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ON THE RANGE OF THE PLANAR X -RAY TRANSFORM ON THE FOURIER LATTICE OF THE TORUS

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ABSTRACT. We find necessary and sufficient conditions on the Fourier coefficients of a function g on the torus to be in the range of the X -ray transform of functions with compact support in the plane.

1. INTRODUCTION

We revisit the range characterization of the classical X -ray transform of a real valued function f compactly supported in the plane. Since the X -ray and Radon transform [33] for planar functions differ merely by the way lines are parameterized, the necessary and sufficient constraints have been long established independently in [9], [11], and [22]. Apart from the symmetry constraints due to the double parameterization of the lines, of specific interest are the moment conditions.

Due to their practical use in noise reduction [37, 38], completion of the data [39, 10, 14, 20], CT-hardware failure diagnosis [30], the range characterization problem has been a continuing subject of research [6, 7, 15, 16, 23]. Some recent work treat directly the discrete X -ray transform problem [21]. Models which account for the attenuation have also been considered in the homogeneous case [19], and in the non-homogeneous case in the breakthrough works [1, 28, 29], and subsequently [27, 4, 3, 17, 34, 24]. The references here are by no means exhaustive.

The corresponding problem for tensors of higher order in non-Euclidean spaces has also been considered in [36, 32, 2, 18], see [31] for a comprehensive review. On simple Riemannian surfaces, the range characterization of the geodesic X -ray transform of compactly supported functions has been established in terms of the scattering relation in the breakthrough work in [32]. The connection between the Euclidean version of the characterization in [32] and the original characterization in [9, 11, 22] was established in [23].

In here we give yet another characterization of the range of the classical X -ray transform in terms of the Fourier coefficients of integrable functions on the torus, where the lines passing through the support of f are parameterized by coordinates on a torus. Although Xf is a function on the torus, our problem differs from the one in [13], where for a given direction (of rational slope) the integration takes place over a finite union of parallel segments in the unit disc.

The thrust of this work are the constraints (2.14) replacing the moment conditions, and the sufficiency part in Theorem 2.1. The result is based on the authors' characterization in [34] and uses a new mapping property (Theorem 4.1) of the Bukhgeim-Hilbert transform corresponding to the A -analytic maps in the sense of [5].

All the details establishing notation and the statement of the main results are in Section 2. In Section 3 we briefly recall existing results on A -analytic maps that are used in the proofs. In

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Section 4 we establish a new mapping property of the Bukhgeim-Hilbert transform, which is key to the proof of our main result in Section 5. To improve the readability of the work, some of the claims are proven in the appendix.

2. PRELIMINARIES AND STATEMENT OF MAIN RESULT

Throughout, f is an integrable, real valued function, of compact support in the plane. Points (x_1, x_2) in the plane are identified by the complex numbers $x_1 + ix_2$, and directions $\boldsymbol{\theta} = (\cos \theta, \sin \theta)$ in the unit sphere \mathbf{S}^1 by $e^{i\theta}$. Upon a translation and scaling, f is assumed supported in the unit disc

$$\Omega = \{z \in \mathbb{C} : |z| < 1\}.$$

The boundary Γ of Ω is the unit circle, but we keep this notation to differentiate from the set \mathbf{S}^1 of directions.

Lines $L_{(\beta, \theta)} := \{e^{i\beta} + se^{i\theta} : s \in \mathbb{R}\}$ intersecting $\overline{\Omega}$ are parametrized in coordinates $\{(e^{i\beta}, e^{i\theta}) : \beta, \theta \in (-\pi, \pi]\}$, and the X -ray transform of f

$$(2.1) \quad Xf(e^{i\beta}, e^{i\theta}) = \int_{L_{(\beta, \theta)}} f ds$$

is understood as a function on the torus $\Gamma \times \mathbf{S}^1$.

Since $L_{(\beta, \theta)} = L_{(2\theta - \beta - \pi, \theta)} = L_{(\beta, \theta + \pi)} = L_{(2\theta - \beta - \pi, \theta + \pi)}$, the set of lines intersecting $\overline{\Omega}$ are quadruply covered when $(e^{i\beta}, e^{i\theta})$ ranges over the entire torus $\Gamma \times \mathbf{S}^1$. Moreover, the following symmetries are satisfied,

$$(2.2) \quad Xf(e^{i\beta}, e^{i\theta}) = Xf(e^{i(2\theta - \beta - \pi)}, e^{i(\theta + \pi)}), \text{ and}$$

$$(2.3) \quad Xf(e^{i\beta}, e^{i\theta}) = Xf(e^{i\beta}, e^{i(\theta + \pi)}), \text{ for } (e^{i\beta}, e^{i\theta}) \in \Gamma \times \mathbf{S}^1;$$

see Figure 1 below.

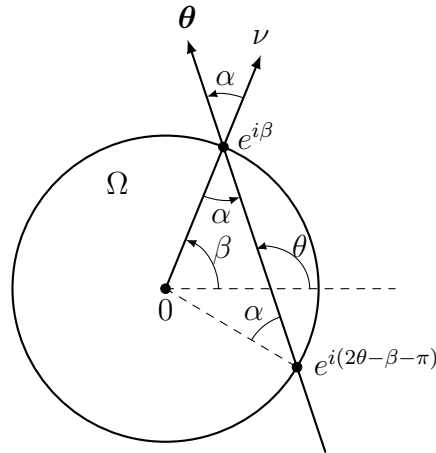


FIGURE 1. Fan-beam coordinates: $e^{i\beta} \in \Gamma$, $e^{i\theta} \in \mathbf{S}^1$, and $\boldsymbol{\theta} = (\cos \theta, \sin \theta)$.

If f is merely integrable in Ω , then Xf may not be integrable on the torus. However, if either

$$(2.4) \quad \text{supp } f \subset \Omega, \text{ or } f \in L^p(\Omega) \text{ for some } p > 2,$$

then $Xf \in L^1(\Gamma \times \mathbf{S}^1)$; see Proposition A.1 in the appendix.

We consider the partition of the torus into three parts: the ‘‘outflux’’ part

$$(2.5) \quad \Gamma_+ := \left\{ (e^{i\beta}, e^{i(\beta+\alpha)}) \in \Gamma \times \mathbf{S}^1 : \beta \in (-\pi, \pi], |\alpha| < \frac{\pi}{2} \right\},$$

the ‘‘influx’’ part

$$(2.6) \quad \Gamma_- := \left\{ (e^{i\beta}, e^{i(\beta+\alpha)}) \in \Gamma \times \mathbf{S}^1 : \beta \in (-\pi, \pi], \frac{\pi}{2} < |\alpha| \leq \pi \right\},$$

and the (Lebesgue negligible) variety $\Gamma_0 := (\Gamma \times \mathbf{S}^1) \setminus (\Gamma_+ \cup \Gamma_-)$ parameterizing the tangent lines to the circle; see Figure 1.

Motivated by (2.2), let $L_{sym}^1(\Omega \times \mathbf{S}^1)$ denote the space of integrable functions g on the torus satisfying the symmetry relation

$$(2.7) \quad g(e^{i\beta}, e^{i\theta}) = g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}), \text{ for a.e. } (e^{i\beta}, e^{i\theta}) \in \Gamma \times \mathbf{S}^1.$$

Since $(e^{i\beta}, e^{i\theta})$ and $(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)})$ are either both in Γ_+ , or both in Γ_- , we can consider the spaces $L_{sym}^1(\Gamma_{\pm})$ of integrable functions on the half-tori satisfying (2.7). Clearly, $g \in L_{sym}^1(\Gamma \times \mathbf{S}^1)$ if and only if its restrictions $g|_{\Gamma_{\pm}} \in L_{sym}^1(\Gamma_{\pm})$. The symmetry relation (2.2) yields $Xf \in L_{sym}^1(\Gamma \times \mathbf{S}^1)$.

