^1(\mathbb{R}^3) : [\gamma_1]u = (\varepsilon^{-2}-1)DN_{\varepsilon\omega}\gamma_0 u\right\}\,. \tag{3.3}$$

There exists $\varepsilon_0 > 0$ sufficiently small that, for all $0 < \varepsilon < \varepsilon_0$, $H_{\varepsilon,\omega}$ is a self-adjoint and semi-bounded operator in $L^2(\mathbb{R}^3)$. Moreover, for any $z \in \mathbb{C}_+$ such that $z^2 \in \varrho(-H_{\varepsilon,\omega}) \cap (\mathbb{C}\backslash[0,+\infty))$, hence at least for any $z \in \mathbb{C}_+\backslash i\mathbb{R}_+$, its resolvent is given by

$$R_z^{\varepsilon,\omega} := (-H_{\varepsilon,\omega} - z^2)^{-1} = R_z - SL_z\left((\varepsilon^{-2}-1)^{-1} + DN_{\varepsilon\omega}S_z\right)^{-1} DN_{\varepsilon\omega}\,\gamma_0 R_z\,, \tag{3.4}$$

where R_z denotes the free resolvent, i.e., $R_z = (-\Delta - z^2)^{-1}$.

Proof. Here, for the sake of brevity, we set

$$M_z^{\varepsilon,\omega} := (\varepsilon^{-2}-1)^{-1} + DN_{\varepsilon\omega}S_z\,. \tag{3.5}$$

By (A.27) and Lemma A.1,

$$\begin{aligned}
M_z^{\varepsilon,\omega} &= (\varepsilon^{-2}-1)^{-1}\left(1 + (\varepsilon^{-2}-1)DN_{\varepsilon\omega}S_z\right)\\
&=\varepsilon^{-2}(\varepsilon^{-2}-1)^{-1}\left(\varepsilon^2 + (1-\varepsilon^2)DN_{\varepsilon\omega}S_z\right)\\
&=\varepsilon^{-2}(\varepsilon^{-2}-1)^{-1}\left(\varepsilon^2 + (1-\varepsilon^2)S_{\varepsilon\omega}^{-1}\left(\frac{1}{2}+K_{\varepsilon\omega}\right)S_z\right)\\
&=\varepsilon^{-2}(\varepsilon^{-2}-1)^{-1}S_{\varepsilon\omega}^{-1}\left(\varepsilon^2 + (1-\varepsilon^2)\left(\frac{1}{2}+K_{\varepsilon z}\right)S_zS_{\varepsilon\omega}^{-1}\right)S_{\varepsilon\omega}\,.
\end{aligned}$$

Let us fix $z \in \mathbb{C}_+$; due to Theorem 2.5 and to the mapping properties of S_z, there exists $\varepsilon_0 > 0$ such that, whenever $0 < \varepsilon < \varepsilon_0$, $M_z^{\varepsilon,\omega}$ has a bounded inverse $(M_z^{\varepsilon,\omega})^{-1} \in \mathcal{L}(H^{-1/2}(\Gamma))$. By $S_z^* = S_{-\bar{z}}$ and $DN_{\varepsilon\omega}^* = DN_{\varepsilon\omega}$,

$$DN_{\varepsilon\omega}(M_z^{\varepsilon,\omega})^* = DN_{\varepsilon\omega}\left((\varepsilon^{-2}-1)^{-1} + S_{-\bar{z}}DN_{\varepsilon\omega})\right) = M_{-\bar{z}}^{\varepsilon,\omega}DN_{\varepsilon\omega}\,,$$

and so

$$(-M_{-\bar{z}}^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega} = DN_{\varepsilon\omega}((-M_z^{\varepsilon,\omega})^*)^{-1} = DN_{\varepsilon\omega}((-M_z^{\varepsilon,\omega})^{-1})^* = \left((-M_z^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}\right)^*\,. \tag{3.6}$$

By the first resolvent identity,

$$SL_w - SL_z = (w^2 - z^2)R_w SL_z\,,$$

and so

$$M_w^{\varepsilon,\omega} - M_z^{\varepsilon,\omega} = (w^2 - z^2)DN_{\varepsilon\omega}\gamma_0 R_w SL_z = (w^2 - z^2)DN_{\varepsilon\omega}SL_{-\bar w}^* SL_z$$

This gives

$$(-M_w^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega} - (-M_z^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega} = (-z^2 - (-w^2))(M_w^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}SL_{-\bar z}^* SL_z(M_w^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}\,. \tag{3.7}$$

Let us remark that (3.6) and (3.7) correspond to [19, relations (2.6) and (2.7)] (be aware of the different notation and convention: our $(-M_w^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}$ corresponds to Λ_z in [19], while our $R_z := (-\Delta - z^2)^{-1}$ is there denoted with R_z^0). Hence [19, Theorem 2.4] applies and we conclude that, for the fixed z

$$\begin{aligned}\widetilde R_z^{\varepsilon,\omega} &:= R_z + SL_z(-M_z^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}\gamma_0 R_z\\ &= R_z - SL_z((\varepsilon^{-2}-1)^{-1} + DN_{\varepsilon\omega}S_z)^{-1}DN_{\varepsilon\omega}\gamma_0 R_z\end{aligned} \tag{3.8}$$

is the resolvent of a self-adjoint operator $\widetilde H_{\varepsilon,\omega}$ in $L^2(\mathbb{R}^3)$ which extends $\Delta|\ker(\gamma_0)$. By [9, Theorem 2.19], such a resolvent formula extends to all $z \in \mathbb{C}_+$ such that $z^2 \in \varrho(-\widetilde H_{\varepsilon,\omega}) \cap (\mathbb{C}\backslash[0,+\infty))$; in particular, by the selfadjointness of $\widetilde H_{\varepsilon,\omega}$, (3.8) holds at least for any $z \in \mathbb{C}_+\backslash i\mathbb{R}_+$.

Let us now show that $\widetilde H_{\varepsilon,\omega} = H_{\varepsilon,\omega}$. By the mapping properties of SL_z (see (A.11)), $SL_z(M_z^{\varepsilon,\omega})^{-1}$ has values in $H^1(\mathbb{R}^3\backslash\Gamma)$ and so, by $[\gamma_0]SL_z = 0$ (see (A.12)), one gets $\mathrm{dom}(\widetilde H_{\varepsilon,\omega}) \subseteq H^1(\mathbb{R}^3)$. By Green's formula (A.9), taking into account the boundary conditions in (3.3), one readily can check that $H_{\varepsilon,\omega}$ is a symmetric operator. Hence, since $\widetilde H_{\varepsilon,\omega} \subset (\Delta|\ker(\gamma_0))^* = \Delta|H^0_\Delta(\mathbb{R}^3\backslash\Gamma)$, it suffices to show that

$$\mathrm{dom}(\widetilde H_{\varepsilon,\omega}) \equiv \{\widetilde u \in H^1(\mathbb{R}^3) : \widetilde u = u_0 - SL_z(M_z^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}\gamma_0 u_0,\ u_0 \in H^2(\mathbb{R}^3)\} \subseteq \mathrm{dom}(H_{\varepsilon,\omega})\,.$$

To this aim, let us notice at first that the identity

$$(-\Delta - z^2)SL_z = (-\Delta - z^2)\mathcal{G}_z * \gamma_0^* = \delta_0 * \gamma_0^* = \gamma_0^*\,,$$

implies: $-\Delta SL_z(x) = z^2 SL_z(x)$ for all $x \notin \Gamma$ and so $\mathrm{dom}(\widetilde H_{\varepsilon,\omega}) \subseteq H^0_\Delta(\mathbb{R}^3\backslash\Gamma)$. Moreover, by (A.12) and the definition (3.5), one gets

$$\begin{aligned}&M_z^{\varepsilon,\omega}[\gamma_1]\widetilde u = -M_z^{\varepsilon,\omega}[\gamma_1]SL_z(M_z^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}\gamma_0 u_0 = DN_{\varepsilon\omega}\gamma_0 u_0\\ &= \big((\varepsilon^{-2}-1)M_z^{\varepsilon,\omega} + 1 - (\varepsilon^{-2}-1)M_z^{\varepsilon,\omega}\big)\,DN_{\varepsilon\omega}\gamma_0 u_0\\ &= \big((\varepsilon^{-2}-1)M_z^{\varepsilon,\omega} - (\varepsilon^{-2}-1)M_z^{\varepsilon,\omega}DN_{\varepsilon\omega}S_z(M_z^{\varepsilon,\omega})^{-1}\big)\,DN_{\varepsilon\omega}\gamma_0 u_0\\ &= (\varepsilon^{-2}-1)M_z^{\varepsilon,\omega}DN_{\varepsilon\omega}\gamma_0\left(u_0 - SL_z(M_z^{\varepsilon,\omega})^{-1}DN_{\varepsilon\omega}\gamma_0 u_0\right)\\ &= (\varepsilon^{-2}-1)M_z^{\varepsilon,\omega}DN_{\varepsilon\omega}\gamma_0\widetilde u\,.\end{aligned}$$

Since $M_z^{\varepsilon,\omega}$ is a bijection, this is equivalent to

$$[\gamma_1]\widetilde u = (\varepsilon^{-2}-1)DN_{\varepsilon\omega}\gamma_0\widetilde u\,.$$

Finally, let us show that $H_{\varepsilon,\omega}$ is semi-bounded. Again by Green's formula (A.9), for any $u \in \mathrm{dom}(H_{\varepsilon,\omega})$ and for any $s \in (0,1/2)$, one gets

$$\langle -H_{\varepsilon,\omega}u, u\rangle_{L^2(\mathbb{R}^3)} = \|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \left(\varepsilon^{-2}-1\right)\langle DN_{\varepsilon\omega}\gamma_0 u, \gamma_0 u\rangle_{H^{-s}(\Gamma),H^s(\Gamma)}\ .$$

By

$$\begin{aligned}\left|\langle DN_{\varepsilon\omega}\gamma_0 u, \gamma_0 u\rangle_{H^{-s}(\Gamma),H^s(\Gamma)}\right| &\le \|DN_{\varepsilon\omega}\|_{H^s(\Gamma),H^{-s}(\Gamma)}\|\gamma_0\|^2_{H^{s+1/2}(\mathbb{R}^3),H^s(\Gamma)}\|u\|^2_{H^{s+1/2}(\mathbb{R}^3)}\\ &\equiv c\,\|u\|^2_{H^{s+1/2}(\mathbb{R}^3)}\ ,\end{aligned}$$

and since for any $a > 0$ there exists $b > 0$ such that

$$\|u\|^2_{H^{s+1/2}(\mathbb{R}^3)} \le a\,\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + b\,\|u\|^2_{L^2(\mathbb{R}^3)}\ ,$$

taking a sufficiently small, we obtain

$$\begin{aligned}\langle -H_{\varepsilon,\omega}u, u\rangle_{L^2(\mathbb{R}^3)} &\ge \left(1 - a\,c\,\left|\varepsilon^{-2}-1\right|\right)\|\nabla u\|^2_{L^2(\mathbb{R}^3)} - b\,c\,\left|\varepsilon^{-2}-1\right|\|u\|^2_{L^2(\mathbb{R}^3)}\\ &\ge -\,b\,c\,\left|\varepsilon^{-2}-1\right|\|u\|^2_{L^2(\mathbb{R}^3)}\ .\end{aligned}$$

■

Remark 3.2 *The jump condition* $[\gamma_0]u = 0$ *holds for* $u \in \mathrm{dom}(H_{\varepsilon,\omega})$ *due to* $\mathrm{dom}(H_{\varepsilon,\omega}) \subseteq H^1(\mathbb{R}^3)$.

Remark 3.3 *The Dirichlet-to-Neumann operator* $DN_{\varepsilon\omega}$ *appearing in both the definitions of* $\mathrm{dom}(H_{\varepsilon,\omega})$ *and* $R_z^{\varepsilon,\omega}$, *is well-defined for any* $\omega > 0$ *and a sufficiently small* $\varepsilon > 0$ *such that* $0 < (\varepsilon\omega)^2 < \lambda_\Omega$, λ_Ω *denoting the smallest eigenvalue of* $-\Delta_\Omega^D$ *(see Subsection A.6 in the Appendix).*

Remark 3.4 *Building upon the theory of singular perturbations presented in [17], the selfadjointness of* $H_{\varepsilon,\omega}$ *could be proved without the assumption* $\varepsilon \ll 1$. *Here we prefer to exploit a less technical construction involving asymptotic estimates for the operator* $(\varepsilon^{-2}-1)^{-1}+DN_{\varepsilon\omega}S_z$*: this allows us to avoid a slightly burdensome abstract framework, while the asymptotic estimates as* $\varepsilon \to 0_+$ *provide the natural tool of the subsequent analysis.*

3.2 Dilation identities

We introduce the smooth map

$$\Phi_\varepsilon(y) = y_0 + \varepsilon\,(y - y_0)\,, \qquad \varepsilon > 0\,, \quad y_0 \in \mathbb{R}^3\,; \tag{3.9}$$

the contracted domain Ω^ε is then defined by

$$\Omega^\varepsilon := \Phi_\varepsilon(\Omega) \equiv \left\{x \in \mathbb{R}^3 : x = y_0 + \varepsilon\,(y - y_0)\,,\ y \in \Omega\right\}\,, \tag{3.10}$$

while its boundary Γ^ε is given by

$$\Gamma^\varepsilon = \Phi_\varepsilon(\Gamma) \equiv \left\{x \in \mathbb{R}^3 : x = y_0 + \varepsilon\,(y - y_0)\,,\ y \in \Gamma\right\}\,.$$

The map Φ_ε and its inverse induce unitary operators on $L^2(\mathbb{R}^3)$ defined by

$$(U_\varepsilon u)\,(x) := \varepsilon^{-3/2}\,u(\Phi_\varepsilon^{-1}(x))\,, \qquad \left(U_\varepsilon^{-1}u\right)(y) := \varepsilon^{3/2}\,u(\Phi_\varepsilon(y))\,. \tag{3.11}$$

By the definition of U_ε one gets

$$\Delta = \varepsilon^{-2} U_\varepsilon \Delta U_\varepsilon^{-1}$$

and hence

$$R_z = \varepsilon^2 U_\varepsilon R_{\varepsilon z} U_\varepsilon^{-1}\,. \tag{3.12}$$

In the next Lemmata, for any linear operator L in spaces of functions on Ω (or Γ) we denote by $L(\varepsilon)$ the corresponding operator in spaces of functions on Ω^ε (or Γ^ε).

