Johann Radon Institute for
Computational and Applied Mathematics
Austrian Academy of Sciences (ÖAW)

Tractability of approximation in the weighted Korobov space in the worst-case setting – a complete picture

A. Ebert, F. Pillichshammer

RICAM-Report 2021-05

Tractability of approximation in the weighted Korobov space in the worst-case setting — a complete picture

Adrian Ebert and Friedrich Pillichshammer*

Abstract

In this paper, we study tractability of L_2-approximation of one-periodic functions from weighted Korobov spaces in the worst-case setting. The considered weights are of product form. For the algorithms we allow information from the class $\Lambda^{\rm all}$ consisting of all continuous linear functionals and from the class $\Lambda^{\rm std}$, which only consists of function evaluations.

We provide necessary and sufficient conditions on the weights of the function space for quasi-polynomial tractability, uniform weak tractability, weak tractability and (σ,τ)-weak tractability. Together with the already known results for strong polynomial and polynomial tractability, our findings provide a complete picture of the weight conditions for all current standard notions of tractability.

Keywords: L_2-approximation; tractability; Korobov space.
MSC 2020: 65D15, 65Y20, 41A25, 41A63.

1 Introduction

We study tractability of L_2-approximation of multivariate one-periodic functions from weighted Korobov spaces of finite smoothness α in the worst-case setting. This problem has already been studied in a vast number of articles and a lot is known for the two information classes $\Lambda^{\rm all}$ and $\Lambda^{\rm std}$, in particular for the primary notions of strong polynomial and polynomial tractability, but also for weak tractability; see, e.g., [4, 5, 12, 13] and also the books [7, 9]. However, there are also some newer tractability notions such as quasi-polynomial tractability (see [1]), (σ,τ)-weak tractability (see [11]) or uniform weak tractability (see [10]) which have not yet been considered for the approximation problem for weighted Korobov spaces. Indeed, in [9, Open Problem 103] Novak and Woźniakowski asked for appropriate weight conditions that characterize quasi-polynomial tractability.

It is the aim of the present paper to close this gap and to provide matching necessary and sufficient conditions for quasi-polynomial, (σ,τ)-weak and uniform weak tractability

*The authors are supported by the Austrian Science Fund (FWF), Projects F5506-N26 (Ebert) and F5509-N26 (Pillichshammer), which are parts of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications".

for both information classes $\Lambda^{\rm all}$ and $\Lambda^{\rm std}$, and therefore to extend and complete the already known picture regarding tractability of L_2-approximation in weighted Korobov spaces. In particular, we show that for the information class $\Lambda^{\rm all}$ the notions of quasi-polynomial tractability, uniform weak tractability and weak tractability are equivalent and any of these holds if and only if the weights become eventually less than one (see Theorem 1). For the class $\Lambda^{\rm std}$ we show that polynomial tractability and quasi-polynomial tractability are equivalent and additionally provide matching sufficient and necessary conditions for the considered notions of weak tractability (see Theorem 3).

The remainder of this article is organized as follows. In Section 2, we recall the underlying function space setting of weighted Korobov spaces with finite smoothness and provide the basics about L_2-approximation for such spaces. Furthermore, we give the definitions of the considered tractability notions. The obtained results are presented in Section 3. Finally, the corresponding proofs can be found in Section 4.

2 Basic definitions

Function space setting

The Korobov space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ with weight sequence $\boldsymbol{\gamma} = (\gamma_j)_{j\geq 1} \in \mathbb{R}^{\mathbb{N}}$ is a reproducing kernel Hilbert space with kernel function $K_{s,\alpha,\boldsymbol{\gamma}} : [0,1]^s \times [0,1]^s \to \mathbb{R}$ given by

$$K_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y}) := \sum_{\boldsymbol{k}\in\mathbb{Z}^s} r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) \exp(2\pi \mathtt{i} \boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}))$$

and corresponding inner product

$$\langle f,g\rangle_{s,\alpha,\boldsymbol{\gamma}} := \sum_{\boldsymbol{k}\in\mathbb{Z}^s} \frac{1}{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})} \widehat{f}(\boldsymbol{k})\, \widehat{g}(\boldsymbol{k}) \quad \text{and} \quad \|f\|_{s,\alpha,\boldsymbol{\gamma}} = \sqrt{\langle f,f\rangle_{s,\alpha,\boldsymbol{\gamma}}}\,.$$