Furthermore, we consider the subspace $L_{sym,odd}^1(\Gamma \times \mathbf{S}^1)$ of functions $g \in L_{sym}^1(\Gamma \times \mathbf{S}^1)$, which, in addition to satisfying (2.7), they are also odd with respect to the angular variable:

$$(2.8) \quad g(e^{i\beta}, e^{i\theta}) = -g(e^{i\beta}, e^{i(\theta+\pi)}).$$

Note that $Xf \notin L_{sym,odd}^1(\Gamma \times \mathbf{S}^1)$, since the symmetry relation (2.3) is broken.

Our main result gives necessary and sufficient conditions for a function $g \in L_{sym,odd}^1(\Gamma \times \mathbf{S}^1)$ to satisfy

$$(2.9) \quad g(e^{i\beta}, e^{i\theta}) = Xf(e^{i\beta}, e^{i\theta}), \text{ a.e. } (e^{i\beta}, e^{i\theta}) \in \Gamma_+,$$

for some real valued $f \in L^1(\Omega)$. Note that (2.8) implies $g = -Xf$ on Γ_- .

The characterization is in terms of the Fourier coefficients

$$(2.10) \quad g_{n,k} := \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(e^{i\beta}, e^{i\theta}) e^{-in\theta} e^{-ik\beta} d\theta d\beta, \quad n, k \in \mathbb{Z},$$

of g on the lattice $\mathbb{Z} \times \mathbb{Z}$. The order of the indexes play a role. Throughout, the first index is the Fourier mode in the angular variable on \mathbf{S}^1 , and we call it an *angular mode*. The second index is the mode in the boundary variable on Γ , and we call it a *boundary mode*.

Theorem 2.1 (Range characterization). *(i) Let $f \in L^1(\Omega)$ be real valued satisfying (2.4), and $g \in L_{sym,odd}^1(\Gamma \times \mathbf{S}^1)$, with*

$$g = Xf \text{ on } \Gamma_+.$$

Then the Fourier coefficients $\{g_{n,k}\}_{n,k \in \mathbb{Z}}$, of g satisfy the following conditions.

$$(2.11) \quad \text{Oddness :} \quad g_{n,k} = 0, \quad \text{for all } n \in \mathbb{Z} \text{ even, and all } k \in \mathbb{Z};$$

$$(2.12) \quad \text{Conjugacy :} \quad g_{-n,-k} = \overline{g_{n,k}}, \quad \text{for all } n, k \in \mathbb{Z};$$

$$(2.13) \quad \text{Symmetry :} \quad g_{n,k} = (-1)^{n+k} g_{n+2k,-k}, \quad \text{for all } n, k \in \mathbb{Z};$$

$$(2.14) \quad \text{Moments :} \quad g_{n,k} = (-1)^k g_{n+2k,-k}, \quad \text{for all } n \leq -1 \text{ odd, and all } k \leq 0.$$

(ii) Let $\{g_{n,k}\}$ be given for all $n \leq -1$ odd, and $k \in \mathbb{Z}$ such that

$$(2.15) \quad \sum_{\substack{n \leq -1 \\ n = \text{odd}}} \langle n \rangle^2 \sum_{k=-\infty}^{\infty} |g_{n,k}| < \infty, \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \langle k \rangle^{1+\mu} \sum_{\substack{n \leq -1 \\ n = \text{odd}}} |g_{n,k}| < \infty, \quad \mu > 1/2.$$

If $\{g_{n,k}\}$ satisfy (2.13) and (2.14), then there exists a real valued $f \in L^1(\Omega) \cap C^\mu(\Omega)$, such that the mapping

$$(2.16) \quad (\Gamma \times \mathbf{S}^1) \ni (e^{i\beta}, e^{i\theta}) \longmapsto 2 \operatorname{Re} \left\{ \sum_{\substack{n \leq -1 \\ n = \text{odd}}} \sum_{k \in \mathbb{Z}} g_{n,k} e^{in\theta} e^{ik\beta} \right\}$$

defines a function in $L^1_{sym,odd}(\Gamma \times \mathbf{S}^1)$, which coincides with Xf on Γ_+ (and with $-Xf$ on Γ_-).

The oddness and conjugacy constraints above in (2.11) and (2.12) are not intrinsic to the X -ray transform. The evenness condition (2.13) merely accounts for each line being doubly parametrized in Γ_+ , and is shared by any function on the torus satisfying the symmetry (2.7); see Lemma A.1 in the appendix.

The thrust of this work are the constraints (2.14) replacing the moment conditions, and the sufficiency part in Theorem 2.1 for functions of finite smoothness. In particular, for n odd, the right hand sides of (2.14) and (2.13) differ by a sign. As a direct consequence, the following holds.

Corollary 2.1. (i) Let $g \in L^1_{sym,odd}(\Gamma \times \mathbf{S}^1)$ coincide with Xf on Γ_+ , and $\{g_{n,k}\}$ be its Fourier coefficients. Then, for all $n \leq -1$ odd,

$$(2.17) \quad g_{n,k} = \begin{cases} 0, & \text{if } k \leq -\frac{n+1}{2}, \\ (-1)^{k+1} \overline{g_{-n-2k,k}}, & \text{if } k \geq \frac{-n+1}{2}, \end{cases}$$

see Figure 2.

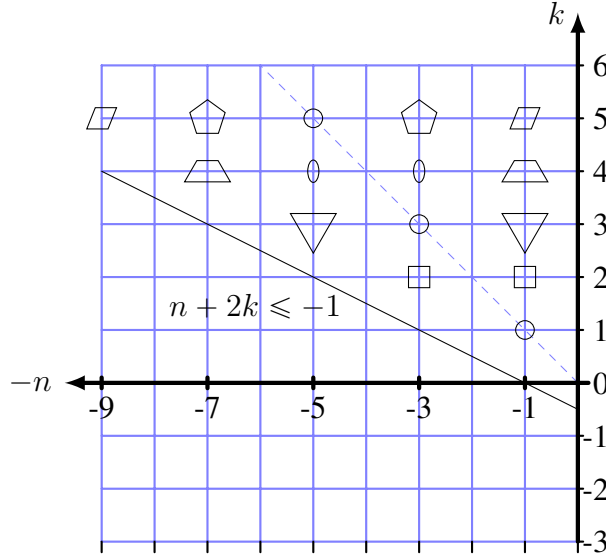


FIGURE 2. f is determined by the odd negative angular modes on or above the diagonal $k = -n$. The diagonal modes $g_{n,-n}$ are real valued. All the odd non-positive angular modes on and below the line $n + 2k = -1$ vanish.

(ii) Let $\{g_{n,k}\}$ be given for $n \leq -1$ odd, and $k \geq \frac{-n-1}{2}$ such that

$$(2.18) \quad \sum_{\substack{n \leq -1 \\ n = \text{odd}}} \langle n \rangle^2 \sum_{k = \frac{-n-1}{2}}^{\infty} |g_{n,k}| < \infty, \quad \text{and} \quad \sum_{k=1}^{\infty} \langle k \rangle^{1+\mu} \sum_{\substack{n = -2k-1 \\ n = \text{odd}}}^{-1} |g_{n,k}| < \infty, \quad \mu > 1/2.$$

If $\{g_{n,k}\}$ satisfies (2.17), then there exists a real valued function $f \in L^1(\Omega) \cap C^\mu(\Omega)$, such that the mapping

$$(2.19) \quad (\Gamma \times \mathbf{S}^1) \ni (e^{i\beta}, e^{i\theta}) \longmapsto 2 \operatorname{Re} \left\{ \sum_{\substack{n \leq -1 \\ n = \text{odd}}} \sum_{k = \frac{-n-1}{2}}^{\infty} g_{n,k} e^{in\theta} e^{ik\beta} \right\}$$

is precisely Xf on Γ_+ , and $-Xf$ on Γ_- .