Lemma 3.5

$$U_\varepsilon \gamma_0^{\mathrm{in/ex}} U_\varepsilon^{-1} = \gamma_0^{\mathrm{in/ex}}(\varepsilon)\,. \tag{3.13}$$

$$U_\varepsilon \gamma_1^{\mathrm{in/ex}} U_\varepsilon^{-1} = \varepsilon\,\gamma_1^{\mathrm{in/ex}}(\varepsilon)\,. \tag{3.14}$$

Proof. The statement is an immediate consequence of the definitions. ■

Lemma 3.6 *Let* $\omega > 0$ *and* $\varepsilon > 0$ *such that* $(\varepsilon\omega)^2 \in \varrho(-\Delta_\Omega^D)$. *Then*

$$U_\varepsilon DN_{\varepsilon\omega} U_\varepsilon^{-1} = \varepsilon\, DN_\omega(\varepsilon)\,. \tag{3.15}$$

Proof. By the definition of $DN_{\varepsilon\omega}$, it results

$$U_\varepsilon DN_{\varepsilon\omega} U_\varepsilon^{-1}\varphi := U_\varepsilon \gamma_1^{\mathrm{in}} u\,, \qquad \begin{cases}(\Delta_\Omega + \varepsilon^2\omega^2)u = 0\,,\\ \gamma_0^{\mathrm{in}} u = U_\varepsilon^{-1}\varphi\,.\end{cases}$$

Setting $\tilde{u} = \varepsilon^2 U_\varepsilon u$ one obtains

$$0 = U_\varepsilon\left(\Delta + \varepsilon^2\omega^2\right)U_\varepsilon^{-1}U_\varepsilon u = \left(U_\varepsilon \Delta U_\varepsilon^{-1} + \varepsilon^2\omega^2\right)U_\varepsilon u = \left(\Delta + \omega^2\right)\varepsilon^2 U_\varepsilon u = \left(\Delta + \omega^2\right)\tilde{u}\,,$$

and

$$\varphi = U_\varepsilon U_\varepsilon^{-1}\varphi = U_\varepsilon \gamma_0^{\mathrm{in}} u = \gamma_0^{\mathrm{in}}(\varepsilon) U_\varepsilon u = \varepsilon^{-2}\gamma_0^{\mathrm{in}}(\varepsilon)\tilde{u}\,.$$

Hence

$$\begin{cases}(\Delta_{\Omega^\varepsilon} + \omega^2)\tilde{u} = 0\,,\\ \gamma_0^{\mathrm{in}}(\varepsilon)\tilde{u} = \varepsilon^2\varphi\,.\end{cases}$$

Using (3.14), this implies

$$U_\varepsilon DN_{\varepsilon\omega} U_\varepsilon^{-1}\varphi = U_\varepsilon \gamma_1^{\mathrm{in}} U_\varepsilon^{-1} U_\varepsilon u = \varepsilon\gamma_1^{\mathrm{in}}(\varepsilon) U_\varepsilon u = \varepsilon^{-1}\,\gamma_1^{\mathrm{in}}(\varepsilon)\tilde{u} = \varepsilon^{-1}\,DN_\omega(\varepsilon)\gamma_0^{\mathrm{in}}(\varepsilon)\tilde{u} = \varepsilon\,DN_\omega(\varepsilon)\varphi\,.$$

■

3.3 The model operator $H_\omega(\varepsilon)$

Let $\omega > 0$ and let $\varepsilon > 0$ be sufficiently small; we define

$$\mathrm{dom}(H_\omega(\varepsilon)) := U_\varepsilon\left(\mathrm{dom}(H_{\varepsilon,\omega})\right), \qquad H_\omega(\varepsilon) := \varepsilon^{-2} U_\varepsilon H_{\varepsilon,\omega} U_\varepsilon^{-1}. \tag{3.16}$$

By Theorem 3.1, $H_\omega(\varepsilon)$ is a well-defined self-adjoint and semi-bounded operator in $L^2(\mathbb{R}^3)$ and, by relations (3.13), (3.14), (3.15), it can be more explicitly defined as the restriction of

$$\Delta : H^0_\Delta(\mathbb{R}^3\backslash\Gamma^\varepsilon) \subset L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

to the domain

$$\mathrm{dom}(H_\omega(\varepsilon)) = \left\{u \in H^0_\Delta(\mathbb{R}^3\backslash\Gamma^\varepsilon) \cap H^1(\mathbb{R}^3) : [\gamma_1(\varepsilon)]\, u = (\varepsilon^{-2} - 1) DN_\omega(\varepsilon)\gamma_0(\varepsilon) u\right\}. \tag{3.17}$$

Notice that the jump condition $[\gamma_0(\varepsilon)]u = 0$ is incorporated into $\mathrm{dom}(H_\omega(\varepsilon)) \subseteq H^1(\mathbb{R}^3)$. Moreover, by (3.4), and Lemmata 3.5 and 3.6, its resolvent

$$R^\omega_z(\varepsilon) := (-H_\omega(\varepsilon) - z^2)^{-1} = \varepsilon^2 U_\varepsilon(-H_{\varepsilon,\omega} - \varepsilon^2 z^2)^{-1} U_\varepsilon^{-1}$$

is given by

$$R^\omega_z(\varepsilon) = R_z - \varepsilon^2 U_\varepsilon SL_{\varepsilon z}((\varepsilon^{-2} - 1)^{-1} + DN_{\varepsilon\omega} S_{\varepsilon z})^{-1} DN_{\varepsilon\omega}\gamma_0 R_{\varepsilon z} U_\varepsilon^{-1}. \tag{3.18}$$

For successive notational convenience, let introduce

$$G_z(\varepsilon) := \varepsilon^{1/2} U_\varepsilon SL_{\varepsilon z}. \tag{3.19}$$

Then, using (2.26) the resolvent formula (3.18) re-writes as

$$R^\omega_z(\varepsilon) = R_z - G_z(\varepsilon)\Lambda^\omega_z(\varepsilon) G_{-\bar z}(\varepsilon)^*. \tag{3.20}$$

Remark 3.7 *By (3.20), the eigenvalues and the resonances of $H_\omega(\varepsilon)$ are those $-z^2$ such that $z \in \mathbb{C}$ is a pole of the map $z \mapsto \Lambda^\omega_z(\varepsilon)$. By the results in Theorem 2.6, this map is analytic in the ball $\{z : |z| < r_0\}$, whenever $\omega \neq \omega_M$, or in $\{z : r_1 < |z| < r_0\}$ whenever $\omega = \omega_M$, provided that ε is small enough depending on ω and $r_1 < r_0$. In the small-ε regime, this shows the absence of eigenvalues/resonances in any open bounded region of the Riemann surface if $\omega \neq \omega_M$, or the absence of eigenvalues/resonances away from the origin if $\omega = \omega_M$.*

The operator $H_\omega(\varepsilon)$ provides a self-adjoint realization of the Laplacian with boundary conditions at the interface Γ^ϵ and, by exploiting its definition and taking into account the boundary conditions appearing in (3.17), for any $f \in L^2(\mathbb{R}^3)$ one gets the resolvent equation

$$\left(-H_\omega(\varepsilon) - z^2\right) u = f \iff \begin{cases} \left(\Delta + z^2\right) u = f\,, & \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\ [\gamma_0(\varepsilon)]\, u = 0\,, & [\gamma_1(\varepsilon)]\, u = (\varepsilon^{-2} - 1) DN_\omega(\varepsilon)\,\gamma_0(\varepsilon)\, u\,. \end{cases} \tag{3.21}$$

Notice that (3.21) is equivalent to

$$H_\omega(\varepsilon) u = f \qquad \iff \qquad \begin{cases} \nabla\cdot(1_{\mathbb{R}^3\backslash\Omega^\varepsilon} + \varepsilon^{-2} 1_{\Omega^\varepsilon})\nabla u = (1_{\mathbb{R}^3\backslash\Omega^\varepsilon} + \varepsilon^{-2} 1_{\Omega^\varepsilon}) f\,, \\ \gamma_0^{\mathrm{in}}(\varepsilon) u = \gamma_0^{\mathrm{ex}}(\varepsilon) u\,, \qquad \varepsilon^{-2}\gamma_1^{\mathrm{in}}(\varepsilon) u = \gamma_1^{\mathrm{ex}}(\varepsilon) u\,. \end{cases}$$

4 Resolvent convergence

This subsection contains our main result; we consider here the norm-resolvent limits of our model operator $H_\omega(\varepsilon)$ as $\varepsilon \to 0_+$. As should be clear from the resolvent formula (3.20), to study the resolvent convergence of $H_\omega(\varepsilon)$, the behavior of $\Lambda^\omega_z(\varepsilon)$ when $\varepsilon \ll 1$ is of pivotal importance. The related asymptotic formula, provided in Theorem 2.6, undergoes a sudden change depending on $\omega \neq \omega_M$ or $\omega = \omega_M$. This mechanism produces a discontinuity of the map $\omega \mapsto R^\omega_z(\varepsilon)$ in the limit $\varepsilon \to 0_+$. In particular, we next show that $H_\omega(\varepsilon)$ has a non-trivial (i.e., different from the free Laplacian) norm resolvent limit as $\varepsilon \to 0_+$ if and only if $\omega = \omega_M$.

Let $\dot H^2_{y_0}(\mathbb{R}^3)$ denote the homogeneous Sobolev space

$$\dot H^2_{y_0}(\mathbb{R}^3) := \left\{u \in \mathcal{C}_b(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\,,\ \Delta u \in L^2(\mathbb{R}^3)\,,\ u(y_0) = 0\right\},$$

and define the operator Δ_{y_0} by (see [2, Chapter I.1])

$$\Delta_{y_0}:\mathrm{dom}(\Delta_{y_0})\subset L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)\,,\qquad \Delta_{y_0}u:=\Delta u_0\equiv\Delta u+q\,\delta_{y_0}\,,\tag{4.1}$$

$$\mathrm{dom}(\Delta_{y_0}):=\left\{u\in L^2(\mathbb{R}^3):u(x)=u_0(x)+q\,\mathcal{G}_0(x-y_0)\,,\ u_0\in\dot H^2_{y_0}(\mathbb{R}^3)\,,\ q\in\mathbb{C}\right\}.\tag{4.2}$$

It belongs to the class of point interactions of the free Laplacian and is a self-adjoint extension of the closed symmetric operator $\Delta|H^2_{y_0}(\mathbb{R}^3)$, where $H^2_{y_0}(\mathbb{R}^3):=\dot H^2_{y_0}(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)=\{u\in H^2(\mathbb{R}^3):u(y_0)=0\}$. With reference to the results and notations in [2, Chapter I.1], Δ_{y_0} corresponds to the operator there denoted by $\Delta_{\alpha,y}$, with $\alpha=0$ and $y=y_0$ (see [2, Theorem 1.1.2]). In particular, denoting with $\mathcal{G}_z$ the Green kernel of R_z,

$$\mathcal{G}_z(x)=\frac{e^{iz|x|}}{4\pi\,|x|}\,,$$

by the well-known Kreĭn resolvent formula results

$$\left(-\Delta_{y_0}-z^2\right)^{-1}u=R_zu+4\pi\,\frac{i}{z}\;\mathcal{G}_z(\cdot-y_0)\,\langle\,\mathcal{G}_{\bar z}(\cdot-y_0),u\rangle_{L^2(\mathbb{R}^3)}\,.\tag{4.3}$$

Hence, the corresponding integral kernel is given by

$$\left(-\Delta_{y_0}-z^2\right)^{-1}(x,y)=\mathcal{G}_z(x-y)+4\pi\,\frac{i}{z}\;\mathcal{G}_z(x-y_0)\,\mathcal{G}_z(y-y_0)\,.$$

Theorem 4.1 *For any $z\in\mathbb{C}_+\backslash i\mathbb{R}_+$ and for any $\varepsilon>0$ sufficiently small, one has*

$$\omega\neq\omega_M\quad\Longrightarrow\quad\left\|R^\omega_z(\varepsilon)-\left(-\Delta-z^2\right)^{-1}\right\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)}\le c\,\varepsilon\,,\tag{4.4}$$

$$\omega=\omega_M\quad\Longrightarrow\quad\left\|R^\omega_z(\varepsilon)-\left(-\Delta_{y_0}-z^2\right)^{-1}\right\|_{L^2(\mathbb{R}^3),L^2(\mathbb{R}^3)}\le c\,\varepsilon^{1/2}\,.\tag{4.5}$$

Proof. By Theorem 2.6,

$$\Lambda^\omega_z(\varepsilon)=\varepsilon^{-2+\alpha_\omega}S^{-1}_{\varepsilon\omega}\left(\beta_{\omega,z}P_0+P_0O(\varepsilon)P_0+O(\varepsilon^2)\right)S_{\varepsilon\omega}DN_{\varepsilon\omega}\,,\tag{4.6}$$

where

$$\alpha_\omega=\begin{cases}1\,, & \omega\neq\omega_M\\ 0\,, & \omega=\omega_M\,,\end{cases}\qquad \beta_{\omega,z}=\begin{cases}(E^0_\omega)^{-1}\,, & \omega\neq\omega_M\,,\\ 4\pi i(c_\Omega\,z)^{-1}\,, & \omega=\omega_M\,.\end{cases}\tag{4.7}$$

By Lemma A.1,

$$S^{-1}_{\varepsilon\omega}=S^{-1}_0+O(\varepsilon)\,,$$

and, by Lemma A.6 (which relies on Lemma A.3),

$$S_{\varepsilon\omega}DN_{\varepsilon\omega}=\widetilde K_0+\varepsilon^2\omega^2K_{(2)}+O\left(\varepsilon^3\right)\,,\qquad \widetilde K_0:=Q_0(1/2+K_0)Q_0\,.$$

Inserting these relations into (4.6), one gets

$$\begin{aligned}\Lambda^\omega_z(\varepsilon)&=\left((S^{-1}_0+O(\varepsilon)\right)\left(\varepsilon^{\alpha_\omega}\omega^2\beta_{\omega,z}P_0K_{(2)}+O\left(\varepsilon^{1+\alpha_\omega}\right)\right)\\ &=\varepsilon^{\alpha_\omega}\omega^2\beta_{\omega,z}S^{-1}_0P_0K_{(2)}+O\left(\varepsilon^{1+\alpha_\omega}\right)\end{aligned}\tag{4.8}$$

By Corollary (A.5),

$$G_z(\varepsilon)=G_z+O(\varepsilon^{1/2})\,,\tag{4.9}$$

where

$$G_z:H^{-1/2}(\Gamma)\to L^2(\mathbb{R}^3)\,,\qquad G_z\phi:=\langle\phi\rangle\,\mathcal{G}^{y_0}_z\,,\tag{4.10}$$

$$\mathcal{G}^{y_0}_z(x):=\mathcal{G}_z(x-y_0)\,,\qquad \langle\phi\rangle:=\langle 1,\phi\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\equiv\langle 1,\phi\rangle_{H^{3/2}(\Gamma),H^{-3/2}(\Gamma)}\,.\tag{4.11}$$

By $G^*_zu=\langle\mathcal{G}^{y_0}_z,u\rangle_{L^2(\mathbb{R}^3)}1$, one gets

$$\mathrm{ran}(G^*_z)=\mathbb{C}=\mathrm{ran}(P_0)=\ker(Q_0)\,,$$

and from Lemma 2.3 follows

$$\omega^2 P_0 K_{(2)} P_0 = -\frac{\omega^2}{\omega_M^2}\, P_0\,. \tag{4.12}$$

Hence, the resolvent formula (3.20) rephrases as

$$\begin{aligned}
R_z^\omega(\varepsilon) - R_z &= -G_z(\varepsilon)\Lambda_z^\omega(\varepsilon)G_{-\bar z}(\varepsilon)^* \\
&= -\left(G_z + O(\varepsilon^{1/2})\right)\left(\varepsilon^{\alpha_\omega}\omega^2\beta_{\omega,z}S_0^{-1}P_0K_{(2)} + O(\varepsilon^{1+\alpha_\omega})\right)\left(G_{-\bar z}^* + O(\varepsilon^{1/2})\right) \\
&= \varepsilon^{\alpha_\omega}\omega^2\beta_{\omega,z}G_zS_0^{-1}P_0K_{(2)}G_{-\bar z}^* + O(\varepsilon^{1/2+\alpha_\omega}) \\
&= \begin{cases} O(\varepsilon)\,, & \omega \neq \omega_M\,, \\ \frac{4\pi i}{c_\Omega\, z}\, G_zS_0^{-1}P_0G_{-\bar z}^* + O(\varepsilon^{1/2})\,, & \omega = \omega_M\,. \end{cases}
\end{aligned} \tag{4.13}$$

The proof is then concluded by (4.3) and the relation

$$G_zS_0^{-1}P_0G_{-\bar z}^*u = \langle S_0^{-1}1, 1\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\,\mathcal{G}_z^{y_0}\langle \mathcal{G}_{-\bar z}^{y_0}, u\rangle_{L^2(\mathbb{R}^3)} = c_\Omega\,\mathcal{G}_z^{y_0}\langle \mathcal{G}_{-\bar z}^{y_0}, u\rangle_{L^2(\mathbb{R}^3)}\,.$$

■

This result allows us to interpret the Minnaert resonance ω_M as the only value of the physical frequency ω such that the frequency-dependent Schrödinger operator $H_\omega(\varepsilon)$ associated to the acoustic scattering problem has a non-trivial resolvent limit.

