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Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

RICAM JOHANN-RADON-INSTITUTE FOR COMPUTATIONAL AND APPLIED MATHEMATICS

Mesoscale Approximation of the Electromagnetic Fields

A. Bouzekri, M. Sini

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Ali Bouzekri[*] Mourad Sini [†]

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Abstract

The point-interaction approximation (or the Foldy-Lax approximation) of the electromagnetic fields generated by a cluster of small scaled inhomogeneities is derived in the mesoscale (i.e. mesoscopic scale) regime, that is when the minimum distance $\boldsymbol{\delta}$ between the particles is proportional to their maximum diameter $\boldsymbol{a}$ in the form $\boldsymbol{\delta} = \boldsymbol{c}_r\, \boldsymbol{a}$ with a positive constant $\boldsymbol{c}_r$ that we call the dilution parameter. The small particles are modeled by anisotropic and variable electric permittivities and/or magnetic permeabilities with possibly complex values. We provide the dominating field (the so-called Foldy-Lax field) with explicit error estimates in terms of the dilution parameter $\boldsymbol{c}_r$ uniformly in terms of the distribution of these inhomogeneities. Such approximations are key steps in different research areas as imaging and material sciences.

Keywords: Electromagnetism, Small Inhomogeneities, Multiple Scattering, Foldy-Lax Approximation, Mesoscale Regime.

AMS subject classification: 35J08, 35Q61, 45Q05.

1 Introduction

Understanding the interaction between waves and matter, such as light and acoustic wave fluctuations or elastic displacements trough inhomogeneous medium, has been of fundamental importance for a long time. Since Rayleigh's and Kirchhoff's pioneering works, it was known that the diffracted wave by small scaled inhomogeneities is dominated by the first multi-poles (poles or dipoles). In modern terminology, the dominating fields are given by (polarized) point sources located inside the particles, which determine the Green's functions' singularities of the corresponding propagator. In this direction, the next key step was achieved by Mie [24] in his full expansion of the electromagnetic field for spherically shaped particles. These formal expansions were later mathematically justified, see for instance [14] in the framework of low frequencies expansions. A further step was achieved in [5] where the full expansion at any order is derived and justified.

These works focused on a single inhomogeneity or well-separated ones, that is to say that only the interaction of a single inhomogeneity and the wave is taken into account, as the diffracted wave from one inhomogeneity is attenuated before reaching the others inhomogeneities. In the presence of multiple and sufficiently close inhomogeneities, the mutual interactions between them and the waves must be taken into account. In this respect, a formal argument to handle such multiple interactions was proposed by Foldy in his seminal work [16]. To state his formulas, he looks at the inhomogeneities as point-like potentials (i.e. Dirac-like). Then, he states a close form of the scattered wave by simply eliminating the singularity on the locations of these potentials. This elimination of the singularity translates a physical motivation saying

[*]Laboratoire de Mathematiques Pures et Appliquees, Universite Mouloud Mammeri, Tizi-Ouzou, Algeria (Email: bouzekri.ali.lmpa@gmail.com).This author thanks the General Directorate of Scientific Research and Technological Development (DGRSDT/MESRS-Algeria) for its financial support.

[†]Radon institute (RICAM), Austrian Academy of Sciences, 69 Altenbergerstrasse, A4040, Linz, Austria (Email: mourad.sini@oeaw.ac.at).This author is partially supported by the Austrian Science Fund (FWF): P28971-N32.

that "the scattering coefficient of each point-like scatterer is proportional to the external field acting on it", which is known as the Foldy assumption. These formal representations of the scattered waves are then stated based on more mathematically sounding arguments than physical intuition by Berezin & Faddeev, see [8], in the framework of the Krein's selfadjoint extension of symmetric operators. Another related method is the so called regularization method (or the renormalization technique) which aims at computing the Green's function, and hence the Schwartz integral operator, in the presence of the collections of inhomogeneities. This idea consists in taking the Fourier transform of the formal equation,"cut" or regularize and invert the related equations, via Weinstein-Aronszajn theorem, in the Fourier domain and then comeback. More details on these ideas can be found in the book [1]. The Faddeev approach was extended to singular potentials supported not only on points but also on curves and surfaces, see [21, 22] for more details. This approach gives us, via the Krein's resolvent representations, exact formulas to represent the scattered waves generated by singular potentials. However, our goal is to deal with a cluster of small scaled inhomogeneities. The intuitive believe is that the dominant part of the generated waves would be reminiscent to the exact formulas described above, for singular potentials supported on the centers of the inhomogeneities, but with scattering coefficients modeled by geometric or contrasts properties of the inhomogeneities. This is called the Foldy-Lax approximation or the point-interaction approximation.

Several methods were proposed in the literature to justify such approximations, see [10, 11, 23, 27] for instance regarding acoustic and elastic waves. Descriptions and relation/differences between these works can be found in [12]. Let us emphasize here that those works dealt with exterior problems (impenetrable inclusions, holes or voids). Regarding electromagnetic waves, very few works are proposed, apart from [28] where both the results and the justifications are quite questionable. In our previous work [9], we considered the case of perfectly conductive inclusions and gave a rigorous justification of the Foldy-Lax approximation, under sufficient conditions imposed on the minimum distance separating them $\boldsymbol{\delta}$ and their maximum diameter $\boldsymbol{a}$, namely $\ln(\boldsymbol{\delta}^{-1})\,\frac{\boldsymbol{a}}{\boldsymbol{\delta}}$ is bounded by a constant depending only on the Lipschitz character of the shapes of the inclusions and the wave number k. The only limitation of this result, to handle the mesoscale regime (i.e. $\boldsymbol{\delta} \sim \boldsymbol{a}$), is the appearance of the term $\ln(\delta^{-1})$. This term appears naturally in the analysis, in that work, which is heavily based on the scales of the related layer potentials knowing that the dyadic Maxwell fundamental solution has a singularity of order 3 (while the ones of Laplace or Lamé have singularities of the order 1).

In the present work, and dealing with the transmission problem, we get rid of this logarithmic term and state the approximation in the mesoscale regime. In addition, anisotropic and eventually complex valued electromagnetic material parameters can be handled as well. The corresponding results for the impenetrable problem, i.e. the perfect conductors case, will be reported elsewhere. To handle the anisotropic transmission problem, we provide a representation of the solution using the electromagnetic Lippmann-Schwinger operator. As a key step in our analysis, we show that, in this mesoscale regime, this operator is invertible. It is worth mentioning that even in the scalar case, i.e. related to the Laplace operator, with a cluster of small obstacles subject to Neumann boundary conditions, the problem was left open to our best knowledge. The approach we follow here definitely handles this case and provides the corresponding Foldy-Lax approximation in the same generality as we are proposing it in this work.

In the case of periodically distributed small inhomogeneities, the homogenization applies, see [7, 18], and provides the equivalent media with averaged materials. As compared to homogenization, the Foldy-Lax approximation has several advantages. The first one is that we have the dominating field (i.e. the Foldy-Lax field) for general (and not only periodic) distributions of the small inhomogeneities. This reduces the complexity of the forward problem to compute the scattered fields by inverting an algebraic system. Second, higher order approximations are possible with more effective dominating fields, i.e. with generalized Foldy-Lax fields. So far, this is not fully justified, but we believe it to be true and we will report on it in the future. Third, as we have freedom in distributing these inhomogeneities, then we can generate not only volumetric equivalent materials but also low dimensional ones as surfaces and curves. This opens the way to applications in low dimensions metamaterials as well, see [2, 3] for instance. As far as the Maxwell model is concerned, the Foldy-Lax approximation provided here shows that one can generate volumetric metamaterials and Gradient-metasurfaces. In this direction, we mention the works [6, 29] where related results are derived in the case of flat and periodic surfaces.

In the next section, we state clearly the model and the obtained results with critical discussions about them.

2 Main results

We deal with the scattering of time-harmonic electromagnetic plane waves at a frequency ω from an anisotropic medium, embedded in constant background, represented by multiply connected, bounded, Lipschitz domain $D := \cup_{m=1}^{\aleph} D_m$ where $\aleph$ is the number of connected components. The Maxwell equations read as follows

$$\begin{cases} \nabla \times \mathcal{E} - i\omega\mu\mathcal{H} = 0, \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\ \nabla \times \mathcal{H} + i\omega\varepsilon\mathcal{E} = \sigma\mathcal{E}, \quad \text{in } \mathbb{R}^3 \setminus \partial D, \\ \mathcal{E} = \mathcal{E}^{\text{in}} + \mathcal{E}^{\text{sc}}, \\ \nu \times \mathcal{E}|_{+} = \nu \times \mathcal{E}|_{-}, \ \nu \times \mathcal{H}|_{+} = \nu \times \mathcal{H}|_{-} \text{ on } \partial D \end{cases} \tag{2.1}$$

with the notation $\partial D := \cup_{m=1}^{\aleph} \partial D_m$, where ε and σ are respectively the electric permittivity and the conductivity and μ corresponds to the magnetic permeability. These parameters can be either real or complex tensor or scalar valued functions. Here $\mathcal{E}^{\text{in}}$ stands for the incident wave, which is an entire solution of maxwell equation. The vector field $\mathcal{E}^{\text{sc}}$ stands for the scattered field. The surrounding background of D is homogeneous with constant parameters ε_0, μ_0 and null conductivity σ. Further, the scattered field must satisfy the radiation conditions

$$\left(\sqrt{\mu_0 \varepsilon_0^{-1}}\right)\mathcal{H}^{\text{sc}}(x) \times \frac{x}{|x|} - \mathcal{E}^{\text{sc}}(x) = O\Big(\frac{1}{|x|^2}\Big). \tag{2.2}$$

Setting $k := \omega\sqrt{\epsilon_0\mu_0}$, $\mathcal{E} := \sqrt{\varepsilon_0 {\mu_0}^{-1}}E$ and $\mathcal{H} := \sqrt{\varepsilon_0 {\mu_0}^{-1}}H$ with $\mu_r := (\mu_0)^{-1}\mu$ and $\varepsilon_r := (\varepsilon_0)^{-1}(\varepsilon + i\sigma/\omega)$, we arrive at

$$\begin{cases} \operatorname{curl} E - ik\mu_r H = 0, \quad \operatorname{curl} H + ik\varepsilon_r E = 0, \quad \text{in } D, \\ \operatorname{curl} E - ikH = 0, \quad \operatorname{curl} H + ikE = 0, \quad \text{in } \mathbb{R}^3 \setminus D, \\ E = E^{\text{sc}} + E^{\text{in}}, \\ \nu \times E|_{-} - \nu \times E|_{+} = \nu \times H|_{-} - \nu \times H|_{+} = 0, \text{ on } \partial D \\ H^{\text{sc}} \times \dfrac{x}{|x|} - E^{\text{sc}} = O\Big(\dfrac{1}{|x|^2}\Big), \quad |x| \to \infty, \end{cases} \tag{$\mathcal{P}$}$$

where the incident field $(E^{\text{in}}, H^{\text{in}})$ satisfies in the whole space the system

$$\begin{cases} \operatorname{curl} E^{\text{in}} - ikH^{\text{in}} = 0, \\ \operatorname{curl} H^{\text{in}} + ikE^{\text{in}} = 0. \end{cases} \tag{2.3}$$

Motivated by applications, typical incident electric fields are plane waves, i.e of the form $E^{\text{in}} := E^{\text{in}}(x, \theta) := P\, e^{ik\theta\cdot x}, x \in \mathbb{R}^3$, where P is the (constant) vector modeling the polarization direction and θ, with $|\theta| = 1$, is the incident direction. The parameters P and θ satisfy the condition $P \cdot \theta = 0$. The related magnetic incident field is then $H^{\text{in}} := H^{\text{in}}(x, \theta) := P \times \theta\, e^{ik\theta\cdot x}/ik$.

Next, we state the conditions on the model ($\mathcal{P}$) under which we derive our results.

I. *Assumptions on the exterior domain.* The permittivity ϵ_0 and the permeability μ_0 of the background are assumed to be scalar, real and positive so that

$$k > 0. \tag{2.4}$$

II. *Assumptions on the cluster of particles.* We suppose that, each connect component of D is given by

$$D_m := \boldsymbol{a}\boldsymbol{\mathcal{D}}_m + \boldsymbol{z}_m, \; m = 1, ..., \aleph, \tag{2.5}$$

where each set $\boldsymbol{\mathcal{D}}_m$, contained in the ball $B_0^{1/2} := B(0, 1/2)$ and contains the origin, is assumed to be a Lipschitz bounded domain. The points $(\boldsymbol{z}_m)_{m=1}^{\aleph}$ are their given locations in $\mathbb{R}^3$ and $\boldsymbol{a} \in \mathbb{R}^+$ is a small parameter measuring the maximum relative radius, that is

$$\boldsymbol{a} = \max_{1 \leq m \leq \aleph} \sup_{x,y \in D_m} d(x, y). \tag{2.6}$$

As, for $m \in \{1, ..., \aleph\}$, we have $D_m \subset B(\boldsymbol{z}_m, \boldsymbol{a}/2)$, we define

$$\boldsymbol{\delta}_{mj} := \min_{\substack{x \in B(\boldsymbol{z}_m, \boldsymbol{a}/2), \\ y \in B(\boldsymbol{z}_j, \boldsymbol{a}/2)}} d(x, y),$$

to be the distance [1] between two bodies $D_m, D_j, m \neq j$, and set

$$\boldsymbol{\delta} := \min_{m \neq j \in \{1, ..., \aleph\}} \boldsymbol{\delta}_{mj}. \tag{2.7}$$

Let us recall that a bounded open connected domain $\boldsymbol{\mathcal{D}}$, is said to be Lipschitz with character $(l_{\partial\boldsymbol{\mathcal{D}}}, L_{\partial\boldsymbol{\mathcal{D}}})$ if for each $x \in \partial\boldsymbol{\mathcal{D}}$ there exist a coordinate system $(y_i)_{i=1,2,3}$, and a truncated cylinder $\mathfrak{C}$ centered at x whose axis is parallel to y_3 with length l satisfying $l_{\partial\boldsymbol{\mathcal{D}}} \leq l \leq 2l_{\partial\boldsymbol{\mathcal{D}}}$, and a Lipschitz function f that is $|f(s_1) - f(s_2)| \leq L_{\partial\boldsymbol{\mathcal{D}}}|s_1 - s_2|$ for every $s_1, s_2 \in \mathbb{R}^2$, such that $\boldsymbol{\mathcal{D}} \cap \mathfrak{C} = \{(y_i)_{i=1,2,3} : y_3 > f(y_1, y_2)\}$ and $\partial\boldsymbol{\mathcal{D}} \cap \mathfrak{C} = \{(y_i)_{i=1,2,3} : y_3 = f(y_1, y_2)\}$.

III. *Assumptions on the permittivity and permeability of each particle.* We assume that the permittivity and permeability of each particle satisfy the following conditions

1. Both μ_r and ε_r are in $\mathbb{W}^{1,\infty}(D)$, with μ_r real valued, and satisfy one of the assumptions
   (a) μ_r and $\Im\varepsilon_r$ are real definite positive with an arbitrary $\Re\varepsilon_r$.
   (b) ε_r and μ_r are both definite positive real symmetric matrices.
2. In addition, we suppose that their contrasts are, with their derivative, essentially uniformly bounded , i.e.

$$(\|\boldsymbol{\mathcal{C}}_B\|_{\mathbb{W}^{1,\infty}(\cup_{m=1}^{\aleph} D_m)})_{B=\varepsilon_r, \, \mu_r} \leq \boldsymbol{c}_\infty, \tag{2.8}$$

   and essentially uniformly coercive, that is, for almost every $x \in D$, we have

$$\begin{aligned} \Re\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}(x) U \cdot \overline{U}\Big) &\geq c_\infty^{\varepsilon-} |U|^2, \\ \boldsymbol{\mathcal{C}}_{\mu_r}(x) U \cdot U &\geq c_\infty^{\mu-} |U|^2, \end{aligned} \tag{2.9}$$

   with positive constants $c_\infty^{\varepsilon-}$ and $c_\infty^{\mu-}$. Here we used the notation for a complex valued 3×3-tensor B

$$\boldsymbol{\mathcal{C}}_B := B - \mathbb{I},$$

   where $\mathbb{I}$ is the identity matrix of $\mathbb{R}^3 \times \mathbb{R}^3$.

[1] That means we will use the balls $B(\boldsymbol{z}_m, \boldsymbol{a}/2)$ to locate D_m, being $[d(D_m, D_j)]^{-q} \equiv [\boldsymbol{\delta}_{ij}]^{-q} = [d(B(\boldsymbol{z}_m, \boldsymbol{a}/2), B(\boldsymbol{z}_j, \boldsymbol{a}/2))]^{-q}$ for any integer q.

IV. *Assumption linking the background, the cluster and the contrasts.* Finally, we need the following assumption that links the distribution of the cluster, through $\boldsymbol{\delta}$ and $\boldsymbol{a}$, the lower and upper bounds of the contrasts and the used wave number k

$$\boldsymbol{c}_r := \frac{\boldsymbol{\delta}}{\boldsymbol{a}} \geq \frac{c_0 2|k|\boldsymbol{c}_\infty^2}{\max\left(c_\infty^{\mu-}, c_\infty^{\varepsilon-}\right)}, \tag{2.10}$$

where c_0 is a positive constant which is independent on the parameters of the model.

The above conditions are quite general and appear naturally to insure a proper estimate for the electromagnetic field when trying to solve the continuous model. These conditions, in particular (2.10), can be relaxed in various ways (see Remark 4.5). A sufficient condition to get the first inequality in (2.9) for the

contrast of the relative electric permittivity is given by

$$\rho^-(\Re \boldsymbol{C}_{\varepsilon_r}) - \rho^+(\Im \boldsymbol{C}_{\varepsilon_r}) > 0,$$

where $\rho^+(A)$ and $\rho^-(A)$ stand respectively, for the largest and the smallest eigenvalue of A. Indeed, we have

$$\begin{aligned}\Re\langle \overline{U}, \boldsymbol{C}_{\varepsilon_r} U\rangle = \langle \Im U, \Re \boldsymbol{C}_{\varepsilon_r} \Im U\rangle + \langle \Re U, \Re \boldsymbol{C}_{\varepsilon_r} \Re U\rangle \\ + \Big(\langle \Re V, \Im \boldsymbol{C}_{\varepsilon_r} \Im U\rangle - \langle \Im U, \Im \boldsymbol{C}_{\varepsilon_r} \Re U\rangle\Big), \\ \geq \rho^-(\Re \boldsymbol{C}_{\varepsilon_r})(\|\Im U\|^2 + \|\Re U\|^2) - 2\rho^+(\Im \boldsymbol{C}_{\varepsilon_r})(\|\Im U\| \|\Re U\|), \\ \geq \Big(\rho^-(\Re \boldsymbol{C}_{\varepsilon_r}) - \rho^+(\Im \boldsymbol{C}_{\varepsilon_r})\Big)\|U\|^2.\end{aligned} \tag{2.11}$$

Under the assumptions (I, II, II, IV) above, the scattering problem $(\mathcal{P})$ is well posed in appropriate spaces (see Proposition 4.2 in Section 4). In addition, we have the following behavior (as spherical-waves) of the scattered electric fields far away from the scatterers D_m's

$$E^{\mathrm{sc}}(x) = \frac{e^{ik|x|}}{|x|}\{E^\infty(\hat{x}) + O(|x|^{-1})\}, \quad |x| \longmapsto \infty, \tag{2.12}$$

and we have a similar behavior for the scattered magnetic field as well

$$H^{\mathrm{sc}}(x) = \frac{e^{ik|x|}}{|x|}\{H^\infty(\hat{x}) + O(|x|^{-1})\}, \quad |x| \longmapsto \infty \tag{2.13}$$

where $(E^\infty(\hat{x}), H^\infty(\hat{x}))$ is the electromagnetic far field pattern in the direction of propagation $\hat{x} := \frac{x}{|x|}$.

We set, for $m \in \{1, ..., \aleph\}$,

$$\boldsymbol{\mathcal{A}}_f^m := \frac{1}{|D_m|}\int_{D_m} f dv, \tag{2.14}$$

and

$$\boldsymbol{\mathcal{A}} f(x) := \sum_{m=1}^{\aleph} \boldsymbol{\mathcal{A}}_f^m \chi_{D_m}(x). \tag{2.15}$$

The notation dv stands for the volume measure in $\mathbb{R}^3$ and $dv(x)$ will be denoted by dx. The surface measure in $\mathbb{R}^3$ will be written as ds and ds_x stands for the variable of integration when specified.