For a complex valued f , since $\operatorname{Re}(Xf) = X(\operatorname{Re}(f))$ and $\operatorname{Im}(Xf) = X(\operatorname{Im}(f))$, Theorem 2.1 and Corollary 2.1 apply to $\operatorname{Re}(f)$ and to $\operatorname{Im}(f)$, respectively.

The feasibility of the constraints (2.17) to data enhancement is subject to future work.

3. L^2 -ANALYTIC MAPS AND THEIR TRACE CHARACTERIZATION

We approach the range characterization via the well-known connection with the transport model, where the unique solution $u(z, \boldsymbol{\theta})$ to the boundary value problem

$$(3.1a) \quad \boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) = 2f(z), \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1,$$

$$(3.1b) \quad u|_{\Gamma_-} = -Xf|_{\Gamma_-}$$

has the trace $u|_{\Gamma \times \mathbf{S}^1}$ in $L^1_{\text{sym, odd}}(\Gamma \times \mathbf{S}^1)$, and $u|_{\Gamma_+} = Xf|_{\Gamma_+}$ on Γ_+ .

The method of proof of (2.14) is based on the characterization in [34] of traces of A -analytic maps in the sense of Bukhgeim [5]. In this section we summarize those existing results used in the proof of Theorem 2.1 in Section 5. These results hold for Ω a strictly convex domain, not necessarily the unit disk.

The advection operator $\boldsymbol{\theta} \cdot \nabla$ in complex notation becomes $e^{-i\theta} \bar{\partial} + e^{i\theta} \partial$, where $\boldsymbol{\theta} = (\cos \theta, \sin \theta)$, and $\bar{\partial} = (\partial_{x_1} + i\partial_{x_2})/2$ and $\partial = (\partial_{x_1} - i\partial_{x_2})/2$ are the Cauchy-Riemann operators.

If $\sum_{n \in \mathbb{Z}} u_n(z) e^{in\theta}$ is the Fourier series expansion in the angular variable $\boldsymbol{\theta}$ of a solution u of (3.1a), then, provided some sufficient decay (to be specified later) of u_n to allow regrouping,

$$(3.2) \quad \boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) = \bar{\partial} u_1(z) + \partial u_{-1}(z) + \sum_{\substack{n \in \mathbb{Z} \\ n \neq 1}} (\bar{\partial} u_n(z) + \partial u_{n-2}(z)) e^{i(n-1)\theta}.$$

By identifying the Fourier modes of the same order, the equation (3.1a) reduces to the system:

$$(3.3) \quad \bar{\partial} u_1(z) + \partial u_{-1}(z) = 2f(z),$$

and

$$(3.4) \quad \bar{\partial} u_n(z) + \partial u_{n-2}(z) = 0, \quad n \neq 1.$$

For a real valued $u(z, \boldsymbol{\theta})$, $u_{-n} = \overline{u_n}$ and the angular dependence is completely determined by the sequence \mathbf{u} of its nonpositive Fourier modes,

$$(3.5) \quad \Omega \ni z \mapsto \mathbf{u}(z) := \langle u_0(z), u_{-1}(z), u_{-2}(z), \dots \rangle.$$

In particular, \mathbf{u} solves the Beltrami-like equation

$$(3.6) \quad \bar{\partial}\mathbf{u}(z) + L^2\partial\mathbf{u}(z) = 0, \quad z \in \Omega,$$

where $L\mathbf{u}(z) = L(u_0(z), u_{-1}(z), u_{-2}(z), \dots) := (u_{-1}(z), u_{-2}(z), \dots)$ denotes the left translation.

Bukhgeim's original theory in [5] shows that solutions of (3.6), called L^2 -analytic, satisfy a Cauchy-like integral formula,

$$(3.7) \quad \mathbf{u}(z) = \mathcal{B}[\mathbf{u}|_\Gamma](z), \quad z \in \Omega,$$

where \mathcal{B} is the Bukhgeim-Cauchy operator acting on $\mathbf{u}|_\Gamma$. We use the formula in [8], where \mathcal{B} is defined component-wise for $n \geq 0$ by

$$(3.8) \quad (\mathcal{B}\mathbf{u})_{-n}(z) := \frac{1}{2\pi i} \int_\Gamma \frac{u_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_\Gamma \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}} \right\} \sum_{j=1}^{\infty} u_{-n-2j}(\zeta) \left(\frac{\bar{\zeta} - \bar{z}}{\zeta - z} \right)^j, \quad z \in \Omega.$$

Similar to the analytic maps, the traces of L^2 -analytic maps on the boundary must satisfy some constraints, which can be expressed in terms of a corresponding Hilbert-like transform introduced in [34]. More precisely, the Bukhgeim-Hilbert transform \mathcal{H} acting on \mathbf{g} ,

$$(3.9) \quad \Gamma \ni z \mapsto (\mathcal{H}\mathbf{g})(z) = \langle (\mathcal{H}\mathbf{g})_0(z), (\mathcal{H}\mathbf{g})_{-1}(z), (\mathcal{H}\mathbf{g})_{-2}(z), \dots \rangle$$

is defined component-wise for $n \geq 0$ by

$$(3.10) \quad (\mathcal{H}\mathbf{g})_{-n}(z) = \frac{1}{\pi} \int_\Gamma \frac{g_{-n}(\zeta)}{\zeta - z} d\zeta + \frac{1}{\pi} \int_\Gamma \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{z}} \right\} \sum_{j=1}^{\infty} g_{-n-2j}(\zeta) \left(\frac{\bar{\zeta} - \bar{z}}{\zeta - z} \right)^j, \quad z \in \Gamma.$$

The theorems below comprise some results in [34, 35]. For $0 < \mu < 1$, $p = 1, 2$, we consider the Banach spaces:

$$(3.11) \quad \begin{aligned} l_\infty^{1,p}(\Gamma) &:= \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \dots \rangle : \|\mathbf{g}\|_{l_\infty^{1,p}(\Gamma)} := \sup_{\xi \in \Gamma} \sum_{j=0}^{\infty} \langle j \rangle^p |g_{-j}(\xi)| < \infty \right\}, \\ C^\mu(\Gamma; l_1) &:= \left\{ \mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \dots \rangle : \sup_{\xi \in \Gamma} \|\mathbf{g}(\xi)\|_{l_1} + \sup_{\substack{\xi, \eta \in \Gamma \\ \xi \neq \eta}} \frac{\|\mathbf{g}(\xi) - \mathbf{g}(\eta)\|_{l_1}}{|\xi - \eta|^\mu} < \infty \right\}, \\ Y_\mu(\Gamma) &:= \left\{ \mathbf{g} : \mathbf{g} \in l_\infty^{1,2}(\Gamma) \text{ and } \sup_{\substack{\xi, \eta \in \Gamma \\ \xi \neq \eta}} \sum_{j=0}^{\infty} \langle j \rangle \frac{|g_{-j}(\xi) - g_{-j}(\eta)|}{|\xi - \eta|^\mu} < \infty \right\}, \end{aligned}$$

where, for brevity, we use the notation $\langle j \rangle = (1 + |j|^2)^{1/2}$. Similarly, we consider $C^\mu(\bar{\Omega}; l_\infty)$, and $C^\mu(\Omega; l_\infty) = \bigcup_{0 < r < 1} C^\mu(\bar{\Omega}_r; l_\infty)$, where for $0 < r < 1$, $\Omega_r = \{z \in \mathbb{C} : |z| < r\}$.