5 Generalized eigenfunctions and asymptotic scattering solutions

Let $\omega, \kappa > 0$; by Theorem 2.6, $\Lambda_\kappa^\omega(\varepsilon)$ is well-defined provided that ε is sufficiently small. We next use this property and consider the stationary scattering problem related to $H_\omega(\varepsilon)$. According to the definitions of Section 3.3, $H_\omega(\varepsilon)$ acts as Δ outside Γ^ε, since $H_\omega(\varepsilon) \subset \Delta|H_\Delta^0(\mathbb{R}^3\backslash\Gamma^\varepsilon)$, while its domain is characterized by the interface conditions

$$[\gamma_0(\varepsilon)]\, u = 0\,, \qquad [\gamma_1(\varepsilon)]\, u = \left(\varepsilon^{-2} - 1\right) DN_{\varepsilon\omega}\gamma_0 u\,.$$

Hence, a generalized eigenfunctions $u_\kappa^\omega(\varepsilon) \in H^2_{\mathrm{loc}}(\mathbb{R}^3\backslash\Gamma^\varepsilon) := H^2(\Omega^\varepsilon) \oplus H^2_{\mathrm{loc}}(\mathbb{R}^3\backslash\overline{\Omega^\varepsilon})$ of $H_\omega(\varepsilon)$ with eigenvalue $-\kappa^2$ solves the problem

$$\begin{cases} \left(\Delta + \kappa^2\right) u_\kappa^\omega(\varepsilon) = 0 \qquad \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\ [\gamma_0(\varepsilon)]\, u_\kappa^\omega(\varepsilon) = 0\,, \\ [\gamma_1(\varepsilon)]\, u_\kappa^\omega(\varepsilon) = \left(\varepsilon^{-2} - 1\right) DN_\omega(\varepsilon)\gamma_0(\varepsilon)u_\kappa^\omega(\varepsilon)\,. \end{cases} \tag{5.1}$$

By the next result, the generalized eigenfunctions of $H_\omega(\varepsilon)$ relate to the scattering solutions and to the functions of the kind $G_\kappa(\varepsilon)\Lambda_\kappa^\omega(\varepsilon)\phi$, with: $\phi \in H^{1/2}(\Gamma)$. Notice that, according to the mapping properties of $\Lambda_\kappa^\omega(\varepsilon)$ and R_κ, the functions

$$G_\kappa(\varepsilon)\Lambda_\kappa^\omega(\varepsilon)\,\phi = \varepsilon^{1/2}U_\varepsilon SL_{\varepsilon\kappa}\Lambda_\kappa^\omega(\varepsilon)\phi = \varepsilon^{1/2}U_\varepsilon R_{\varepsilon\kappa} * \gamma_0^*\Lambda_\kappa^\omega(\varepsilon)\phi\,,$$

belong to the weighted Sobolev space $H^2_{-\alpha}(\mathbb{R}^3)$, $\alpha > 1/2$.

Theorem 5.1 *Let $\kappa > 0$ and $u_\kappa^{\mathrm{in}} \in H^2_{-\alpha}(\mathbb{R}^3)$, $\alpha > 1/2$, be a solution of the homogeneous Helmholtz equation*

$$\left(\Delta + \kappa^2\right) u_\kappa^{\mathrm{in}} = 0\,. \tag{5.2}$$

The scattering problem

$$\begin{cases} \left(\Delta + \kappa^2\right)\left(u_\kappa^{\mathrm{in}} + u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)\right) = 0 \qquad \text{in } \mathbb{R}^3\backslash\Gamma^\varepsilon\,, \\ [\gamma_0(\varepsilon)]\, u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon) = 0\,, \\ [\gamma_1(\varepsilon)]\, u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon) = \left(\varepsilon^{-2} - 1\right) DN_\omega(\varepsilon)\gamma_0(\varepsilon)\left(u_\kappa^{\mathrm{in}} + u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)\right)\,, \\ \lim_{|x|\to\infty} |x|\left(\frac{x}{|x|}\cdot\nabla - i\kappa\right) u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon) = 0\,, \end{cases} \tag{5.3}$$

admits an unique solution in $H^2_{-\alpha}(\mathbb{R}^3\backslash\Gamma^\varepsilon)$ given by

$$u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon) := -G_\kappa(\varepsilon)\Lambda_\kappa^\omega(\varepsilon)\,\gamma_0\left(u_\kappa^{\mathrm{in}} \circ \Phi_\varepsilon\right)\,. \tag{5.4}$$

Proof. We proceed in two steps: at first we consider a dilated problem with interface conditions assigned on Γ and prove the result in this setting. Then, we discuss (5.3) by using the dilation mapping.

Let $\psi_\kappa^{\rm in} \in H^2_{-\alpha}(\mathbb{R}^3)$ be a solution of the Helmholtz equation (5.2) and consider the dilated scattering problem

$$\begin{cases} \left(\Delta+\kappa^2\right)\left(\psi_\kappa^{\rm in}+\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)\right)=0\,, \qquad \text{in } \mathbb{R}^3\backslash\Gamma\,, \\ [\gamma_0]\,\psi_{\kappa,\omega}^{\rm sc}\,(\varepsilon)=0\,, \\ [\gamma_1]\,\psi_{\kappa,\omega}^{\rm sc}\,(\varepsilon)=\left(\varepsilon^{-2}-1\right) DN_{\varepsilon\omega}\gamma_0\left(\psi_\kappa^{\rm in}+\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)\right)\,, \\ \lim_{|x|\to\infty}|x|\left(\frac{x}{|x|}\cdot\nabla-i\kappa\right)\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)=0\,, \end{cases} \tag{5.5}$$

Notice that, due to (3.15), $DN_\omega(\varepsilon)$ is here replaced by $DN_{\varepsilon\omega}$. We next proceed as in [18, Lemmata 5.1 and 5.3], where a similar problem involving abstract boundary conditions were discussed. Let us look for a solution of the form: $\psi_{\kappa,\omega}^{\rm sc}(\varepsilon):=-\varepsilon^{-1}SL_\kappa\Lambda^\omega_{\kappa/\varepsilon}\,(\varepsilon)\,\gamma_0\psi_\kappa^{\rm in}$. By [18, Lemma 5.3], $\psi_{\kappa,\omega}^{\rm sc}\,(\varepsilon)$ satisfies the Sommerfeld radiation condition in (5.5). Since the distributions $\gamma_0^*\phi$, $\phi\in H^{1/2}(\Gamma)$, are supported on Γ, from

$$\left(\Delta+k^2\right)SL_\kappa\phi=\left(\Delta+k^2\right)R_\kappa\gamma_0^*\phi=-\gamma_0^*\phi\,,$$

and (5.2) there follows

$$\left(\Delta+k^2\right)\left(\psi_\kappa^{\rm in}+\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)\right)=0\,, \qquad \text{in } \mathbb{R}^3\backslash\Gamma\,.$$

The boundary conditions in (5.5) follows by proceeding along the same lines as in the proof of Theorem 3.1 (see the calculations there involving the function $\widetilde{u}$); by (A.12) and $\psi_\kappa^{\rm in}\in H^2_{-\alpha}(\mathbb{R}^3)$ it results $[\gamma_0]\,\psi_{\kappa,\omega}^{\rm sc}\,(\varepsilon)=0$ and

$$[\gamma_1]\,\psi_{\kappa,\omega}^{\rm sc}\,(\varepsilon)=-\varepsilon^{-1}\,[\gamma_1]\,SL_\kappa\Lambda^\omega_{\kappa/\varepsilon}(\varepsilon)\gamma_0\psi_\kappa^{\rm in}=\varepsilon^{-1}\Lambda^\omega_{\kappa/\varepsilon}(\varepsilon)\gamma_0\psi_\kappa^{\rm in}\,.$$

Furthermore, from

$$\begin{aligned}&\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\left(\psi_\kappa^{\rm in}+\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)\right)=\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\left(\psi_\kappa^{\rm in}-\varepsilon^{-1}SL_\kappa\Lambda^\omega_{\kappa/\varepsilon}\,(\varepsilon)\,\gamma_0\psi_\kappa^{\rm in}\right)\\ &=\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\psi_\kappa^{\rm in}-\varepsilon^{-1}\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0SL_\kappa\Lambda^\omega_{\kappa/\varepsilon}(\varepsilon)\gamma_0\psi_\kappa^{\rm in}\\ &=\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\psi_\kappa^{\rm in}-\varepsilon^{-1}\left(1+\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}S_\kappa\right)\Lambda^\omega_{\kappa/\varepsilon}(\varepsilon)\gamma_0\psi_\kappa^{\rm in}+\varepsilon^{-1}\Lambda^\omega_{\kappa/\varepsilon}(\varepsilon)\gamma_0\psi_\kappa^{\rm in}\\ &=\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\psi_\kappa^{\rm in}-\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\psi_\kappa^{\rm in}+\varepsilon^{-1}\Lambda^\omega_{\kappa/\varepsilon}(\varepsilon)\gamma_0\psi_\kappa^{\rm in}=\varepsilon^{-1}\Lambda^\omega_{\kappa/\varepsilon}(\varepsilon)\gamma_0\psi_\kappa^{\rm in}\,,\end{aligned}$$

there follows

$$[\gamma_1]\,\psi_{\kappa,\omega}^{\rm sc}\,(\varepsilon)=\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\left(\psi_\kappa^{\rm in}+\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)\right)\,.$$

Hence, $\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)$ solves the dilated scattering problem (5.5). To conclude this part of the proof, we need to show that such a solution is unique. Let us assume that v_κ solve the same scattering problem; then, the difference $w_\kappa:=\psi_{\kappa,\omega}^{\rm sc}(\varepsilon)-v_\kappa$ solves the exterior Helmholtz equation

$$(\Delta_{\mathbb{R}^3\backslash\overline{\Omega}}+\kappa^2)w_\kappa=0 \tag{5.6}$$

and satisfies the radiation condition

$$\lim_{|x|\to\infty}|x|\left(\frac{x}{|x|}\cdot\nabla-i\kappa\right)w_\kappa=0\,. \tag{5.7}$$

Let $R>0$ such that $\Omega\subset B_R=\{x\in\mathbb{R}^3:|x|<R\}$; by [20, eq. (9.19)] there follows

$$\lim_{R\to\infty}\int_{|x|=R}|w_\kappa(x)|^2\,d\sigma(x)=0\,,$$

and, by Rellich's Lemma (see, e.g., [20, Lemma 9.8]), this entails $w_\kappa=0$ in $\mathbb{R}^3\backslash\overline{B_R}$. The Green identity in $B_R\backslash\overline{\Omega}$ then yields

$$\int_{B_R\backslash\overline{\Omega}}|\nabla w_\kappa(x)|^2dx-\int_{\Gamma^\varepsilon}\overline{\gamma_0 w_\kappa}(x)\,\gamma_1 w_\kappa(x)\,d\sigma(x)=\kappa^2\int_{B_R\backslash\overline{\Omega}}|w_\kappa(x)|^2\,dx$$

and hence

$$\mathrm{Im}\int_{\Gamma}\overline{\gamma_0 w_\kappa}(x)\,\gamma_1 w_\kappa(x)\,d\sigma(x)=0\,.$$

By [20, Lemma 9.9], this gives $w_\kappa(x)=0$ whenever $x\in\mathbb{R}^3\backslash\overline{\Omega}$. Since $[\gamma_0]w_\kappa=0$, the boundary conditions in (5.5) imply $\gamma_0^{\mathrm{in}}w_\kappa=\gamma_1^{\mathrm{in}}w_\kappa=0$ and so w_κ solves the interior Helmholtz equation with both zero Dirichlet and Neumann boundary conditions. This implies $w_\kappa(x)=0$ whenever $x\in\Omega$.