We recall the Green's function for the Helmholtz operator (i.e. the fundamental solution for the Helmholtz equation)

$$\Phi_k(x, y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \quad x \neq y,$$

and the electromagnetic dyadic Green's function

$$\Pi(x,y) := k^2\Phi_k(x,y)\mathbb{I} + \nabla_x\nabla_x\Phi_k(x,y) = k^2\Phi_k(x,y)\mathbb{I} - \nabla_x\nabla_y\Phi_k(x,y), \quad x \neq y. \tag{2.16}$$

Finally, for a given matrix function B, we set

$$[\mathcal{P}^B_{D_m}] := \int_{D_m} \nabla V^B_m \ (B - \mathbb{I})dv, \tag{2.17}$$

with V^B_m standing for the solution the following integral equation

$$V^B_m - \operatorname{div}\int_{D_m} \frac{1}{|x-y|}(B-\mathbb{I})\nabla V^B_m(y)dy = (y - \boldsymbol{z}_m), \quad y \in D_m. \tag{2.18}$$

In the whole text B^T stands for the transpose of the matrix B.

Our main results are stated in the following theorem.

Theorem 2.1. *Under the assumptions (I, II, II, IV) above, the problem $\mathcal{P}$ has one and only one solution. In addition, the far field of the scattered wave admits the following expansions:*

- *If both ε_r and μ_r are symmetric, then we have*

$$\begin{aligned} E^\infty(\hat{x}) = \sum_{m=1}^{\aleph} \Bigg(& \frac{k^2}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \Big(\mathcal{R}^{\varepsilon_r}_m \times \hat{x}\Big) + \frac{ik}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \mathcal{Q}^{\mu_r}_m \Bigg) \\ & + O\Big(\frac{|k|(2|k|+1)}{\boldsymbol{c}_r^3}\Big[\frac{1}{\boldsymbol{c}_r^4} + \boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})| + \boldsymbol{a}\Big]\Big), \end{aligned} \tag{2.19}$$

where $(\mathcal{R}^{\varepsilon_r}_m, \mathcal{Q}^{\mu_r}_m)_{m=1}^{\aleph}$ is the solution of the following invertible linear system

$$\begin{aligned} [\mathcal{P}^{\mu_r}_{D_m}]^{-1}\mathcal{Q}^{\mu_r}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{Q}^{\mu_r}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{R}^{\varepsilon_r}_j\Big] + H^{in}(\boldsymbol{z}_m) \\ & \qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad m = 1,...,\aleph. \\ [\mathcal{P}^{\varepsilon_r}_{D_m}]^{-1}\mathcal{R}^{\varepsilon_r}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{R}^{\varepsilon_r}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{Q}^{\mu_r}_j\Big] + E^{in}(\boldsymbol{z}_m). \end{aligned} \tag{2.20}$$

- *If on the contrary, either ε_r or μ_r is not symmetric then, with $(\boldsymbol{\mathcal{A}}^m_{\varepsilon_r})_{m=1}^{\aleph}$ and $(\boldsymbol{\mathcal{A}}^m_{\mu_r})_{m=1}^{\aleph}$ standing for their respective average over each inclusion D_m, we have*

$$\begin{aligned} E^\infty(\hat{x}) = \sum_{m=1}^{\aleph} \Bigg(& \frac{k^2}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \Big(\mathcal{R}_m \times \hat{x}\Big) + \frac{ik}{4\pi} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \hat{x} \times \mathcal{Q}_m \Bigg) \\ & + O\Big(\frac{|k|(2|k|+1)}{\boldsymbol{c}_r^3}\Big[\frac{1}{\boldsymbol{c}_r^4} + \boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})| + \boldsymbol{a}\Big]\Big), \end{aligned} \tag{2.21}$$

where, in this case, $\Big(\mathcal{R}_m, \mathcal{Q}_m\Big)_{m=1}^{\aleph}$ is the unique solution of the following linear system

$$\begin{aligned} [\mathcal{T}^{\boldsymbol{\mathcal{A}}^{m\,T}_{\mu_r}}_{D_m}]^{-1}\mathcal{Q}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{Q}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{R}_j\Big] + H^{in}(\boldsymbol{z}_m), \\ & \qquad\qquad\qquad\qquad\qquad\qquad \textit{for} \quad m = 1,...,\aleph. \\ [\mathcal{T}^{\boldsymbol{\mathcal{A}}^{m\,T}_{\varepsilon_r}}_{D_m}]^{-1}\mathcal{R}_m &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{R}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{Q}_j\Big] + E^{in}(\boldsymbol{z}_m), \end{aligned} \tag{2.22}$$

and the tensors above are given by

$$[\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}]^{-1} := \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r^T}} [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}]^{-1} (\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}})^{-1},$$

and

$$[\mathcal{T}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^T}}]^{-1} := \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r^T}} [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^T}}]^{-1} (\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}})^{-1}.$$

Before providing the proof of of the above result, we would like to address some remarks.

Remark 2.2. *1. In the error of approximation, the constant appearing in the Landau notation are bounded by the largest ratio of the eigenvalues of both ε_r and μ_r.*

2. In our opinion, and in the current form of the algebraic systems, it is hard to improve the approximation error order, except maybe for rotation invariant geometries by using the fundamental Newton's theorem for fields that are also rotation invariant.

3. The tensor that appears could be explicitly calculated for simple geometries (sphere, ellipsoid), for more details see [4].

4. It is also possible to evaluate $[\mathcal{P}_{D_m}^{\mu_r}]$ and $[\mathcal{P}_{D_m}^{\varepsilon_r}]$, using boundary integral equation when both ε_r^T and μ_r^T are symmetric definite positive matrices, see section 3 here after.

The rest of the paper is organized as follows. In Section 3, we introduce the polarization tensors and summarize the needed properties to establish the main result. In Section 4, we discuss the well posedness and solve the continuous model using an integral equation approach based on the well known Lippmann-Schwinger representation. Precisely, we show that the system of integral equations

$$\begin{pmatrix} E \\ H \end{pmatrix} - \begin{pmatrix} (k^2 + \nabla \operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}} & +ik \operatorname{curl} \mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\mu_r}} \\ -ik \operatorname{curl} \mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}} & (k^2 + \nabla \operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\mu_r}} \end{pmatrix} \begin{pmatrix} E(x) \\ H(x) \end{pmatrix} = \begin{pmatrix} E^{\text{in}} \\ H^{\text{in}} \end{pmatrix},$$

is invertible. Here, we have

$$\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_B}(V) := \sum_{m=1}^{\aleph} \mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_B}(V) := \sum_{m=1}^{\aleph} \int_{D_m} \Phi_k(x,y)\boldsymbol{\mathcal{C}}_B V(y)dy, \tag{2.23}$$

where, for $B := \varepsilon_r$ (or $B := \mu_r$) and $V := E$ (or $V := H$). Hence, we have the estimate $\|(E,H)\|_{\mathbb{L}^2(D)\times\mathbb{L}^2(D)} \leq C\|(E^{\text{in}}, H^{\text{in}})\|_{\mathbb{L}^2(D)\times\mathbb{L}^2(D)}$.

Section 5 is devoted to the derivation of the approximations and the associated linear algebraic systems. In Proposition 5.1, with a first order expansion, we give the approximation

$$E^\infty(\hat{x}) = \frac{k^2}{4\pi}\hat{x} \times \sum_{m=1}^{\aleph} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big[\int_{D_m} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E dv \ \times \hat{x} + \int_{D_m} \boldsymbol{\mathcal{C}}_{\mu_r} H dv\Big] + Error, \tag{2.24}$$

for the far field. Then, using the Maxwell equations in the interior of $\cup_{m=1}^{\aleph} D_m$ with an appropriate choice of the testing fields (U_m^1, U_m^2), we estimate the unknowns $\int_{D_m} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E dv$ and $\int_{D_m} \boldsymbol{\mathcal{C}}_{\mu_r} H dv$ as follows

$$\int_{D_m} \boldsymbol{\mathcal{C}}_{\mu_r} H dv = \int_{D_m} U_m^1\Big(H - (k^2 + \nabla \operatorname{div})\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H) + ik \operatorname{curl} \mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E)\Big)dv$$

$$= \Big[\int_{D_m} U_m^1 dv\Big] \sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \Bigg(\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j)\Big[\int_{D_j} \boldsymbol{\mathcal{C}}_{\mu_r} H dv\Big] - ik\nabla\Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \Big[\int_{D_j} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E dv\Big]\Bigg) + H^{\text{in}}(\boldsymbol{z}_m) + Error_m,$$

and

$$\int_{D_m} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E dv = \int_{D_m} U_m^2 \Big(E - (k^2 + \nabla \operatorname{div}) \mathcal{S}_{D_m}^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E) - ik \operatorname{curl} \mathcal{S}_{D_m}^{k, \boldsymbol{\mathcal{C}}_{\mu_r}}(H) \Big) dv$$

$$= \Big[\int_{D_m} U_m^2 dv \Big] \sum_{\substack{j \geq 1 \\ j \neq m}}^{\aleph} \left(\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \Big[\int_{D_j} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E dv \Big] + ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \Big[\int_{D_j} \boldsymbol{\mathcal{C}}_{\mu_r} H dv \Big] \right) + E^{\mathrm{in}}(\boldsymbol{z}_m) + Error_m.$$

In Proposition 5.2, we show that the above linear system, without the error and (Q_m, R_m) in place of $(\int_{D_m} \boldsymbol{\mathcal{C}}_{\mu_r} H dv, \int_{D_m} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E dv)$, admits a unique solution and satisfies a certain estimate which provides us with the following estimations

$$\Big(\sum_{m=1}^{\aleph} \Big| R_m - \int_{D_m} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E dv \Big|^2 \Big)^{\frac{1}{2}} \leq (constant > 0) \Big(\sum_{m=1}^{\aleph} |Error_m|^2 \Big)^{\frac{1}{2}}$$

and

$$\Big(\sum_{m=1}^{\aleph} \Big| Q_m - \int_{D_m} \boldsymbol{\mathcal{C}}_{\mu_r} H dv \Big|^2 \Big)^{\frac{1}{2}} \leq (constant > 0) \Big(\sum_{m=1}^{\aleph} |Error_m|^2 \Big)^{\frac{1}{2}}.$$

To end the proof, we add and subtract the terms (Q_m, R_m) in the far field approximation (2.24) and use the last estimations above.

A short appendix is added at the end of the manuscript to include some needed technical tools. In particular, we derive a useful counting lemma which provides us with estimates for the approximations of the interaction potentials solely in terms of $\boldsymbol{\delta}$ and $\boldsymbol{a}$.

3 Anisotropic polarization tensor

The polarization tensors appear naturally in the asymptotic expansion of the scattered fields (see Chapter 4 [4]). They can be described as quantities that carry information about the geometry of the scatterer and the material parameters. Technically speaking, in the present context where we have anisotropic and variable materials, it is a normalized averaged response of a single inhomogeneity in the presence of an electromagnetic field. In this subsection, we describe some of their mathematical properties which are important for the sequel.

Before doing so, we need to introduce some notations. Let us recall the vector Newtonian operator

$$\mathcal{S}_D^k(V) := \int_D \Phi_k(x, y) V(y) dy, \tag{3.1}$$

which is defined, for V in $\mathbb{L}^2(D)$ and maps continuously $\mathbb{L}^2(D)$ into $\mathbb{H}^2(D)$, precisely

$$\big\| \mathcal{S}_D^k(V) \big\|_{\mathbb{H}^2(D)} \leq \boldsymbol{c}_{2,k} \|V\|_{\mathbb{L}^2(D)}, \tag{3.2}$$

(see Theorem 9.11 [17]). The constant $c_{2,k}$ remains independent of the diameter of D. To show it, it suffices to write, for a ball $B(z, R)$ with $z \in D$ and a sufficiently large radius R to have $D \subset B(z, R)$,

$$\mathcal{S}_D^k(V) = \mathcal{S}_{B(z,R)}^k(V \chi_D)$$

which provides us the following estimate

$$\|\mathcal{S}_D^k(V)\|_{\mathbb{H}^2(D)} \leq \|\mathcal{S}_{B(z,R)}^k(V \chi_D)\|_{\mathbb{H}^2(B(z,R))} \leq \boldsymbol{c}_{2,k} \|V \chi_D\|_{\mathbb{L}^2(B(z,R))}. \tag{3.3}$$

We will also write, for an essentially bounded tensor B,

$$\mathcal{S}_D^{k,B}(V) := \mathcal{S}_D^k(BV), \tag{3.4}$$

for V in $\mathbb{L}^2(D)$.
For a given complex valued tensor B with a coercive contrast, that is

$$\Re(\boldsymbol{\mathcal{C}}_B(x)U\cdot\overline{U}) \geq c^{B-}|U|^2, \text{ with } c^{B-}>0, \tag{3.5}$$

we set

$$[\mathcal{P}^B_{D_m}] := \int_{D_m} \nabla V^B_m \, \boldsymbol{\mathcal{C}}_B \, dv = \int_{D_m} \nabla V^B_m \, (B-\mathbb{I}) \, dv, \tag{3.6}$$

where $V^B_m := V^{\mathrm{sc},B}_m + (x-\boldsymbol{z}_m)$ stands for the solution of the following problem

$$\begin{cases} \Delta V^B_m = 0 \text{ in } \mathbb{R}^3 \setminus D_m, \\ \operatorname{div}(B\nabla V^B_m) = 0 \text{ in } D_m, \\ V^B_m = V^{\mathrm{sc},B}_m + (x-z_i), \\ V^B_m|_- - V^B_m|_+ = 0 \text{ on } \partial D_m, \\ \nu\cdot B\nabla V^B_m|_- - \nu\cdot\nabla V^B_m|_+ = 0, \\ V^{\mathrm{sc}}_m \to 0, |x|\to\infty. \end{cases} \tag{$\mathcal{P}r^{Ani}(1)$}$$

The problem $(\mathcal{P}r^{Ani}(1))$ can be solved using the following Lippmann-Schwinger integral equation with $V_l := (V^B_m)_l = V^B_m \cdot e_l$, for $l=1,2,3$,

$$V_l - \operatorname{div}\mathcal{S}^{0,\boldsymbol{\mathcal{C}}_B}_{D_m}(\nabla V_l) = (x-\boldsymbol{z}_m)\cdot e_l. \tag{3.7}$$

The following proposition summarizes the needed properties.

Proposition 3.1. *Every solution to $(\mathcal{P}r^{Ani}(1))$ satisfies the following estimate*

$$\|V^B_m\|_{\mathbb{H}^1(D_m)} \leq \left(\frac{\sqrt{\min(1,c^{B-})}}{2} - \boldsymbol{a}^2\|\boldsymbol{\mathcal{C}}_B\|_{\mathbb{L}^\infty(D_m)}\sqrt{\frac{3}{\pi}}\right)^{-1}(\|\boldsymbol{\mathcal{C}}_B\|_{\mathbb{L}^\infty(D_m)}+1)\boldsymbol{a}^{3/2}, \tag{3.8}$$

furthermore, whenever $B \in \mathbb{W}^{1,\infty}(D_m)$ and for sufficiently small $\boldsymbol{a}$, the tensor (3.6) *behaves like $\boldsymbol{\mathcal{C}}_B$ in terms of positive or negative definiteness and symmetry, namely*

$$([\mathcal{P}^B_{D_m}]U,\overline{U}) > 0, \text{ whenever } (\boldsymbol{\mathcal{C}}_B U,\overline{U}) > 0, \tag{3.9}$$

and

$$\Re([\mathcal{P}^B_{D_m}]U,\overline{U}) > 0, \text{whenever } \Re(\boldsymbol{\mathcal{C}}_B U,\overline{U}) > 0. \tag{3.10}$$

Also, with $\widehat{B}(s) := B(\boldsymbol{a}s+z_m)$ for $s\in\boldsymbol{\mathcal{D}}_m$, the following scaling property holds true

$$[\mathcal{P}^B_{D_m}] = \boldsymbol{a}^3[\mathcal{P}^{\widehat{B}}_{\boldsymbol{\mathcal{D}}_m}]. \tag{3.11}$$

Before stating the proof, we need to introduce the anisotropic polarization tensor, defined in ([4], p.121-122), as

$$\left([\boldsymbol{\mathcal{P}}^B_{\partial D_m}]\right)_{ij} = \int_{\partial D_m} \nu\cdot(\boldsymbol{\mathcal{C}}_B e_j)(\Theta_m)_i|_- ds,$$

where Θ_m solves, for a fixed $m\in\{1,...,\aleph\}$, the following transmission problem

$$\begin{cases} \Delta\Theta = 0 \text{ in } \mathbb{R}^3\setminus D_m, \\ \operatorname{div}(B\nabla\Theta) = 0 \text{ in } D_m, \\ \Theta|_- - \Theta|_+ = (x-\boldsymbol{z}_m) \text{ on } \partial D_m, \\ \nu\cdot B\nabla\Theta|_- - \nu\cdot\nabla\Theta|_+ = \nu\cdot\nabla(x-\boldsymbol{z}_m), \\ \Theta\to 0, |x|\to\infty. \end{cases} \tag{$\mathcal{P}r^{Ani}$}$$

Obviously, we have

$$\Theta_m = \begin{cases} V_m^{\mathrm{sc},B} \text{ in } \mathbb{R}^3 \setminus D_m, \\ V_m^{\mathrm{sc},B} + (x - \boldsymbol{z}_m) \text{ in } D_m. \end{cases} \tag{3.12}$$

Hence, as[2]

$$\begin{aligned} \Big([\boldsymbol{\mathcal{P}}_{\partial D_m}^B]\Big)_{ij} &= \int_{\partial D_m} \nu \cdot (\boldsymbol{\mathcal{C}}_B e_j)(\Theta_m)_i|_- ds = \int_{D_m} \mathrm{div}\Big((\boldsymbol{\mathcal{C}}_B e_j)\ (\Theta_m \cdot e_i)\Big) dv, \\ &= \int_{D_m} ((\nabla \Theta_m)^T e_i) \cdot (\boldsymbol{\mathcal{C}}_B e_j) dv + \int_{D_m} (\Theta_m \cdot e_i) \,\mathrm{div}(\boldsymbol{\mathcal{C}}_B e_j) dv. \end{aligned}$$

we get the following identity

$$[\boldsymbol{\mathcal{P}}_{\partial D_m}^B] = [\mathcal{P}_{D_m}^B] + \int_{D_m} \Theta_m \otimes \mathrm{div}(\boldsymbol{\mathcal{C}}_B^T) dv, \tag{3.13}$$

or

$$[\mathcal{P}_{D_m}^B] = [\boldsymbol{\mathcal{P}}_{\partial D_m}^B] - \int_{D_m} \Theta_m \otimes \mathrm{div}(\boldsymbol{\mathcal{C}}_B^T) dv. \tag{3.14}$$

Here, the divergence is applied to each line of the tensor considered as a vector field.

With the above notations, we have the following result (see Theorem 3.4 of [19] for details).

Lemma 3.2. *The polarization tensor $[\boldsymbol{\mathcal{P}}_{\partial D_m}^B]$ behaves like $\boldsymbol{\mathcal{C}}_B$ in term of positive or negative definiteness and symmetry.*

Proof. (of Proposition 3.1) The operator $[I - \mathrm{div}\, \mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} \nabla]$ is one-to-one on $\mathbb{H}^1(D_m)$, provided that $\boldsymbol{\mathcal{C}}_B$ is definite-positive, and it satisfies the following estimates (see Lemma 1 [20].)

$$\begin{aligned} \|[I - \mathrm{div}\, \mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} \nabla](V)\|_{\mathbb{H}^{1,\boldsymbol{\mathcal{C}}_B}(D_m)} &:= \int_{D_m} [I - \mathrm{div}\, \mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} \nabla](V) \cdot \overline{V} dx \\ &+ \int_{D_m} \nabla [I - \mathrm{div}\, \mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} \nabla](V) \cdot \overline{\boldsymbol{\mathcal{C}}_B \nabla V} dx \geq \frac{\|V\|_{\mathbb{H}^{1,\boldsymbol{\mathcal{C}}_B}(D_m)}}{2}. \end{aligned} \tag{3.15}$$

We have

$$\|[\mathrm{div}(\mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} - \mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_B}) \nabla V_l]\|_{\mathbb{H}^1(D_m)} \leq \frac{3}{\sqrt{\pi}} \boldsymbol{a}^2 \|V_l\|_{\mathbb{H}^1(D_m)}, \tag{3.16}$$

since, due to (A.3) and (A.4), we have the following estimates

$$\begin{cases} |[\mathrm{div}(\mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} - \mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_B}) \nabla](V)(x)| \leq \dfrac{1}{4\pi} \displaystyle\int_{D_m} |\nabla V_l(y)|\, dy \leq \dfrac{1}{\sqrt{3}} \dfrac{\boldsymbol{a}^{\frac{3}{2}}}{4} \|V_l\|_{\mathbb{H}^1(D_m)} \\ |\nabla [\mathrm{div}(\mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} - \mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_B}) \nabla](V)(x)| \leq \displaystyle\int_{D_m} \dfrac{1}{4\pi |x-y|} \Big[1 + \big(2 + \dfrac{1}{|x-y|}\big)|x-y|\Big] |\nabla V_l(y)|\, dy, \\ \qquad \leq \displaystyle\int_{D_m} \dfrac{1}{2\pi} \Big[1 + \dfrac{1}{|x-y|}\Big] |\nabla V_l(y)|\, dy \leq \dfrac{2}{\sqrt{\pi}} \sqrt{\boldsymbol{a}} \|V_l\|_{\mathbb{H}^1(D_m)}. \end{cases} \tag{3.17}$$

Obviously, we have

$$\int_{D_m} \frac{1}{|x-y|^2}\, dy \leq \lim_{r \to 0} \int_{B(y,\boldsymbol{a}) \setminus B(y,r)} \frac{1}{|x-y|^2}\, dy \leq \lim_{r \to 0} \int_0^{2\pi} \int_0^{\pi} \int_r^{\boldsymbol{a}} \frac{1}{R^2} R^2 dR \sin(\theta) d\theta\, d\phi \leq 2\pi \boldsymbol{a}. \tag{3.18}$$

We write

$$(x - z_i) = [I - \mathrm{div}\, \mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_B} \nabla](V) = [I - \mathrm{div}\, \mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} \nabla](V) + [\mathrm{div}(\mathcal{S}_{D_m}^{i,\boldsymbol{\mathcal{C}}_B} - \mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_B}) \nabla](V), \tag{3.19}$$

[2] Recall that, for a given matrix B we have $Be_j \cdot e_i = (B)_{ij}$

then from (3.15) and (3.16), we get

$$\|(x-z_i)\|_{\mathbb{H}^{1,\mathcal{C}_B}(D_m)} \geq \|[I - \operatorname{div}\mathcal{S}^{i,\mathcal{C}_B}_{D_m}\nabla](V)\|_{\mathbb{H}^{1,\mathcal{C}_B}(D_m)} - \|[\operatorname{div}(\mathcal{S}^{i,\mathcal{C}_B}_{D_m} - \mathcal{S}^{0,\mathcal{C}_B}_{D_m})\nabla](V)\|_{\mathbb{H}^{1,\mathcal{C}_B}(D_m)},$$
$$\geq \frac{\|V\|_{\mathbb{H}^{1,\mathcal{C}_B}(D_m)}}{2} - \frac{\sqrt{3}\boldsymbol{a}^2}{\sqrt{\pi}}\|\mathcal{C}_B\|_{\mathbb{L}^\infty(D_m)}\|V\|_{\mathbb{H}^1(D_m)}.$$

The remaining part of the proof is due to the fact that, for $\boldsymbol{a}$ sufficiently small, $[\mathcal{P}^B_{D_m}]$ inherits its property from $[\boldsymbol{\mathcal{P}}^B_{\partial D_m}]$ through (3.14) and Lemma 3.2.