Theorem 3.1. *Let $0 < \mu < 1$. Let $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \dots \rangle$ be a sequence valued map defined on the boundary Γ and \mathcal{B} be the Bukhgeim-Cauchy operator acting on \mathbf{g} as in (3.8).*

(i) *If $\mathbf{g} \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$, then $\mathbf{u} := \mathcal{B}\mathbf{g} \in C^{1,\mu}(\Omega; l_\infty) \cap C(\bar{\Omega}; l_\infty)$ is L^2 -analytic in Ω .*

(ii) *Moreover, if $\mathbf{g} \in Y_\mu(\Gamma)$ for $\mu > 1/2$, then $\mathcal{B}\mathbf{g} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\bar{\Omega}; l_1)$.*

For the proof of Theorem 3.1 (i) we refer to [34, Theorem 3.1], and for part (ii) we refer to [35, Proposition 2.3].

Key to the proof of the constraints (2.14) is the following characterization of traces of L^2 -analytic maps.

Theorem 3.2. *Let $0 < \mu < 1$, and let \mathcal{H} be the Bukhgeim-Hilbert transform in (3.10).*

(i) *If $\mathbf{g} \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$ is the boundary value of an L^2 -analytic function, then $\mathcal{H}\mathbf{g} \in C^\mu(\Gamma; l_\infty)$ and satisfies*

$$(3.12) \quad (I + i\mathcal{H})\mathbf{g} = \mathbf{0}.$$

(ii) *If $\mathbf{g} \in Y_\mu(\Gamma)$ for $\mu > 1/2$, satisfies (3.12), then there exists an L^2 -analytic function $\mathbf{u} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\bar{\Omega}; l_1)$, such that*

$$(3.13) \quad \mathbf{u}|_\Gamma = \mathbf{g}.$$

For the proof of Theorem 3.2 we refer to [34, Proposition 3.1, Theorem 3.2, Corollary 4.1, and Proposition 4.2].

4. PROPERTIES OF THE BUKHGEIM-HILBERT TRANSFORM ON THE FOURIER LATTICE

In this section, we present a new mapping property of the Bukhgeim-Hilbert transform. In here Ω is the unit disk, and Γ is its unit circle boundary.

Given $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \dots \rangle \in l_\infty(\mathbb{N}; L^1(\Gamma))$, we consider the Fourier coefficients of its components

$$(4.1) \quad g_{-n,k} := \frac{1}{2\pi} \int_{-\pi}^{\pi} g_{-n}(e^{i\beta}) e^{-ik\beta} d\beta, \text{ for all } n \geq 0, \text{ and } k \in \mathbb{Z}.$$

Theorem 4.1. *Let $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \dots \rangle \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$, $0 < \mu < 1$, and $g_{-n,k}$ be the Fourier coefficients of its components as in (4.1). Let $\mathcal{H}\mathbf{g} = \langle (\mathcal{H}\mathbf{g})_0, (\mathcal{H}\mathbf{g})_{-1}, (\mathcal{H}\mathbf{g})_{-2}, \dots \rangle$ be the Bukhgeim-Hilbert transform acting on \mathbf{g} as defined in (3.10). Then $\mathcal{H}\mathbf{g} \in C^\mu(\Gamma; l_\infty)$, and the Fourier coefficients $(\mathcal{H}\mathbf{g})_{-n,k} := \frac{1}{2\pi} \int_0^{2\pi} (\mathcal{H}\mathbf{g})_{-n}(e^{i\beta}) e^{-ik\beta} d\beta$, for $n \geq 0, k \in \mathbb{Z}$, of its components satisfy*

$$(4.2) \quad (-i)(\mathcal{H}\mathbf{g})_{-n,k} = \begin{cases} g_{-n,k} & \text{if } k \geq 0, \\ -g_{-n,k} + 2(-1)^k g_{-n+2k,-k} & \text{if } k \leq -1. \end{cases}$$

Proof. Since $\mathbf{g} = \langle g_0, g_{-1}, g_{-2}, \dots \rangle \in l_\infty^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1)$, by Theorem 3.2 (i), $\mathcal{H}\mathbf{g} \in C^\mu(\Gamma; l_\infty)$.

For $\xi = e^{i\beta}$ and $\zeta = e^{i\alpha}$ on Γ ,

$$\frac{d\zeta}{\zeta - \xi} = \frac{ie^{i\alpha} d\alpha}{e^{i\alpha} - e^{i\beta}}, \quad \frac{d\zeta}{\zeta - \xi} - \frac{d\bar{\zeta}}{\bar{\zeta} - \bar{\xi}} = id\alpha, \quad \left(\frac{\bar{\zeta} - \bar{\xi}}{\zeta - \xi} \right) = -e^{-i(\beta+\alpha)},$$

and the components $(\mathcal{H}\mathbf{g})_n$, $n \geq 0$, rewrite as

$$(4.3) \quad (-i)(\mathcal{H}\mathbf{g})_{-n}(e^{i\beta}) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{g_{-n}(e^{i\alpha})}{e^{i\alpha} - e^{i\beta}} e^{i\alpha} d\alpha + \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k e^{-i\beta k} \int_{-\pi}^{\pi} g_{-n-2k}(e^{i\alpha}) e^{-i\alpha k} d\alpha.$$

Since $g_{-n}(e^{i\alpha}) = \sum_{k=-\infty}^{\infty} g_{-n,k} e^{ik\alpha}$, by using Lemma A.2 (in the appendix), the first term on the right-hand-side of (4.3) becomes

$$\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{g_{-n}(e^{i\alpha})}{e^{i\alpha} - e^{i\beta}} e^{i\alpha} d\alpha = \sum_{k=-\infty}^{\infty} g_{-n,k} \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{i(k+1)\alpha}}{e^{i\alpha} - e^{i\beta}} d\alpha = \sum_{k=-\infty}^{\infty} \text{sgn}(k) g_{-n,k} e^{ik\beta}.$$

The last term of (4.3),

$$\frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k e^{-i\beta k} \int_{-\pi}^{\pi} g_{-n-2k}(e^{i\alpha}) e^{-i\alpha k} d\alpha = \sum_{k=1}^{\infty} 2(-1)^k g_{-n-2k,k} e^{-i\beta k}.$$

Thus, the right-hand-side of (4.3) is

$$\sum_{k=-\infty}^{\infty} \operatorname{sgn}(k) g_{-n,k} e^{ik\beta} + \sum_{k=-\infty}^{-1} 2(-1)^k g_{-n+2k,-k} e^{i\beta k},$$

and the Fourier coefficients $(\mathcal{H}g)_{-n,k}$ satisfy (4.2). \square

Using the Fourier representation (4.2) of the Bukhgeim-Hilbert transform (applied twice), it is now easy to see that \mathcal{H} enjoys the idempotent property.

Corollary 4.1. *Let $0 < \mu < 1$. If $g \in l_{\infty}^{1,1}(\Gamma) \cap C^{\mu}(\Gamma; l_1)$, then $\mathcal{H}^2 g = -g$.*

5. PROOF OF THEOREM 2.1

(i) **Necessity:** Since g is angularly odd, (2.11) holds. Since g is real valued, (2.12) holds. The identities (2.13) follow by direct calculation, see Lemma A.1.