Let us next consider (5.3); setting $u_{\kappa,\omega}(\varepsilon):=u_\kappa^{\mathrm{in}}+u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)$ and using the identities (3.13), (3.14), (3.15) and: $U_\varepsilon^{-1}\Delta U_\varepsilon=\varepsilon^{-2}\Delta$, we get

$$\begin{cases}\varepsilon^{-2}\left(-\Delta-\varepsilon^2k^2\right)U_\varepsilon^{-1}u_{\kappa,\omega}(\varepsilon)=0\,, & \text{in }\mathbb{R}^3\backslash\Gamma^\varepsilon\,,\\[2mm] [\gamma_0(\varepsilon)]\,U_\varepsilon^{-1}u_{\kappa,\omega}(\varepsilon)=0\,, & \\[2mm] [\gamma_1(\varepsilon)]\,U_\varepsilon^{-1}u_{\kappa,\omega}(\varepsilon)=\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0U_\varepsilon^{-1}u_{\kappa,\omega}(\varepsilon)\,. & \end{cases}\tag{5.8}$$

Then, the function

$$\psi_{\varepsilon\kappa,\omega}(\varepsilon):=\varepsilon^{-2}U_\varepsilon^{-1}u_{\kappa,\omega}(\varepsilon)\,,\tag{5.9}$$

solves the problem

$$\begin{cases}\left(-\Delta-\varepsilon^2k^2\right)\psi_{\varepsilon\kappa,\omega}(\varepsilon)=0\,, & \text{in }\mathbb{R}^3\backslash\Gamma\,,\\[2mm] [\gamma_0(\varepsilon)]\,\psi_{\varepsilon\kappa,\omega}(\varepsilon)=0\,, & \\[2mm] [\gamma_1(\varepsilon)]\,\psi_{\varepsilon\kappa,\omega}(\varepsilon)=\left(\varepsilon^{-2}-1\right)DN_{\varepsilon\omega}\gamma_0\psi_{\varepsilon\kappa,\omega}(\varepsilon)\,. & \end{cases}$$

Let $\psi_{\varepsilon\kappa,\omega}^{\mathrm{sc}}(\varepsilon):=\varepsilon^{-2}U_\varepsilon^{-1}u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)$; since $u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)$ fulfills the radiation conditions in (5.3), it follows

$$\lim_{|x|\to\infty}\left(\hat{x}\cdot\nabla\mp i\varepsilon\kappa\right)\psi_{\varepsilon\kappa,\omega}^{\mathrm{sc}}(\varepsilon)=\lim_{|x|\to\infty}\left(\hat{x}\cdot\nabla\mp i\varepsilon\kappa\right)\varepsilon^{-2}\left(U_\varepsilon^{-1}u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)\right)(x)$$

$$=\varepsilon^{-1/2}\lim_{|x|\to\infty}\left(\hat{x}\cdot\nabla\mp i\varepsilon\kappa\right)u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)\left(y_0+\varepsilon\,(x-y_0)\right)$$

$$=\varepsilon^{1/2}\lim_{|x|\to\infty}\left(\hat{x}\cdot\left(\nabla u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)\right)\left(y_0+\varepsilon\,(x-y_0)\right)\mp i\kappa u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)\left(y_0+\varepsilon\,(x-y_0)\right)\right)=0\,.$$

Hence, a scaled radiation condition holds for $\psi_{\varepsilon\kappa,\omega}^{\mathrm{sc}}(\varepsilon)$. Moreover, from (5.2) follows

$$\left(-\Delta-\varepsilon^2\kappa^2\right)\varepsilon^{-2}U_\varepsilon^{-1}u_\kappa^{\mathrm{in}}=U_\varepsilon^{-1}\left(-\Delta-\kappa^2\right)u_\kappa^{\mathrm{in}}=0\,.$$

Then, $\psi_{\varepsilon\kappa}^{\mathrm{in}}:=\varepsilon^{-2}U_\varepsilon^{-1}u_\kappa^{\mathrm{in}}$ is a solution of the Helmholtz equation at energy $\varepsilon^2\kappa^2$. Therefore, the total field $\psi_{\varepsilon\kappa,\omega}(\varepsilon)=\psi_{\varepsilon\kappa}^{\mathrm{in}}+\psi_{\varepsilon\kappa,\omega}^{\mathrm{sc}}(\varepsilon)$ solves the dilated scattering problem (5.5) at energy $\varepsilon^2\kappa^2$, whose unique solution, by the first part of the proof, writes as

$$\psi_{\varepsilon\kappa,\omega}^{\mathrm{sc}}(\varepsilon)=-\varepsilon^{-1}SL_{\varepsilon\kappa}\Lambda_\kappa^\omega(\varepsilon)\,\gamma_0\psi_{\varepsilon\kappa}^{\mathrm{in}}\,.$$

From (5.9) there follows

$$u_{\kappa,\omega}(\varepsilon)=\varepsilon^2U_\varepsilon\psi_{\varepsilon\kappa,\omega}(\varepsilon)=\varepsilon^2U_\varepsilon\left(\psi_{\varepsilon\kappa}^{\mathrm{in}}-\varepsilon^{-1}SL_{\varepsilon\kappa}\Lambda_\kappa^\omega(\varepsilon)\,\gamma_0\psi_{\varepsilon\kappa}^{\mathrm{in}}\right)$$

$$=\varepsilon^2U_\varepsilon\left(\varepsilon^{-2}U_\varepsilon^{-1}u_{\varepsilon\kappa}^{\mathrm{in}}-\varepsilon^{-1}SL_{\varepsilon\kappa}\Lambda_\kappa^\omega(\varepsilon)\,\gamma_0\varepsilon^{-2}U_\varepsilon^{-1}u_\kappa^{\mathrm{in}}\right)$$

$$=u_{\varepsilon\kappa}^{\mathrm{in}}-\varepsilon^{-1}U_\varepsilon SL_{\varepsilon\kappa}\Lambda_\kappa^\omega(\varepsilon)\,\gamma_0U_\varepsilon^{-1}u_\kappa^{\mathrm{in}}\,.$$

Using the definition (3.19), this leads us to

$$u_{\kappa,\omega}^{\mathrm{sc}}(\varepsilon)=-\varepsilon^{-3/2}G_\kappa(\varepsilon)\Lambda_\kappa^\omega(\varepsilon)\gamma_0U_\varepsilon^{-1}u_\kappa^{\mathrm{in}}\,,$$

and from

$$\gamma_0U_\varepsilon^{-1}u_\kappa^{\mathrm{in}}=\varepsilon^{3/2}\,\gamma_0\left(u_\kappa^{\mathrm{in}}\circ\Phi_\varepsilon\right)\,,$$

the representation (5.4) follows. ■

Remark 5.2 *According to* (5.1), *the solution* $u^{\rm in}_\kappa + u^{\rm sc}_{\kappa,\omega}(\varepsilon)$ *in* (5.3) *can be equivalently defined as the unique generalized eigenfunction of* $H_\omega(\varepsilon)$ *with eigenvalue* $-\kappa^2$ *such that* $u^{\rm sc}_{\kappa,\omega}(\varepsilon)$ *satisfies the (outgoing) Sommerfeld radiation condition.*

According to the above Remark, the next result is the analogous of Theorem 4.1 as regards the behavior of generalized eigenfunctions of $H_\omega(\varepsilon)$ whenever $\varepsilon \ll 1$:

Theorem 5.3 *For any* $\kappa > 0$ *and for any* $\varepsilon > 0$ *sufficiently small, let* $u^\omega_\kappa(\varepsilon) := u^{\rm in}_\kappa + u^{\rm sc}_{\kappa,\omega}(\varepsilon)$ *be as in Theorem 5.1. Then, one has*

$$\omega \neq \omega_M \quad\Longrightarrow\quad \|u^\omega_\kappa(\varepsilon) - u^\omega_\kappa\|_{L^2_{-\alpha}(\mathbb{R}^3)} \le c\,\varepsilon^{3/2}\,, \tag{5.10}$$

$$\omega = \omega_M \quad\Longrightarrow\quad \|u^\omega_\kappa(\varepsilon) - \widehat{u}_\kappa\|_{L^2_{-\alpha}(\mathbb{R}^3)} \le c\,\varepsilon^{1/2}\,, \tag{5.11}$$

where $\alpha > 1/2$,

$$u^\omega_\kappa(x) := u^{\rm in}_\kappa(x) + \varepsilon\,\frac{c_\Omega\,\omega^2}{\omega_M^2 - \omega^2}\; u^{\rm in}_\kappa(y_0)\,\mathcal{G}_\kappa(x-y_0)\,, \tag{5.12}$$

$$\widehat{u}_\kappa(x) := u^{\rm in}_\kappa(x) + 4\pi\,\frac{i}{\kappa}\; u^{\rm in}_\kappa(y_0)\,\mathcal{G}_\kappa(x-y_0)\,, \tag{5.13}$$

and the estimates hold uniformly with respect to the choice of $u^{\rm in}_\kappa$ *in any bounded subset of* $H^2_{-\alpha}(\mathbb{R}^3)$.

Proof. By (A.30) applied to $\psi_\varepsilon = u^{\rm in}_\kappa \circ \Phi_\varepsilon$ one gets

$$\|u^{\rm in}_\kappa \circ \Phi_\varepsilon - u^{\rm in}_\kappa(y_0)\|^2_{H^2(B)} \le c_{\alpha,B}\,\varepsilon^{1/2}\,\|u^{\rm in}_\kappa\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,,$$

where $\alpha > 1/2$ and $B \subset \mathbb{R}^3$ is any star-shaped bounded open set. Since $\gamma_0 \in \mathcal{L}(H^2(B), H^{3/2}(\Gamma))$ and $u^{\rm in}_\kappa \in H^2_{-\alpha}(\mathbb{R}^3)$, it follows

$$\|\gamma_0\left(u^{\rm in}_\kappa \circ \Phi_\varepsilon - u^{\rm in}_\kappa(y_0)\right)\|_{H^{1/2}(\Gamma)} \le c\,\varepsilon^{1/2}\|u^{\rm in}_\kappa\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,, \tag{5.14}$$

and hence

$$\gamma_0\left(u^{\rm in}_\kappa \circ \Phi_\varepsilon\right) = u^{\rm in}_\kappa(y_0) + O(\varepsilon^{1/2})\,.$$

We next proceed along the same lines as in the proof of Theorem 4.1. By (4.7), (4.8), (5.14) and (2.17), it results

$$\Lambda^\omega_\kappa(\varepsilon)\gamma_0\left(u^{\rm in}_\kappa \circ \Phi_\varepsilon\right) = \left(\varepsilon^{\alpha_\omega}\omega^2\beta_{\omega,\kappa}S_0^{-1}P_0K_{(2)} + O\left(\varepsilon^{1+\alpha_\omega}\right)\right)\left(u^{\rm in}_\kappa(y_0) + O(\varepsilon^{1/2})\right)$$

$$= \begin{cases} \left(\varepsilon\frac{\omega^2\omega_M^2}{\omega^2-\omega^2}S_0^{-1}P_0K_{(2)} + O(\varepsilon^2)\right)\left(u^{\rm in}_\kappa(y_0) + O(\varepsilon^{1/2})\right)\,, & \omega \neq \omega_M\,, \\[2ex] \left(\frac{\omega_M^2 4\pi i}{c_\Omega\,\kappa}S_0^{-1}P_0K_{(2)} + O(\varepsilon)\right)\left(u^{\rm in}_\kappa(y_0) + O(\varepsilon^{1/2})\right)\,, & \omega = \omega_M\,, \end{cases}$$

and, by (4.12), we get

$$\Lambda^\omega_\kappa(\varepsilon)\gamma_0\left(u^{\rm in}_\kappa \circ \Phi_\varepsilon\right) = \begin{cases} -\varepsilon\frac{\omega^2}{\omega_M^2-\omega^2}\,u^{\rm in}_\kappa(y_0)S_0^{-1}(1) + O(\varepsilon^{3/2})\,, & \omega \neq \omega_M\,, \\[2ex] -\frac{4\pi i}{c_\Omega\,\kappa}\,u^{\rm in}_\kappa(y_0)S_0^{-1}(1) + O(\varepsilon^{1/2})\,, & \omega = \omega_M\,. \end{cases} \tag{5.15}$$

The expansion (4.9) implies

$$G_\kappa(\varepsilon)S_0^{-1}(1) = G_\kappa S_0^{-1}(1) + O(\varepsilon^{1/2}) = \langle 1, S_0^{-1}(1)\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\,\mathcal{G}^{y_0}_\kappa + O(\varepsilon^{1/2}) = c_\Omega\,\mathcal{G}^{y_0}_\kappa + O(\varepsilon^{1/2})\,. \tag{5.16}$$

Finally, combining (5.4) with (5.15) and (5.16), one gets (5.10)-(5.13). ■

Remark 5.4 *Suppose* $u^{\rm in}_\kappa(x)$ *is a plane wave with direction* $\hat\theta$ *and frequency* κ, *i.e.:* $u^{\rm in}_\kappa(x) = e^{i\omega\hat\theta\cdot x}$ *. Then, in consistency with Theorem 4.1,* $\widehat{u}_\kappa$ *in (5.13) is a generalized eigenfunction with eigenvalue* $-\kappa^2$ *of the self-adjoint operator* Δ_{y_0} *defined in* (4.1) *and* (4.2) *(see [2, equation 1.4.11], there* $\alpha = 0$ *and* $y = y_0$*).*

6 The acoustic scattering in the asymptotic regime

We are now in the position to state our results for the acoustic scattering problem. As pointed out in Section 1.2, the acoustic equation

$$\begin{cases} \left(\nabla\cdot(1_{\mathbb{R}^3\backslash\Omega^\varepsilon}+\varepsilon^{-2}1_{\Omega^\varepsilon})\nabla+\omega^2\left(1_{\mathbb{R}^3\backslash\Omega^\varepsilon}+\varepsilon^{-2}1_{\Omega^\varepsilon}\right)\right)u_\omega\left(\varepsilon\right)=0\,, \\ \gamma_0^{\rm in}(\varepsilon)u_\omega\left(\varepsilon\right)=\gamma_0^{\rm ex}(\varepsilon)u_\omega(\varepsilon)\,, \qquad \varepsilon^{-2}\gamma_1^{\rm in}(\varepsilon)u_\omega\left(\varepsilon\right)=\gamma_1^{\rm ex}(\varepsilon)u_\omega(\varepsilon)\,, \end{cases} \tag{6.1}$$

is equivalent to the generalized eigenvalue problem for the operator $H_\omega(\varepsilon)$ at energy ω^2. By Theorem 5.1, the corresponding scattering problem is well posed and the diffusion of an incident wave $u^{\rm in}_\omega$ with frequency $\omega>0$ is described by the outgoing radiating solution (5.4) and allows the asymptotic expansions provided in Theorem 5.3. The next result resumes these conclusions.

Corollary 6.1 *Let $u^{\rm in}_\omega\in H^2_{-\alpha}(\mathbb{R}^3)$, $\alpha>1/2$, be a solution of the homogeneous Helmholtz equation*

$$\left(\Delta+\omega^2\right)u^{\rm in}_\omega=0\,. \tag{6.2}$$

The boundary value problem (6.1) admits an unique solution $u_\omega(\varepsilon)\in H^2_{-\alpha}(\mathbb{R}^3\backslash\Gamma^\varepsilon)$, $\alpha>1/2$, such that $u^{\rm sc}_\omega(\varepsilon):=u_\omega(\varepsilon)-u^{\rm in}_\omega$ satisfies the outgoing Sommerfeld radiation condition. The scattered field represents as

$$u^{\rm sc}_\omega(\varepsilon):=-G_\omega(\varepsilon)\Lambda^\omega_\omega(\varepsilon)\,\gamma_0\left(u^{\rm in}_\omega\circ\Phi_\varepsilon\right)\,, \tag{6.3}$$

i.e., $u_\omega(\varepsilon)$ is a generalized eigenfunction of $H_\omega(\varepsilon)$ with eigenvalue $-\omega^2$. Moreover, for any $\varepsilon>0$ sufficiently small, one has

$$\omega\neq\omega_M \quad\Longrightarrow\quad \begin{cases} \left(u^{\rm sc}_\omega(\varepsilon)\right)(x)=\varepsilon\,\frac{\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\;u^{\rm in}_\omega(y_0)\,\frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|}+\left(r_\omega(\varepsilon)\right)(x)\,, \\ \|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)}\leq c\,\varepsilon^{3/2}\,, \qquad \alpha>1/2\,, \end{cases} \tag{6.4}$$

$$\omega=\omega_M \quad\Longrightarrow\quad \begin{cases} \left(u^{\rm sc}_\omega(\varepsilon)\right)(x)=\frac{4\pi i}{\omega}\;u^{\rm in}_\omega(y_0)\,\frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|}+\left(r_\omega\left(\varepsilon\right)\right)(x)\,, \\ \|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)}\leq c\,\varepsilon^{1/2}\,, \qquad \alpha>1/2\,, \end{cases} \tag{6.5}$$

where the estimates hold uniformly with respect to $u^{\rm in}_\omega$ in any bounded subset of $H^2_{-\alpha}(\mathbb{R}^3)$.