Here, the Fourier coefficients are given by

$$\widehat{f}(\boldsymbol{k}) = \int_{[0,1]^s} f(\boldsymbol{x}) \exp(-2\pi\mathtt{i}\boldsymbol{k}\cdot\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

and the used decay function equals $r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^s r_{\alpha,\gamma_j}(k_j)$ with $\alpha > 1$ (the so-called smoothness parameter of the space) and

$$r_{\alpha,\gamma}(k) := \begin{cases} 1 & \text{for } k = 0, \\ \frac{\gamma}{|k|^\alpha} & \text{for } k \in \mathbb{Z}\setminus\{0\}. \end{cases}$$

The kernel $K_{s,\alpha,\boldsymbol{\gamma}}$ is well defined for $\alpha > 1$ and for all $\boldsymbol{x},\boldsymbol{y}\in[0,1]^s$, since

$$|K_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{x},\boldsymbol{y})| \leq \sum_{\boldsymbol{k}\in\mathbb{Z}^s} r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^s (1+2\zeta(\alpha)\gamma_j) < \infty,$$

where $\zeta(\cdot)$ is the Riemann zeta function (note that $\alpha > 1$ and hence $\zeta(\alpha) < \infty$).

Furthermore, we assume in this article that the weights satisfy $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots \geq 0$.

Approximation in the weighted Korobov space

We consider the operator $\mathrm{APP}_s : \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}} \to L_2([0,1]^s)$ with $\mathrm{APP}_s(f) = f$ for all $f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$. In order to approximate APP_s with respect to the L_2-norm $\|\cdot\|_{L_2}$ over $[0,1]^s$, we will employ linear algorithms $A_{n,s}$ that use n information evaluations and are of the form

$$A_{n,s}(f) = \sum_{i=1}^{n} T_i(f)\, g_i \quad \text{for } f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}} \tag{1}$$

with functions $g_i \in L_2([0,1]^s)$ and bounded linear functionals $T_i \in \mathcal{H}^*_{s,\alpha,\boldsymbol{\gamma}}$ for $i = 1, \ldots, n$. We will assume that the considered functionals T_i belong to some permissible class of information Λ. In particular, we study the class Λ^{all} consisting of the entire dual space $\mathcal{H}^*_{s,\alpha,\boldsymbol{\gamma}}$ and the class Λ^{std}, which consists only of point evaluation functionals. Remember that $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ is a reproducing kernel Hilbert space, which means that point evaluations are continuous linear functionals and therefore Λ^{std} is a subclass of Λ^{all}.

The worst-case error of an algorithm $A_{n,s}$ as in (1) is then defined as

$$e(A_{n,s}) := \sup_{\substack{f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}} \\ \|f\|_{s,\alpha,\boldsymbol{\gamma}} \le 1}} \|\mathrm{APP}_s(f) - A_{n,s}(f)\|_{L_2}$$

and the n-th minimal worst-case error with respect to the information class Λ is given by

$$e(n, \mathrm{APP}_s; \Lambda) := \inf_{A_{n,s} \in \Lambda} e(A_{n,s}).$$

We are interested in how the approximation error of algorithms $A_{n,s}$ depends on the number of used information evaluations n and how it depends on the problem dimension s. To this end, we define the so-called information complexity as

$$n(\varepsilon, \mathrm{APP}_s; \Lambda) := \min\{n \in \mathbb{N}_0 \ : \ e(n, \mathrm{APP}_s; \Lambda) \le \varepsilon\}$$

with $\varepsilon \in (0,1)$ and $s \in \mathbb{N}$. We note that it is well known and easy to see that the initial error equals one for the considered problem and therefore there is no need to distinguish between the normalized and the absolute error criterion.

Notions of tractability

In order to characterize the dependency of the information complexity on the dimension s and the error threshold ε, we will study several notions of tractability which are given in the following definition.