Recall that

$$(5.1) \quad g = \begin{cases} Xf, & \text{on } \Gamma_+, \\ -Xf, & \text{on } \Gamma_- \end{cases}$$

is the trace on $\Gamma \times \mathbf{S}^1$ of the unique solution of the boundary value problem

$$(5.2a) \quad \boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) = 2f(z), \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1,$$

$$(5.2b) \quad u|_{\Gamma_-} = -Xf|_{\Gamma_-}.$$

Moreover, u must be angularly odd, $u(z, \boldsymbol{\theta}) = -u(z, -\boldsymbol{\theta})$: Indeed, if $u^{odd}(z, \boldsymbol{\theta}) := \frac{1}{2} [u(z, \boldsymbol{\theta}) - u(z, -\boldsymbol{\theta})]$ denotes the angularly odd part of u , then

$$\boldsymbol{\theta} \cdot \nabla u^{odd}(z, \boldsymbol{\theta}) = \frac{1}{2} [\boldsymbol{\theta} \cdot \nabla u(z, \boldsymbol{\theta}) + (-\boldsymbol{\theta}) \cdot \nabla u(z, -\boldsymbol{\theta})] = 2f(z),$$

and $u^{even} := u - u^{odd}$ solves

$$(5.3a) \quad \boldsymbol{\theta} \cdot \nabla u^{even}(z, \boldsymbol{\theta}) = 0, \quad (z, \boldsymbol{\theta}) \in \Omega \times \mathbf{S}^1,$$

$$(5.3b) \quad u^{even}|_{\Gamma_-} = 0.$$

Since (5.3) has the unique solution $u^{even} \equiv 0$, $u = u^{odd}$ in $\overline{\Omega} \times \mathbf{S}^1$.

We will first prove (2.14) for $f \in C_0^2(\Omega)$ real valued of compact support in Ω . The result for $f \in L^1(\Omega)$ follows by a density argument.

Since f is real valued, the solution $u(z, \boldsymbol{\theta})$ of (5.2) is also real valued, and its Fourier modes in the angular variable occur in conjugates, $u_{-n}(z) = \overline{u_n(z)}$, $n \in \mathbb{Z}$. Let \mathbf{u} be the sequence valued map of the non-positive odd Fourier modes

$$(5.4) \quad \Omega \ni z \mapsto \mathbf{u}(z) := \langle u_{-1}(z), u_{-3}(z), u_{-5}(z), \dots \rangle.$$

Since $f \in C_0^2(\Omega)$, $u \in C^2(\overline{\Omega} \times \mathbf{S}^1)$, in particular $\mathbf{u} \in C^1(\overline{\Omega}; l_1)$.

Since $u \in C^2(\overline{\Omega} \times \mathbf{S}^1)$, its trace $g = u|_{\Gamma \times \mathbf{S}^1} \in C^2(\Gamma \times \mathbf{S}^1)$. We define the sequence valued map \mathbf{g} on the boundary Γ by

$$(5.5) \quad \Gamma \ni e^{i\beta} \mapsto \mathbf{g}(e^{i\beta}) := \langle g_{-1}(e^{i\beta}), g_{-3}(e^{i\beta}), g_{-5}(e^{i\beta}), \dots \rangle,$$

where $g_{-n}(e^{i\beta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\beta}, e^{i\theta}) e^{in\theta} d\theta$ is the $(-n)$ -th Fourier coefficients in the angular variable of the function g .

Since $g = u|_{\Gamma \times \mathbf{S}^1}$, $\mathbf{g} = \mathbf{u}|_{\Gamma} \in C^1(\Gamma; l_1) \subset l_{\infty}^{1,1}(\Gamma) \cap C^{\mu}(\Gamma; l^1)$, $\mu > 1/2$.

By (3.6), \mathbf{u} is L^2 -analytic in Ω , and, then, \mathbf{g} is the boundary value of an L^2 -analytic map.

By the necessity part in Theorem 3.2, we have $\mathcal{H}\mathbf{g} \in C^{\mu}(\Gamma; l_{\infty})$ and

$$(5.6) \quad (I + i\mathcal{H})\mathbf{g} = \mathbf{0},$$

where \mathcal{H} is the Bukhgeim-Hilbert operator in (3.10). In particular,

$$(5.7) \quad ([I + i\mathcal{H}]\mathbf{g})_{n,k} = 0,$$

for all $n \leq -1$ odd, and $k \in \mathbb{Z}$.

By Theorem 4.1, the Fourier coefficients $(\mathcal{H}\mathbf{g})_{n,k}$, for $n \leq -1$ odd, $k \in \mathbb{Z}$, satisfy (4.2), and thus

$$(5.8) \quad ([I + i\mathcal{H}]\mathbf{g})_{n,k} = \begin{cases} 0 & \text{if } k \geq 0, \\ 2g_{n,k} - 2(-1)^k g_{n+2k,-k} & \text{if } k \leq -1. \end{cases}$$

From (5.7), the Fourier coefficients of \mathbf{g} must satisfy (2.14):

$$(5.9) \quad g_{n,k} = (-1)^k g_{n+2k,-k}, \text{ for all } n \leq -1 \text{ odd, and } k \leq -1.$$

Equation (2.14) for $k = 0$ is trivially satisfied.

The proof for $f \in L^1(\Omega)$ follows from the density of $C_0^2(\Omega)$ in $L^1(\Omega)$.

(ii) **Sufficiency:** Given the double sequence $\{g_{n,k}\}$ for $n \leq -1$ odd, and $k \in \mathbb{Z}$, we construct a real valued function f in Ω such that the map on the torus $\begin{cases} Xf \text{ on } \Gamma_+, \\ -Xf \text{ on } \Gamma_- \end{cases}$ has its Fourier coefficients equal to $\{g_{n,k}\}$.

Define first the sequence valued map on the boundary Γ by

$$(5.10) \quad \Gamma \ni e^{i\beta} \mapsto \mathbf{g}^{\text{odd}}(e^{i\beta}) := \langle g_{-1}(e^{i\beta}), g_{-3}(e^{i\beta}), g_{-5}(e^{i\beta}), \dots \rangle,$$

where for each $n \leq -1$ odd,

$$(5.11) \quad g_n(e^{i\beta}) := \sum_{k=-\infty}^{\infty} g_{n,k} e^{ik\beta}.$$

We construct the sequence valued map $\mathbf{u}^{\text{odd}}(z)$ inside Ω , by the Bukhgeim-Cauchy Integral formula (3.8), namely

$$(5.12) \quad \mathbf{u}^{\text{odd}}(z) = \langle u_{-1}(z), u_{-3}(z), u_{-5}(z), \dots \rangle := \mathcal{B}[\mathbf{g}^{\text{odd}}](z), \quad z \in \Omega.$$

Using the decay assumption (2.15), by Lemma A.4 in the appendix, $\mathbf{g}^{\text{odd}} \in l_{\infty}^{1,2}(\Gamma) \cap C^{1,\mu}(\Gamma; l^1)$, and, in particular, $\mathbf{g}^{\text{odd}} \in Y_{\mu}(\Gamma)$ for $\mu > 1/2$.