□

Finally, we have the following (obvious) statements.

Proposition 3.3. *For $U^{\mathcal{C}_B}_m := \mathcal{C}_B\nabla V^B_m$, we have*

$$\int_{D_m} U^{\mathcal{C}_B}_m dv = \int_{D_m} \mathcal{C}_B\nabla V^B_m dv = [\mathcal{P}^B_{D_m}], \tag{3.20}$$

for $\mathcal{C}_B$ symmetric. If B is a constant matrix and not necessarily symmetric, we have

$$\int_{D_m} U^{\mathcal{C}_B}_m dv = \mathcal{C}_B[\mathcal{P}^B_{D_m}]\mathcal{C}_B^{-1}. \tag{3.21}$$

4 Well-posedness and a priori estimates

In this section, we address the question of well-posedness of the problem $(\mathcal{P})$. In the first subsection, we discuss different possible conditions on ε_r and μ_r under which the uniqueness is guarantied. Firstly, we develop the classical approach based on the usual Green's identity for complex valued tensors that satisfy some coercivity conditions. Then we summarize known results based on the unique continuation property for symmetric real valued and definite positive parameters.
The second subsection is devoted to solving the problem, using a Lippmann-Schwinger integral representation, which answers the existence issue and provides us with an a priori estimate for the total field under more restrictive conditions (that can be relaxed in various ways, see Remark 4.5).

4.1 Uniqueness

The following lemma (see Corollary 1.2 of both [26] and [15]) on the unique continuation principle for the Maxwell system will be useful in the sequel.

Lemma 4.1. *Let the electromagnetic field (E, H) solves the Maxwell system, in an open and connected set D, with real symmetric definite positive tensors ε_r and μ_r in $\mathbb{W}^{1,\infty}(D)$. Assume that both E and H are in $L^2(D)$ and we have $E \equiv H \equiv 0$ in $B(x_0, r) \subset D$, for some $r_0 > 0$ and $x_0 \in D$. Then $E \equiv H \equiv 0$ in D.*

Proposition 4.2. *For $k > 0$ the problem $(\mathcal{P})$ admits a unique solution under one of the following conditions*

1. *If μ_r, ε_r are in $\mathbb{L}^\infty(D)$ and satisfy one of the assumptions*
   - *(a) μ_r (or $-\mu_r$) and $\Im\varepsilon_r$ are real definite positive with an arbitrary $\Re\varepsilon_r$.*
   - *(b) ε_r and μ_r are differently signed with $\Im\varepsilon_r = 0$ (i.e. μ_r is real strictly positive whenever ε_r is real strictly negative and vice versa).*
2. *If ε_r and μ_r are in $\mathbb{W}^{1,\infty}(D)$ and both are definite positive real symmetric matrix.*

Proof. Let $(E = E_1 - E_2,\ H = H_1 - H_2)$ be the difference of two solutions of the problem $(\mathcal{P})$. Obviously both the normal trace of E and H are continuous across the boundary and satisfy the Silver–Müller radiation condition, hence, due to Green's formula inside D for the interior trace, we have

$$\int_{\partial D} \nu \times E|_\pm \cdot \overline{H}|_\pm ds = ik\int_D \mu_r H\cdot\overline{H}dv + \overline{ik}\int_D E\cdot\varepsilon_r\overline{E}dv = ik\Big(\int_D \mu_r H\cdot\overline{H}dv - \int_D E\cdot\overline{\varepsilon_r E}dv\Big). \tag{4.1}$$

Taking the imaginary part and then the real part, we obtain

$$\Im\int_{\partial D}\nu\times E|_+\cdot\overline{H}|_+ds = k\Bigg[\int_D\Big(\mu_r\Re H\cdot\Re H+\mu_r\Im H\cdot\Im H\Big)dv+\int_D(\Im(\varepsilon_r-\varepsilon_r^T))\Im E\cdot\Re E dv \\ -\int_D\big(\Re\varepsilon_r\Re E\cdot\Re E+\Re\varepsilon_r\Im E\cdot\Im E\big)dv\Bigg], \tag{4.2}$$

and

$$\Re\int_{\partial D}\nu\times E|_+\cdot\overline{H}|_+ds = k\Bigg[\int_D(\mu_r-\mu_r^T)\Re H\cdot\Im H dv+\int_D(\Re(\varepsilon_r-\varepsilon_r^T))\Re E\cdot\Im E dv \\ -\int_D\big(\Im\varepsilon_r\Re E\cdot\Re E+\Im\varepsilon_r\Im E\cdot\Im E\big)dv\Bigg]. \tag{4.3}$$

We write, with $(\varepsilon_r)_{ij}$ standing for the components of ε_r,

$$(\varepsilon_r-\varepsilon_r^T)V = \overbrace{\begin{pmatrix}(\varepsilon_r)_{32}-(\varepsilon_r)_{23}\\(\varepsilon_r)_{13}-(\varepsilon_r)_{31}\\(\varepsilon_r)_{21}-(\varepsilon_r)_{12}\end{pmatrix}}^{U(\varepsilon_r):=}\times V.$$

As $\Im(H)\times\Re(H)=\Im(E)\times\Re(E)=0$,[3] we get for the integrand of the first integral of the right-hand side of (4.3)

$$(\mu_r-\mu_r^T)\Im H\cdot\Re H = (U(\varepsilon_r)\times\Im(H))\cdot\Re H = 0.$$

A similar observation concerning the second integral of the right-hand side of (4.3) implies that

$$\Re\int_{\partial D}\nu\times E|_+\cdot\overline{H}|_+ds = -k\int_D\big(\Im\varepsilon_r\Re E\cdot\Re E+\Im\varepsilon_r\Im E\cdot\Im E\big)dv. \tag{4.4}$$

As for all the cases stated in Proposition 4.2, $\Im\varepsilon_r$ is non-negative, we get $\Re\int_{\partial D}\nu\times E|_+\cdot\overline{H}|_+ds\le 0$. The Rellich lemma (see Theorem 6.11. [13]) induces that

$$E\equiv H\equiv 0 \text{ in } \mathbb{R}^3\setminus D. \tag{4.5}$$

The next step is to show that $E\equiv H\equiv 0$ in D. For this, we proceed quite differently according to the three case stated in Proposition 4.2.

1. We suppose that ε_r and μ_r are as in the first statement 1.(a). Again from (4.4) we get[4]

$$Cst_{\Im\varepsilon_r}\|E\|_{\mathbb{L}^2(D)^2}\le\int_D\big(\Im\varepsilon_r\Re E\cdot\Re E+\Im\varepsilon_r\Im E\cdot\Im E\big)dv = 0.$$

which, with the Maxwell equations,[5] guaranties that $E\equiv H\equiv 0$ in D.

2. For the case 1.(b), we combine (4.1) and (4.5) to obtain

$$ik\Big(\int_D\mu_r H\cdot\overline{H}dv-\int_D E\cdot\overline{\varepsilon_r E}dv\Big) = \int_{\partial D}\nu\times E|_-\cdot\overline{H}|_-ds = \int_{\partial D}\nu\times E|_+\cdot\overline{H}|_+ds = 0$$

and taking the imaginary parts, we have

$$0=\Im\int_{\partial D}\nu\times E|_+\cdot\overline{H}|_+ds = k\Bigg[\int_D\Big(\mu_r\Re H\cdot\Re H+\mu_r\Im H\cdot\Im H\Big)dv-\int_D\big(\Re\varepsilon_r\Re E\cdot\Re E+\Re\varepsilon_r\Im E\cdot\Im E\big)dv\Bigg].$$

[3] Due to the fact that $H\times H=E\times E=0$.

[4] The constant $Cst_{\Im\varepsilon_r}>0$ is the one that guaranties the definite positiveness.

[5] As $H=(\mu_r)^{-1}\operatorname{curl}E/ik=0$ in D.

As $k>0$ and ε_r and μ_r are real valued and differently signed, then we deduce that $E\equiv H\equiv 0$ in D.

3. Let us consider the case (2). As $D_m \subset B(\boldsymbol{z}_m, \frac{\boldsymbol{a}}{2}) \subsetneq B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})$, we first extend both $\varepsilon_r/_{D_m}$ and $\mu_r/_{D_m}$[6] from elements of $\mathbb{W}^{1,\infty}(D_m)$ to $\tilde{\varepsilon_r}/_{B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})}$, $\tilde{\mu_r}/_{B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})}$ in $\mathbb{W}^{1,\infty}(B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2}))$.
Let $\tilde{E}$ and $\tilde{H}$ be defined by

$$\tilde{E} := \begin{cases} E, \text{ in } D = \cup_{m=1}^{\aleph} D_m \\ 0, \text{ in } \cup_{m=1}^{\aleph} B(\boldsymbol{z}_m, \dfrac{\boldsymbol{a}+\boldsymbol{\delta}}{2}) \setminus D_m \end{cases} \tag{4.6}$$

and

$$\tilde{H} := \begin{cases} H, \text{ in } D = \cup_{m=1}^{\aleph} D_m \\ 0, \text{ in } \cup_{m=1}^{\aleph} B(\boldsymbol{z}_m, \dfrac{\boldsymbol{a}+\boldsymbol{\delta}}{2}) \setminus D_m. \end{cases} \tag{4.7}$$

With $(E,\, H)$ as in the beginning of the proof, we claim that $(\tilde{E}, \tilde{H})$ solves the anisotropic Maxwell system with parameters $\tilde{\mu_r}, \tilde{\varepsilon_r}$ in $\cup_{m=1}^{m} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})$ in the sense of distribution.
Indeed, let $\phi \in \mathcal{C}_0^{\infty}(\cup_{m=1}^{\aleph} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2}))$. By the Schwarz duality pairing, we have

$$< \operatorname{curl} \tilde{E}, \phi >=< \tilde{E}, \operatorname{curl} \phi >= \int_{\cup_{m=1}^{m} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})} \tilde{E} \cdot \operatorname{curl} \phi \, dv, \qquad \text{as } \tilde{E} \in \mathbb{L}^2(\cup_{m=1}^{\aleph} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2}))$$

and by (4.6)

$$< \operatorname{curl} \tilde{E}, \phi >= \int_{\cup_{m=1}^{m} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})} \tilde{E} \cdot \operatorname{curl} \phi \, dv = \int_{D=\cup_{m=1}^{\aleph} D_m} E \cdot \operatorname{curl} \phi \, dv. \tag{4.8}$$

As we have (4.5), then

$$0 = \int_{\partial D} \nu \times E \cdot \phi \, ds = \int_D \operatorname{curl} E \cdot \phi \, dv - \int_D E \cdot \operatorname{curl} \phi \, dv. \tag{4.9}$$

From (4.8), (4.9) and (4.7), we get

$$< \operatorname{curl} \tilde{E}, \phi >= \int_D \operatorname{curl} E \cdot \phi \, dv = \int_D ik\mu_r H \cdot \phi \, dv = \int_{\cup_{m=1}^{m} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})} ik\tilde{\mu}_r \tilde{H} \cdot \phi \, dv, \tag{4.10}$$

which means that $\operatorname{curl} \tilde{E} = ik\tilde{\mu}_r\tilde{H}$ in $\cup_{m=1}^{m} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2})$ in the distribution sense. With a similar argument, we get $\operatorname{curl} H = -ik\tilde{\varepsilon}_r\tilde{E}$ in the distribution sense as well. As by construction E and H are in $\mathbb{L}^2(\cup_{m=1}^{m} B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2}))$ and $\tilde{H} \equiv \tilde{E} \equiv 0$ in each $B(\boldsymbol{z}_m, \frac{\boldsymbol{a}+\boldsymbol{\delta}}{2}) \setminus D_m, m = 1, ..., \aleph$, and each of these sets contains an open ball, we get, by Lemma 4.1, that $\tilde{E} \equiv \tilde{H} \equiv 0$ in each D_m, $m = 1, ..., \aleph$. This ends the proof. □

4.2 Lippmann-Schwinger integral formulation and a priori estimates.

We recall that the contrast of magnetic permeability and electric permittivity, which are assumed to be respectively real and complex valued 3×3-tensor, are, with their derivatives, essentially uniformly bounded and coercive (i.e. both conditions (2.8) and (2.9) are satisfied). Le us introduce the Lippmann-Schwinger integral operator $\mathcal{LS}^{(\varepsilon_r,\mu_r)}$ defined by

$$\mathcal{LS}^{(\varepsilon_r,\mu_r)} := \begin{pmatrix} I - (k^2 + \nabla \operatorname{div}) \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}} & -ik \operatorname{curl} \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\mu_r}} \\ +ik \operatorname{curl} \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\varepsilon_r}} & I - (k^2 + \nabla \operatorname{div}) \mathcal{S}_D^{k, \boldsymbol{\mathcal{C}}_{\mu_r}} \end{pmatrix}. \tag{4.11}$$

In the next proposition, we study the invertibility of the following system of integral equations

[6]By $(B/_{D_m})_{\{B=\varepsilon_r,\mu_r\}}$ we mean the restriction of B to D_m.

$$\mathcal{LS}^{(\varepsilon_r,\mu_r)}\begin{pmatrix}E\\H\end{pmatrix}=\begin{pmatrix}E^{\text{in}}\\H^{\text{in}}\end{pmatrix}\tag{$\mathcal{M}_{\mathcal{L}.\mathcal{S}}$}$$

and compare its solution to the one of the following system of integral equations

$$\mathcal{LS}^{(\boldsymbol{\mathcal{A}}\varepsilon_r,\boldsymbol{\mathcal{A}}\mu_r)}\begin{pmatrix}E_{\mathcal{A}}\\H_{\mathcal{A}}\end{pmatrix}=\begin{pmatrix}E^{\text{in}}\\H^{\text{in}}\end{pmatrix}\tag{$\mathcal{A}-\mathcal{M}_{\mathcal{L}.\mathcal{S}}$}$$

where $\boldsymbol{\mathcal{A}}\varepsilon_r$ and $\boldsymbol{\mathcal{A}}\mu_r$ stand for the averages of the parameters ε_r and μ_r respectively, i.e. (2.14).

Proposition 4.3. *Under the conditions* (2.9) *and* (2.10)*, the operator* $\mathcal{LS}^{(\varepsilon_r,\mu_r)}$ *is an isomorphism of* $\mathbb{H}(\text{curl},\cup_{m=1}^{\aleph}D_m)$ *and the solution of the problem* ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$) *satisfies following estimates*

$$\begin{aligned}\|E\|_{\mathbb{L}^2(\cup_{m=1}^{\aleph}D_m))}&\leq\frac{5\boldsymbol{c}_\infty}{4c_\infty^{\varepsilon-}}\|(E^{in},H^{in})\|_{\mathbb{L}^2(\cup_{m=1}^{\aleph}D_m)\times\mathbb{L}^2(\cup_{m=1}^{\aleph}D_m)},\\\|H\|_{\mathbb{L}^2(\cup_{m=1}^{\aleph}D_m))}&\leq\frac{5\boldsymbol{c}_\infty}{4c_\infty^{\mu-}}\|(E^{in},H^{in})\|_{\mathbb{L}^2(\cup_{m=1}^{\aleph}D_m)\times\mathbb{L}^2(\cup_{m=1}^{\aleph}D_m)}.\end{aligned}\tag{4.12}$$

Besides, for $\boldsymbol{\mathcal{C}}_{\mu_r}$, $\boldsymbol{\mathcal{C}}_{\varepsilon_r}$ *in* $\mathbb{W}^{1,\infty}(\cup_{i=1^m}D_m)$*, with the condition* (2.8)*, and* $(E_{\mathcal{A}},H_{\mathcal{A}})$ *solution of* ($\mathcal{A}-\mathcal{M}_{\mathcal{L}.\mathcal{S}}$)*, we have*

$$\begin{aligned}\|E_{\mathcal{A}}-E\|_{\mathbb{L}^2(\cup D_m)}&\leq\boldsymbol{c}_{2,k}c_\infty\boldsymbol{a}\Big(\|H\|_{\mathbb{L}^2(\cup D_m)}+\|E\|_{\mathbb{L}^2(\cup D_m)}\Big),\\\|H_{\mathcal{A}}-H\|_{\mathbb{L}^2(\cup D_m)}&\leq\boldsymbol{c}_{2,k}c_\infty\boldsymbol{a}\Big(\|H\|_{\mathbb{L}^2(\cup D_m)}+\|E\|_{\mathbb{L}^2(\cup D_m)}\Big).\end{aligned}\tag{4.13}$$

Before providing the proof of this proposition, let us mention that any solution of the equation ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$) solves the problem ($\mathcal{P}$). Hence combining Proposition 4.2 and Proposition 4.12, we deduce that, under the assumptions of Theorem 2.1, the problem ($\mathcal{P}$) has one and only ones solution and it satisfies the a priori estimates (4.12) and (4.13).

In the proof of this proposition, we use the following notations

$$\mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_B}:=(k^2+\nabla\operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_B}\text{ and }\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_B}:=ik\operatorname{curl}\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_B}$$

where $\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_B}$ is the vector Newtonian operator, see (2.23). In addition, we sometimes use the following short notations for the corresponding potentials

$$\mathcal{N}_{D,V}^{k,\boldsymbol{\mathcal{C}}_B}:=\mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_B}(V)\text{ and }\mathcal{M}_{D,V}^{k,\boldsymbol{\mathcal{C}}_B}:=\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_B}(V)$$

where V is a given vector field.

We start with the following lemma.