Definition 1. *Consider the approximation problem* $\mathrm{APP} = (\mathrm{APP}_s)_{s \ge 1}$ *for the information class* Λ*. We say we have:*

(a) *Polynomial tractability* (PT) *if there exist non-negative numbers* τ, σ, C *such that*

$$n(\varepsilon, \mathrm{APP}_s; \Lambda) \le C\, \varepsilon^{-\tau} s^{\sigma} \quad \textit{for all} \quad s \in \mathbb{N},\ \varepsilon \in (0,1).$$

(b) *Strong polynomial tractability* (SPT) *if there exist non-negative numbers* τ, C *such that*

$$n(\varepsilon, \mathrm{APP}_s; \Lambda) \le C\, \varepsilon^{-\tau} \quad \text{for all} \quad s \in \mathbb{N},\, \varepsilon \in (0,1). \tag{2}$$

The infimum over all exponents $\tau \ge 0$ *such that* (2) *holds for some* $C \ge 0$ *is called the exponent of strong polynomial tractability and is denoted by* $\tau^*(\Lambda)$.

(c) *Weak tractability* (WT) *if*

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_s; \Lambda)}{s + \varepsilon^{-1}} = 0.$$

(d) *Quasi-polynomial tractability* (QPT) *if there exist non-negative numbers* t, C *such that*

$$n(\varepsilon, \mathrm{APP}_s; \Lambda) \le C\, \exp(t\,(1+\ln s)(1+\ln \varepsilon^{-1})) \quad \text{for all} \quad s \in \mathbb{N},\, \varepsilon \in (0,1). \tag{3}$$

The infimum over all exponents $t \ge 0$ *such that* (3) *holds for some* $C \ge 0$ *is called the exponent of quasi-polynomial tractability and is denoted by* $t^*(\Lambda)$.

(e) (σ, τ)*-weak tractability* ((σ,τ)-WT) *if there exist positive* σ, τ *such that*

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\ln n(\varepsilon, \mathrm{APP}_s; \Lambda)}{s^\sigma + \varepsilon^{-\tau}} = 0.$$

(f) *Uniform weak tractability* (UWT) *if* (σ,τ)*-weak tractability holds for all* $\sigma, \tau \in (0,1]$.

We obviously have the following hierarchy of tractability notions:

$$\mathrm{SPT} \Rightarrow \mathrm{PT} \Rightarrow \mathrm{QPT} \Rightarrow \mathrm{UWT} \Rightarrow (\sigma,\tau)\text{-WT} \quad \text{for all } (\sigma,\tau) \in (0,1]^2.$$

Furthermore, WT coincides with (σ,τ)-WT for $(\sigma,\tau) = (1,1)$.

For more information about tractability of multivariate problems we refer to the three volumes [7, 8, 9] by Novak and Woźniakowski.

3 The results

Here we state our results about quasi-polynomial-, weak- and uniform weak tractability of approximation in the weighted Korobov space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ for information from Λ^{all}. In order to provide a complete picture of all instances at a glance, we also include the already known results for (strong) polynomial tractability which were first proved by Wasilkowski and Woźniakowski in [12].

Theorem 1. *Consider the approximation problem* $\mathrm{APP} = (\mathrm{APP}_s)_{s\ge1}$ *for the information class* Λ^{all} *and let* $\alpha > 1$. *Then we have the following conditions:*

1. *(Cf. [12]) Strong polynomial tractability for the class* Λ^{all} *holds if and only if* $s_{\boldsymbol{\gamma}} < \infty$, *where for* $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$ *the sum exponent* $s_{\boldsymbol{\gamma}}$ *is defined as*

$$s_{\boldsymbol{\gamma}} = \inf \left\{ \kappa > 0 \ : \ \sum_{j=1}^{\infty} \gamma_j^{\kappa} < \infty \right\},$$

with the convention that $\inf \emptyset = \infty$. *In this case the exponent of strong polynomial tractability is*

$$\tau^*(\Lambda^{\mathrm{all}}) = 2 \max \left(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha} \right).$$

2. *(Cf. [12]) Strong polynomial tractability and polynomial tractability for the class* Λ^{all} *are equivalent.*

3. *Quasi-polynomial tractability, uniform weak tractability and weak tractability for the class* Λ^{all} *are equivalent and hold if and only if the weights* $(\gamma_j)_{j \geq 1}$ *become eventually less than 1, i.e., if and only if there exists some index* $j_0 \in \mathbb{N}$ *such that* $\gamma_1 = \cdots = \gamma_{j_0} = 1$ *and* $\gamma_{j_0+1} < 1$. *Setting* $\gamma_* := \gamma_{j_0+1}$, *we then have* $\gamma_j \leq \gamma_* < 1$ *for all* $j > j_0$.