By Theorem 3.1 (ii), the constructed $\mathbf{u}^{\text{odd}} \in C^{1,\mu}(\Omega; l^1) \cap C^{\mu}(\overline{\Omega}; l^1)$ is L^2 -analytic in Ω ,

$$(5.13) \quad \bar{\partial}u_n + \partial u_{n-2} = 0, \quad n \leq -1, \quad n \text{ odd}.$$

While \mathbf{u}^{odd} constructed in (5.12) is L^2 -analytic, in general, its trace $\mathbf{u}^{\text{odd}}|_{\Gamma}$ need not be equal to \mathbf{g}^{odd} . It is at this point that the constraints (2.14) come into play. By using the constraints (2.14),

$$g_{n,k} = (-1)^k g_{n+2k,-k}, \quad \text{for } n \leq -1 \text{ odd, and } k \leq -1,$$

in (4.2), we obtain that, for each $n \leq -1$, n odd, and $k \in \mathbb{Z}$, $([I + i\mathcal{H}]\mathbf{g}^{\text{odd}})_{n,k} = 0$. Thus, $[I + i\mathcal{H}]\mathbf{g}^{\text{odd}} = \mathbf{0}$, and the sufficiency part of Theorem 3.2 applies to yield

$$(5.14) \quad \mathbf{u}^{\text{odd}}|_{\Gamma} = \mathbf{g}^{\text{odd}}.$$

All of the positive Fourier modes u_n, g_n for $n \geq 1$ odd are constructed by conjugation,

$$(5.15) \quad u_n := \overline{u_{-n}}, \quad \text{in } \Omega,$$

$$(5.16) \quad g_n := \overline{g_{-n}}, \quad \text{on } \Gamma.$$

Also, by conjugating (5.13) we note that the positive Fourier modes satisfy

$$\bar{\partial}u_{n+2} + \partial u_n = 0, \quad n \geq 1, \text{ } n \text{ odd.}$$

Moreover, using (5.14) they extend continuously to Γ and

$$u_n|_{\Gamma} = \overline{u_{-n}}|_{\Gamma} = \overline{g_{-n}} = g_n, \quad n \geq 1, \text{ } n \text{ odd.}$$

In summary, we have shown that

$$(5.17) \quad \bar{\partial}u_n + \partial u_{n-2} = 0, \quad \text{for all odd integers } n \neq 1,$$

$$(5.18) \quad u_n|_{\Gamma} = g_n, \quad \text{for all odd integers } n.$$

Define next the following three real valued functions:

$$(5.19) \quad u^{\text{odd}}(z, \boldsymbol{\theta}) := \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} u_n(z) e^{in\theta}, \quad (z, \theta) \in \Omega \times \mathbf{S}^1,$$

$$(5.20) \quad g(z, \boldsymbol{\theta}) := \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} g_n(z) e^{in\theta}, \quad (z, \theta) \in \Gamma \times \mathbf{S}^1,$$

and

$$(5.21) \quad f(z) := \Re(\partial u_{-1}(z)), \quad z \in \Omega.$$

Note that $u_{-1} \in C^{1,\mu}(\Omega)$, yields $f \in C^\mu(\Omega)$.

We are left to prove that g defined in (5.20) coincides with Xf on Γ_+ .

Since only odd modes are used in the angular variable, $g(z, \cdot)$ is an odd function, and thus satisfy (2.8). We claim that g also satisfies the symmetry condition (2.7). Indeed,

$$\begin{aligned}
g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}) &= \sum_{\substack{m=-\infty \\ m=\text{odd}}}^{\infty} \sum_{p \in \mathbb{Z}} g_{m,p} e^{im(\theta+\pi)} e^{ip(2\theta-\beta-\pi)} \\
&= \sum_{\substack{m=-\infty \\ m=\text{odd}}}^{\infty} \sum_{p \in \mathbb{Z}} (-1)^{m+p} g_{m,p} e^{i(2p+m)\theta} e^{-ip\beta} \\
&\stackrel{\text{change variables}}{=} \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} \sum_{k \in \mathbb{Z}} (-1)^{n+k} g_{n+2k,-k} e^{in\theta} e^{ik\beta} \\
(5.22) \quad &\stackrel{(2.13)}{=} \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} \sum_{k \in \mathbb{Z}} g_{n,k} e^{in\theta} e^{ik\beta} = g(e^{i\beta}, e^{i\theta}).
\end{aligned}$$

Since $\mathbf{u}^{\text{odd}} \in C^{1,\mu}(\Omega; l^1) \cap C^\mu(\overline{\Omega}; l^1)$, by [34, Corollary 4.1] and [34, Proposition 4.1 (iii)], we conclude that $u^{\text{odd}} \in C^{1,\mu}(\Omega \times \mathbf{S}^1) \cap C^\mu(\overline{\Omega} \times \mathbf{S}^1)$. In particular, for each $\boldsymbol{\theta} \in \mathbf{S}^1$, the trace of $u^{\text{odd}}(\cdot, \boldsymbol{\theta})$ satisfies

$$(5.23) \quad u^{\text{odd}}(\cdot, \boldsymbol{\theta})|_\Gamma = \left(\sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} u_n e^{in\theta} \right) \Big|_\Gamma = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} (u_n|_\Gamma) e^{in\theta} = \sum_{\substack{n=-\infty \\ n=\text{odd}}}^{\infty} g_n e^{in\theta} = g(\cdot, \boldsymbol{\theta}),$$

where the third equality above uses (5.18). Since $u^{\text{odd}} \in C^\mu(\overline{\Omega} \times \mathbf{S}^1)$, its trace $g \in C^\mu(\Gamma \times \mathbf{S}^1)$.

Since $u^{\text{odd}} \in C^{1,\mu}(\Omega \times \mathbf{S}^1)$, the formal calculation in (3.2) is now justified for each $z \in \Omega$. From (5.17) and (5.21), we obtain

$$(5.24) \quad \boldsymbol{\theta} \cdot \nabla u^{\text{odd}}(z, \boldsymbol{\theta}) = 2 \operatorname{Re}(\partial u_{-1}(z)) + 2 \operatorname{Re} \left(\sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} (\bar{\partial} u_{-n}(z) + \partial u_{-n-2}(z)) e^{-in\theta} \right) = 2f(z).$$

In the following calculation we let $l(\beta, \theta) = |e^{i\beta} - e^{i(2\theta-\beta-\pi)}|$ denote the length of the chord in Figure 1, and use the geometric equality $e^{i\beta} - l(\beta, \theta)e^{i\theta} = e^{i(2\theta-\beta-\pi)}$. For each $(e^{i\beta}, e^{i\theta}) \in \Gamma_+$,

$$\begin{aligned}
2g(e^{i\beta}, e^{i\theta}) &= g(e^{i\beta}, e^{i\theta}) - g(e^{i\beta}, e^{i(\theta+\pi)}) \\
&= g(e^{i\beta}, e^{i\theta}) - g(e^{i(2\theta-\beta-\pi)}, e^{i\theta}) \\
&= u^{\text{odd}}(e^{i\beta}, e^{i\theta}) - u^{\text{odd}}(e^{i(2\theta-\beta-\pi)}, e^{i\theta}) \\
&= \int_{-l(\beta,\theta)}^0 \boldsymbol{\theta} \cdot \nabla u^{\text{odd}}(e^{i\beta} + te^{i\theta}, \boldsymbol{\theta}) dt \\
(5.25) \quad &= \int_{-l(\beta,\theta)}^0 2f(e^{i\beta} + te^{i\theta}) dt = 2[Xf](e^{i\beta}, e^{i\theta}),
\end{aligned}$$

where the first equality uses $g(e^{i\beta}, \cdot)$ is angularly odd, the second equality uses the symmetry relation (5.22) with θ replaced by $\theta + \pi$, the third equality uses (5.23), the fourth equality is

the fundamental theorem of calculus, the fifth equality uses (5.24), and the last equality uses the support of f in Ω .

Therefore $g = Xf$ on Γ_+ , and, since g is angularly odd, $g = -Xf$ on Γ_- .

The equation (5.25) also shows that f integrates along all lines in the direction of θ , and thus $f \in L^1(\Omega)$. □

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APPENDIX A. ELEMENTARY RESULTS

To improve the readability, we moved the proof of the more elementary claims to this section. The presentation follows the order of their occurrence. Recall $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ is the complex unit disc, $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ is its boundary, and \mathbf{S}^1 is the set of unit directions.