Lemma 4.4. *The potential* $\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_B}$ *solves*

$$\big(\operatorname{curl}^2\mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_B}+\alpha^2\mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_B}\big)(V)=(i\alpha)^2\boldsymbol{\mathcal{C}}_BV$$

and we have, for $\alpha>0$,

$$-\alpha^2\int_D\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_B}\cdot\overline{\boldsymbol{\mathcal{C}}_BV}dv\geq\|\operatorname{curl}\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_B}\|_{\mathbb{L}^2(\mathbb{R}^3)}+\alpha^2\|\mathcal{N}_{D,V}^{i\alpha,\boldsymbol{\mathcal{C}}_B}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}\geq0.\tag{4.14}$$

Further, both $\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_B}$ *and* $\mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_B}-\mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_B}$ *are compact operators, and it holds that*

$$\int_D(\mathcal{N}_D^{i\alpha,\boldsymbol{\mathcal{C}}_B}-\mathcal{N}_D^{k,\boldsymbol{\mathcal{C}}_B})(V)\cdot\overline{\boldsymbol{\mathcal{C}}_BV}dv\leq\frac{|k|(|k|+1)}{4\pi}\Big((|k|+5)\boldsymbol{a}^{\frac52}+2^3c_0\frac{|k|+5}{(1+2\boldsymbol{c}_r)^2\boldsymbol{c}_r}\Big)\|\boldsymbol{\mathcal{C}}_BV\|^2_{\mathbb{L}^2(D)},\tag{4.15}$$

$$\int_D\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_{B_1}}(V_1)\cdot\overline{\boldsymbol{\mathcal{C}}_{B_2}V_2}dv\leq\Big((\frac12+\frac k4\boldsymbol{a})\boldsymbol{a}+c_0\frac{(1+|k|)}{4\pi\boldsymbol{c}_r^3}\Big)\|\boldsymbol{\mathcal{C}}_{B_2}V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{B_1}V_1\|_{\mathbb{L}^2(D)},\tag{4.16}$$

where, we recall that, $\boldsymbol{c}_r=\boldsymbol{\delta}/\boldsymbol{a}$ *and* B, B_1, B_2 *are indifferently complex or real valued matrices.*

Proof. First, we write

$$\begin{aligned}\alpha^2\int_D(\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\overline{\boldsymbol{\mathcal{C}}_B V})dv &= -\int_D \mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\Big(\mathrm{curl}^2\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}+\alpha^2\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}\Big)dx,\\ &= -\int_D \mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}^2\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}dx-\alpha^2\|\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(D)}.\end{aligned} \tag{4.17}$$

Due to Green's formula inside D, we have

$$\int_{\partial D}\nu\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}ds=\|\mathrm{curl}\,\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(D)}-\int_D\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}^2\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}dx. \tag{4.18}$$

As, outside of D, $\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}$ satisfies the Maxwell equation for $k=i\alpha$, a direct application of Green's identity outside of D, for a sufficiently large $R>0$,[7] implies that

$$\begin{aligned}&\int_{\partial B_R}\nu\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}ds\\ &\quad-\int_{\partial D}\nu\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}ds=\|\mathrm{curl}\,\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(B_R\setminus D)}+\alpha^2\|\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(B_R\setminus D)}\end{aligned} \tag{4.19}$$

which guaranties[8] that

$$-\Re\Big(\int_{\partial D}\nu\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}ds\Big)\geq\|\mathrm{curl}\,\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}+\alpha^2\|\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}.$$

Again, we have from (4.19),

$$\Im\Big(\int_{\partial B_R}\nu_{\partial B_R}\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}\;ds\Big)-\Im\Big(\int_{\partial D}\nu\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}\;ds\Big)=0,$$

which, with the exponential decay, gives

$$-\int_{\partial D}\nu\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}ds\geq\|\mathrm{curl}\,\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}+\alpha^2\|\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}. \tag{4.20}$$

Inserting the above inequality in (4.18) gives

$$\|\mathrm{curl}\,\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}+\alpha^2\|\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(\mathbb{R}^3\setminus D)}\leq-\|\mathrm{curl}\,\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\|^2_{\mathbb{L}^2(D)}+\int_D\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}^2\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}dv.$$

Replacing this last inequality in (4.17) ends the proof of (4.14).

Let us now prove (4.16) and (4.15). We write, for $\alpha=1$,

$$\begin{aligned}\Big\|[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D,V}]\Big\|_{\mathbb{L}^2(D)}\leq&\Big(\sum_{m=1}^{\aleph}\Big\|[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D_m,V}]\Big\|^2_{\mathbb{L}^2(D_m)}\Big)^{\frac{1}{2}}\\ &+\Big(\sum_{m=1}^{\aleph}\Big\|[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D\setminus D_m,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D\setminus D_m,V}]\Big\|_{\mathbb{L}^2(D_m)}\Big)^{\frac{1}{2}}.\end{aligned} \tag{4.21}$$

Due to (A.2) we have $[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D/D_m,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D/D_m,V}](x)=Int_1^m+Int_2^m$ where

$$Int_1^m:=k^2\int_{D\setminus D_m}\frac{e^{ik|x-y|}}{4\pi|x-y|}(\boldsymbol{\mathcal{C}}_B V)(y)dy-\int_{D\setminus D_m}\frac{e^{-|x-y|}}{4\pi|x-y|}(\boldsymbol{\mathcal{C}}_B V)(y)dy$$

[7] The continuity of the normal trace of $\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}$ across ∂D is due the facts that the operator $\nu\times\nabla$ is an isomorphism from $\mathbb{H}^s(\partial D)\setminus\mathbb{R}$ to $\mathbb{H}^{s-1}(\partial D)\setminus\mathbb{R}$ (see [25]), and that $\mathrm{div}\,\mathcal{S}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_D(\cdot)$ has a continuous Dirichlet trace.

[8] Obviously $\Re\int_{\partial B_R}\nu\times\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}\cdot\mathrm{curl}\,\overline{\mathcal{N}^{i\alpha,\boldsymbol{\mathcal{C}}_B}_{D,V}}ds\longrightarrow 0$, as R grows, due to the exponential decay of the kernel for $\alpha>0$.

and

$$Int_2^m := \int_{D\setminus D_m}\int_0^1 \frac{e^{\left((ik+1)t-1\right)|x-y|}\left((ik+1)t-1\right)}{4\pi(ik-1)^{-1}|x-y|}\Big[I+\Big((ik+1)t-1-\frac{1}{|x-y|}\Big)\frac{(x-y)^{\overset{2}{\otimes}}}{|x-y|}\Big]dt(\boldsymbol{\mathcal{C}}_B V)(y)dy.$$

We have, using Hölder's inequality, [9]

$$|Int_1^m| \leq \frac{(|k|^2+1)}{4\pi}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{\boldsymbol{a}^{\frac{3}{2}}}{\boldsymbol{\delta}_{mj}}\|\boldsymbol{\mathcal{C}}_B V\|_{\mathbb{L}^2(D_j)},$$

and

$$\begin{aligned}|Int_2^m| &\leq \int_{D\setminus D_m}\frac{|k|(|k|+1)}{4\pi}\Big[\frac{2}{|x-y|}+|k|\Big]|\boldsymbol{\mathcal{C}}_B V|(y)dy,\\ &\leq \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{|k|(|k|+1)}{4\pi}\Big[\frac{2}{\boldsymbol{\delta}_{jm}}+|k|\Big]\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_B V\|_{\mathbb{L}^2(D_j)}.\end{aligned}$$

Using (A.6), of Lemma A.1, and Hölder's inequality for the second step, we obtain

$$\begin{aligned}\sum_{m=1}^{\aleph}\Big\|\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D/D_m,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D/D_m,V}\Big\|^2_{\mathbb{L}^2(D_m)} &\leq \sum_{m=1}^{\aleph}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{|k|(|k|+1)}{4\pi}\Big[\frac{3}{\boldsymbol{\delta}_{jm}}+|k|\Big]\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_B V\|_{\mathbb{L}^2(D_j)}\Big)^2\boldsymbol{a}^3,\\ &\leq c_0\Big(\frac{(|k|+1)}{4\pi|k|^{-1}}\Big)^2\aleph\boldsymbol{a}^6\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\frac{9}{\boldsymbol{\delta}_{jm}^2}+\frac{2|k|}{\boldsymbol{\delta}_{jm}}+|k|^2\Big]\ \|\boldsymbol{\mathcal{C}}_B V\|^2_{\mathbb{L}^2(D_j)} \qquad (4.22)\\ &\leq c_0\Big(\frac{(|k|+1)}{4\pi|k|^{-1}}\Big)^2\aleph\boldsymbol{a}^6\Big[\frac{9\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{2\aleph^{\frac{2}{3}}|k|}{\boldsymbol{\delta}}+\aleph|k|^2\Big]\ \|\boldsymbol{\mathcal{C}}_B V\|^2_{\mathbb{L}^2(D)}.\end{aligned}$$

Hence

$$\begin{aligned}&\sum_{m=1}^{\aleph}\Big\|\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D/D_m,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D/D_m,V}\Big\|^2_{\mathbb{L}^2(D_m)}\\ &\leq c_0\Big(\frac{(|k|^2+|k|)}{4\pi}\Big)^2\Big[\frac{9\boldsymbol{a}^6}{(\boldsymbol{a}/2+\boldsymbol{\delta})^4\boldsymbol{\delta}^2}+\frac{2\boldsymbol{a}^6|k|}{(\boldsymbol{a}/2+\boldsymbol{\delta})^5\boldsymbol{\delta}}+\frac{\boldsymbol{a}^6|k|^2}{(\boldsymbol{a}/2+\boldsymbol{\delta})^6}\Big]\ \|\boldsymbol{\mathcal{C}}_B V\|^2_{\mathbb{L}^2(D)}\\ &\leq c_0\Big(\frac{(|k|^2+|k|)}{4\pi}\Big)^2\Big[\frac{9(2^4)}{(1+2\boldsymbol{c}_r)^4\boldsymbol{c}_r^2}+\frac{2^6|k|}{(1+2\boldsymbol{c}_r)^5\boldsymbol{c}_r}+\frac{2^6|k|^2}{(1+2\boldsymbol{c}_r)^6}\Big]\ \|\boldsymbol{\mathcal{C}}_B V\|^2_{\mathbb{L}^2(D)} \qquad (4.23)\\ &\leq 2^3c_0\Big(\frac{(|k|^2+|k|)}{4\pi}\Big)^2\Big[\frac{|k|^2+4|k|+18}{(1+2\boldsymbol{c}_r)^4\boldsymbol{c}_r^2}\Big]\ \|\boldsymbol{\mathcal{C}}_B V\|^2_{\mathbb{L}^2(D)}.\end{aligned}$$

We deal now with the first term of the right-hand side of (4.21). We write

$$\begin{aligned}\Big|[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D_m,V}](x)\Big| &\leq \int_{D_m}\frac{|k|(|k|+1)}{4\pi}\Big[\Big(\frac{3}{|x-y|}+|k|\Big)\Big]|\boldsymbol{\mathcal{C}}_B V|(y)dy\\ &\leq \frac{|k|(|k|+1)}{4\pi}\Big[\Big(\int_{D_m}\frac{3}{|x-y|^2}dy\Big)^{\frac{1}{2}}+|k|\boldsymbol{a}^{\frac{3}{2}}\Big]\|\boldsymbol{\mathcal{C}}_B V\|_{\mathbb{L}^2(D_m)}\end{aligned}$$

[9] We have considered $|k|>1$.

and, as done in (3.18), we obtain

$$\sum_{m=1}^{\aleph}\Big\|[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D_m,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D_m,V}](x)\Big\|^2_{\mathbb{L}^2(D_m)} \leq \sum_{m\geq 1}^{\aleph}\boldsymbol{a}^3\Big(\frac{|k|(|k|+1)}{4\pi}\Big)^2\Big[\sqrt{6\pi\boldsymbol{a}}+|k|\boldsymbol{a}^{\frac{3}{2}}\Big]^2\|\boldsymbol{\mathcal{C}}_B V\|^2_{\mathbb{L}^2(D_m)}$$
$$\leq\Big(\frac{|k|(|k|+1)}{4\pi}\Big)^2\Big[\sqrt{6\pi\boldsymbol{a}}+|k|\boldsymbol{a}^{\frac{3}{2}}\Big]^2\boldsymbol{a}^3\|\boldsymbol{\mathcal{C}}_B V\|^2_{\mathbb{L}^2(\cup_{m\geq 1}D_m)}. \tag{4.24}$$

Gathering (4.23) and (4.24) gives us

$$\Big\|[\mathcal{N}^{i,\boldsymbol{\mathcal{C}}_B}_{D,V}-\mathcal{N}^{k,\boldsymbol{\mathcal{C}}_B}_{D,V}]\Big\|_{\mathbb{L}^2(D)} \leq \frac{|k|(|k|+1)}{4\pi}\Big((|k|+5)\boldsymbol{a}^{\frac{5}{2}}+2^3c_0\frac{|k|+5}{(1+2\boldsymbol{c_r})^2\boldsymbol{c_r}}\Big)\|\boldsymbol{\mathcal{C}}_B V\|_{\mathbb{L}^2(D)}. \tag{4.25}$$

To derive (4.16), we write

$$\sum_{m=1}^{\aleph}\int_{D_m}\mathcal{M}^{k,\boldsymbol{\mathcal{C}}_{B_1}}_{D,V_1}\cdot\overline{\boldsymbol{\mathcal{C}}_{B_2}V_2}dv=\sum_{m=1}^{\aleph}\int_{D_m}\mathcal{M}^{k,\boldsymbol{\mathcal{C}}_{B_1}}_{D_m,V_1}\cdot\overline{\boldsymbol{\mathcal{C}}_{B_2}V_2}dv+\sum_{m=1}^{\aleph}\int_{D_m}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\mathcal{M}^{k,\boldsymbol{\mathcal{C}}_{B_1}}_{D_j,V_1}\cdot\overline{\boldsymbol{\mathcal{C}}_{B_1}V_2}dv. \tag{4.26}$$

Concerning the first term of the right hand side, we use the following Young's convolution inequality

$$\left|\int_{\mathbb{R}^3}u(x)(v*w)(x)dx\right|\leq\|u\|_{\mathbb{L}^2(\mathbb{R}^3)}\|w\|_{\mathbb{L}^2(\mathbb{R}^3)}\|v\|_{\mathbb{L}^1(\mathbb{R}^3)}$$

with $u(x)=|\boldsymbol{\mathcal{C}}_{B_2}V_2|(x)\chi_{D_m}(x)$, $v(x)=|\nabla\Phi_k|(x)\chi_{B(0,\boldsymbol{a})}(x)$, and $w(x)=|\boldsymbol{\mathcal{C}}_{B_1}V_1|(x)\chi_{D_m}(x)$,[10] to get

$$\sum_{m=1}^{\aleph}\int_{R^3}|\boldsymbol{\mathcal{C}}_{B_2}V_2|(x)\chi_{D_m}(x)\int_{R^3}|\nabla\phi_k(x-y)|\chi_{B(0,\boldsymbol{a})}(x-y)\;|\boldsymbol{\mathcal{C}}_{B_1}V_1|(y)\chi_{D_m}(y)dy\;dx$$
$$\leq\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{B_2}V_2\chi_{D_m}\|_{\mathbb{L}^2(\mathbb{R}^3)}\|\boldsymbol{\mathcal{C}}_{B_1}V_1\chi_{D_m}\|_{\mathbb{L}^2(\mathbb{R}^3)}\big\|\nabla\Phi_k\chi_{B(0,\boldsymbol{a})}\big\|_{\mathbb{L}^1(\mathbb{R}^3)}$$
$$\leq\|\boldsymbol{\mathcal{C}}_{B_2}V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{B_1}V_1\|_{\mathbb{L}^2(D)}\int_{B(0,\boldsymbol{a})}\frac{1}{4\pi}\Big(\frac{1}{|x-y|^2}+\frac{|k|}{|x-y|}\Big)dx$$
$$\leq(\frac{1}{2}+\frac{k}{4}\boldsymbol{a})\boldsymbol{a}\|\boldsymbol{\mathcal{C}}_{B_2}V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{B_1}V_1\|_{\mathbb{L}^2(D)}. \tag{4.27}$$

The second term in the right hand side of (4.26) is smaller than

$$S_1:=\sum_{m=1}^{\aleph}\int_{D_m}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\int_{D_j}\frac{1}{\boldsymbol{\delta}_{mj}}(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|)\;|\boldsymbol{\mathcal{C}}_{B_1}V_1|dv\;|\boldsymbol{\mathcal{C}}_{B_2}V_2|dv, \tag{4.28}$$

thus, similar calculations as those done above, using (A.6) of Lemma A.1, gives successively

$$S_1\leq\sum_{m=1}^{\aleph}\boldsymbol{a}^3\|\boldsymbol{\mathcal{C}}_{B_2}V_2\|_{\mathbb{L}^2(D_m)}\Big(c_0(\frac{\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{|k|\aleph^{\frac{2}{3}}}{\boldsymbol{\delta}})\Big)^{\frac{1}{2}}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}(\frac{1}{\boldsymbol{\delta}^2_{mj}}+\frac{|k|}{\boldsymbol{\delta}_{mj}})\|\boldsymbol{\mathcal{C}}_{B_1}V_1\|^2_{\mathbb{L}^2(D_j)}\Big)^{\frac{1}{2}}$$
$$\leq\Big(c_0(\frac{\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{|k|\aleph^{\frac{2}{3}}}{\boldsymbol{\delta}})\Big)^{\frac{1}{2}}\boldsymbol{a}^3\left(\|\boldsymbol{\mathcal{C}}_{B_2}V_2\|^2_{\mathbb{L}^2(D)}\sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}(\frac{1}{\boldsymbol{\delta}^2_{mj}}+\frac{|k|}{\boldsymbol{\delta}_{mj}})\|\boldsymbol{\mathcal{C}}_{B_1}V_1\|^2_{\mathbb{L}^2(D_j)}\right)^{\frac{1}{2}}$$
$$\leq c_0(\frac{\aleph^{\frac{1}{3}}}{\boldsymbol{\delta}^2}+\frac{|k|\aleph^{\frac{2}{3}}}{\boldsymbol{\delta}})\boldsymbol{a}^3\|\boldsymbol{\mathcal{C}}_{B_2}V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{B_1}V_1\|_{\mathbb{L}^2(D)},$$

[10] Notice that $D_m\subset B(\boldsymbol{z}_m,\frac{\boldsymbol{a}}{2})\subset B(y,\boldsymbol{a})$ whenever $y\in D_m$ and $\chi_{B(0,\boldsymbol{a})}(x-y)=\chi_{B(y,\boldsymbol{a})}(x)$.

which is

$$\left|\sum_{m=1}^{\aleph}\int_{D_m}\mathcal{M}_D^{k,\boldsymbol{\mathcal{C}}_{B_1}}(V_1)\cdot\boldsymbol{\mathcal{C}}_{B_2}V_2dv\right|\leq c_0\frac{(1+|k|)}{4\pi\boldsymbol{c}_r^3}\|\boldsymbol{\mathcal{C}}_{B_2}V_2\|_{\mathbb{L}^2(D)}\|\boldsymbol{\mathcal{C}}_{B_1}V_1\|_{\mathbb{L}^2(D)}. \tag{4.29}$$

□

Proof. (Proposition 4.3) Consider the equation ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$), which is, with the notation of Lemma 4.4 and $\langle\cdot,\cdot\rangle$ standing for the scalar product in $\mathbb{L}^2(D)$,

$$\begin{aligned}\big\langle E,\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle-\big\langle\mathcal{N}_{D,E}^{i\alpha,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle-\big\langle\mathcal{N}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}-\mathcal{N}_{D,E}^{i\alpha,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle-\big\langle ik\mathcal{M}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle&=\big\langle E^{\text{in}}\cdot\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle,\\ \big\langle H,\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle-\big\langle\mathcal{N}_{D,H}^{i\alpha,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle-\big\langle\mathcal{N}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}-\mathcal{N}_{D,H}^{i\alpha,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle+\big\langle ik\mathcal{M}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle&=\big\langle H^{\text{in}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle,\end{aligned}$$

and with (4.14), taking the real part of the above equations, we get

$$\begin{aligned}\Re\big\langle E,\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle-\Re\big\langle\mathcal{N}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}-\mathcal{N}_{D,E}^{i\alpha,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle-\Re\big\langle ik\mathcal{M}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle&\leq\Re\big\langle E^{\text{in}}\cdot\overline{\boldsymbol{\mathcal{C}}_{\varepsilon_r}E}\big\rangle,\\ \Re\big\langle H,\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle-\Re\big\langle\mathcal{N}_{D,H}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}-\mathcal{N}_{D,H}^{i\alpha,\boldsymbol{\mathcal{C}}_{\mu_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle+\Re\big\langle ik\mathcal{M}_{D,E}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle&\leq\Re\big\langle H^{\text{in}},\overline{\boldsymbol{\mathcal{C}}_{\mu_r}H}\big\rangle.\end{aligned} \tag{4.30}$$

With the estimations (4.15), (4.16) of Lemma 4.4 and the assumption (2.9) we get[11]

$$\begin{aligned}&c_\infty^{\varepsilon-}\|E\|^2-c_\infty^2\frac{|k|(|k|+1)}{4\pi}\Big(\boldsymbol{a}^{\frac{3}{2}}+2^3c_0\frac{|k|+5}{(1+2\boldsymbol{c}_r)^2\boldsymbol{c}_r}\Big)\|E\|^2\\ &\qquad\qquad-\boldsymbol{c}_\infty^2|k|\Big(\boldsymbol{a}+c_0\frac{(1+|k|)}{4\pi\boldsymbol{c}_r^3}\Big)\|H\|\|E\|\leq\big\langle E^{\text{in}}\cdot\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\big\rangle,\\ &c_\infty^{\mu-}\|H\|^2-c_\infty^2\frac{|k|(|k|+1)}{4\pi}\Big(\boldsymbol{a}^{\frac{3}{2}}+2^3c_0\frac{|k|+5}{(1+2\boldsymbol{c}_r)^2\boldsymbol{c}_r}\Big)\|H\|^2\\ &\qquad\qquad-\boldsymbol{c}_\infty^2|k|\Big(\boldsymbol{a}+c_0\frac{(1+|k|)}{4\pi\boldsymbol{c}_r^3}\Big)\|H\|\|E\|\leq\big\langle H^{\text{in}},\boldsymbol{\mathcal{C}}_{\mu_r}H\big\rangle,\end{aligned}$$

which, under the conditions

$$\boldsymbol{c}_r\geq\frac{c_02|k|\boldsymbol{c}_\infty^2}{\max(c_\infty^{\varepsilon-},c_\infty^{\mu-})},\quad\frac{|k|(|k|+1)}{4\pi}\boldsymbol{a}<1,\ \text{and}\ \boldsymbol{a}\leq\frac{1}{16}, \tag{4.31}$$

gives

$$\|E\|^2-\frac{1}{4\pi}\|H\|\|E\|\leq\frac{\boldsymbol{c}_\infty}{c_\infty^{\varepsilon-}}\|E^{\text{in}}\|\|E\|,$$

and

$$\|H\|^2-\frac{1}{4\pi}\|H\|\|E\|\leq\frac{\boldsymbol{c}_\infty}{c_\infty^{\mu-}}\|H^{\text{in}}\|\|H\|.$$