4. *If we have quasi-polynomial tractability, then the exponent of quasi-polynomial tractability equals*

$$t^*(\Lambda^{\mathrm{all}}) = 2 \max \left(\frac{1}{\alpha}, \frac{1}{\ln \gamma_*^{-1}} \right),$$

where γ_* *is the first weight in the weight sequence* $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$ *that is strictly less than* 1. *If* $\gamma_* = 0$, *then we set* $(\ln \gamma_*^{-1})^{-1} := 0$.

Remark 2. We remark that in [7] a different formulation of the necessary and sufficient condition for weak tractability is given. In particular, according to [7, Theorem 5.8] the approximation problem $\mathrm{APP} = (\mathrm{APP}_s)_{s \geq 1}$ for Λ^{all} is weakly tractable if and only if

$$\lim_{s+\varepsilon^{-1} \to \infty} \frac{k(\varepsilon, s, \boldsymbol{\gamma})}{s + \varepsilon^{-1}} = 0, \tag{4}$$

where $k(\varepsilon, s, \boldsymbol{\gamma})$ is defined as the element $k \in \{1, \ldots, s\}$ such that

$$\prod_{j=1}^{k} \gamma_j > \varepsilon^2 \quad \text{and} \quad \prod_{j=1}^{k+1} \gamma_j \leq \varepsilon^2.$$

If such a k does not exist, we set $k(\varepsilon, s, \boldsymbol{\gamma}) = s$. In the following, we show that this condition is equivalent to our condition that the weights γ_j become eventually less than 1.

Assume that there exists an index $j_0 \in \mathbb{N}$ such that $\gamma_1 = \cdots = \gamma_{j_0} = 1$ and $\gamma_{j_0+1} =: \gamma_* < 1$. Then we see that for $k > j_0$ we have

$$\prod_{j=1}^{k+1} \gamma_j = \prod_{j=j_0+1}^{k+1} \gamma_j \leq \prod_{j=j_0+1}^{k+1} \gamma_* = \gamma_*^{k-j_0+1}.$$

For given $\varepsilon > 0$ assume now w.l.o.g. that s is large enough such that there exists a $k_* \in \{1, \ldots, s\}$ with $\gamma_*^{k_*-j_0+1} \leq \varepsilon^2$. This implies in particular that $k(\varepsilon, s, \boldsymbol{\gamma}) \leq k_*$. Elementary transformations show that the inequality $\gamma_*^{k_*-j_0+1} \leq \varepsilon^2$ is equivalent to

$$k_* \leq \frac{2 \ln \varepsilon^{-1}}{\ln \gamma_*^{-1}} + j_0 - 1.$$

Therefore, we obtain that

$$\lim_{s+\varepsilon^{-1} \to \infty} \frac{k(\varepsilon, s, \boldsymbol{\gamma})}{s + \varepsilon^{-1}} \leq \lim_{s+\varepsilon^{-1} \to \infty} \frac{k_*}{s + \varepsilon^{-1}} \leq \lim_{s+\varepsilon^{-1} \to \infty} \frac{\frac{2 \ln \varepsilon^{-1}}{\ln \gamma_*^{-1}} + j_0 - 1}{s + \varepsilon^{-1}} = 0$$

and thus the condition in (4) is satisfied.

On the other hand, assume that (4) is satisfied but $\gamma_j = 1$ for all $j \in \mathbb{N}$. Then, according to the definition we obviously have that $k(\varepsilon, s, \boldsymbol{\gamma}) = s$ for all $\varepsilon \in (0, 1)$. But then, we have for fixed $\varepsilon \in (0, 1)$ that

$$\lim_{s \to \infty} \frac{k(\varepsilon, s, \boldsymbol{\gamma})}{s + \varepsilon^{-1}} = \lim_{s \to \infty} \frac{s}{s + \varepsilon^{-1}} = 1$$

and this contradicts (4). Hence the γ_j have to become eventually less than 1. $\square$

In the next theorem we present the respective conditions for tractability of approximation in the weighted Korobov space for the information class Λ^{std}. In order to provide a detailed overview, we also include the already known results for (strong) polynomial tractability, see, e.g., [5].