Let us next denote with $u^\infty_\omega(\varepsilon)$ the far-field pattern (a.k.a. the scattering amplitude) related to the solutions $u^{\rm sc}_\omega(\varepsilon)$ by (see, e.g., [15, Theorem 1.4])

$$(u^{\rm sc}_\omega(\varepsilon))(x)=\frac{e^{i\omega|x|}}{4\pi\,|x|}\,(u^\infty_\omega(\varepsilon))(\hat{x})+O(|x|^{-2})$$

as $|x|\to+\infty$, uniformly with respect to $\hat{x}\in\mathbb{S}^2$, $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$. By (3.11), (3.19), (6.3) and (A.13), one gets

$$\begin{aligned}(u^{\rm sc}_\omega(\varepsilon))(x)=&-\varepsilon^{-1}\big\langle\gamma_0\big(\mathcal{G}_{-\varepsilon\omega}^{\Phi_\epsilon^{-1}(x)}\big),\Lambda^\omega_\omega(\varepsilon)\,\gamma_0(u^{\rm in}_\omega\circ\Phi_\varepsilon)\big\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\\ =&\big\langle\gamma_0(\mathcal{G}^x_{-\omega}\circ\Phi_\varepsilon),\Lambda^\omega_\omega(\varepsilon)\,\gamma_0(u^{\rm in}_\omega\circ\Phi_\varepsilon)\big\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\,.\end{aligned}$$

Then, since the far-field pattern of $\mathcal{G}^y_\omega$ is given by $e^{-i\omega\,\hat{x}\cdot y}$, we get

$$(u^\infty_\omega(\varepsilon))(\hat{x})=-\big\langle e^{i\omega\,\hat{x}\cdot\Phi_\varepsilon(\cdot)},\Lambda^\omega_\omega(\varepsilon)\,\gamma_0(u^{\rm in}_\omega\circ\Phi_\varepsilon)\big\rangle_{H^{1/2}(\Gamma),H^{-1/2}(\Gamma)}\,. \tag{6.6}$$

The expansions provided in Theorem 5.3 are next rephrased in terms of the far-field patterns as follows:

Lemma 6.2 *Let $u^{\rm sc}_\omega(\varepsilon)=u_\omega(\varepsilon)-u^{\rm in}_\omega$ be as in Corollary 6.1. Then*

$$\omega\neq\omega_M \quad\Longrightarrow\quad \begin{cases} (u^\infty_\omega(\varepsilon))(\hat{x})=\varepsilon\,\frac{\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\,u^{\rm in}_\omega(y_0)\,e^{-i\omega\,\hat{x}\cdot y_0}+\left(r^\infty_\omega(\varepsilon)\right)(\hat{x})\,, \\ \|r^\infty_\omega(\varepsilon)\|_{L^2(\mathbb{S}^2)}\leq c\,\varepsilon^{3/2}\,, \end{cases} \tag{6.7}$$

$$\omega=\omega_M \quad\Longrightarrow\quad \begin{cases} (u^\infty_\omega(\varepsilon))(\hat{x})=\frac{4\pi i}{\omega}\,u^{\rm in}_\omega(y_0)\,e^{-i\omega\,\hat{x}\cdot y_0}+\left(r^\infty_\omega(\varepsilon)\right)(\hat{x})\,, \\ \|r^\infty_\omega(\varepsilon)\|_{L^2(\mathbb{S}^2)}\leq c\,\varepsilon^{1/2}\,, \end{cases} \tag{6.8}$$

Proof. The expansion (5.15) gives

$$\Lambda_\omega^\omega(\varepsilon)\gamma_0\left(u_\omega^{\rm in}\circ\Phi_\varepsilon\right)=\begin{cases}-\varepsilon\,\frac{\omega^2}{\omega_M^2-\omega^2}\,u_\omega^{\rm in}(y_0)S_0^{-1}(1)+O(\varepsilon^{3/2})\,, & \omega\neq\omega_M\,,\\ -\frac{4\pi i}{c_\Omega\,\omega}\,u_\omega^{\rm in}(y_0)\,S_0^{-1}(1)+O(\varepsilon^{1/2})\,, & \omega=\omega_M\,.\end{cases}\tag{6.9}$$

By the definition of Φ_ε (see 3.9),

$$e^{-i\omega\,\hat x\cdot\Phi_\varepsilon(y)}=e^{-i\omega\,\hat x\cdot y_0}+\varepsilon F_{\varepsilon,\omega}(\hat x,y)\,,\tag{6.10}$$

where $F_{\varepsilon,\omega}\in\mathcal{C}^\infty(\mathbb{S}^2\times\Gamma)$ and

$$\sup_{0\le\omega\le\omega_0,\,0\le\varepsilon\le\varepsilon_0,\,\hat x\in\mathbb{S}^2,\,y\in\Gamma}|F_{\varepsilon,\omega}(\hat x,y)|\le c\,.\tag{6.11}$$

The proof is then concluded by inserting (6.9) and (6.10) in (6.6) and by the bound (6.11).

■

6.1 Quasi-resonant asymptotic scattering solutions

The estimates in the expansions provided in Corollary 6.1 are frequency-dependent and so they are useless as regards an accurate descriptions of the transitions between the two different asymptotic scattering regimes as the frequency ω approaches the Minnaert one ω_M. In this section we provide more refined estimates which are uniform with respect to the frequency ω. Their proof relies on ω-uniform estimates on the ε-expansion of the operator $\Lambda_\omega^\omega(\varepsilon)$ which we provide at first.

Theorem 6.3 *Let $c_M>0$, $\mathcal{I}_M\subset\mathbb{R}_+$ be a bounded interval containing ω_M and E_ω^0, E_ω^1 be given by (2.17), (2.21). For $\varepsilon>0$ sufficiently small the expansion*

$$\Lambda_\omega^\omega(\varepsilon)=\frac{1}{\varepsilon}\,S_{\varepsilon\omega}^{-1}\left(\frac{1}{E_\omega^0}\left(\frac{E_\omega^0}{E_\omega^0+E_\omega^1\,\varepsilon}\,P_0+P_0O(\varepsilon)P_0\right)+O(\varepsilon^2)\right)S_{\varepsilon\omega}DN_{\varepsilon\omega}\,,\tag{6.12}$$

holds uniformly w.r.t. ω in $\{\omega\in\mathcal{I}_M:|\omega-\omega_M|\ge c_M\,\varepsilon\}$, i.e.,

$$\sup_{\omega\in\mathcal{I}_M:|\omega-\omega_M|\ge c_M\,\varepsilon}\|O(\varepsilon^j)\|_{H^{1/2}(\Gamma),H^{1/2}(\Gamma)}\le C_M\varepsilon^j\,,\tag{6.13}$$

with C_M depending only on c_M.

Proof. From (2.30) follows

$$\Lambda_\omega^\omega(\varepsilon)=\varepsilon(1-\varepsilon^2)S_{\varepsilon\omega}^{-1}\left(\varepsilon^2+\left(1-\varepsilon^2\right)\left(\frac{1}{2}+K_{\varepsilon\omega}\right)\right)^{-1}S_{\varepsilon\omega}DN_{\varepsilon\omega}\,.$$

Thus, by (2.9), $\mathbb{M}(\varepsilon)$ in (2.2) has the following components:

$$M_{00}(\varepsilon)=P_0\left(E_\omega^0\varepsilon^2+E_\omega^1\varepsilon^3+\varepsilon^2O((\varepsilon\omega)^2)\right)P_0\,,$$

$$M_{01}(\varepsilon)=P_0O((\varepsilon\omega)^2)Q_0\,,\qquad M_{10}(\varepsilon)=Q_0O((\varepsilon\omega)^2)P_0\,,$$

$$M_{11}(\varepsilon)=Q_0\left(1/2+K_0+\varepsilon^2O((\varepsilon\omega)^0)\right)Q_0\,.$$

The requirement $|\omega-\omega_M|\ge c_M\,\varepsilon$ is equivalent to $\varepsilon/E_\omega^0=O_u(1)$, where $O_u\left(\varepsilon^j\right)$ means that the corresponding estimate is uniform with respect to ω. It follows

$$\begin{aligned}M_{00}(\varepsilon)=&E_\omega^0\varepsilon^2P_0\left(1+E_\omega^1\,\varepsilon/E_\omega^0+O_u(\varepsilon)\varepsilon/E_\omega^0\right)P_0\\ =&E_\omega^0\varepsilon^2P_0\left(1+E_\omega^1\,\varepsilon/E_\omega^0+O_u(\varepsilon)\right)P_0\,,\end{aligned}$$

$$M_{01}(\varepsilon)=P_0O_u(\varepsilon^2)Q_0\,,\qquad M_{10}(\varepsilon)=Q_0O_u(\varepsilon^2)P_0\,,$$

$$M_{11}(\varepsilon)=Q_0\left(1/2+K_0+O_u(\varepsilon^2)\right)Q_0\,.$$

Then,

$$C_{00}(\varepsilon)=E_\omega^0\varepsilon^2P_0\left(1+E_\omega^1\,\varepsilon/E_\omega^0+O_u(\varepsilon)\right)P_0$$

and, by Lemma 2.1,

$$C_{00}(\varepsilon)^{-1} = \frac{1}{\varepsilon^2}\,\frac{1}{E^0_\omega}\,\left(\left(1 + E^1_\omega\,\varepsilon/E^0_\omega\right)^{-1}\,P_0 + P_0 O_u(\varepsilon)P_0\right)\,. \tag{6.14}$$

Notice that $1 + E^1_\omega\,\varepsilon/E^0_\omega \neq 0$ since $E^1_\omega \in i\mathbb{R}$. Then, by (6.14), one gets, as in the proof of point (1) in Theorem 2.5,

$$\varepsilon^2\mathbb{M}(\varepsilon)^{-1} = \begin{bmatrix} (E^0_\omega)^{-1}\,\left(\left(1 + E^1_\omega\,\varepsilon/E^0_\omega\right)^{-1}\,P_0 + P_0 O_u(\varepsilon)P_0\right) & P_0 O(\varepsilon^2)Q_0 \\ Q_0 O(\varepsilon^2)P_0 & Q_0 O(\varepsilon^2)Q_0 \end{bmatrix}\,.$$

This entails

$$\varepsilon^2\left(\varepsilon^2 + \left(1-\varepsilon^2\right)(1/2 + K_{\varepsilon\omega})\right)^{-1} = \frac{1}{E^0_\omega}\,\left(\left(1 + E^1_\omega\,\varepsilon/E^0_\omega\right)^{-1}\,P_0 + P_0 O_u(\varepsilon)P_0\right) + O_u(\varepsilon^2)\,.$$

The proof is then concluded by proceeding as in Theorem 2.6. ∎

The result of Theorem 6.3 allows to improve our analysis of the asymptotic acoustic scattering including the quasi-resonant regime $|\,\omega - \omega_M| \gtrsim \varepsilon$.

Theorem 6.4 *Let $c_M > 0$, $\mathcal{I}_M \subset \mathbb{R}_+$ be a bounded interval containing ω_M. Let $u^{\rm in}_\omega \in H^2_{-\alpha}(\mathbb{R}^3)$, $\alpha > 1/2$, be a solution of the homogeneous Helmholtz equation (6.2) and $u_\omega(\varepsilon) = u^{\rm sc}_\omega(\varepsilon) + u^{\rm in}_\omega$ be the unique solution of the problem (6.1) defined in Corollary 6.1. For $\varepsilon > 0$ sufficiently small, the expansion*

$$\left(u^{\rm sc}_\omega(\varepsilon)\right)(x) = \frac{\varepsilon\,\omega^2\,c_\Omega}{\omega^2_M - \omega^2}\,\left(1 - i\,\frac{\omega}{4\pi}\,\frac{\varepsilon\,\omega^2\,c_\Omega}{\omega^2_M - \omega^2}\right)^{-1} u^{\rm in}_\omega(y_0)\,\frac{e^{i\omega|x-y_0|}}{4\pi|x-y_0|} + \left(r_\omega(\varepsilon)\right)(x)\,, \tag{6.15}$$

$$\|r_\omega(\varepsilon)\|_{L^2_{-\alpha}(\mathbb{R}^3)} \le c\,\frac{\varepsilon^{3/2}}{\omega^2_M - \omega^2}\,,\qquad \alpha > 1/2\,. \tag{6.16}$$

holds uniformly both w.r.t. ω in $\{\omega \in \mathcal{I}_M : |\omega - \omega_M| \ge c_M\,\varepsilon\}$ and $u^{\rm in}_\omega$ in any bounded subset of $H^2_{-\alpha}(\mathbb{R}^3)$.

Proof. We proceed as in the proofs of Theorem 5.3. Since, by Lemma A.1, $S^{-1}_{\varepsilon\omega} = S^{-1}_0 + O(\varepsilon\omega)$ and, by Lemma A.6, $S_{\varepsilon\omega}DN_{\varepsilon\omega} = Q_0(1/2 + K_0)Q_0 + (\varepsilon\omega)^2K_{(2)} + O((\varepsilon\omega)^3)$, one has

$$S^{-1}_{\varepsilon\omega} = S^{-1}_0 + O_u(\varepsilon)$$

and

$$S_{\varepsilon\omega}DN_{\varepsilon\omega} = Q_0(1/2 + K_0)Q_0 + (\varepsilon\omega)^2K_{(2)} + O_u(\varepsilon^3)\,,$$

where $O_u(\varepsilon^\lambda)$ means that the corresponding estimate holds – in the appropriate norm – uniformly with respect to ω in $\{\omega \in \mathcal{I}_M : |\omega - \omega_M| \ge c_M\,\varepsilon\}$. Since $\varepsilon/E^0_\omega = O_u(1)$, combining such relations with (6.12), one gets

$$\Lambda^\omega_\omega(\varepsilon) = \frac{\varepsilon}{E^0_\omega}\,\frac{E^0_\omega}{E^0_\omega + E^1_\omega\,\varepsilon}\;\omega^2 S^{-1}_0 P_0 K_{(2)} + P_0 O_u(\varepsilon)P_0 + O_u(\varepsilon^2)\,. \tag{6.17}$$

From (4.12), (5.14) and (6.17) there follows

$$\begin{aligned}
&\Lambda^\omega_\omega(\varepsilon)\gamma_0\left(u^{\rm in}_\omega \circ \Phi_\varepsilon\right) = \\
&= \left(\frac{\varepsilon}{E^0_\omega}\,\frac{E^0_\omega}{E^0_\omega + E^1_\omega\,\varepsilon}\;\omega^2 S^{-1}_0 P_0 K_{(2)} + P_0 O_u(\varepsilon)P_0 + O_u(\varepsilon^2)\right)\left(u^{\rm in}_\omega(y_0) + \left\|u^{\rm in}_\omega\right\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,O_u(\varepsilon^{1/2})\right) \\
&= -\frac{\varepsilon}{E^0_\omega + E^1_\omega\,\varepsilon}\left(u^{\rm in}_\omega(y_0)\frac{\omega^2}{\omega^2_M}\;S^{-1}_0(1) + \left\|u^{\rm in}_\omega\right\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,O_u(\varepsilon^{1/2})\right) + u^{\rm in}_\omega\,(y_0)\,O_u(\varepsilon) \\
&\quad + O_u(\varepsilon^2)\left(u^{\rm in}_\omega\,(y_0) + \|u^{\rm in}_\omega\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,O(\varepsilon^{1/2})\right)\,.
\end{aligned}$$