More precisely,

$$\|E\|\leq\frac{5\boldsymbol{c}_\infty}{4c_\infty^{\varepsilon-}}\Big(\|E^{\text{in}}\|+\frac{1}{4\pi}\|H^{\text{in}}\|\Big), \tag{4.32}$$

and

$$\|H\|\leq\frac{5\boldsymbol{c}_\infty}{4c_\infty^{\mu-}}\Big(\|H^{\text{in}}\|+\frac{1}{4\pi}\|E^{\text{in}}\|\Big). \tag{4.33}$$

Concerning the estimate (4.13), identifying both left hand sides of the eqs. ($\mathcal{M}_{\mathcal{L}.\mathcal{S}}$) and ($\mathcal{A}-\mathcal{M}_{\mathcal{L}.\mathcal{S}}$), we have

$$\begin{aligned}(H_{\mathcal{A}}-H)&-(k^2+\nabla\operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{A}}_{\boldsymbol{c}_{\mu_r}}}(H_{\mathcal{A}}-H)+ik\operatorname{curl}\mathcal{S}_D^{k,\boldsymbol{\mathcal{A}}_{\boldsymbol{c}_{\varepsilon_r}}}(E_{\mathcal{A}}-E)\\ &=-(k^2+\nabla\operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\mu_r}-\boldsymbol{\mathcal{A}}_{\boldsymbol{c}_{\mu_r}}}(H)+ik\operatorname{curl}\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}-\boldsymbol{\mathcal{A}}_{\boldsymbol{c}_{\varepsilon_r}}}(E),\end{aligned}$$

[11] Assuming that $|k|\boldsymbol{a}\leq 1$.

then, due to estimates (4.12) for the solution of Lippmann-Schwinger integral equation, we obtain

$$\begin{aligned}\|H_{\mathcal{A}} - H\|_{\mathbb{L}^2(D)} &\leq \left\| -(k^2 + \nabla \operatorname{div})\mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\mu_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}}(H) + ik \operatorname{curl} \mathcal{S}_D^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}}(E) \right\|_{\mathbb{L}^2(D)}, \\ &\leq \boldsymbol{c}_{2,k} \Big(\left\| (\boldsymbol{\mathcal{C}}_{\mu_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}})H \right\|_{\mathbb{L}^2(D)} + \left\| (\boldsymbol{\mathcal{C}}_{\varepsilon_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}})E \right\|_{\mathbb{L}^2(D)} \Big), \\ &\leq \boldsymbol{c}_{2,k} c_\infty \boldsymbol{a} \Big(\|H\|_{\mathbb{L}^2(D)} + \|E\|_{\mathbb{L}^2(D)} \Big).\end{aligned}$$

□

Remark 4.5. *1. We can consider either both real tensors or complex valued ones for the relative electric permittivity and magnetic permeability inducing similar assumption concerning the corresponding contrast in* (2.9)*.*

2. *We could improve the condition on the ratio $\boldsymbol{a}/\boldsymbol{\delta}$ by taking a larger α. But this would increase the constant that appears in the estimation* (4.12) *which will result in the worsening of the error of approximation in the latter calculation.*
3. *We can also weaken the condition* (2.9) *using a Helmholtz decomposition. As the divergence free part can be chosen with a vanishing normal trace (see [25]), we would be able to obtain similar estimate as in* (3.15) *for the gradient part which is the $\mathbb{L}^2$-norm-dominant term of the decomposition.*

5 Field approximation and the related linear system

Let (E, H) be the solution of the problem $(\mathcal{M}_{\mathcal{L}.\mathcal{S}})$ and set

$$\mathcal{Q}_m^{\mu_r} := \int_{D_m} \boldsymbol{\mathcal{C}}_B H \, dv, \quad \mathcal{R}_m^{\varepsilon_r} := \int_{D_m} \boldsymbol{\mathcal{C}}_B E \, dv. \tag{5.1}$$

Analogously, let $(E_{\mathcal{A}},\ H_{\mathcal{A}})$ be the solution of problem $(\mathcal{A} - \mathcal{M}_{\mathcal{L}.\mathcal{S}})$ and set

$$\mathcal{Q}_m := \int_{D_m} H_{\mathcal{A}} \, dv, \quad \mathcal{R}_m := \int_{D_m} E_{\mathcal{A}} \, dv. \tag{5.2}$$

The vector fields defined by $U_m^{\mu_r^T} := \boldsymbol{\mathcal{C}}_{\mu_r}^T \nabla V_m^{\mu_r^T}$ and $U_m^{\varepsilon_r^T} := \boldsymbol{\mathcal{C}}_{\varepsilon_r}^T \nabla V_m^{\varepsilon_r^T}$, where $V_m^{\mu_r^T}$ and $V_m^{\varepsilon_r^T}$ are the respective solutions of $(\mathcal{P}r^{Ani}(1))$ with $B = \mu_r^T$ and $B = \varepsilon_r^T$, satisfy, respectively,

$$U_m^{\mu_r^T} - \boldsymbol{\mathcal{C}}_{\mu_r}^T \nabla \operatorname{div} \mathcal{S}_{D_m}^0 (U_m^{\mu_r^T}) = \boldsymbol{\mathcal{C}}_{\mu_r}^T \tag{5.3}$$

and

$$U_m^{\varepsilon_r^T} - \boldsymbol{\mathcal{C}}_{\mu_r}^T \nabla \operatorname{div} \mathcal{S}_{D_m}^0 (U_m^{\varepsilon_r^T}) = \boldsymbol{\mathcal{C}}_{\varepsilon_r}^T. \tag{5.4}$$

Due to Proposition 3.1, we should get

$$\|U_m^{\mu_r^T}\|_{\mathbb{L}^2(D_m)} \leq \|\boldsymbol{\mathcal{C}}_{\mu_r}^T \nabla V\|_{\mathbb{L}^2(D_m)} \leq c_\infty \boldsymbol{a}^{\frac{3}{2}}, \tag{5.5}$$

and

$$\|U_m^{\varepsilon_r^T}\|_{\mathbb{L}^2(D_m)} \leq \|\boldsymbol{\mathcal{C}}_{\varepsilon_r}^T \nabla V\|_{\mathbb{L}^2(D_m)} \leq c_\infty \boldsymbol{a}^{\frac{3}{2}}, \tag{5.6}$$

where, for $\boldsymbol{a}$ sufficiently small,

$$c_\infty := \max_{m \leq \aleph} \sup_{x \in D_m,\, B = \varepsilon_r, \mu_r.} \left(\frac{\sqrt{\min(1, c^{B-})}}{2} - \boldsymbol{a}^2 \|B\|_{\mathbb{L}^\infty(D_m)} \sqrt{\frac{3}{\pi}} \right)^{-1} (\|B\|_{\mathbb{L}^\infty(D_m)} + 2)^2.$$

As consequences of the scaling (3.11) and the behaviors (3.9), (3.10) of Proposition 3.1, we have

$$\lambda_B^- \boldsymbol{a}^3 |V|^2 \leq [\mathcal{P}_{D_m}^B] V \cdot V \leq \lambda_B^+ \boldsymbol{a}^3 |V|^2 \qquad \text{whenever } B \in \mathbb{W}^{1,\infty}(D_m, \mathbb{R}^3 \times \mathbb{R}^3), \tag{5.7}$$

and

$$\lambda_B^- \boldsymbol{a}^3 |V|^2 \leq \Re\Big([\mathcal{P}_{D_m}^B] V \cdot \overline{V} \Big) \leq \lambda_B^+ \boldsymbol{a}^3 |V|^2 \qquad \text{whenever } B \in \mathbb{W}^{1,\infty}(D_m, \mathbb{C}^3 \times \mathbb{C}^3), \tag{5.8}$$

where $\lambda_B^+ := \max_m \lambda_{B,m}^+$, $\lambda_B^- := \min_m \lambda_{B,m}^-$ and $(\lambda_{B,m}^{\pm})_{m=1}^{\aleph}$ are constants that satisfy the previous inequalities for each D_m.

For U and V in $\mathbb{L}^2(D_m)$, we set

$$Er_m(U,V) := O\Bigg(\sum_{\substack{(j\geq 1) \\ j\neq m}}^{\aleph} \Bigg[\frac{\|V\|_{\mathbb{L}^2(D_j)}}{\boldsymbol{\delta}_{mj}^4} + \Big(\frac{3|k|}{\boldsymbol{\delta}_{mj}^3} + \frac{3|k|^2+1}{\boldsymbol{\delta}_{mj}^2} + \frac{|k|(|k|^2+1)}{\boldsymbol{\delta}_{mj}} \Big) \Big(\|U\|_{\mathbb{L}^2(D_j)} + \|V\|_{\mathbb{L}^2(D_j)} \Big) \Bigg] \boldsymbol{a}^{\frac{11}{2}} \Bigg) \tag{5.9}$$

then, using the estimate (A.6) of Lemma A.1, we obtain

$$\sum_{m=1}^{\aleph} Er_m(U,V)^2 = O\Bigg(\|U\|_{\mathbb{L}^2(D)}^2 \frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8} + \Big(\|U\|_{\mathbb{L}^2(D)}^2 + \|V\|_{\mathbb{L}^2(\cup_m D_m)}^2 \Big) \frac{(|k|+2)^3 \boldsymbol{a}^{11} |\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6} \Bigg). \tag{5.10}$$

The far field of the scattered wave is given by, see ([13], (6.26)-(6.27) p. 199),

$$E^\infty(\hat{x}) = \frac{k^2}{4\pi} \hat{x} \times \Big(\int_{\cup_{m=1}^{\aleph} D_m} e^{-ik\hat{x}\cdot y} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E(y) dy \times \hat{x} \Big) + \frac{ik}{4\pi} \hat{x} \times \int_{\cup_{m=1}^{\aleph} D_m} e^{-ik\hat{x}\cdot y} \boldsymbol{\mathcal{C}}_{\mu_r} H(y) dy. \tag{5.11}$$

Furthermore, for the scattering of plane wave, (4.12) gives, with (4.32) and (4.33),

$$\|E\|_{\mathbb{L}^2(D)} \leq \frac{5\boldsymbol{c}_\infty}{4c_\infty^{\varepsilon^-}} \Big(|P| + \frac{|k|}{4\pi} |\theta \times P| \Big) \Big(\frac{1}{\boldsymbol{c}_r} \Big)^{\frac{3}{2}} \tag{5.12}$$

and

$$\|H\|_{\mathbb{L}^2(D)} \leq \frac{5\boldsymbol{c}_\infty}{4c_\infty^{\varepsilon^-}} \Big(\frac{1}{4\pi} |P| + |k| |\theta \times P| \Big) \Big(\frac{1}{\boldsymbol{c}_r} \Big)^{\frac{3}{2}}. \tag{5.13}$$

With the above estimates, we will next prove the following results.

Proposition 5.1. *Under the assumptions of Theorem 2.1, we have the following approximations for the far fields of the scattered waves uniformly for $\hat{x} \in \mathbb{S}^2$.*

1. *If $\boldsymbol{\mathcal{C}}_{\mu_r}$, $\boldsymbol{\mathcal{C}}_{\varepsilon_r}$ are symmetric*

$$E^\infty(\hat{x}) = \sum_{m=1}^{\aleph} \Bigg(\frac{k^2}{4\pi} e^{-ik\hat{x}\cdot \boldsymbol{z}_m} \hat{x} \times \Big(\mathcal{R}_m^{\varepsilon_r} \times \hat{x} \Big) + \frac{ik}{4\pi} e^{-ik\hat{x}\cdot \boldsymbol{z}_m} \hat{x} \times \mathcal{Q}_m^{\mu_r} \Bigg) + \Big(\frac{|k|^3}{\boldsymbol{c}_r^3} \boldsymbol{a} \Big). \tag{5.14}$$

2. *If $\boldsymbol{\mathcal{C}}_{\mu_r}$, $\boldsymbol{\mathcal{C}}_{\varepsilon_r}$ are not symmetric*

$$\begin{aligned} E^\infty(\hat{x}) = \sum_{m=1}^{\aleph} \Bigg(&\frac{k^2}{4\pi} e^{-ik\hat{x}\cdot \boldsymbol{z}_m} \hat{x} \times \Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r} \mathcal{R}_m \times \hat{x} \Big) + \frac{ik}{4\pi} e^{-ik\hat{x}\cdot \boldsymbol{z}_m} \hat{x} \times \boldsymbol{\mathcal{C}}_{\mu_r} \mathcal{Q}_m \Bigg) \\ &+ O\Bigg(\frac{|k|^3 + |k|^2}{\boldsymbol{c}_r^3} \boldsymbol{a} + \frac{|k| \boldsymbol{c}_\infty (\boldsymbol{c}_\infty \boldsymbol{c}_{2,k} + 1)}{\boldsymbol{c}_r^3} \boldsymbol{a} \Bigg). \end{aligned} \tag{5.15}$$

Here $\left(\mathcal{R}_m^{\varepsilon_r}\right)_{m=1}^{\aleph}$ *and* $\left(\mathcal{Q}_m^{\mu_r}\right)_{m=1}^{\aleph}$ *are solutions of the following linear algebraic system*

$$\begin{aligned}
\mathcal{Q}_m^{\mu_r} &= [\mathcal{P}_{D_m}^{\mu_r}]\bigg(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{Q}_j^{\mu_r} - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{R}_j^{\varepsilon_r}\Big] + H^{in}(\boldsymbol{z}_m)\bigg)\\
&\quad + Er_m(H,E) + O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}} + |k|^2\boldsymbol{a}^4\Big),\\
\mathcal{R}_m^{\varepsilon_r} &= [\mathcal{P}_{D_m}^{\varepsilon_r}]\bigg(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\mathcal{R}_j^{\varepsilon_r} + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\mathcal{Q}_j^{\mu_r}\Big] + \cdot E^{in}(\boldsymbol{z}_m)\bigg)\\
&\quad + Er_m(E,H) + O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}} + |k|^2\boldsymbol{a}^4\Big),
\end{aligned} \tag{5.16}$$

and $\left(\mathcal{R}_m\right)_{m=1}^{\aleph}$ *and* $\left(\mathcal{Q}_m\right)_{m=1}^{\aleph}$ *are solutions the following linear algebraic system*

$$\begin{aligned}
\mathcal{Q}_m &= [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}](\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r^T}})^{-1}\bigg(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\mu_r}}\mathcal{Q}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\mathcal{R}_j\Big] + H^{in}(\boldsymbol{z}_m)\bigg)\\
&\quad + Er_m(H_{\mathcal{A}},E_{\mathcal{A}}) + O(k^2\|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}),\\
\mathcal{R}_m &= [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^T}}](\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r^T}})^{-1}\bigg(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\mathcal{R}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\mu_r}}\mathcal{Q}_j\Big] + E^{in}(\boldsymbol{z}_m)\bigg)\\
&\quad + Er_m(E_{\mathcal{A}},H_{\mathcal{A}}) + O(k^2\|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}).
\end{aligned} \tag{5.17}$$

Proof. From $(\mathcal{M}_{\mathcal{L}.\mathcal{S}})$, we have

$$\begin{aligned}
&H - (k^2+\nabla\operatorname{div})\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H)\\
&\quad + ik\operatorname{curl}\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E) = \sum_{j\geq 1 j\neq m}^{\aleph}\Big[(k^2+\nabla\operatorname{div})\mathcal{S}_{D_j}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H) - ik\operatorname{curl}\mathcal{S}_{D_j}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E)\Big] + H^{\text{in}},
\end{aligned} \tag{5.18}$$

and

$$\begin{aligned}
&E - (k^2+\nabla\operatorname{div})\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E)\\
&\quad - ik\operatorname{curl}\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H) = \sum_{j\geq 1 j\neq m}^{\aleph}\Big[(k^2+\nabla\operatorname{div})\mathcal{S}_{D_j}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E) + ik\operatorname{curl}\mathcal{S}_{D_j}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H)\Big] + E^{\text{in}},
\end{aligned} \tag{5.19}$$

in each D_m, $m = 1, ..., \aleph$.

- (*Derivation of* (5.16)) Multiplying the left hand side of (5.18), by $U_m^{\mu_r^T}$ and integrating over D_m, we derive

$$\begin{aligned}
&\int_{D_m} U_m^{\mu_r^T}(x)\cdot\Big(H - (k^2+\nabla\operatorname{div})\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H) + ik\operatorname{curl}\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E)\Big)(x)dx\\
&\qquad = \int_{D_m} U_m^{\mu_r^T}\cdot\big[I - \nabla\operatorname{div}\mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_{\mu_r}}\big](H)(x)dx - k^2\int_{D_m} U_m^{\mu_r^T}(x)\cdot\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}(H)(x)dx\\
&\qquad\quad - \int_{D_m} U_m^{\mu_r^T}(x)\cdot\Big(\nabla\operatorname{div}\Big[\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}} - \mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_{\mu_r}}\Big](H) - ik\operatorname{curl}\mathcal{S}_{D_m}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}(E)\Big)(x)dx,\\
&\qquad = \int_{D_m} U_m^{\mu_r^T}\cdot\big[I - \nabla\operatorname{div}\mathcal{S}_{D_m}^{0,\boldsymbol{\mathcal{C}}_{\mu_r}}\big](H)(x)dx\\
&\qquad\quad + O\Big(k^2\boldsymbol{a}^{7/2}\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)} + \boldsymbol{a}^{5/2}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\Big).
\end{aligned} \tag{5.20}$$

Indeed, due to the third identity (A.4), for $\alpha = 0$, it is obvious that

$$\Big|\nabla \operatorname{div}\big[\mathcal{S}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m} - \mathcal{S}^{0,\boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m}\big](H)(x)\Big|$$

$$\leq \Big|\int_{D_m}\Big(\int_0^1 \frac{e^{\big((ik)t\big)|(x-y)|}k^2 t}{4\pi|x-y|}\Big[\mathbb{I} + \big(ikt - \frac{1}{|x-y|}\big)\frac{(x-y)^{\overset{2}{\otimes}}}{|x-y|}\Big]dt\Big)(\boldsymbol{\mathcal{C}}_{\mu_r}H)(y)dy\Big|,$$

$$\leq \int_{D_m}\Big[\frac{2k^2}{4\pi|x-y|} + \frac{k^3}{4\pi}\Big]\ |(\boldsymbol{\mathcal{C}}_{\mu_r}H)(y)|dy,$$

$$\leq \Big[\frac{k^2}{2\pi}\Big(\lim_{r\to 0}\int_{B(y,\boldsymbol{a})\setminus B(\boldsymbol{z}_m,r)}\frac{1}{|x-y|^2}dy\Big)^{\frac{1}{2}} + \frac{k^3}{4\pi}\boldsymbol{a}^{\frac{3}{2}}\Big]\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)},$$

$$\leq \Big[\frac{k^2}{2\pi}\pi\boldsymbol{a}^{\frac{1}{2}} + \frac{k^3}{4\pi}\boldsymbol{a}^{\frac{3}{2}}\Big]\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)},$$

hence,

$$\Big\|\nabla \operatorname{div}\big[\mathcal{S}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m} - \mathcal{S}^{0,\boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m}\big](H)\Big\|_{\mathbb{L}^2(D_m)} \leq \frac{\pi}{3}\Big[\frac{k^2}{\sqrt{2\pi}} + \frac{k^3}{4\pi}\boldsymbol{a}\Big]\boldsymbol{a}^2\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)}.$$

We also have, for similar reasons,

$$\Big\|k^2\big[\mathcal{S}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m}\big](H)\Big\|_{\mathbb{L}^2(D_m)} \leq \frac{\pi}{3}\Big[\frac{k^2}{2\pi}\sqrt{2\pi}\boldsymbol{a}^{\frac{1}{2}}\Big]\boldsymbol{a}^{\frac{3}{2}}\big\|\boldsymbol{\mathcal{C}}_{\mu_r}H\big\|_{\mathbb{L}^2(D_m)}.$$