Theorem 3. *Consider multivariate approximation* $\mathrm{APP} = (\mathrm{APP}_s)_{s \geq 1}$ *for the information class* Λ^{std} *and* $\alpha > 1$. *Then we have the following conditions:*

1. *(Cf. [5]) Strong polynomial tractability for the class* Λ^{std} *holds if and only if*

$$\sum_{j=1}^{\infty} \gamma_j < \infty$$

(which is equivalent to $s_{\boldsymbol{\gamma}} \leq 1$*). In this case the exponent of strong polynomial tractability satisfies*

$$\tau^*(\Lambda^{\mathrm{std}}) = 2 \max\left(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha}\right).$$

2. *(Cf. [5]) Polynomial tractability for the class* Λ^{std} *holds if and only if*

$$\limsup_{s \to \infty} \frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < \infty.$$

3. *Polynomial and quasi-polynomial tractability for the class* Λ^{std} *are equivalent.*

4. Weak tractability for the class Λ^{std} holds if and only if

$$\lim_{s\to\infty} \frac{1}{s} \sum_{j=1}^{s} \gamma_j = 0. \tag{5}$$

5. Weak (σ,τ)-tractability for the class Λ^{std} holds if and only if

$$\lim_{s\to\infty} \frac{1}{s^{\sigma}} \sum_{j=1}^{s} \gamma_j = 0. \tag{6}$$

6. Uniform weak tractability for the class Λ^{std} holds if and only if

$$\lim_{s\to\infty} \frac{1}{s^{\sigma}} \sum_{j=1}^{s} \gamma_j = 0 \quad \textit{for all } \sigma \in (0,1]. \tag{7}$$

The proofs of the statements in Theorems 1 and 3 are given in the next section.

The results in Theorems 1 and 3 provide a complete characterization for tractability of approximation in the weighted Korobov space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ with respect to all commonly studied notions of tractability and the two information classes Λ^{all} and Λ^{std}. We summarize the conditions in a concise table (Table 1) below.

	Λ^{all}	Λ^{std}
SPT	$s_{\boldsymbol{\gamma}} < \infty$	$\sum_{j=1}^{\infty} \gamma_j < \infty$
PT	$s_{\boldsymbol{\gamma}} < \infty$	$\limsup_{s\to\infty} \frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < \infty$
QPT	$\exists\, j_0 \in \mathbb{N} : \gamma_j < 1 \quad \forall j > j_0$	$\limsup_{s\to\infty} \frac{1}{\ln s} \sum_{j=1}^{s} \gamma_j < \infty$
UWT	$\exists\, j_0 \in \mathbb{N} : \gamma_j < 1 \quad \forall j > j_0$	$\lim_{s\to\infty} \frac{1}{s^{\sigma}} \sum_{j=1}^{s} \gamma_j = 0 \ \forall \sigma \in (0,1]$
(σ,τ)-WT	$\exists\, j_0 \in \mathbb{N} : \gamma_j < 1 \quad \forall j > j_0$	$\lim_{s\to\infty} \frac{1}{s^{\sigma}} \sum_{j=1}^{s} \gamma_j = 0$
WT	$\exists\, j_0 \in \mathbb{N} : \gamma_j < 1 \quad \forall j > j_0$	$\lim_{s\to\infty} \frac{1}{s} \sum_{j=1}^{s} \gamma_j = 0$

Table 1: Overview of the conditions for tractability of approximation in $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$.

4 The proofs

In this section we present the proofs of Theorem 1 and Theorem 3.

The information class Λ^{all}

It is commonly known that the n-th minimal worst-case errors $e(n, \mathrm{APP}_s; \Lambda)$ are directly related to the eigenvalues of the self-adjoint operator

$$W_s := \mathrm{APP}_s^* \mathrm{APP}_s : \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}} \to \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}.$$

In the following lemma, we derive the eigenpairs of the operator W_s. For this purpose, we define, for $\boldsymbol{x} \in [0,1]^s, \boldsymbol{k} \in \mathbb{Z}^s$, the vectors $e_{\boldsymbol{k}}(\boldsymbol{x}) = e_{\boldsymbol{k},\alpha,\boldsymbol{\gamma}}(\boldsymbol{x}) := \sqrt{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})}\, \exp(2\pi \mathtt{i} \boldsymbol{k} \cdot \boldsymbol{x})$.