By the definitions of E^0_ω, E^1_ω and ω^2_M, results

$$\frac{\varepsilon}{E^0_\omega + E^1_\omega\,\varepsilon} = \frac{\varepsilon\omega^2_M}{\omega^2_M - \omega^2}\left(1 - i\,\frac{\omega^3\varepsilon}{\omega^2_M - \omega^2}\,\frac{c_\Omega}{4\pi}\right)^{-1} = O_u(1)\,,$$

and the r.h.s. simplifies to

$$\Lambda^{\omega}_{\omega}(\varepsilon)\gamma_0\left(u^{\rm in}_{\omega}\circ\Phi_{\varepsilon}\right)=$$

$$=-\frac{\varepsilon\omega^2}{\omega_M^2-\omega^2}\left(1-i\,\frac{\omega^3\varepsilon}{\omega_M^2-\omega^2}\,\frac{c_\Omega}{4\pi}\right)^{-1}u^{\rm in}_{\omega}(y_0)S_0^{-1}(1)+\left\|u^{\rm in}_{\omega}\right\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,O_u(\varepsilon^{1/2})\,. \tag{6.18}$$

Since $\mathcal{I}_M\ni\omega\mapsto\|R_\omega\|_{L^2_\alpha(\mathbb{R}^3),H^2_{-\alpha}(\mathbb{R}^3)}$ is continuous (see Subsection A.2), by Corollary A.5, one gets

$$G_\omega(\varepsilon)=G_\omega+O_u(\varepsilon^{1/2})\,.$$

Therefore, from

$$u^{\rm sc}_{\omega}(\varepsilon)=-G_\omega(\varepsilon)\Lambda^{\omega}_{\omega}(\varepsilon)\gamma_0\left(u^{\rm in}_{\kappa}\circ\Phi_{\varepsilon}\right)\,,$$

combining the above expansions, we obtain

$$u^{\rm sc}_{\omega}(\varepsilon)=\Big(G_\omega+O_u(\varepsilon^{1/2})\Big)\left(\frac{\varepsilon\omega^2}{\omega_M^2-\omega^2}\left(1-i\,\frac{\omega^3\varepsilon}{\omega_M^2-\omega^2}\,\frac{c_\Omega}{4\pi}\right)^{-1}u^{\rm in}_{\omega}\,(y_0)\,S_0^{-1}(1)+\left\|u^{\rm in}_{\omega}\right\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,O_u(\varepsilon^{1/2})\right)\,.$$

The definition of G_ω and in particular: $G_\omega S_0^{-1}\,(1)=c_\Omega\,\mathcal{G}_\omega\,(\cdot-y_0)=c_\Omega\,\mathcal{G}^{y_0}_\omega$ (see (5.16)), leads to the expansion

$$u^{\rm sc}_{\omega}(\varepsilon)=\frac{\varepsilon\omega^2}{\omega_M^2-\omega^2}\left(1-i\,\frac{\omega^3\varepsilon}{\omega_M^2-\omega^2}\,\frac{c_\Omega}{4\pi}\right)^{-1}u^{\rm in}_{\omega}(y_0)c_\Omega\,\mathcal{G}^{y_0}_\omega+\left\|u^{\rm in}_{\omega}\right\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,O_u(\varepsilon^{1/2})\,,$$

which corresponds to our statement after noticing that $\varepsilon/\left(\omega_M^2-\omega^2\right)=O_u(1)$. ■

Building on (6.17), similar computations to the ones provided in the proof of Lemma 6.2 lead to the asymptotic expansion of the far-field pattern in the quasi-resonant regime:

Lemma 6.5 *Let $c_M>0$, $\mathcal{I}_M\subset\mathbb{R}_+$ and $u^{\rm sc}_\omega(\varepsilon)=u_\omega(\varepsilon)-u^{\rm in}_\omega$ be defined as in Theorem 6.4. The far-field pattern associated to $u^{\rm sc}_\omega(\varepsilon)$ has the expansion*

$$(u^{\infty}_{\omega}(\varepsilon))(\hat x)=\varepsilon\,\frac{\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\left(1-i\,\frac{\omega}{4\pi}\,\frac{\varepsilon\,\omega^2\,c_\Omega}{\omega_M^2-\omega^2}\right)^{-1}u^{\rm in}_{\omega}(y_0)\,e^{-i\omega\,\hat x\cdot y_0}+\left(r_\omega(\varepsilon)\right)(\hat x)\,, \tag{6.19}$$

$$\|r_\omega(\varepsilon)\|_{L^2(\mathbb{S}^2)}\le c\,\frac{\varepsilon^{3/2}}{\omega_M^2-\omega^2}\,,\qquad \alpha>1/2\,, \tag{6.20}$$

where the estimates holds uniformly both w.r.t. ω in $\{\omega\in\mathcal{I}_M:|\omega-\omega_M|\ge c_M\,\varepsilon\}$ and $u^{\rm in}_\omega$ in any bounded subset of $H^2_{-\alpha}(\mathbb{R}^3)$.

Proof. The proof is the same as for Lemma 6.2, it suffice to replace (6.9) with (6.18) and use $\varepsilon/\left(\omega_M^2-\omega^2\right)=O_u(1)$.
■

A Resolvent analysis, boundary integral operators and operator expansions

A.1 (Weighted) Sobolev spaces

Given $\Omega \subset \mathbb{R}^3$ open and bounded, with smooth boundary Γ, we adopt the notation

$$\Omega_{\text{in}} = \Omega\,, \qquad \Omega_{\text{ex}} = \mathbb{R}^3 \backslash \overline{\Omega}\,.$$

The symbols $H^s(\mathbb{R}^3)$, $H^s(\Omega_{\text{in}})$, $H^s(\Omega_{\text{ex}})$, $H^s(\Gamma)$, $s \in \mathbb{R}$, denote the usual scales of Sobolev-Hilbert spaces of function on $\mathbb{R}^3$, Ω_{in}, Ω_{ex} and Γ respectively (see, e.g., [20]). We use the notation

$$H^s(\mathbb{R}^3\backslash\Gamma) := H^s(\Omega_{\text{in}}) \oplus H^s(\Omega_{\text{ex}})\,.$$

Let $\langle x \rangle := (1+|x|^2)^{1/2}$ and $\alpha \in \mathbb{R}$. Then we define the weighted L^2-space by

$$L^2_\alpha(\mathbb{R}^3) := \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^3) : \|u\|_{L^2_\alpha(\mathbb{R}^3)} < +\infty \right\}, \quad \|u\|_{L^2_\alpha(\mathbb{R}^3)} := \| \langle x \rangle^\alpha u \|_{L^2(\mathbb{R}^3)}\,.$$

The weighted Sobolev spaces of positive integer order ℓ are defined by

$$H^\ell_\alpha(\mathbb{R}^3) = \left\{ u \in L^2_\alpha(\mathbb{R}^3) : \|u\|_{H^\ell_\alpha(\mathbb{R}^3)} < +\infty \right\}, \quad \|u\|^2_{H^\ell_\alpha(\mathbb{R}^3)} := \sum_{|k|\le \ell} \|D^k u\|^2_{L^2_\alpha(\mathbb{R}^3)}.$$

If $s > 0$ is not integer, $H^s_\alpha(\mathbb{R}^3)$ is defined via interpolation and for $s < 0$ we define $H^s_\alpha(\mathbb{R}^3)$ as the dual of $H^{-s}_{-\alpha}(\mathbb{R}^3))$.

The spaces $L^2_\alpha(\Omega_{\text{ex}})$ and $H^s_\alpha(\Omega_{\text{ex}})$ are defined in a similar way. One has

$$L^2_\alpha(\mathbb{R}^3) = L^2(\Omega_{\text{in}}) \oplus L^2_\alpha(\Omega_{\text{ex}})$$

and we set

$$H^s_\alpha(\mathbb{R}^3\backslash\Gamma) := H^s(\Omega_{\text{in}}) \oplus H^s_\alpha(\Omega_{\text{ex}})\,.$$

A.2 The free resolvent

Let Δ be the distributional Laplacian; whenever restricted to $H^2(\mathbb{R}^3)$, it is a self-adjoint operator in $L^2(\mathbb{R}^3)$ and its resolvent

$$R_z := \left(-\Delta - z^2\right)^{-1}\,, \qquad z \in \mathbb{C}_+\,, \tag{A.1}$$

provides a map $R_z \in \mathcal{L}(H^s(\mathbb{R}^3), H^{s+2}(\mathbb{R}^3))$ for any $s \ge 0$. For any $u \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $z \in \mathbb{C}_+$ one has the integral representation

$$R_z u(x) = \int_{\mathbb{R}^3} \mathcal{G}_z(x-y) u(y)\, dy\,, \qquad \mathcal{G}_z(x) := \frac{e^{iz|x|}}{4\pi\,|x|}\,. \tag{A.2}$$

R_z in (A.1) extends to a map $R_z \in \mathcal{L}(H^s(\mathbb{R}^3), H^{s+2}(\mathbb{R}^3))$ for any real s; moreover, $\mathbb{C}_+ \ni z \mapsto R_z$ is a $\mathcal{L}(H^s(\mathbb{R}^3), H^{s+2}(\mathbb{R}^3))$-valued continuous map for any real s. By the resolvent identity

$$R_z - R_w = (z^2 - w^2) R_w R_z\,,$$

the latter entails that $\mathbb{C}_+ \ni z \mapsto R_z$ is a $\mathcal{L}(H^s(\mathbb{R}^3), H^{s+2}(\mathbb{R}^3))$-valued analytic map for any real s.

By the Limiting Absorption Principle (see, e.g., [16, Theorem 18.3]), $\mathbb{C}_+ \ni z \mapsto R_z$ extends to a map $\overline{\mathbb{C}_+} \ni z \mapsto R_z$ defined as

$$z \mapsto \begin{cases} (-\Delta - z^2)^{-1}\,, & z \in \mathbb{C}_+ \\ \lim_{\delta \to 0_+} \left(-\Delta - (\kappa + i\delta)^2\right)^{-1}\,, & z = \kappa \in \mathbb{R}\,. \end{cases} \tag{A.3}$$

The above limit exists in $\mathcal{L}(H^{-s}_\alpha(\mathbb{R}^3), H^{-s+2}_{-\alpha}(\mathbb{R}^3))$ for any $s \in [0,2]$, where $\alpha > 1/2$ whenever $\kappa \neq 0$, or $\alpha > 1$ if $\kappa = 0$; moreover, $\overline{\mathbb{C}_+}\backslash\{0\} \ni z \mapsto R_z$ is continuous as a $\mathcal{L}(H^{-s}_\alpha(\mathbb{R}^3), H^{-s+2}_{-\alpha}(\mathbb{R}^3))$-valued map for any $\alpha > 1/2$ and $\overline{\mathbb{C}_+} \ni z \mapsto R_z$ is continuous as a $\mathcal{L}(H^{-s}_\alpha(\mathbb{R}^3), H^{-s+2}_{-\alpha}(\mathbb{R}^3))$-valued map for any $\alpha > 1$.

We extend $z \mapsto R_z$ in (A.3) to the whole $\mathbb{C}$ by

$$R_z u := \mathcal{G}_z * u\,, \qquad z \in \mathbb{C}\,. \tag{A.4}$$

The two definitions (A.3) and (A.4) agree when $z \in \overline{\mathbb{C}_+}$, while the integral representation (A.2) still holds for $z \in \overline{\mathbb{C}_-}$ and $u \in \mathcal{D}(\mathbb{R}^3) \equiv \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^3)$. Since $\mathcal{G}_z \in L^1_{\text{loc}}(\mathbb{R}^3) \subset \mathcal{D}'(\mathbb{R}^3)$, R_z in (A.4) belongs to $\mathcal{L}(\mathcal{E}'(\mathbb{R}^3),\mathcal{D}'(\mathbb{R}^3))$ (see, e.g., [25, Theorem 27.6]). Since the series

$$\mathcal{G}_z = \mathcal{G}_0 + \sum_{n=1}^{+\infty} \mathcal{G}_{(n)}\, z^n\,, \qquad \mathcal{G}_{(n)}(x) := \frac{1}{4\pi}\,\frac{i^n}{n!}\,|x|^{n-1} \tag{A.5}$$

converges in $\mathcal{D}'(\mathbb{R}^3)$ and the map $f \mapsto f * u$ belongs to $\mathcal{L}(\mathcal{D}'(\mathbb{R}^3))$ for any $u \in \mathcal{E}'(\mathbb{R}^3)$ (see, e.g., [25, Theorem 27.6]), one has

$$R_z = R_0 + \sum_{n=1}^{+\infty} R_{(n)}\, z^n\,, \qquad R_{(n)}u := \mathcal{G}_{(n)} * u\,, \tag{A.6}$$

and the series strongly converges in $\mathcal{L}(\mathcal{E}'(\mathbb{R}^3),\mathcal{D}'(\mathbb{R}^3))$.