Regarding the third term of (5.20), we have

$$\Big|\int_{D_m}U^{\mu_r^T}_m(x)\cdot\Big(ik\operatorname{curl}\mathcal{S}^{k,\boldsymbol{\mathcal{C}}_{\varepsilon_r}}_{D_m}(E)\Big)(x)dx\Big| \leq \|U^{\mu_r^T}_m\|_{\mathbb{L}^2(D_m)}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\int_{B(0,\boldsymbol{a})}|\nabla\Phi_k(x)|dx,$$

$$\leq \boldsymbol{c}_{\mu_r}\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\frac{1}{4\pi}\int_{B(0,\boldsymbol{a})}\Big(\frac{1}{|x-y|^2} + \frac{|k|}{|x-y|}\Big)dx,$$

$$\leq \boldsymbol{c}_{\mu_r}(\frac{1}{2} + \frac{k}{4}\boldsymbol{a})\boldsymbol{a}^{\frac{5}{2}}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}.$$

Hence, we get from (5.20)

$$\begin{aligned}
&\int_{D_m}U^{\mu_r^T}_m\cdot\Big[H - (k^2 + \nabla\operatorname{div})\mathcal{S}^{k,\boldsymbol{\mathcal{C}}_{\mu_r}}_{D_m}(H) + ik\operatorname{curl}\mathcal{S}^{k,Q^{\boldsymbol{\mathcal{C}}}(\varepsilon_r)}_{D_m}(E)\Big]dx \\
&\quad = \int_{D_m}\Big[I - \boldsymbol{\mathcal{C}}^T_{\mu_r}\nabla\operatorname{div}\mathcal{S}^0_{D_m}\Big](U^{\mu_r^T}_m)\cdot H\ dv + O(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}), \\
&\quad = \int_{D_m}\boldsymbol{\mathcal{C}}^T_{\mu_r}H\ dv + O(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}).
\end{aligned} \tag{5.21}$$

Multiplying the right hand side of (5.18) by $U^{\mu_r^T}_m$ and integrating over D_m, gives

$$\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\int_{D_m}U^{\mu_r^T}_m(x)\cdot\Big[\int_{D_j}\Pi_k(x,y)(\boldsymbol{\mathcal{C}}_{\mu_r}H)(y)dy - ik\int_{D_j}\nabla\Phi_k(x,y)\times(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E)(y)dy\Big]dx$$

$$+ \int_{D_m}U^{\mu_r^T}_m\cdot H^{\mathrm{in}}dv.$$

Using (A.1), we derive at the first order

$$\begin{aligned}
&\int_{D_m}U^{\mu_r^T}_m dv\cdot\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\int_{D_j}(\boldsymbol{\mathcal{C}}_{\mu_r}H)(y)dy - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\int_{D_j}(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E)(y)dy\Big] \\
&+ \int_{D_m}U^{\mu_r^T}_m dv\ H^{\mathrm{in}}(\boldsymbol{z}_j) + O(|k|^2\boldsymbol{a}^4) + 2Er_m(H,E),
\end{aligned} \tag{5.22}$$

which gives, when the contrasts are symmetric, joined with (5.21), the first approximation in (5.16). Analogous calculations give the second one starting from (5.19), and multiplying by $U_m^{\varepsilon_r^T}$ as defined in (5.4).

- (*Derivation of* (5.17)) When, in the other hand, both $\boldsymbol{\mathcal{C}}_{\varepsilon_r}$ and $\boldsymbol{\mathcal{C}}_{\mu_r}$ are in $\mathbb{W}^{1,\infty}(\cup_{m=1}^{\aleph} D_m)$, and not necessarily symmetric, we get, with a first order approximation, for the combined eqs. (5.21) and (5.22) of the corresponding integral formulation of the Problem $(\mathcal{A} - \mathcal{M}_{\mathcal{L}.\mathcal{S}})$

$$\int_{D_m} \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}^T H_{\mathcal{A}}\, dv + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}})$$
$$= \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}\mu_r^T} dv \cdot \sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \int_{D_j} (\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}})(y) dy - ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \int_{D_j} (\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}})(y) dy\Big]$$
$$+ \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}\mu_r^T} \cdot H^{\text{in}} dv + Er_m^{(E_{\mathcal{A}}, H_{\mathcal{A}})}(H_{\mathcal{A}}).$$

We rewrite it as

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}^T \mathcal{R}_m + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}})$$
$$= \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}\mu_r^T} dv \cdot \sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} \mathcal{R}_m - ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} \mathcal{Q}_m\Big]$$
$$+ \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}\mu_r^T} \cdot H^{\text{in}} dv + Er_m^{(E_{\mathcal{A}}, H_{\mathcal{A}})}(H_{\mathcal{A}}),$$

or more precisely

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r^T}} \mathcal{R}_m + O(k^2 \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} H_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{7}{2}} + \|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)} \boldsymbol{a}^{\frac{5}{2}})$$
$$= \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r^T}} [\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}] \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r^T}}^{-1} \cdot \sum_{\substack{j\geq 1 \\ j\neq m}}^{\aleph} \Big[\Pi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}} \mathcal{R}_m - ik \nabla \Phi_k(\boldsymbol{z}_m, \boldsymbol{z}_j) \times \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}} \mathcal{Q}_m\Big]$$
$$+ \int_{D_m} U_m^{\boldsymbol{\mathcal{A}}\mu_r^T} dv \cdot H^{\text{in}}(\boldsymbol{z}_m) + Er_m^{(E_{\mathcal{A}}, H_{\mathcal{A}})}(H_{\mathcal{A}}) + O(|k|^2 \boldsymbol{a}^4).$$

as in the symmetric case. The slights changes, for the second equation, are retrieved in the error of approximation.

- (*Derivation of* (5.14)) Concerning the far field approximation (5.14), a first order approximation gives

$$E^\infty(\hat{x}) = \sum_{m=1}^{\aleph} \Big(\frac{k^2}{4\pi} \hat{x} \times \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \boldsymbol{\mathcal{C}}_{\varepsilon_r} E(y) \times \hat{x} dy + \frac{ik}{4\pi} \hat{x} \times \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m} \boldsymbol{\mathcal{C}}_{\mu_r} H(y) dy\Big)$$
$$+ \sum_{m=1}^{\aleph} \Bigg(\frac{k^2}{4\pi} \hat{x} \times \Big(\int_{D_m} (e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot\boldsymbol{z}_m}) \boldsymbol{\mathcal{C}}_{\varepsilon_r} E(y) \times \hat{x} dy \tag{5.23}$$
$$+ \frac{ik}{4\pi} \int_{D_m} (e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot\boldsymbol{z}_m}) \boldsymbol{\mathcal{C}}_{\mu_r} H(y) dy\Big)\Bigg).$$

As

$$\big|(e^{-ik\hat{x}\cdot y} - e^{-ik\hat{x}\cdot\boldsymbol{z}_m})\big| \leq \Big|ik\hat{x} \int_0^1 e^{-ik\hat{x}\cdot(ty+(1-t)z_m)} dt \cdot (y - \boldsymbol{z}_m)\Big| \leq |k| \boldsymbol{a}, \tag{5.24}$$

we get

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\varepsilon_r}E(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\mu_r}H(y)dy\Big)$$
$$+ O\left(\sum_{m=1}^{\aleph}\frac{|k|^3+|k|^2}{4\pi}\boldsymbol{a}\Big(\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)} + \boldsymbol{a}^{\frac{3}{2}}\left\|\boldsymbol{\mathcal{C}}_{\mu_r}H\right\|_{\mathbb{L}^2(D_m)}\Big)\right) \tag{5.25}$$

then follows

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\varepsilon_r}E(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{C}}_{\mu_r}H(y)dy\Big)$$
$$+ O\left(\frac{|k|^3+|k|^2}{4\pi}(\aleph\boldsymbol{a}^3)^{\frac{1}{2}}\Big(\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(\cup D_m)} + \left\|\boldsymbol{\mathcal{C}}_{\mu_r}H\right\|_{\mathbb{L}^2(\cup D_m)}\Big)^{\frac{1}{2}}\boldsymbol{a}\right). \tag{5.26}$$

The conclusion immediately follows using the estimations (5.12) and (5.13).

The approximation (5.15) could be achieved by adding-subtracting to (5.26) above the following expression

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}(y)dy\Big).$$

Precisely

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}(y)\times\hat{x}dy + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}(y)dy\Big)$$
$$+ \sum_{m=1}^{\aleph}\left(\frac{k^2}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)\times\hat{x}dy\right.$$
$$\left. + \frac{ik}{4\pi}\hat{x}\times\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\mu_r}H - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}\Big)(y)dy\right) + O\left(\frac{|k|^3+|k|^2}{\boldsymbol{c}_r^3}\boldsymbol{a}\right). \tag{5.27}$$

Obviously

$$\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)dy = \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big[\boldsymbol{\mathcal{C}}_{\varepsilon_r}\Big(E - E_{\mathcal{A}}\Big)\Big](y)dy$$
$$+ \int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big[\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r} - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\Big)E_{\mathcal{A}}\Big](y)dy,$$

hence

$$\sum_{m=1}^{\aleph}\left|\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)dy\right| \leq \sum_{m=1}^{\aleph}\boldsymbol{a}^{\frac{3}{2}}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}\|_{\mathbb{L}^{\infty}(D_m)}\|E - E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)}$$
$$+ \sum_{m=1}^{\aleph}\boldsymbol{a}^{\frac{3}{2}}\boldsymbol{a}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}\|_{\mathbb{W}^{\infty}(D_m)}\|E_{\mathcal{A}}\|_{\mathbb{L}^2(D_m)}.$$

Using Hölder's inequality, with both (4.13) and similar (5.12) and (5.13) for $E_{\mathcal{A}}$, we get

$$\sum_{m=1}^{\aleph}\left|\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\varepsilon_r}E - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}E_{\mathcal{A}}\Big)(y)dy\right| = O\left(\frac{\boldsymbol{c}_{\infty}^2\boldsymbol{c}_{2,k} + \boldsymbol{c}_{\infty}}{\boldsymbol{c}_r^3}\boldsymbol{a}\right). \tag{5.28}$$

With similar computations, we get

$$\sum_{m=1}^{\aleph}\left|\int_{D_m} e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\Big(\boldsymbol{\mathcal{C}}_{\mu_r}H - \boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}H_{\mathcal{A}}\Big)(y)dy\right| = O\left(\frac{\boldsymbol{c}_\infty^2\boldsymbol{c}_{2,k}+\boldsymbol{c}_\infty}{\boldsymbol{c}_r^3}\boldsymbol{a}\right). \tag{5.29}$$

Injecting both of the above estimations in (5.27) gives the results.

□

Proposition 5.2. *Both the linear systems*

$$\begin{aligned}
[\mathcal{P}_{D_m}^{\mu_r^T}]^{-1}\widehat{\mathcal{Q}}_m^{\mu_r} &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\widehat{\mathcal{Q}}_j^{\mu_r} - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\widehat{\mathcal{R}}_j^{\varepsilon_r}\Big] + \widehat{H}^{in}(\boldsymbol{z}_m),\\
[\mathcal{P}_{D_m}^{\varepsilon_r^T}]^{-1}\widehat{\mathcal{R}}_m^{\varepsilon_r} &= \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\widehat{\mathcal{R}}_j^{\varepsilon_r} + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\widehat{\mathcal{Q}}_j^{\mu_r}\Big] + \widehat{E}^{in}(\boldsymbol{z}_m),
\end{aligned} \tag{5.30}$$

and

$$\begin{aligned}
\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r^T}}[\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}]^{-1}\widehat{\mathcal{Q}}_m &= \Bigg(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_j\Big] + \widehat{H}^{in}(\boldsymbol{z}_m)\Bigg),\\
\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r^T}}[\mathcal{P}_{D_m}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^T}}]^{-1}\widehat{\mathcal{R}}_m &= \Bigg(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times\boldsymbol{\mathcal{A}}^j_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_j\Big] + \widehat{E}^{in}(\boldsymbol{z}_m)\Bigg),
\end{aligned} \tag{5.31}$$

are invertible, provided

$$\frac{\boldsymbol{\delta}}{\boldsymbol{a}} = \boldsymbol{c}_r \geq 3|k|\lambda^+_{(\varepsilon_r,\mu_r)}. \tag{5.32}$$

For $|k|>1$ we have the following estimates

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}|^2\Big)^{\frac{1}{2}} \leq \frac{9\lambda^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3}{8}\left(\Big(\sum_{m=1}^{\aleph}|H^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}} + \frac{1}{3}\Big(\sum_{m=1}^{\aleph}|E^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\right), \tag{5.33}$$

and

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}_m^{\varepsilon_r}|^2\Big)^{\frac{1}{2}} \leq \frac{9\lambda^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3}{8}\left(\frac{1}{3}\Big(\sum_{m=1}^{\aleph}|H^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}} + \Big(\sum_{m=1}^{\aleph}|E^{in}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\right). \tag{5.34}$$

Furthermore $(\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_m, \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_m)_{m=1}^{\aleph}$ *satisfy similar estimates.*

Proof. We deal with the case of real valued coefficients (i.e the estimates (5.7) are fulfilled). When either one of them or both are complex valued, we procceed in the same way taking the real parts in the the coming derivations.

- (*First linear system*) We start by writing the system (5.30) as follows

$$\begin{aligned}
\boldsymbol{\mathcal{Q}}_m - \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_m}^{\varepsilon_r^T}]\boldsymbol{\mathcal{R}}_j\Big] &= H^{\text{in}}(\boldsymbol{z}_m),\\
\boldsymbol{\mathcal{R}}_m - \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_m}^{\varepsilon_r}]\boldsymbol{\mathcal{R}}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j\Big] &= E^{\text{in}}(\boldsymbol{z}_m),
\end{aligned} \tag{5.35}$$

where $\boldsymbol{\mathcal{Q}}_j := [\mathcal{P}_{D_j}^{\mu_r^T}]^{-1}\widehat{\mathcal{Q}}_j^{\mu_r}$ and $\boldsymbol{\mathcal{R}}_j := [\mathcal{P}_{D_j}^{\varepsilon_r^T}]^{-1}\widehat{\mathcal{R}}_j^{\varepsilon_r}$ for $j \in \{1, ..., \aleph\}$.
Due to the mean value property for harmonic functions, we have

$$\begin{aligned}
&|B(0,\boldsymbol{\delta}/4)|^2 \sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_0(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\Big]\\
&\qquad = \sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\int_{B(\boldsymbol{z}_j,\boldsymbol{\delta}/4)}\Big[\Pi_0(x,y)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\Big]dydx,\\
&\qquad = \sum_{m=1}^{\aleph}\Bigg(\int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\int_{B(\boldsymbol{z},\rho)}\Big[\Pi_0(x,y)\sum_{j\geq 1}^{\aleph}\chi_{D_j}(x)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\Big]dydx\\
&\qquad - \int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\int_{B(\boldsymbol{z}_m,\boldsymbol{\delta}/4)}\Big[\Pi_0(x,y)\chi_{D_m}(x)[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\Big]dydx\Bigg),
\end{aligned} \tag{5.36}$$

for some $\boldsymbol{z}$ and ρ such that $\boldsymbol{z} \in \cup_{m=1}^{\aleph}(B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4} := B(\boldsymbol{z}_m, \boldsymbol{\delta}/4)) \subset B(\boldsymbol{z}, \rho)$.
We also get, for $Q(x) = \sum_{j\geq 1}^{\aleph}\chi_{D_j}(x)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j$,

$$\begin{aligned}
&-|B_0^{\boldsymbol{\delta}/4}|^2 \sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_0(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\Big]\\
&\qquad = -\int_{B(\boldsymbol{z},\rho)}\mathcal{N}^{0,I}_{B(\boldsymbol{z},\rho)}(Q)\cdot Q dv + \sum_{m=1}^{\aleph}\int_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}}\int_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}}\Big[\Pi_0(x,y)[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\Big]dydx,
\end{aligned}$$

which give

$$-\sum_{\substack{m,j=1\\ j\neq m}}^{\aleph}\Big[\Pi_0(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\Big] \geq \frac{\sum_{m=1}^{\aleph}}{|B_0^{\boldsymbol{\delta}/4}|^2}\int_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}}\mathcal{N}^{0,I}_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}}(Q)\cdot Q dv \tag{5.37}$$

With this in mind we have, multiplying each side of the first equation of (5.35) by $[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m$ and summing over m,

$$\begin{aligned}
&\sum_{m=1}^{\aleph}\boldsymbol{\mathcal{Q}}_m\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m - \sum_{\substack{m,j=1\\ j\neq m}}^{\aleph}\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\\
&+\sum_{\substack{m,j=1\\ j\neq m}}^{\aleph} ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_m}^{\varepsilon_r^T}]\boldsymbol{\mathcal{R}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m = \sum_{m=1}^{\aleph}H^{\text{in}}(\boldsymbol{z}_m)\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m.
\end{aligned}$$

Using (5.37), we get

$$\begin{aligned}
&\sum_{m=1}^{\aleph}\boldsymbol{\mathcal{Q}}_m\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m - \sum_{\substack{m,j=1\\ j\neq m}}^{\aleph}(\Pi_k-\Pi_0)(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{P}_{D_j}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_j[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\\
&+\sum_{\substack{m,j=1\\ j\neq m}}^{\aleph} ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{P}_{D_j}^{\varepsilon_r^T}]\boldsymbol{\mathcal{R}}_j\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m\\
&+\frac{\sum_{m=1}^{\aleph}}{|B_0^{\boldsymbol{\delta}/4}|^2}\int_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}}\mathcal{N}^{0,I}_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}}(Q)\cdot Q dv \leq \sum_{m=1}^{\aleph}H^{\text{in}}(\boldsymbol{z}_m)\cdot[\mathcal{P}_{D_m}^{\mu_r^T}]\boldsymbol{\mathcal{Q}}_m.
\end{aligned} \tag{5.38}$$

As we have, see ([17], Theorem 9.9),

$$\left\| \int_{\mathbb{R}^3} \Pi_0(\cdot, y) \chi_{B(\boldsymbol{z}_m,1)}(y) dy \right\|_{\mathbb{L}^2(\mathbb{R}^3)} = \left\| \chi_{B(\boldsymbol{z}_m,1)} \right\|_{\mathbb{L}^2(\mathbb{R}^3)} \tag{5.39}$$

we get, with $S_{\aleph}(\Pi_0)$ standing for the third term of the left-hand side of (5.38),[12]

$$\begin{aligned}
\frac{\sum_{m=1}^{\aleph}}{|B_0^{\boldsymbol{\delta}/4}|^2} \int_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}} \mathcal{N}^{0,I}_{B_{\boldsymbol{z}_m}^{\boldsymbol{\delta}/4}}(Q) \cdot Q dv &= \frac{\sum_{m=1}^{\aleph}}{(\boldsymbol{\delta}/4)^3 |B_0^1|^2} \int_{B(\boldsymbol{z}_m,1)} \int_{B(\boldsymbol{z}_m,1)} \Big[\Pi_0(x,y) [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m \Big] dy dx, \\
&\leq \frac{\sum_{m=1}^{\aleph}}{(\boldsymbol{\delta}/4)^3 |B_0^1|^2} \left\| \int_{B_{\boldsymbol{z}_m}^1} \Pi_0(x,y) [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m dy \right\|_{\mathbb{L}^2(B_{\boldsymbol{z}_m}^1)} \left\| [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m \right\|_{\mathbb{L}^2(B_{\boldsymbol{z}_m}^1)} \\
&\leq \frac{3 * 4^2}{(\pi \boldsymbol{\delta}^3)} \sum_{m=1}^{\aleph} |[\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m|^2,
\end{aligned}$$

which, replaced in (5.38) together with (A.4) for $\alpha = 0$, give

$$\begin{aligned}
&\sum_{m=1}^{\aleph} \boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m - \sum_{\substack{m,j=1 \\ j \neq m}}^{\aleph} \left(|k|^2 \Big(\frac{2}{\boldsymbol{\delta}_{mj}} + |k| \Big) + \frac{|k|^2}{\boldsymbol{\delta}_{mj}} \right) \left| [\mathcal{P}_{D_j}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_j \right| \left| [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m \right| \\
&- \sum_{\substack{m,j=1 \\ j \neq m}}^{\aleph} \left(\frac{|k|}{\boldsymbol{\delta}_{mj}} \Big(\frac{1}{\boldsymbol{\delta}_{mj}} + |k| \Big) \right) \left| [\mathcal{P}_{D_j}^{\varepsilon_r^T}] \boldsymbol{\mathcal{R}}_j \right| \left| [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m \right| \\
&- \frac{3 * 4^2}{\pi \boldsymbol{\delta}^3} \sum_{m=1}^{\aleph} |[\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m|^2 \leq \sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)| \, |[\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m|.
\end{aligned} \tag{5.40}$$