Proposition A.1. *Let $f \in L^1(\Omega)$. If*

$$\text{supp } f \subset \{z : |z| \leq \sqrt{1 - \delta^2}\}, \quad 0 < \delta < 1, \quad \text{or} \quad f \in L^p(\Omega), \quad p > 2,$$

then $Xf \in L^1(\Gamma \times \mathbf{S}^1)$.

Proof. For $e^{i\beta} \in \Gamma$ and $e^{i\theta} \in \mathbf{S}^1$, the X -ray transform of f (extended by 0 outside Ω) is given by

$$Xf(e^{i\beta}, e^{i\theta}) = \int_{-\infty}^{\infty} f(e^{i\beta} + te^{i\theta}) dt = \int_{-\infty}^{\infty} f_{\theta}(e^{i(\beta-\theta)} + t) dt,$$

where

$$f_{\theta}(z) := f(ze^{i\theta})$$

is obtained from f by a rotation by angle θ .

We estimate

$$\begin{aligned} \|Xf\|_{L^1(\Gamma \times \mathbf{S}^1)} &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |Xf(e^{i\beta}, e^{i\theta})| d\beta d\theta \\ &\leq \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} |f_{\theta}(e^{i(\beta-\theta)} + t)| dt d\beta d\theta \\ &\stackrel{\alpha=\beta-\theta}{=} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi/2}^{\pi/2} \int_{-\infty}^{\infty} |f_{\theta}(e^{i\alpha} + t)| dt d\alpha d\theta \\ &\stackrel{s=\sin \alpha}{=} \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-1}^1 \frac{|f_{\theta}(\sqrt{1-s^2} + t + is)|}{\sqrt{1-s^2}} ds dt d\theta \\ &\stackrel{u=t+\sqrt{1-s^2}}{=} \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-1}^1 \frac{|f_{\theta}(u + is)|}{\sqrt{1-s^2}} ds du d\theta \\ (A.1) \quad &= \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-1}^1 \int_{-1}^1 \frac{|f_{\theta}(u + is)|}{\sqrt{1-s^2}} ds du d\theta, \end{aligned}$$

where the last equality uses $\text{supp } f_{\theta} \subset \Omega$.

If $\text{supp} f \subset \{z : |z| \leq \sqrt{1 - \delta^2}\}$, then

$$\begin{aligned} \|Xf\|_{L^1(\Gamma \times \mathbf{S}^1)} &\leq \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-1}^1 \int_{-\sqrt{1-\delta^2}}^{\sqrt{1-\delta^2}} \frac{|f_{\theta}(u + is)|}{\sqrt{1-s^2}} ds du d\theta \\ &\leq \frac{1}{2\pi^2} \frac{1}{\delta} \int_{-\pi}^{\pi} \|f_{\theta}\|_{L^1(\Omega)} d\theta = \frac{1}{\pi\delta} \|f\|_{L^1(\Omega)}. \end{aligned}$$

Next we consider $f \in L^p(\Omega)$, $p > 2$. Let also $T := (-1, 1) \times (-1, 1)$ denote the unit square. Since $\Omega \subset T$, for every $\theta \in (-\pi, \pi]$, $f_{\theta} \in L^p(T)$, and $\|f_{\theta}\|_{L^p(T)} = \|f\|_{L^p(\Omega)}$.

Let $q = \frac{p}{p-1}$ be the conjugate index of p . Since $p > 2$, $q < 2$, the map $T \ni (u, s) \mapsto \frac{1}{\sqrt{1-s^2}}$ is in $L^q(T)$, and $\left\| \frac{1}{\sqrt{1-(\cdot)^2}} \right\|_{L^q(T)} = 2 \left\| \frac{1}{\sqrt{1-(\cdot)^2}} \right\|_{L^q(-1,1)}$. An application of the Hölder's inequality in (A.1) yields

$$\|Xf\|_{L^1(\Gamma \times \mathbf{S}^1)} \leq \frac{1}{\pi^2} \int_{-\pi}^{\pi} \left\| \frac{1}{\sqrt{1-(\cdot)^2}} \right\|_{L^q(-1,1)} \|f_{\theta}\|_{L^p(\Omega)} d\theta = \frac{2}{\pi} \left\| \frac{1}{\sqrt{1-(\cdot)^2}} \right\|_{L^q(T)} \|f\|_{L^p(\Omega)}.$$

□

Lemma A.1. *Let g be an integrable function on the torus satisfying the symmetry relation*

$$(A.2) \quad g(e^{i\beta}, e^{i\theta}) = g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}), \text{ for } (e^{i\beta}, e^{i\theta}) \in \Gamma \times \mathbf{S}^1,$$

and $g_{n,k}$'s be its Fourier coefficients. Then

$$(A.3) \quad g_{n,k} = (-1)^{n+k} g_{n+2k, -k}, \text{ for all } n, k \in \mathbb{Z}.$$

Proof. Indeed,

$$\begin{aligned} g_{n,k} &= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(e^{i\beta}, e^{i\theta}) e^{-in\theta} e^{-ik\beta} d\theta d\beta \\ &\stackrel{(A.2)}{=} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(e^{i(2\theta-\beta-\pi)}, e^{i(\theta+\pi)}) e^{-in\theta} e^{-ik\beta} d\theta d\beta \\ &\stackrel{\gamma=\theta+\pi}{=} (-1)^n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_0^{2\pi} g(e^{i(2\gamma-\beta-3\pi)}, e^{i\gamma}) e^{-in\gamma} e^{-ik\beta} d\gamma d\beta \\ &\stackrel{\text{periodicity}}{=} (-1)^n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(e^{i(2\gamma-\beta-\pi)}, e^{i\gamma}) e^{-in\gamma} e^{-ik\beta} d\gamma d\beta \\ &\stackrel{\alpha=2\gamma-\beta-\pi}{=} (-1)^n \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{2\gamma}^{2\gamma-2\pi} g(e^{i\alpha}, e^{i\gamma}) e^{-ik(2\gamma-\alpha-\pi)} e^{-in\gamma} (-d\alpha) d\gamma \\ &= (-1)^{n+k} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{2\gamma-2\pi}^{2\gamma} g(e^{i\alpha}, e^{i\gamma}) e^{-i(n+2k)\gamma} e^{ik\alpha} d\alpha d\gamma \\ &\stackrel{\text{periodicity}}{=} (-1)^{n+k} \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} g(e^{i\alpha}, e^{i\gamma}) e^{-i(n+2k)\gamma} e^{ik\alpha} d\alpha d\gamma \\ &= (-1)^{n+k} g_{n+2k, -k}. \end{aligned}$$

□

The following result is used in the proof of Theorem 4.1.

Lemma A.2. For $\beta \in (-\pi, \pi]$,

$$(A.4) \quad \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{in\alpha}}{e^{i\alpha} - e^{i\beta}} d\alpha = \begin{cases} e^{i(n-1)\beta} & \text{if } n \geq 1, \\ -e^{i(n-1)\beta} & \text{if } n \leq 0. \end{cases}$$

Proof. Let $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ be the complex unit disc, and $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ be its boundary.