A.3 Trace maps

Here we recall some well known definitions and results about traces in Sobolev spaces (see, e.g., [20]). The zero and first-order traces on Γ are defined on smooth functions as

$$\gamma_0 u = u|_\Gamma\,, \qquad \gamma_1 u = \nu\cdot\nabla u|\Gamma\,, \tag{A.7}$$

where ν is the exterior unit normal to Γ, and extend to bounded linear operators

$$\gamma_0 \in \mathcal{L}(H^s(\mathbb{R}^3), H^{s-\frac12}(\Gamma))\,,\quad s>\frac12\,, \qquad \gamma_1 \in \mathcal{L}(H^s(\mathbb{R}^3), H^{s-\frac32}(\Gamma))\,,\quad s>\frac32\,. \tag{A.8}$$

The one-sided trace maps

$$\gamma_0^{\text{in/ex}} \in \mathcal{L}(H^s(\Omega_{\text{in/ex}}), H^{s-\frac12}(\Gamma))\,,\quad s>\frac12\,, \qquad \gamma_1^{\text{in/ex}} \in \mathcal{L}(H^s(\Omega_{\text{in/ex}}), H^{s-\frac32}(\Gamma))\,,\quad s>\frac32\,,$$

defined on smooth (up to the boundary) functions by

$$\gamma_0^{\text{in/ex}} u_{\text{in/ex}} = u_{\text{in/ex}}|\Gamma\,, \qquad \gamma_1^{\text{in/ex}} u_{\text{in/ex}} = \nu\cdot\nabla u_{\text{in/ex}}|\Gamma\,,$$

can be extended to

$$\gamma_0^{\text{in/ex}} \in \mathcal{L}(H^0_\Delta(\Omega_{\text{in/ex}})), H^{-\frac12}(\Gamma))\,, \qquad \gamma_1^{\text{in/ex}} \in \mathcal{L}(H^0_\Delta(\Omega_{\text{in/ex}})), H^{-\frac32}(\Gamma))\,,$$

where

$$H^0_\Delta(\Omega_{\text{in/ex}}) := \{u_{\text{in/ex}} \in L^2(\Omega_{\text{in/ex}}) : \Delta u_{\text{in/ex}} \in L^2(\Omega_{\text{in/ex}})\}\,,$$
$$\|u_{\text{in/ex}}\|^2_{H^0_\Delta(\Omega_{\text{in/ex}})} := \|\Delta u_{\text{in/ex}}\|^2_{L^2(\Omega_{\text{in/ex}})} + \|u_{\text{in/ex}}\|^2_{L^2(\Omega_{\text{in/ex}})}\,.$$

Setting

$$\Delta^{\max}_{\Omega_{\text{in/ex}}} := \Delta|H^0_\Delta\left(\Omega_{\text{in/ex}}\right)\,,$$

by the "half" Green formula (see [20, Theorem 4.4]), one has, for any $u, v \in H^1(\Omega_{\text{in/ex}}) \cap H^0_\Delta(\Omega_{\text{in/ex}})$,

$$\begin{aligned}&\langle -\Delta^{\max}_{\Omega_{\text{in/ex}}} u_{\text{in/ex}}, v_{\text{in/ex}}\rangle_{L^2(\Omega_{\text{in/ex}})} \\ =&\langle \nabla u_{\text{in/ex}}, \nabla v_{\text{in/ex}}\rangle_{L^2(\Omega_{\text{in/ex}})} + \epsilon_{\text{in/ex}}\langle \gamma_1^{\text{in/ex}} u_{\text{in/ex}}, \gamma_0^{\text{in/ex}} v_{\text{in/ex}}\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\,,\end{aligned} \tag{A.9}$$

where $\epsilon_{\text{in}} = -1$ and $\epsilon_{\text{ex}} = 1$. Setting

$$H^0_\Delta(\mathbb{R}^3\backslash\Gamma) := H^0_\Delta(\Omega_{\text{in}}) \oplus H^0_\Delta(\Omega_{\text{ex}})\,, \tag{A.10}$$

the extended traces allow to define

$$\gamma_\ell \in \mathcal{L}(H^0_\Delta(\mathbb{R}^3\backslash\Gamma), H^{-\frac12-\ell}(\Gamma))\,, \qquad [\gamma_\ell] \in \mathcal{L}(H^0_\Delta(\mathbb{R}^3\backslash\Gamma), H^{-\frac12-\ell}(\Gamma))\,,\quad \ell = 0,1\,,$$

by

$$\gamma_\ell u := \frac12\left(\gamma_\ell^{\text{in}}\,(u|\Omega_{\text{in}}) + \gamma_\ell^{\text{ex}}\,(u|\Omega_{\text{ex}})\right)\,, \qquad [\gamma_\ell]u := \gamma_\ell^{\text{ex}}\,(u|\Omega_{\text{ex}}) - \gamma_\ell^{\text{in}}\,(u|\Omega_{\text{in}})\,.$$

Notice that the maps $\gamma_\ell|H^2(\mathbb{R}^3\backslash\Gamma)$, $\ell = 0,1$, coincide with the ones in (A.8) when restricted to $H^2(\mathbb{R}^3)$.

These operators can be further extended to $H^2_\alpha(\mathbb{R}^3\backslash\Gamma)$, $\alpha < 0$, by

$$\gamma_\ell^{\text{in/ex}} u_{\text{in/ex}} := \gamma_\ell^{\text{in/ex}}\left(\chi u_{\text{in/ex}}\right)\,,\quad \ell = 0,1\,,$$

where χ belongs to $\mathcal{C}^\infty_{\text{comp}}(\Omega^c)$ and $\chi = 1$ on a neighborhood of Γ.

A.4 The single layer boundary operator

From (A.8) follows that γ_0^* is a bounded mapping: $H^{1/2-s}(\Gamma) \to H^{-s}(\mathbb{R}^3)$ for any $s > 1/2$. Since $\gamma_0^*\varphi$ has bounded support, results: $\gamma_0^* \in \mathcal{L}\left(H^{1/2-s}(\Gamma), \mathcal{E}'(\mathbb{R}^3)\right)$. Let $z \in \mathbb{C}$; by $R_z \in \mathcal{L}\left(\mathcal{E}'(\mathbb{R}^3), \mathcal{D}'(\mathbb{R}^3)\right)$, we get: $R_z\gamma_0^* \in \mathcal{L}\left(H^{1/2-s}(\Gamma), \mathcal{D}'(\mathbb{R}^3)\right)$, $s > 1/2$. This defines the well known single layer operator

$$SL_z = R_z\gamma_0^* .$$

Let recall from [20, Corollary 6.14] that the mapping properties

$$\chi SL_z \in \mathcal{L}(H^s(\Gamma), H^{s+3/2}(\mathbb{R}^3\backslash\Gamma)), \qquad s > -1 \tag{A.11}$$

and the jump relations

$$[\gamma_0]\, SL_z = 0, \qquad [\gamma_1]\, SL_z = -1, \tag{A.12}$$

hold for any $\chi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^3)$ and $z \in \mathbb{C}$. Moreover, SL_z has the integral representation

$$SL_z\phi = \int_\Gamma \mathcal{G}_z(\cdot - y)\, \phi(y)\, d\sigma(y), \tag{A.13}$$

where σ denotes the surface measure. When $z \in \mathbb{C}_+$, the mapping properties of R_z, the identity: $SL_z = (\gamma_0 R_{-\bar{z}})^*$ and a duality argument allow to improve (A.11) as follows

$$SL_z \in \mathcal{L}\left(H^s(\Gamma), H^{s+3/2}(\mathbb{R}^3)\right), \qquad s \geq -3/2, \qquad z \in \mathbb{C}_+ .$$

Next we define the single layer boundary operator

$$S_z := \gamma_0 SL_z .$$

By [20, Theorem 7.2], $S_z \in \mathcal{L}(H^s(\Gamma), H^{s+1}(\Gamma))$ for any real s. The operator S_0 plays a central role in our construction. By [20, Corollary 8.13], $S_0 : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma))$ is (self-adjoint) positive and bounded from below:

$$\langle\phi, S_0\phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} \geq c_0\, \|\phi\|^2_{H^{-1/2}(\Gamma)}, \qquad c_0 > 0. \tag{A.14}$$

Hence $S_0^{-1} \in \mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\mathbb{R}^3))$ provides an inner product in $H^{1/2}(\Gamma)$ defined by

$$\langle\phi, \varphi\rangle_{S_0^{-1}} := \langle S_0^{-1}\phi, \varphi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)} . \tag{A.15}$$

By (A.14), the inner product in (A.15) induces a norm $\|\cdot\|_{S_0^{-1}}$ on $H^{1/2}(\Gamma)$ which is equivalent to the original one.

Lemma A.1 *$\mathbb{C} \ni z \mapsto S_z$ is a $\mathcal{L}\big(H^{-1/2}(\Gamma), H^{1/2}(\mathbb{R}^3)\big)$-valued analytic map and*

$$S_z = S_0 + \sum_{n=1}^{+\infty} S_{(n)}\, z^n, \tag{A.16}$$

where $S_{(n)}$ has the integral representation

$$S_{(n)}\phi(x) = \frac{1}{4\pi}\frac{i^n}{n!}\int_\Gamma |x-y|^{n-1}\phi(y)\, d\sigma(y), \tag{A.17}$$

and the series converges in $\mathcal{L}_{HS}(H^{-1/2}(\Gamma), H^{1/2}(\mathbb{R}^3))$. Let D_Ω be the discrete set $D_\Omega := \{z \in \mathbb{C} : z^2 \in \sigma(-\Delta^D_\Omega)\}$. Then: $\mathbb{C}\backslash D_\Omega \ni z \mapsto S_z^{-1}$ is a $\mathcal{L}\big(H^{1/2}(\Gamma), H^{-1/2}(\mathbb{R}^3)\big)$-valued analytic map.

Proof. Let $\{\phi_k^\pm\}_1^\infty \subset \mathcal{C}^\infty(\Gamma)$ be the orthonormal basis in $H^{\pm 1/2}(\Gamma)$ defined by $\phi_k^\pm := (-\Delta_{LB}+1)^{\mp 1/4}\varphi_k$, where $\{\varphi_k\}_1^\infty \subset \mathcal{C}^\infty(\Gamma)$ is the set of normalized eigenfunctions of the self-adjoint operator $\Delta_{LB} : H^2(\Gamma) \subset L^2(\Gamma) \to L^2(\Gamma)$. Here Δ_{LB} denotes the Laplace-Beltrami operator on the surface Γ with respect to the Riemannian metric induced by the embedding $\Gamma \subset \mathbb{R}^3$. For any couple of indices (i, j), one has

$$\langle\phi_i^+, S_{(n)}\phi_j^-\rangle_{H^{1/2}(\Gamma)} = \langle\phi_i^-, S_{(n)}\phi_j^-\rangle_{L^2(\Gamma)} = \int_{\Gamma\times\Gamma} \phi_i^-(x)\, \mathcal{G}_{(n)}(x-y)\, \phi_j^-(y)\, d\sigma(x)d\sigma(y)$$

$$= \langle\phi_i^- \otimes \phi_j^-, \mathcal{R}_{(n)}\rangle_{L^2(\Gamma)\otimes L^2(\Gamma)} = \big\langle\varphi_i \otimes \varphi_j, \big((-\Delta_{LB}+1)^{1/4} \otimes (-\Delta_{LB}+1)^{1/4}\big)\mathcal{R}_{(n)}\big\rangle_{L^2(\Gamma)\otimes L^2(\Gamma)},$$

where $\mathcal{R}_{(n)}(x,y) := \mathcal{G}_{(n)}(x-y)$. Therefore $S_{(n)}$ is a Hilbert-Schmidt operator and its Hilbert-Schmidt norm is estimated by (the penultimate inequality follows from [13, Proposition 4.3])

$$\begin{aligned}
&\|S_{(n)}\|_{HS}^2 = \sum_{k=1}^{\infty} \|S_{(n)}\phi_k^-\|_{H^{1/2}}^2 \\
&= \sum_{i,j=1}^{\infty} \left|\langle \varphi_i \otimes \varphi_i, \left((-\Delta_{LB}+1)^{1/4} \otimes (-\Delta_{LB}+1)^{1/4}\right)\mathcal{R}_{(n)}\rangle_{L^2(\Gamma)\otimes L^2(\Gamma)}\right|^2 \\
&= \|((-\Delta_{LB}+1)^{1/4} \otimes (-\Delta_{LB}+1)^{1/4})\mathcal{R}_{(n)}\|_{L^2(\Gamma)\otimes L^2(\Gamma)}^2 \\
&\le c\,\|\mathcal{R}_{(n)}\|_{H^1(\Gamma\times\Gamma)}^2 \\
&\le c\,\frac{|\Gamma|^2}{(n!)^2}\left((d_\Omega^{n-1})^2 + ((n-1)\,d_\Omega^{n-2})^2\right).
\end{aligned}$$

Hence the series

$$\widetilde{S}_z := \sum_{n=2}^{+\infty} S_{(n)}\, z^n$$

converges in $\mathcal{L}_{HS}(H^{1/2}(\Gamma))$ for any $z \in \mathbb{C}$ and defines the $\mathcal{L}_{HS}(H^{1/2}(\Gamma))$-valued analytic map $\mathbb{C} \ni z \mapsto \widetilde{S}_z$. By (A.5) and (A.13), one has $\langle \phi_i, \widetilde{S}_z\phi_j\rangle_{H^{1/2}(\Gamma)} = \langle \phi_i, (S_z - S_0)\phi_j\rangle_{H^{1/2}(\Gamma)}$ for any $z \in \mathbb{C}$ and for any couple (i,j). Therefore $S_z = S_0 + \widetilde{S}_z$ for any $z \in \mathbb{C}$.

By [20, Theorem 7.6], S_z is Fredholm with zero index; by [20, Theorem 7.5], $\ker(S_z) \neq \{0\}$ is equivalent to the existence of non-trivial solutions of $(\Delta_\Omega^D + z^2)u = 0$. Hence $S_z^{-1} \in \mathcal{L}(H^{-1/2}(\Gamma), H^{1/2}(\mathbb{R}^3))$ for any $z \in \mathbb{C}\backslash D_\Omega$. The proof is then concluded by Lemma A.1 and by the analyticity of the inverse (see [24, Theorem 5.1]). ■

A.5 The Neumann-Poincaré operator

Proceeding as before we observe from (A.8) that $\gamma_1^* \in \mathcal{L}\left(H^{3/2-s}(\Gamma), \mathcal{E}'(\mathbb{R}^3)\right)$ for $s > 3/2$. Then, for $z \in \mathbb{C}$ we get: $R_z\gamma_0^* \in \mathcal{L}\left(H^{1/2-s}(\Gamma), \mathcal{D}'(\mathbb{R}^3)\right)$, $s > 1/2$. This defines the double layer operator

$$DL_z = R_z\gamma_1^*\,.$$

Let recall from [20, Corollary 6.14] that the mapping properties

$$\chi DL_z \in \mathcal{L}(H^s(\Gamma), H^{s+1/2}(\mathbb{R}^3\backslash\Gamma))\,, \qquad s > 0$$

hold for any $\chi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^3)$ and $z \in \mathbb{C}$. Moreover, DL_z has the integral representation

$$DL_z\phi = \int_\Gamma \nu(y)\cdot\nabla_y\mathcal{G}_z(\cdot - y)\phi(y)\,d\sigma(y)\,.$$

Next we define the *Neumann-Poincaré* boundary operator

$$K_z := \gamma_0 DL_z\,.$$

By [20, Theorem 7.2], $K_z \in \mathcal{L}(H^s(\Gamma))$ for any real s. The next Lemma resumes the spectral properties of K_0 (see, e.g., [22, Section 4]):

Lemma A.2 *K_0 is a compact operator in $L^2(\Gamma)$ and $\sigma(K_0) \subseteq [-1/2, 1/2)$; $\lambda_0 = -1/2$ is a simple eigenvalue and the corresponding eigenfunction is $\phi_0 = 1$.*

Let $P_0 : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ be the orthogonal (w.r.t. the inner product $\langle\cdot,\cdot\rangle_{S_0^{-1}}$) projector onto $\mathbb{C}$, i.e. onto the subspace generated by the eigenfunction ϕ_0:

$$P_0\phi := c_\Omega^{-1}\langle\phi_0, \phi\rangle_{S_0^{-1}}\,\phi_0 \equiv c_\Omega^{-1}\langle S_0^{-1}1, \phi\rangle_{H^{-1/2}(\Gamma),H^{1/2}(\Gamma)}\,1\,. \tag{A.18}$$

Denoting then by Q_0 the orthogonal projector onto $\text{ran}(P_0)^\perp$, i.e.