We get from the above inequality, with computations similar to those carried out in (4.22),

$$\begin{aligned}
&\sum_{m=1}^{\aleph} \boldsymbol{\mathcal{Q}}_m \cdot [\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m - \Big(\frac{3|k|^2}{(\boldsymbol{a}/2 + \boldsymbol{\delta})^{\frac{1}{2}} \boldsymbol{\delta}} + \frac{|k|(|k|^2 + 1)}{(\boldsymbol{a}/2 + \boldsymbol{\delta})^{\frac{3}{2}}} \Big) \frac{1}{(\boldsymbol{a}/2 + \boldsymbol{\delta})^{\frac{3}{2}}} \sum_{m=1}^{\aleph} |[\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m|^2 \\
&- \left(\frac{|k|}{\boldsymbol{\delta}} \Big(\frac{1}{(\boldsymbol{a}/2 + \boldsymbol{\delta}) \boldsymbol{\delta}} + \frac{|k|}{(\boldsymbol{a}/2 + \boldsymbol{\delta})^2} \Big) \right) \Big(\sum_{m=1}^{\aleph} |[\mathcal{P}_{D_m}^{\varepsilon_r^T}] \boldsymbol{\mathcal{R}}_m|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |[\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m|^2 \Big)^{\frac{1}{2}} \\
&- \frac{3 * 4^2}{\pi \boldsymbol{\delta}^3} \sum_{m=1}^{\aleph} |[\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m|^2 \leq \sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)| |[\mathcal{P}_{D_m}^{\mu_r^T}] \boldsymbol{\mathcal{Q}}_m|,
\end{aligned}$$

then, with the original notations, we get

$$\begin{aligned}
&\sum_{m=1}^{\aleph} [\mathcal{P}_{D_m}^{\mu_r^T}]^{-1} \widehat{\mathcal{Q}}_m^{\mu_r} \cdot \widehat{\mathcal{Q}}_m^{\mu_r} - \Big(\frac{3\sqrt{2}|k|^2}{(1 + 2\boldsymbol{c}_r)^{\frac{1}{2}} \boldsymbol{c}_r} + \frac{\sqrt{2}^3 |k|(|k|^2 + 1)}{(1 + 2\boldsymbol{c}_r)^{\frac{3}{2}}} \Big) \frac{\sqrt{2}^3}{(1 + 2\boldsymbol{c}_r)^{\frac{3}{2}}} \frac{\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2}{\boldsymbol{a}^3} \\
&- \left(\frac{|k|}{\boldsymbol{c}_r} \Big(\frac{2}{(1 + 2\boldsymbol{c}_r)} + \frac{2|k|}{(1 + \boldsymbol{c}_r)^2} \Big) \right) \frac{\Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{R}}_m^{\varepsilon_r}|^2 \Big)^{\frac{1}{2}} \Big(\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2 \Big)^{\frac{1}{2}}}{\boldsymbol{a}^3} \\
&- \frac{3 * 4^2}{\pi \boldsymbol{c}_r} \frac{\sum_{m=1}^{\aleph} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2}{\boldsymbol{a}^3} \leq \sum_{m=1}^{\aleph} |H^{\text{in}}(\boldsymbol{z}_m)| |\mathcal{Q}_m^{\mu_r}|.
\end{aligned} \tag{5.41}$$

Due to (5.7), considering that, both

$$\frac{1}{\boldsymbol{a}^3 \lambda^{+}_{(\varepsilon_r, \mu_r)}} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2 \leq [\mathcal{P}_{D_m}^{\mu_r^T}]^{-1} \widehat{\mathcal{Q}}_m^{\mu_r} \cdot \widehat{\mathcal{Q}}_m^{\mu_r} \leq \frac{1}{\boldsymbol{a}^3 \lambda^{-}_{(\varepsilon_r, \mu_r)}} |\widehat{\mathcal{Q}}_m^{\mu_r}|^2 \tag{5.42}$$

[12] Here we re-scaled the variables of integration.

and

$$\frac{1}{\boldsymbol{a}^3\lambda^+_{(\varepsilon_r,\mu_r)}}|\widehat{\mathcal{R}}^{\varepsilon_r}_m|^2 \leq [\mathcal{P}^{\varepsilon_r^T}_{D_m}]^{-1}\widehat{\mathcal{R}}^{\varepsilon_r}_m \cdot \widehat{\mathcal{R}}^{\varepsilon_r}_m \leq \frac{1}{\boldsymbol{a}^3\lambda^-_{(\varepsilon_r,\mu_r)}}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2 \tag{5.43}$$

are satisfied, both (5.41) and the obtained result from repeating the same computation for the second equation, gives, with $\boldsymbol{c}_r \geq 3|k|\lambda^+_{(\varepsilon_r,\mu_r)}$ and $|k|>1$,

$$(\frac{7}{9}-\boldsymbol{a}^3)\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2 - \frac{2}{9}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\varepsilon_r}_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}} \leq \lambda^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3\sum_{m=1}^{\aleph}|H^{\mathrm{in}}(\boldsymbol{z}_m)||\widehat{\mathcal{Q}}^{\mu_r}_m|,$$

and

$$(\frac{7}{9}-\boldsymbol{a}^3)\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\mu_r}_m|^2 - \frac{2}{9}\Big(\sum_{m=1}^{\aleph}|\mathcal{Q}^{\varepsilon_r}_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}} \leq \lambda^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3\sum_{m=1}^{\aleph}|E^{\mathrm{in}}(\boldsymbol{z}_m)||\widehat{\mathcal{Q}}^{\mu_r}_m|.$$

These estimates, for $\boldsymbol{a} \leq \frac{1}{3}$, give

$$\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2 - \frac{1}{3}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\varepsilon_r}_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}} \leq \lambda^+_{(\varepsilon_r,\mu_r)}\frac{\boldsymbol{a}^3}{3}\Big(\sum_{m=1}^{\aleph}|H^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}},$$

and

$$\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\mu_r}_m|^2 - \frac{1}{3}\Big(\sum_{m=1}^{\aleph}|\mathcal{Q}^{\varepsilon_r}_m|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}} \leq \lambda^+_{(\varepsilon_r,\mu_r)}\frac{\boldsymbol{a}^3}{3}\Big(\sum_{m=1}^{\aleph}|E^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}}.$$

Then follows the conclusion, namely

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}} \leq \lambda^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3\Big(\sum_{m=1}^{\aleph}|H^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}} + \frac{1}{3}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\varepsilon_r}_m|^2\Big)^{\frac{1}{2}},$$

and

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}^{\mu_r}_m|^2\Big)^{\frac{1}{2}} \leq \lambda^+_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3\Big(\sum_{m=1}^{\aleph}|E^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}} + \frac{1}{3}\Big(\sum_{m=1}^{\aleph}|\mathcal{Q}^{\varepsilon_r}_m|^2\Big)^{\frac{1}{2}}.$$

- (*Non symmetric case*). We could deduce the estimate from the above computations writing down the linear system as follows. First, we set, for $m \in \{1,...,\aleph\}$,

$$\begin{aligned}
[\mathcal{T}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}_{D_m}]^{-1} :=& \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r^T}}[\mathcal{P}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}_{D_m}]^{-1}(\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}})^{-1},\\
[\mathcal{T}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^T}}_{D_m}]^{-1} :=& \boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r^T}}[\mathcal{P}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^T}}_{D_m}]^{-1}(\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}})^{-1},
\end{aligned}$$

and, this time,

$$\boldsymbol{\mathcal{R}}_m := [\mathcal{T}^{\boldsymbol{\mathcal{A}}^m_{\varepsilon_r^T}}_{D_m}]^{-1}\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_m, \tag{5.44}$$

$$\boldsymbol{\mathcal{Q}}_m := [\mathcal{T}^{\boldsymbol{\mathcal{A}}^m_{\mu_r^T}}_{D_m}]^{-1}\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_m, \tag{5.45}$$

the system (5.31) can, hence, be written as

$$\begin{aligned}
\boldsymbol{\mathcal{Q}}_m - &\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{T}^{\boldsymbol{\mathcal{A}}\mu_r^T}_{D_j}]\boldsymbol{\mathcal{Q}}_j - ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{T}^{\boldsymbol{\mathcal{A}}\varepsilon_r^T}_{D_m}]\boldsymbol{\mathcal{R}}_j\Big] = H^{\mathrm{in}}(\boldsymbol{z}_m),\\
\boldsymbol{\mathcal{R}}_m - &\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big[\Pi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)[\mathcal{T}^{\boldsymbol{\mathcal{A}}\varepsilon_r^T}_{D_m}]\boldsymbol{\mathcal{R}}_j + ik\nabla\Phi_k(\boldsymbol{z}_m,\boldsymbol{z}_j)\times[\mathcal{T}^{\boldsymbol{\mathcal{A}}\mu_r^T}_{D_m}]\boldsymbol{\mathcal{Q}}_j\Big] = E^{\mathrm{in}}(\boldsymbol{z}_m).
\end{aligned} \tag{5.46}$$

Hence, following the same lines as in the proof for the first linear system, we get

$$\begin{aligned}\Big(\sum_{m=1}^{\aleph}|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\mu_r}}\widehat{\mathcal{Q}}_m|^2\Big)^{(1/2)} &\leq \frac{9\lambda^{+}_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3}{8}\left(\Big(\sum_{m=1}^{\aleph}|H^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}+\frac{1}{3}\Big(\sum_{m=1}^{\aleph}|E^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\right),\\ \Big(\sum_{m=1}^{\aleph}|\boldsymbol{\mathcal{A}}_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\widehat{\mathcal{R}}_m|^2\Big)^{(1/2)} &\leq \frac{9\lambda^{+}_{(\varepsilon_r,\mu_r)}\boldsymbol{a}^3}{8}\left(\frac{1}{3}\Big(\sum_{m=1}^{\aleph}|H^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}+\Big(\sum_{m=1}^{\aleph}|E^{\mathrm{in}}(\boldsymbol{z}_m)|^2\Big)^{\frac{1}{2}}\right),\end{aligned} \tag{5.47}$$

with unchanged condition on

$$\boldsymbol{c}_r = 3|k|\lambda^{+}_{(\varepsilon_r,\mu_r)}, |k|>1. \tag{5.48}$$

For the last statement, it suffices to observe, that $\boldsymbol{\mathcal{C}}_B$ and its transpose $\boldsymbol{\mathcal{C}}_B^T$ share the same eigenvalues, hence, leave those of $[\mathcal{P}^{A^T}_{D_m}]^{-1}$ identical to those of $[\mathcal{T}^{A^T}_{D_m}]^{-1}$.

□

Proof. (Of Theorem 2.1)

- We start with (2.19), noticing that, for $(\widehat{\mathcal{Q}}^{\mu_r}_m, \widehat{\mathcal{R}}^{\mu_r}_m)_{m=1}^{\aleph}$ solution of (5.30) and $(\mathcal{Q}^{\mu_r}_m, \mathcal{R}^{\mu_r}_m)_{m=1}^{\aleph}$ solution of (5.16) then $(\widehat{\mathcal{Q}}^{\mu_r}_m - \mathcal{Q}^{\mu_r}_m, \widehat{\mathcal{R}}^{\mu_r}_m - \mathcal{R}^{\mu_r}_m)_{m=1}^{\aleph}$ solves (5.30) with

$$\widehat{H}^{\mathrm{in}}(\boldsymbol{z}_m) = [\mathcal{P}^{\mu_r^T}_{D_m}]^{-1}\left(Er_m(H,E)+O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}+k^2\boldsymbol{a}^4\Big)\right) \tag{5.49}$$

and

$$\widehat{E}^{\mathrm{in}}(\boldsymbol{z}_m) = [\mathcal{P}^{\varepsilon_r^T}_{D_m}]^{-1}\left(Er_m(E,H)+O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}+k^2\boldsymbol{a}^4\Big)\right). \tag{5.50}$$

Thus, with the estimates (5.33), (5.34), (5.42) and the aid of (5.10) for the third inequality hereafter, we have

$$\begin{aligned}&\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m-\mathcal{Q}^{\mu_r}_m|^2\Big)^{\frac{1}{2}}\\ &\leq \frac{9\lambda^{+}_{(\varepsilon_r,\mu_r)}}{8\lambda^{-}_{(\varepsilon_r,\mu_r)}}\left[\left\{\sum_{m=1}^{\aleph}\Big(Er_m(H,E)+O\big(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}+k^2\boldsymbol{a}^4\big)\Big)^2\right\}^{\frac{1}{2}}\right.\\ &\qquad\left.+\frac{1}{3}\left\{\sum_{m=1}^{\aleph}\Big(Er_m(E,H)+O\big(k^2\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}+|k|\boldsymbol{a}^4\big)\Big)^2\right\}^{\frac{1}{2}}\right],\end{aligned}$$

then

$$\begin{aligned}&\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}^{\mu_r}_m-\mathcal{Q}^{\mu_r}_m|^2\Big)^{\frac{1}{2}}\\ &\leq \frac{9\lambda^{+}_{(\varepsilon_r,\mu_r)}}{8\lambda^{-}_{(\varepsilon_r,\mu_r)}}\left[\Big(\sum_{m=1}^{\aleph}Er_m(H,E)^2\Big)^{\frac{1}{2}}+\left(\sum_{m=1}^{\aleph}O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}+k^2\boldsymbol{a}^4\Big)^2\right)^{\frac{1}{2}}\right.\\ &\qquad\left.+\Big(\sum_{m=1}^{\aleph}Er_m(E,H)^2\Big)^{\frac{1}{2}}+\left(\sum_{m=1}^{\aleph}O\Big(k^2\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{7}{2}}+\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|_{\mathbb{L}^2(D_m)}\boldsymbol{a}^{\frac{5}{2}}+|k|\boldsymbol{a}^4\Big)^2\right)^{\frac{1}{2}}\right],\end{aligned}$$

and then

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}_m^{\mu_r}|^2\Big)^{\frac{1}{2}}$$

$$\leq \frac{9\lambda^+_{(\varepsilon_r,\mu_r)}}{8\lambda^-_{(\varepsilon_r,\mu_r)}}\Bigg[O\Bigg(\|H\|^2_{\mathbb{L}^2(D)}\frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8}+\Big(\|E\|^2_{\mathbb{L}^2(D)}+\|H\|^2_{\mathbb{L}^2(D)}\Big)\frac{(|k|+2)^3\boldsymbol{a}^{11}|\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6}\Bigg)^{\frac{1}{2}}$$

$$+O\Bigg(k^2(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^7)^{\frac{1}{2}}+(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^5)^{\frac{1}{2}}\Bigg)$$

$$+O\Bigg(\|E\|^2_{\mathbb{L}^2(D)}\frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8}+\Big(\|E\|^2_{\mathbb{L}^2(D)}+\|H\|^2_{\mathbb{L}^2(D)}\Big)\frac{(|k|+2)^3\boldsymbol{a}^{11}|\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6}\Bigg)^{\frac{1}{2}}$$

$$+O\Bigg(k^2(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\varepsilon_r}E\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^7)^{\frac{1}{2}}+(\sum_{m=1}^{\aleph}\|\boldsymbol{\mathcal{C}}_{\mu_r}H\|^2_{\mathbb{L}^2(D_m)}\boldsymbol{a}^5)^{\frac{1}{2}}\Bigg)+O\Big(\boldsymbol{a}^{\frac{5}{2}}\Big)\Bigg].$$

With the estimates (5.12) and (5.13), for the scattering of plane waves, we obtain

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}_m^{\mu_r}|^2\Big)^{\frac{1}{2}}\leq\frac{4\lambda^+_{(\varepsilon_r,\mu_r)}}{8\lambda^-_{(\varepsilon_r,\mu_r)}}\times\frac{\boldsymbol{c}_\infty(|k|+1)}{\min(c_\infty^{\varepsilon-},c_\infty^{\mu-})\boldsymbol{c}_r^{\frac{3}{2}}}\Bigg[O\Bigg(\frac{\boldsymbol{a}^{11}}{\boldsymbol{\delta}^8}+\frac{(|k|+2)^3\boldsymbol{a}^{11}|\ln(\boldsymbol{\delta})|}{\boldsymbol{\delta}^6}\Bigg)^{\frac{1}{2}} +O\big(k^2c_\infty\boldsymbol{a}^{7/2}+c_\infty\boldsymbol{a}^{5/2}\big)\Bigg], \tag{5.51}$$

hence,

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}_m^{\mu_r}|^2\Big)^{\frac{1}{2}}=O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3/2}}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\boldsymbol{a}^{3/2}\Big), \tag{5.52}$$

and with similar computations follows

$$\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}_m^{\mu_r}-\mathcal{R}_m^{\mu_r}|^2\Big)^{\frac{1}{2}}=O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3/2}}\Bigg[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Bigg]\boldsymbol{a}^{3/2}\Big), \tag{5.53}$$

with

$$\boldsymbol{c}_{(\varepsilon_r,\mu_r)}:=\frac{4\lambda^+_{(\varepsilon_r,\mu_r)}}{8\lambda^-_{(\varepsilon_r,\mu_r)}}\times\frac{\boldsymbol{c}_\infty(|k|+1)}{\min(c_\infty^{\varepsilon-},c_\infty^{\mu-})}.$$

Recalling the approximation (5.14), we have

$$\begin{aligned}E^\infty(\hat{x})&=\sum_{m=1}^{\aleph}\Bigg(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\mathcal{R}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\mathcal{Q}_m^{\mu_r}\Bigg)+O\big(\boldsymbol{a}\big),\\&=\sum_{m=1}^{\aleph}\Bigg(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\widehat{\mathcal{R}}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\widehat{\mathcal{Q}}_m^{\mu_r}\Bigg)+O\big(\boldsymbol{a}\big)\\&\quad+\Big(\sum_{m=1}^{\aleph}\frac{|k|^4}{16\pi^2}\big|e^{-i2k\hat{x}\cdot z_m}\big|\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{Q}}_m^{\mu_r}-\mathcal{Q}^{\mu_r}|^2\Big)^{\frac{1}{2}}\\&\quad+\Big(\sum_{m=1}^{\aleph}\frac{|k|^4}{16\pi^2}\big|e^{-i2k\hat{x}\cdot z_m}\big|\Big)^{\frac{1}{2}}\Big(\sum_{m=1}^{\aleph}|\widehat{\mathcal{R}}_m^{\varepsilon_r}-\mathcal{R}^{\varepsilon_r}|^2\Big)^{\frac{1}{2}},\end{aligned} \tag{5.54}$$

which, with the estimations (5.52) and (5.53), reduces to

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\widehat{\mathcal{R}}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\widehat{\mathcal{Q}}_m^{\mu_r}\Big)+O\big(\boldsymbol{a}\big) + \frac{|k|}{2\pi}\frac{1}{\boldsymbol{\delta}^{3/2}}O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3/2}}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\boldsymbol{a}^{3/2}\Big), \tag{5.55}$$

or

$$E^{\infty}(\hat{x}) = \sum_{m=1}^{\aleph}\Big(\frac{k^2}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\Big(\widehat{\mathcal{R}}_m^{\varepsilon_r}\times\hat{x}\Big)+\frac{ik}{4\pi}e^{-ik\hat{x}\cdot\boldsymbol{z}_m}\hat{x}\times\widehat{\mathcal{Q}}_m^{\mu_r}\Big)+O\big(\boldsymbol{a}\big) + \frac{|k|}{2\pi}O\Big(\frac{\boldsymbol{c}_{(\varepsilon_r,\mu_r)}(|k|+1)}{\boldsymbol{c}_r^{3}}\Big[\frac{1}{\boldsymbol{c}_r^4}+\boldsymbol{a}\,|\ln(\boldsymbol{c}_r\boldsymbol{a})|+\boldsymbol{a}\Big]\Big). \tag{5.56}$$

- The approximation (2.21) can be justified in a similar way replacing, respectively, $\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\varepsilon_r}}\mathcal{R}_m$ and $\boldsymbol{\mathcal{A}}^m_{\boldsymbol{\mathcal{C}}_{\mu_r}}\mathcal{Q}_m$ by $\mathcal{R}_m$ and $\mathcal{Q}_m$.