I. Let $n \geq 0$. For $z \in \Omega$, apply the Cauchy integral formula to $z \mapsto z^n$, to get

$$(A.5) \quad z^n = \frac{1}{2\pi i} \int_{\Gamma} \frac{\zeta^n}{\zeta - z} d\zeta.$$

Let $z \mapsto z_0 \in \Gamma$, with $|z| < 1$, and apply Plemelj-Sokhotski formula (e.g [25, Chap. 2, Section 17]) on both sides of (A.5),

$$\begin{aligned} z_0^n &= \frac{1}{2} z_0^n + \frac{1}{2\pi i} \int_{\Gamma} \frac{\zeta^n}{\zeta - z_0} d\zeta, \\ z_0^n &= \frac{1}{\pi i} \int_{\Gamma} \frac{\zeta^n}{\zeta - z_0} d\zeta, \end{aligned}$$

where the integral is understood in the Cauchy principal value sense. Now parametrize the unit circle $\partial\Omega$. In parametric form, let $z_0 = e^{i\beta}$, $\zeta = e^{i\alpha}$, then $d\zeta = ie^{i\alpha} d\alpha$, and

$$e^{in\beta} = \frac{1}{\pi i} \int_{-\pi}^{\pi} \frac{e^{in\alpha} i e^{i\alpha}}{e^{i\alpha} - e^{i\beta}} d\alpha = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{i(n+1)\alpha}}{e^{i\alpha} - e^{i\beta}} d\alpha, \quad n \geq 0,$$

Thus,

$$e^{i(n-1)\beta} = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{in\alpha}}{e^{i\alpha} - e^{i\beta}} d\alpha, \quad n \geq 1.$$

II. For $n \geq 0$, we apply the Cauchy integral formula to $z \mapsto z^{-(n+1)}$ analytic in $(\mathbb{C} \cup \infty) - \Omega$,

$$(A.6) \quad z^{-(n+1)} = -\frac{1}{2\pi i} \int_{\Gamma} \frac{\zeta^{-(n+1)}}{\zeta - z} d\zeta,$$

where the integral is counter-clockwise, hence the (-) sign to account for the exterior of the disc. Let $z \mapsto z_0 \in \Gamma$, with $|z| > 1$, and apply the exterior Plemelj-Sokhotski formula (e.g [25, Chap. 2, Section 17]) on both sides of (A.6), to get

$$(A.7) \quad \begin{aligned} z_0^{-(n+1)} &= -\left[-\frac{1}{2} z_0^{-(n+1)} + \frac{1}{2\pi i} \int_{\Gamma} \frac{z^{-(n+1)}}{z - z_0} dz \right], \\ -z_0^{-(n+1)} &= \frac{1}{\pi i} \int_{\Gamma} \frac{\zeta^{-(n+1)}}{\zeta - z_0} d\zeta, \end{aligned}$$

where the integral is understood in the Cauchy principal value sense. Now parametrize the unit circle by $\zeta = e^{i\alpha}$, and let $z_0 = e^{i\beta}$. Then (A.7) becomes

$$-e^{-i(n+1)\beta} = \frac{1}{\pi i} \int_{-\pi}^{\pi} \frac{e^{-i(n+1)\alpha} i e^{i\alpha}}{e^{i\alpha} - e^{i\beta}} d\alpha = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{e^{-in\alpha}}{e^{i\alpha} - e^{i\beta}} d\alpha, \quad n \geq 0.$$

The change $n \rightarrow -n$, in the above integral show (A.4) for $n \leq 0$.

□

The following result is a direct consequence of the uniform convergence. Recall that S^1 denotes the unit circle.

Lemma A.3. *Let $(X, \|\cdot\|)$ be a Banach space, $\{a_n\}_{n \in \mathbb{Z}}$ be a sequence in X , and $p \geq 0$ integer. If $\sum_{n \in \mathbb{Z}} \langle n \rangle^p \|a_n\| < \infty$, then $\mathbf{S}^1 \ni \zeta \mapsto \sum_{n \in \mathbb{Z}} a_n \zeta^n$ defines a C^p map on \mathbf{S}^1 with values in X .*

For the following result, we recall some of the spaces in (3.11), for $0 < \mu < 1$, $p = 1, 2$:

$$(A.8) \quad \begin{aligned} l_\infty^{1,p}(\Gamma) &:= \left\{ \mathbf{g} := \langle g_0, g_{-1}, g_{-2}, \dots \rangle : \|\mathbf{g}\|_{l_\infty^{1,p}(\Gamma)} := \sup_{\xi \in \Gamma} \sum_{j=0}^{\infty} \langle j \rangle^p |g_{-j}(\xi)| < \infty \right\}, \\ C^\mu(\Gamma; l_1) &:= \left\{ \mathbf{g} := \langle g_0, g_{-1}, g_{-2}, \dots \rangle : \sup_{\xi \in \Gamma} \|\mathbf{g}(\xi)\|_{l_1} + \sup_{\substack{\xi, \eta \in \Gamma \\ \xi \neq \eta}} \frac{\|\mathbf{g}(\xi) - \mathbf{g}(\eta)\|_{l_1}}{|\xi - \eta|^\mu} < \infty \right\}, \end{aligned}$$

where, for brevity, we use the notation $\langle j \rangle = (1 + |j|^2)^{1/2}$.

Lemma A.4. *Let $\{g_{-n,k}\}_{n \geq 0, k \in \mathbb{Z}}$ be a double sequence satisfying the decay*

$$(A.9) \quad \sum_{n=0}^{\infty} \langle n \rangle^2 \sum_{k=-\infty}^{\infty} |g_{-n,k}| < \infty, \quad \text{and} \quad \sum_{k=-\infty}^{\infty} \langle k \rangle^{1+\mu} \sum_{n=0}^{\infty} |g_{-n,k}| < \infty, \quad \text{for some } 0 \leq \mu \leq 1.$$

Let $\mathbf{g}(\zeta) := \langle g_0(\zeta), g_{-1}(\zeta), g_{-2}(\zeta), \dots \rangle$, for $\zeta \in \Gamma = \{z : |z| = 1\}$, and where for each $n \geq 0$, $g_{-n}(\zeta) := \sum_{k=-\infty}^{\infty} g_{-n,k} \zeta^k$. Then $\mathbf{g} \in l_\infty^{1,2}(\Gamma) \cap C^{1,\mu}(\Gamma; l^1)$.

Proof. The fact that $\mathbf{g} \in l_\infty^{1,2}(\Gamma)$ follows from the first bound in (A.9):

$$\|\mathbf{g}\|_{l_\infty^{1,2}(\Gamma)} = \sup_{\zeta \in \Gamma} \sum_{n=0}^{\infty} \langle n \rangle^2 |g_{-n}(\zeta)| \leq \sum_{n=0}^{\infty} \langle n \rangle^2 \sum_{k=-\infty}^{\infty} |g_{-n,k}| < \infty.$$

We prove that $\mathbf{g} \in C^{1,\mu}(\Gamma; l^1)$ by interpolation between $C^1(\Gamma; l^1)$, and $C^{1,1}(\Gamma; l^1)$. For each $k \in \mathbb{Z}$, consider the l^1 sequence $\mathbf{F}(k) := \{g_{-n,k}\}_{n \geq 0}$. By the second bound in (A.9), we have

$$\sum_{k \in \mathbb{Z}} \langle k \rangle^{1+\mu} \|\mathbf{F}(k)\|_{l^1} = \sum_{k=-\infty}^{\infty} \langle k \rangle^{1+\mu} \sum_{n=0}^{\infty} |g_{-n,k}| < \infty.$$

Since $\{\langle k \rangle^{1+\mu} \mathbf{F}(k)\}_{k \in \mathbb{Z}} \in l^1(\mathbb{Z}; l^1(\mathbb{N}_n))$, by applying Lemma A.3 for $X = l^1(\mathbb{N}_n)$, and $p = 1 + \mu$, with $\mu = 0$, we conclude $\mathbf{g} \in C^1(\Gamma; l^1)$, whereas with $\mu = 1$, we obtain $\mathbf{g} \in C^{1,1}(\Gamma; l^1)$. The result for $0 < \mu < 1$ follows by interpolation. \square

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