$$Q_0\phi := \phi - P_0\phi\,, \tag{A.19}$$

$K_0 \in \mathcal{L}(H^{1/2}(\Gamma))$ has the decomposition

$$K_0 = P_0K_0P_0 + Q_0K_0Q_0 = -\frac{1}{2}\,P_0 + Q_0K_0Q_0\,. \tag{A.20}$$

Lemma A.3 *$\mathbb{C} \ni z \mapsto K_z$ is a $\mathcal{L}(H^{1/2}(\Gamma))$-valued analytic map and*

$$K_z = K_0 + K_{(2)}\, z^2 + \sum_{n=3}^{+\infty} K_{(n)}\, z^n\,, \tag{A.21}$$

where the series converges in $\mathcal{L}_{HS}(H^{1/2}(\Gamma))$. On smooth functions, $K_{(n)}$ has the integral representation

$$K_{(n)}\phi(x) = -(n-1)\,\frac{1}{4\pi}\,\frac{i^n}{n!}\int_\Gamma \nu(y)\cdot(x-y)|x-y|^{n-3}\phi(y)\,d\sigma(y)\,. \tag{A.22}$$

Proof. We proceed as in the proof of Lemma A.1; see that proof for the definitions of $\phi_k^{\pm}$. For any couple of indices (i,j), one has

$$\begin{aligned}\langle \phi_i^+, K_{(n)}\phi_j^+\rangle_{H^{1/2}(\Gamma)} &= \langle \phi_i^-, K_{(n)}\phi_j^+\rangle_{L^2(\Gamma)} = \int_{\Gamma\times\Gamma}\phi_i^-(x)\,\nu(y)\cdot\nabla_y\mathcal{G}_{(n)}(x-y)\,\phi_j^+(y)\,d\sigma(x)d\sigma(y)\\ =&\langle \phi_i^-\otimes\phi_j^+, \mathcal{R}_n'\rangle_{L^2(\Gamma)\otimes L^2(\Gamma)} = \langle \varphi_i\otimes\varphi_j, ((-\Delta_{LB}+1)^{1/4}\otimes(-\Delta_{LB}+1)^{-1/4})\,\mathcal{R}_n'\rangle_{L^2(\Gamma)\otimes L^2(\Gamma)}\,,\end{aligned}$$

where

$$\mathcal{R}_n'(x,y) := -(n-1)\,\frac{1}{4\pi}\,\frac{i^n}{n!}\,\nu(y)\cdot(x-y)|x-y|^{n-3}\,. \tag{A.23}$$

Therefore, for any $n \geq 3$,

$$\begin{aligned}\|K_{(n)}\|_{HS}^2 &= \sum_{k=1}^{\infty}\|K_{(n)}\phi_k^+\|_{H^{1/2}}^2\\ &= \sum_{i,j=1}^{\infty}\left|\langle\varphi_i\otimes\varphi_j, ((-\Delta_{LB}+1)^{1/4}\otimes(-\Delta_{LB}+1)^{-1/4})\mathcal{R}_n'\rangle_{L^2(\Gamma)\otimes L^2(\Gamma)}\right|^2\\ &= \|((-\Delta_{LB}+1)^{1/4}\otimes(-\Delta_{LB}+1)^{-1/4})\mathcal{R}_n'\|_{L^2(\Gamma)\otimes L^2(\Gamma)}^2\\ &\leq \|\mathcal{R}_n'\|_{H^1(\Gamma\times\Gamma)}\|\mathcal{R}_n'\|_{L^2(\Gamma\times\Gamma)}\\ &\leq c\,\frac{|\Gamma|^2}{(n!)^2}\,(n-1)^2\,d_\Omega^{n-2}\,\left(d_\Omega^{n-2}+d_\Omega^{n-3}+(n-3)d_\Omega^{n-4}\right)\,.\end{aligned}$$

The proof is then concluded as in Lemma A.1. ■

A.6 The Dirichlet-to-Neumann operator

In this section $z \in \mathbb{C}$ is such that $z^2 \in \varrho(-\Delta_\Omega^D)$. The Dirichlet-to-Neumann operator related to the interior Helmholtz equation is defined by

$$DN_z\varphi := \gamma_1^{\text{in}}u\,, \qquad \begin{cases}\left(\Delta_\Omega + z^2\right)u = 0\,,\\ \gamma_0^{\text{in}}u = \varphi\,.\end{cases} \tag{A.24}$$

As is well known (see, e.g., [20, Theorem 4.10]) the solution exists and is unique. By elliptic regularity (see, e.g., [20, Theorem 4.21]), DN_z extends to a pseudo-differential operator of order one on the whole scale of Sobolev spaces $H^s(\Gamma)$:

$$DN_z \in \mathcal{L}(H^s(\Gamma), H^{s-1}(\Gamma))\,. \tag{A.25}$$

If $z^2 \in \mathbb{R}$, then DN_z is self-adjoint as unbounded operator between the dual couple $H^s(\Gamma)$-$H^{-s}(\Gamma)$, $DN_z : H^{s+1}(\Gamma) \subset H^{-s}(\Gamma) \to H^s(\Gamma)$ (see e.g. in [8, Sec. 2 and Example 4.9]).

By [20, Theorem 7.5], u in (A.24) is uniquely determined by the solution of the boundary integral equation

$$S_z\gamma_1^{\text{in}}u = \left(\frac{1}{2} + K_z\right)\varphi \tag{A.26}$$

so that $DN_z \in\mathcal{L}(H^{1/2}(\Gamma), H^{-1/2}(\Gamma))$ has the representation

$$DN_z = S_z^{-1}\left(\frac{1}{2} + K_z\right)\,. \tag{A.27}$$

By Lemma (A.3) and Lemma (A.1), (A.27) entails that $\mathbb{C}\backslash D_\Omega \ni z \mapsto DN_z$ is a $\mathcal{L}\big(H^{1/2}(\Gamma), H^{-1/2}(\mathbb{R}^3)\big)$-valued analytic map.

A.7 Further auxiliary operator expansions

Lemma A.4 *Let* $\operatorname{Im} z \geq 0$, $z \neq 0$, *let* $y_0 \in \mathbb{R}^3$ *and define the linear operator* $\Psi_{\varepsilon,z}$ *by*

$$(\Psi_{\varepsilon,z}u)(y) = (R_z u)\,(y_0 + \varepsilon\,(y - y_0)) - (R_z u)\,(y_0)\,. \tag{A.28}$$

Then, for any star-shaped, bounded open set $B \subset \mathbb{R}^3$, *one has the estimate*

$$\|\Psi_{\varepsilon,z}\|_{L^2_\alpha(\mathbb{R}^3),H^2(B)} \leq c_{\alpha,B}\,\|R_z\|_{L^2_\alpha(\mathbb{R}^3),H^2_{-\alpha}(\mathbb{R}^3)}\,\varepsilon^{1/2}\,, \tag{A.29}$$

where $\alpha = 0$ *whenever* $\operatorname{Im} z > 0$ *and* $\alpha > 1/2$ *whenever* $\operatorname{Im} z = 0$.

Proof. Without loss of generality we can suppose that $y_0 = 0$. Given $u \in L^2_\alpha(\mathbb{R}^3)$, let us set

$$\psi_\varepsilon(y) := \psi(\varepsilon y)\,, \qquad \psi := R_z u\,.$$

By $R_z \in \mathcal{L}(L^2_\alpha(\mathbb{R}^3), H^2_{-\alpha}(\mathbb{R}^3))$ (see Subsection A.2), one has

$$\|\psi\|^2_{H^2_{-\alpha}(\mathbb{R}^3)} \leq \|R_z\|_{L^2_\alpha(\mathbb{R}^3),H^2_{-\alpha}(\mathbb{R}^3)}\,\|u\|^2_{L^2_\alpha(\mathbb{R}^3)}$$

and so it suffices to show that

$$\begin{aligned}\|\psi_\varepsilon - \psi_0\|^2_{H^2(B)} &= \sum_{1\leq i,j\leq 3}\|\partial^2_{ij}\psi_\varepsilon\|^2_{L^2(B)} + \sum_{1\leq i\leq 3}\|\partial_i\psi_\varepsilon\|^2_{L^2(B)} + \|\psi_\varepsilon - \psi_0\|^2_{L^2(B)} \\ &\leq\, c_{\alpha,B}\,\|\psi\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,\varepsilon^{1/2}\,.\end{aligned} \tag{A.30}$$

The latter is consequence of the following estimates:
1)

$$\begin{aligned}\|\partial^2_{ij}\psi_\varepsilon\|^2_{L^2(B)} &= \int_B|\partial^2_{ij}\psi_\varepsilon(y)|^2dy = \varepsilon^4\!\int_B|\partial^2_{ij}\psi(\varepsilon y)|^2dy = \varepsilon\!\int_{B_\varepsilon}|\partial^2_{ij}\psi(y)|^2dy \leq \varepsilon\!\int_B|\partial^2_{ij}\psi(y)|^2dy \\ &\leq \varepsilon\,\|\langle x\rangle^\alpha\|_{L^\infty(B)}\!\int_B|\partial^2_{ij}\psi(y)|^2\langle y\rangle^{-\alpha}\,dy \leq \varepsilon\,\|\langle x\rangle^\alpha\|_{L^\infty(B)}\,\|\partial^2_{ij}\psi\|^2_{L^2_{-\alpha}(\mathbb{R}^3)} \leq \varepsilon\,\|\langle x\rangle^\alpha\|_{L^\infty(B)}\|\psi\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,;\end{aligned}$$

2) by the Sobolev embedding $H^1(B) \subset L^6(B)$,

$$\begin{aligned}\|\partial_i\psi_\varepsilon\|^2_{L^2(B)} &\leq |B|^{\frac{3}{2}}\left(\int_B|\partial_i\psi_\varepsilon(y)|^6\,dy\right)^{\frac{1}{3}} = |B|^{\frac{3}{2}}\left(\varepsilon^6\int_B|\partial_i\psi(\varepsilon y|^6\,dy\right)^{\frac{1}{3}} = |B|^{\frac{3}{2}}\varepsilon\,\|\partial_i\psi\|^2_{L^6(B_\varepsilon)} \\ &\leq |B|^{\frac{3}{2}}\varepsilon\,\|\partial_i\psi\|^2_{L^6(B)} \leq c\,|B|^{\frac{3}{2}}\varepsilon\,\|\partial_i\psi\|^2_{H^1(B)} \leq c\,|B|^{\frac{3}{2}}\varepsilon\,\|\psi\|^2_{H^2(B)} \leq c\,|B|^{\frac{3}{2}}\varepsilon\,\|\langle x\rangle^\alpha\|_{L^\infty(B)}\|\psi\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,;\end{aligned}$$

3) by the continuous embedding of $H^2(B)$ into the space of Hölder-continuous functions of order $\frac{1}{2}$,

$$\|\psi_\varepsilon - \psi_0\|^2_{L^2(B)} = \int_B|\psi(\varepsilon y) - \psi(0)|^2\,dy \leq c\,\varepsilon\int_B|y|\,dy\;\|\psi\|^2_{H^2(B)} \leq c\,\varepsilon\,\|\langle x\rangle^\alpha\|_{L^\infty(B)}\int_B|y|\,dy\;\|\psi\|^2_{H^2_{-\alpha}(\mathbb{R}^3)}\,.$$

■

Corollary A.5 *Let* $\operatorname{Im} z \geq 0$, $z \neq 0$, *let* $y_0 \in \mathbb{R}^3$ *and define the linear operators* $\Xi_{\varepsilon,z}$ *and* $\Phi_{\varepsilon,z}$ *by*

$$\Xi_{\varepsilon,z}u = \gamma_0 R_{\varepsilon z}U_\varepsilon^{-1}u - \varepsilon^{-1/2}\,(R_z u)\,(y_0)\,,$$

$$\Phi_{\varepsilon,z}\,\phi = U_\varepsilon R_{\varepsilon z}\gamma_0^*\phi - \varepsilon^{-1/2}\langle\phi\rangle\;\mathcal{G}_z\,, \qquad \langle\phi\rangle := \langle 1,\phi\rangle_{H^{3/2}(\Gamma),H^{-3/2}(\Gamma)}\,.$$

Then, for any $\epsilon > 0$,

$$\|\Xi_{\varepsilon,z}\|_{L^2_\alpha(\mathbb{R}^3),H^{3/2}(\Gamma)} \leq c_{\alpha,B}\,\|\gamma_0\|_{H^2(B),H^{3/2}(\Gamma)}\,\|R_z\|_{L^2_\alpha(\mathbb{R}^3),H^2_{-\alpha}(\mathbb{R}^3)} \tag{A.31}$$

and

$$\|\Phi_{\varepsilon,z}\|_{H^{-3/2}(\Gamma),L^2_\alpha(\mathbb{R}^3)} \leq c_{\alpha,B}\,\|\gamma_0\|_{H^2(B),H^{3/2}(\Gamma)}\,\|R_z\|_{L^2_\alpha(\mathbb{R}^3),H^2_{-\alpha}(\mathbb{R}^3)}\,, \tag{A.32}$$

where B *is any star-shaped open and bounded set such that* $B \supset \overline{\Omega}$, $c_{\alpha,B}$ *and* α *are the same as in Lemma A.4.*

Proof. By (3.12), one has

$$(R_{\varepsilon z}U_\varepsilon^{-1}u)(y) = \varepsilon^{-2}(U_\varepsilon^{-1}R_z u)(y) = \varepsilon^{-1/2}(R_z u)(y_0 + \varepsilon(y - y_0))$$

and so, by (A.28),

$$\Xi_{\varepsilon,z}u = \gamma_0 R_{\varepsilon z}U_\varepsilon^{-1}u - \varepsilon^{-1/2}\,(R_z u)\,(y_0) = \varepsilon^{-1/2}\gamma_0\Psi_{\varepsilon,z}\,.$$

Hence, whenever $B \supset \overline{\Omega}$, one gets

$$\|\Xi_{\varepsilon,z}\|_{L^2_\alpha(\mathbb{R}^3),H^{3/2}(\Gamma)} \le \|\gamma_0\|_{H^2(B),H^{3/2}(\Gamma)}\,\varepsilon^{-1/2}\|\Psi_{\varepsilon,z}\|_{L^2_\alpha(B),H^2(B)}$$

and (A.31) follows from (A.29). By

$$\langle u, U_\varepsilon R_{\varepsilon z}\gamma_0^*\phi\rangle_{L^2(\mathbb{R}^3)} = \left\langle U_\varepsilon^{-1}\gamma_0 R_{-\varepsilon\bar{z}}u, \phi\right\rangle_{H^{3/2}(\Gamma),H^{-3/2}(\Gamma)}$$

and by

$$\langle u, \langle\phi\rangle\mathcal{G}_z\rangle_{L^2(\mathbb{R}^3)} = \langle\phi\rangle(R_z u)(y_0) = \langle(R_{-\bar{z}}u)(y_0), \phi\rangle_{H^{3/2}(\Gamma),H^{-3/2}(\Gamma)}\ ,$$

one gets

$$\Phi_{z,\varepsilon} = \Xi^*_{-\bar{z},\varepsilon}$$

and so (A.32) is consequence of (A.31). ■

Lemma A.6 *For any $z \in \mathbb{C}$, one has*

$$S_z DN_z = Q_0\left(\frac{1}{2} + K_0\right)Q_0 + K_{(2)}\,z^2 + O(|z|^3)\,.$$

Proof. By (A.27), there follows $S_z DN_z = (1/2 + K_z)$. By (A.3), one has

$$S_z DN_z = \frac{1}{2} + K_0 + K_{(2)}\,z^2 + O(|z|^3)\,.$$

The proof is then concluded by (A.20). ■

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