□

A Appendix

A.1 Green function approximations

A simple application of mean value theorem gives

$$\begin{aligned}
\big(\Phi_k(x,y)-\Phi_k(\boldsymbol{z}_m,y)\big) &= \int_0^1 \nabla\Phi_k(tx+(1-t)\boldsymbol{z}_m,y)dt\circ(x-\boldsymbol{z}_m) = O\Big(\frac{1}{\boldsymbol{\delta}_{mj}}\big(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|\big)\boldsymbol{a}\Big),\\
\nabla\big(\Phi_k(x,y)-\Phi_k(\boldsymbol{z}_m,y)\big) &= \int_0^1 D^2\Phi_k(tx+(1-t)\boldsymbol{z}_m,y)dt\circ(x-\boldsymbol{z}_m) = O\Big(\frac{1}{\boldsymbol{\delta}_{mj}}\big(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|\big)^2\boldsymbol{a}\Big),\\
\nabla\nabla\big(\Phi_k(x,y)-\Phi_k(\boldsymbol{z}_m,y)\big) &= \int_0^1 D^3\Phi_k(tx+(1-t)\boldsymbol{z}_m,y)dt\circ(x-\boldsymbol{z}_m) = O\Big(\frac{1}{\boldsymbol{\delta}_{mj}}\big(\frac{1}{\boldsymbol{\delta}_{mj}}+|k|\big)^3\boldsymbol{a}\Big).
\end{aligned} \tag{A.1}$$

whenever $y\in D_j$ and $j\neq m$. We also have the following first order expansion of the Green's function[13]

$$[\Phi_k-\Phi_{i\alpha}](x) = \frac{(ik-\alpha)}{4\pi}\int_{[0,1]} e^{\big((ik)t-(1-t)\alpha\big)|x|}\,dt, \tag{A.2}$$

$$\nabla[\Phi_k-\Phi_{i\alpha}](x) = \Big[\frac{(ik-\alpha)}{4\pi}\int_{[0,1]} e^{(ikt-(1-t)\alpha)|x|}\,\big(ikt+(1-t)\alpha\big)dt\Big]\,\frac{x}{|x|}, \tag{A.3}$$

$$[\otimes\nabla]^2[\Phi_k-\Phi_{i\alpha}](x) = \int_0^1 \frac{e^{(ikt-(1-t)\alpha)|x|}\big(ikt+(1-t)\alpha\big)}{4\pi\,(ik-\alpha)^{-1}|x|}\Big[I+\big(ikt+(1-t)\alpha-\frac{1}{|x|}\big)\frac{[\otimes x]^2}{|x|}\Big]dt. \tag{A.4}$$

A.2 Counting lemma

Lemma A.1. *For any non negative function g we have*

$$\sum_{j\geq 1, j\neq m}^{\aleph} g(\boldsymbol{\delta}_{mj}) \leq 48 \sum_{\substack{1\leq l\leq\aleph\\ O\leq i\leq k\leq l}} g\Big(\big[\big(l^2+k^2+i^2\big)^{\frac{1}{2}}(\boldsymbol{c}_r+1)-1\big]\boldsymbol{a}\Big), \tag{A.5}$$

[13] By $[\otimes x]^p$ we mean the p-times repeated tensor product of x.

and for any non negative sequence $(\alpha_j)_{j=1}^{\aleph}$, *we have*

$$\sum_{m=1}^{\aleph}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^q}\Big)^2 \leq \Big(\frac{c_0}{\boldsymbol{\delta}^q}\sum_{l=1}^{\aleph^{\frac{1}{3}}} l^{2-q}\Big)^2\sum_{j=1}^{\aleph}\alpha_j^2. \tag{A.6}$$

where c_0 *is a positive number.*

Proof. We first address the following observation. The worst case (i.e the maximum number of inhomogeneities around a fixed one, let's say $\boldsymbol{z}_m$) is described by the tessellation of a sphere of diameter $n\boldsymbol{\delta}$, $n \in \mathbb{N}$, by equilateral triangles of side $\boldsymbol{\delta} + \boldsymbol{a}$, where the vertices stands for the position of the inhomogeneities. Therefore, we will only consider a periodically distributed of these inhomogeneities, since the results will differ by a multiplicative constant. The reason is that there exists a bounded homeomorphism, which is a radial projection, between the two configurations. To justify this last statement, let $CU(0, r)$ be a cube of fixed side r centered at the origin whose sides are parallel to the coordinates axis.
For $y = (y_1, y_2, y_3)$, we set $\tau_r(y) := \frac{r}{2|y|_\infty} y$ where $|y|_\infty := \sup_{j=1,2,3}|y_j|$. Let $B(0, r/2)$ be the ball of center the origin and radius $r/2$. It is obvious that $B(0, r/2) \subset CU(0, r)$. In addition, for any y we have $\tau_r(y) \in \partial CU(0, r)$. Indeed, as $\partial CU(0, r)$ is the union of six truncated plans, namely $\{x \in \mathbb{R}^3;\ x_i = \pm r/2, |x_j|_{j\neq i} \leq r/2\}_{i=1,2,3}$, and $|y|_\infty = y_i$, for some $i \in \{1, 2, 3\}$, then putting

$$x_i := \frac{r}{2}\frac{y_i}{|y|_\infty} = \pm r/2,\ x_j := \frac{r}{2}\frac{y_j}{|y|_\infty} \text{ for } j \neq i$$

guaranties us that $x := \tau_r(y) \in \cup_{i=1}^3\{x \in \mathbb{R}^3;\ x_i = \pm r/2, |x_j|_{j\neq i} \leq r/2\}$.

Let us prove that $\tau : \partial B(0, r/2) \to \partial CU(0, r)$ is a bounded homeomorphism.

To show the injectivity of τ, let y^1, y^2 be any two points such that $y^1/|y^1|_\infty = y^2/|y^2|_\infty$ then we have, for $i = 1, 2, 3$, $y_i^1/|y^1|_\infty = y_i^2/|y^2|_\infty$, squaring and summing the previous identity gives $|y^2|_\infty/|y^1|_\infty = 1$ which guaranties the injectivity τ_r.

Concerning the surjectivity, it is sufficient to set, for x in the truncated plan $P_i^\pm := \{x \in \mathbb{R}^3;\ x_i = \pm r/2, |x_j|_{j\neq i} \leq r/2\}$,

$$y := \pi(x) := \frac{r}{2|x|_2}x.$$

Obviously $y \in B(0, r/2)$. Let us fix $x \in P_i^+$ for a certain i. As $|y|_\infty = r/2|x_i|/|x|_2 = r^2/(4|x|_2)$, we have

$$\tau(\pi(x)) = \tau(y) = \frac{r}{2}\frac{y}{|y|_\infty} = x.$$

Similar reasoning gives us $\pi\tau(y) = y$ for any $y \in \partial B(0, r/2)$, to conclude $\tau^{-1} = \pi$.

It remains to show that τ and τ^{-1} are continuous. For this purpose, noticing that for any $y \in \partial B(0, r/2)$ we have $|y|_\infty \geq r/2\sqrt{3}$, we obtain

$$\begin{aligned}\big|\tau(y^1) - \tau(y^2)\big|_2 = \frac{r}{2}\Big|\frac{y^1}{|y^1|_\infty} - \frac{y^2}{|y^2|_\infty}\Big|_2 \leq \frac{r}{2}\Bigg|\frac{|y^2|_\infty y^1 - |y^1|_\infty y^2}{|y^1|_\infty|y^2|_\infty}\Bigg|_2 &\leq 6/r\Big(|y^2|_\infty|y^1 - y^2|_2 + |y^2|_2|y^1 - y^2|_\infty\Big)\\ &\leq 6/r\Big(\frac{r}{2}|y^1 - y^2|_2 + \frac{r}{2}|y^1 - y^2|_2\Big)\end{aligned}$$

which is $|\tau(y^1) - \tau(y^2)|_2 \leq 3|y^1 - y^2|_2$. With similar computations, for $x^1, x^2 \in \partial CU(0, r)$, we get

$$\big|\tau^{-1}(x^1) - \tau^{-1}(x^2)\big|_2 = \big|\pi(x^1) - \pi(x^2)\big|_2 =\leq \sqrt{3}\big|x^1 - x^2\big|_2.$$

Hence $\tau : \partial B(0, r/2) \to \partial CU(0, r)$ is a bounded homeomorphism.

Now let $\boldsymbol{z}_i, \boldsymbol{z}_j$ be in $\partial B(\boldsymbol{z}_m, r)$ and set $\boldsymbol{z}_j' = \tau(\boldsymbol{z}_j)$ and $\boldsymbol{z}_i' = \tau(\boldsymbol{z}_i)$. Hence, as we have the continuity of τ^{-1},

$$d(\boldsymbol{z}_j, \boldsymbol{z}_i) = \big|\tau^{-1}(\tau(\boldsymbol{z}_j)) - \tau^{-1}(\tau(\boldsymbol{z}_i))\big|_2 \leq \sqrt{3}|\tau(\boldsymbol{z}_j) - \tau(\boldsymbol{z}_i)|_2 = \sqrt{3}d(\boldsymbol{z}_j', \boldsymbol{z}_i') \leq 3\sqrt{3}d(\boldsymbol{z}_j, \boldsymbol{z}_i).$$

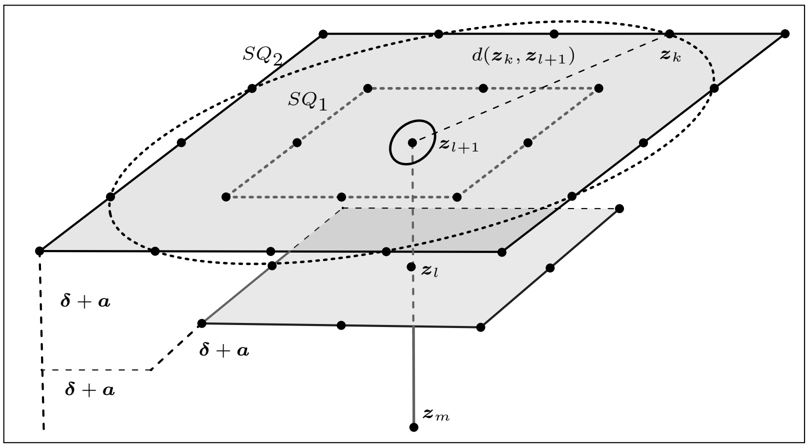

Figure 1: Disposition of the faces F_l.

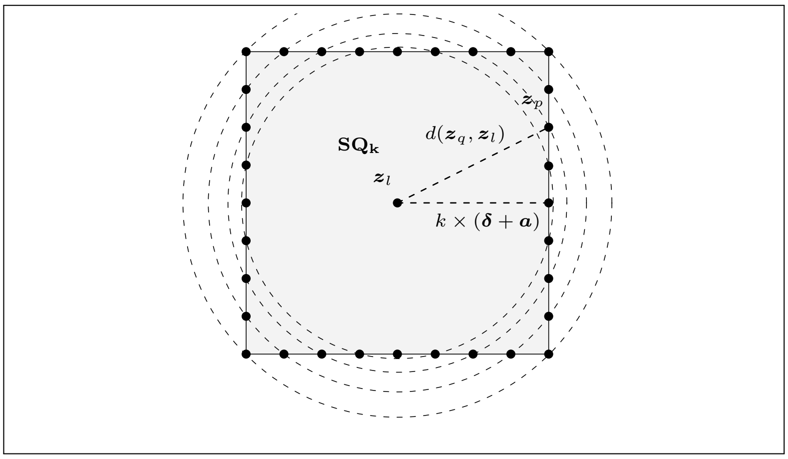

Figure 2: Counting on the square SQ_k.

From a given position $\boldsymbol{z}_m$, we split the space into equidistant cubes (CU_l), centered at $\boldsymbol{z}_m$, such that each of its faces support some of the $(\boldsymbol{z}_j)_{j\geq 1, j\neq m}^{\aleph}$ with $d(CU_l, CU_{l+1}) = \boldsymbol{\delta} + \boldsymbol{a}$.

There is at most $O(\aleph^{\frac{1}{3}})$ of such cubes. Indeed, let p denotes the number of cubes, for a given $l \in \{1, ..., p\}$, we have

$$d(CU_l, \boldsymbol{z}_m) = l(\boldsymbol{\delta} + \boldsymbol{a}).$$

As there are six faces on CU_l, we also have a total surface of

$$|\partial(CU_l)| := 6(2l(\boldsymbol{\delta} + \boldsymbol{a}))^2.$$

Now, on the surface of a given cube CU_l, each inhomogeneity occupies a total surface of

$$S(B(\boldsymbol{z}_m, \boldsymbol{a} + \boldsymbol{\delta})) := |\partial(CU_l) \cap B(\boldsymbol{z}_m, \boldsymbol{a} + \boldsymbol{\delta})| = \pi(\boldsymbol{a} + \boldsymbol{\delta})^2$$

hence, each cube may contain $6(2l(\boldsymbol{\delta} + \boldsymbol{a}))^2/\pi(\boldsymbol{a} + \boldsymbol{\delta})^2 = 24l^2/\pi$ inhomogeneities on its surface. Then

$$(\aleph - 1) = \sum_{\boldsymbol{z}_j, j\neq m} 1 = \sum_{l=1}^{p} \text{cardinal}\{\boldsymbol{z}_j \in CU_l\} \geq \frac{24}{\pi} \sum_{l=1}^{p} l^2 \geq p(p+1)(2p+1)$$

which means that p is at most of the order $\aleph^{1/3}$.

Now, if $(F_l)_{l=1}^p$ stands, respectively, for one of the faces of $(CU_l)_{l=1}^p$, chosen to have the same orientation (i.e $d(F_{l\pm1}, F_l) = \boldsymbol{a} + \boldsymbol{\delta}$ (see fig. 1), then, for z_l standing for the orthogonal projection of $\boldsymbol{z}_m$ on F_l, the distance from a point $\boldsymbol{z}_j \in F_l$ to $\boldsymbol{z}_m$ is[14]

$$d(\boldsymbol{z}_m, \boldsymbol{z}_j) = \sqrt{d(\boldsymbol{z}_m, F_l)^2 + d(\boldsymbol{z}_j, \boldsymbol{z}_l)^2}, \text{ with } d(\boldsymbol{z}_m, F_l) = d(\boldsymbol{z}_m, \boldsymbol{z}_l) = l(\boldsymbol{\delta} + \boldsymbol{a}). \tag{A.7}$$

Analogously, we split each face F_l with concentric squares $(SQ_k)_{k=1}^l$, centered at $\boldsymbol{z}_l$ (the orthogonal projection of $\boldsymbol{z}_m$ on F_l). There is 4 or at most 8 locations that are equidistant from a given square SQ_k to $\boldsymbol{z}_l$ which corresponds to the intersections of a circle and a square sharing the same center. Similarly, for a point $z_p \in SQ_k$, we get, with z_k standing for the orthogonal projection of $\boldsymbol{z}_l$ on one sides of SQ_k,

$$d(\boldsymbol{z}_p, \boldsymbol{z}_l) = \sqrt{(d(\boldsymbol{z}_p, \boldsymbol{z}_k)^2 + d(SQ_k, \boldsymbol{z}_l)^2)}, \text{ with } d(SQ_k, \boldsymbol{z}_l) = k(\boldsymbol{\delta} + \boldsymbol{a}). \tag{A.8}$$

As

$$d(B^{\boldsymbol{a}}_{\boldsymbol{z}_m}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j}) = d(\boldsymbol{z}_m, \boldsymbol{z}_j) - \boldsymbol{a},$$

and for a non negative function g, with (A.7), we get

$$\begin{aligned}
\sum_{j(\neq m)=1}^{\aleph} g\big(d(B^{\boldsymbol{a}}_{\boldsymbol{z}_m}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j})\big) = \sum_{j(\neq m)=1}^{\aleph} g\big(d(\boldsymbol{z}_m, \boldsymbol{z}_j) - \boldsymbol{a}\big) &= \sum_{l=1}^{\aleph^{\frac{1}{3}}} 6 \sum_{\boldsymbol{z}_j \in F_l} g\big(d(\boldsymbol{z}_j, \boldsymbol{z}_m) - \boldsymbol{a}\big),\\
&= 6 \sum_{l=1}^{\aleph^{\frac{1}{3}}} \sum_{z_j \in F_l} g\Big(\big(d(F_l, \boldsymbol{z}_m)^2 + d(\boldsymbol{z}_l, \boldsymbol{z}_j)^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big)\\
&\leq 6 \sum_{l=1}^{\aleph^{\frac{1}{3}}} \sum_{k=1}^{l} 8 \sum_{z_p \in SQ_k} g\Big(\big(l^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + d(\boldsymbol{z}_l, \boldsymbol{z}_p)^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big),
\end{aligned}$$

with (A.8), we get

$$\begin{aligned}
\sum_{j(\neq m)=1}^{\aleph} g\big(d(B^{\boldsymbol{a}}_{\boldsymbol{z}_m}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j})\big) &\leq 6 \times 8 \sum_{l=1}^{\aleph^{\frac{1}{3}}} \sum_{k=1}^{l} \sum_{p=1}^{k} g\Big(\big(l^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + d(\boldsymbol{z}_l, SQ_k)^2 + d(\boldsymbol{z}_p, \boldsymbol{z}_k)^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big),\\
&\leq 6 \times 8 \sum_{l\geq 1}^{\aleph^{\frac{1}{3}}} \sum_{k=1}^{l} \sum_{p=1}^{k} g\Big(\big(l^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + k^2(\boldsymbol{\delta} + \boldsymbol{a})^2 + p^2(\boldsymbol{\delta} + \boldsymbol{a})^2\big)^{\frac{1}{2}} - \boldsymbol{a}\Big),
\end{aligned}$$

which guaranties that

$$\sum_{j(\neq m)=1}^{\aleph} g\big(d(B^{\boldsymbol{a}}_{\boldsymbol{z}_m}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j})\big) \leq 6 \times 8 \sum_{\substack{1 \leq l \leq \aleph^{\frac{1}{3}}\\ 1 \leq p \leq k \leq l}} g\Big(\big[\big(l^2 + k^2 + p^2\big)^{\frac{1}{2}} (\boldsymbol{c}_r + 1) - 1\big]\boldsymbol{a}\Big). \tag{A.9}$$

To derive (A.6), we start from (A.9), with $g(x) = 1/x^q$, and get

$$\begin{aligned}
\sum_{j(\neq m)=1}^{\aleph} \frac{1}{\big(d(B^{\boldsymbol{a}}_{\boldsymbol{z}_m}, B^{\boldsymbol{a}}_{\boldsymbol{z}_j})\big)^q} &\leq 6 \times 8 \sum_{\substack{1 \leq l \leq \aleph^{\frac{1}{3}}\\ 1 \leq p \leq k \leq l}} \frac{1}{\Big(\big[\big(l^2 + k^2 + p^2\big)^{\frac{1}{2}} (\boldsymbol{c}_r + 1) - 1\big]\boldsymbol{a}\Big)^q},\\
&\leq 6 \times 8 \sum_{1 \leq l \leq \aleph^{\frac{1}{3}}} \sum_{k=1}^{l} k \frac{1}{\Big(\big[\big(l^2 + k^2\big)^{\frac{1}{2}} (\boldsymbol{c}_r + 1) - 1\big]\boldsymbol{a}\Big)^q},\\
&\leq \frac{6 \times 8}{\boldsymbol{a}^q} \sum_{1 \leq l \leq \aleph^{\frac{1}{3}}} l^2 \frac{1}{\Big(\big[l(\boldsymbol{c}_r + 1) - 1\big]\Big)^q},
\end{aligned}$$

[14]Being the segment $[\boldsymbol{z}_j, \boldsymbol{z}_l]$ orthogonal to $[\boldsymbol{z}_m, \boldsymbol{z}_l]$

hence

$$\sum_{j(\neq m)=1}^{\aleph} \frac{1}{\boldsymbol{\delta}_{mj}^{q}} \leq \sum_{j(\neq m)=1}^{\aleph} \frac{1}{\left(d(B_{\boldsymbol{z}_m}^{\boldsymbol{a}}, B_{\boldsymbol{z}_j}^{\boldsymbol{a}})\right)^{q}} \leq \frac{c_0}{(\boldsymbol{c}_r \boldsymbol{a})^{q}} \sum_{1 \leq l \leq \aleph^{\frac{1}{3}}} l^2 \frac{1}{l^q}. \tag{A.10}$$

Using (A.10) and Hölder's inequality for the inner sum, we obtain

$$\begin{aligned}
\sum_{m=1}^{\aleph}\Big(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph} \frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{q}}\Big)^2 &\leq \sum_{m=1}^{\aleph}\left(\left[\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big(\frac{\alpha_j}{\boldsymbol{\delta}_{mj}^{\frac{q}{2}}}\Big)^2\right]^{\frac{1}{2}}\left[\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big(\frac{1}{\boldsymbol{\delta}_{mj}^{\frac{q}{2}}}\Big)^2\right]^{\frac{1}{2}}\right)^2,\\
&\leq \sum_{m=1}^{\aleph}\left(\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\Big(\frac{\alpha_j^2}{\boldsymbol{\delta}_{mj}^{q}}\Big)\ \sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{1}{\boldsymbol{\delta}_{mj}^{q}}\right),\\
&\leq \frac{c_0}{\boldsymbol{\delta}^q}\sum_{l=1}^{\aleph^{\frac{1}{3}}} l^{(2-q)} \sum_{m=1}^{\aleph}\sum_{\substack{j\geq 1\\ j\neq m}}^{\aleph}\frac{\alpha_j^2}{\boldsymbol{\delta}_{mj}^{q}}.
\end{aligned}$$

The proof ends with

$$\sum_{m=1}^{\aleph}\sum_{j\geq 1, m\neq j}^{\aleph}\frac{\alpha_j^2}{\boldsymbol{\delta}_{mj}^{q}} = \sum_{j=1}^{\aleph}\alpha_j^2 \sum_{\substack{m\geq 1\\ m\neq j}}\frac{1}{\boldsymbol{\delta}_{mj}^{q}}.$$

□

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