Lemma 4. *The eigenpairs of the operator W_s are $(r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}), e_{\boldsymbol{k}})$ with $\boldsymbol{k} \in \mathbb{Z}^s$.*

This result is well known; see, e.g., [7, p. 215]. For the sake of completeness we add the following short proof.

Proof of Lemma 4. We find that for any $f, g \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ we have

$$\langle \mathrm{APP}_s(f), \mathrm{APP}_s(g) \rangle_{L_2} = \langle f, \mathrm{APP}_s^* \mathrm{APP}_s(g) \rangle_{s,\alpha,\boldsymbol{\gamma}} = \langle f, W_s(g) \rangle_{s,\alpha,\boldsymbol{\gamma}}$$

and hence, due to the orthonormality of the Fourier basis functions,

$$\langle e_{\boldsymbol{k}}, W_s(e_{\boldsymbol{h}}) \rangle_{s,\alpha,\boldsymbol{\gamma}} = \langle e_{\boldsymbol{k}}, e_{\boldsymbol{h}} \rangle_{L_2} = \sqrt{r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})\, r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})}\, \delta_{\boldsymbol{k},\boldsymbol{h}},$$

where the Kronecker delta $\delta_{\boldsymbol{k},\boldsymbol{h}}$ is 1 if $\boldsymbol{k} = \boldsymbol{h}$, and 0 otherwise. For $\boldsymbol{k} = \boldsymbol{h}$ this gives $\langle e_{\boldsymbol{k}}, W_s(e_{\boldsymbol{k}}) \rangle_{s,\alpha,\boldsymbol{\gamma}} = r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k})$ which in turn implies that

$$W_s(e_{\boldsymbol{h}}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^s} \langle W_s(e_{\boldsymbol{h}}), e_{\boldsymbol{k}} \rangle_{s,\alpha,\boldsymbol{\gamma}}\, e_{\boldsymbol{k}} = r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{h})\, e_{\boldsymbol{h}}$$

and proves the lemma. $\square$

In order to exploit the relationship between the eigenvalues of W_s and the information complexity, we define the set

$$\mathcal{A}(\varepsilon, s) := \{\boldsymbol{k} \in \mathbb{Z}^s \ : \ r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) > \varepsilon^2\}.$$

It is commonly known (see [7]) that then the following identity holds

$$n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) = |\mathcal{A}(\varepsilon, s)|.$$

We will use this fact also in the proof of Theorem 1, which is presented below.

Proof of Theorem 1. We prove the necessary and sufficient conditions for each of the listed notions of tractability. For the sake of completeness, we also include the proofs for items 1 and 2 of Theorem 1.

1. In order to give a necessary and sufficient condition for strong polynomial tractability for Λ^{all}, we use a criterion from [7, Section 5.1]. From [7, Theorem 5.2] we find that the problem APP is strongly polynomially tractable for Λ^{all} if and only if there exists a $\tau > 0$ such that

$$\sup_{s\in\mathbb{N}} \left(\sum_{\boldsymbol{k}\in\mathbb{Z}^s} (r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^\tau \right)^{1/\tau} < \infty \tag{8}$$

and then $\tau^*(\Lambda^{\mathrm{all}}) = \inf\{2\tau \ : \ \tau \text{ satisfies (8)}\}$.

Assume that $s_{\boldsymbol{\gamma}} < \infty$. Then take τ such that $\tau > \max(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha})$ and thus $\sum_{j=1}^\infty \gamma_j^\tau$ is finite. For the sum in (8) we then obtain

$$\begin{aligned} \sum_{\boldsymbol{k}\in\mathbb{Z}^s} (r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^\tau &= \prod_{j=1}^s \left(\sum_{k=-\infty}^{\infty} (r_{\alpha,\gamma_j}(k))^\tau \right) \\ &= \prod_{j=1}^s \left(1 + 2\gamma_j^\tau \sum_{k=1}^\infty \frac{1}{k^{\alpha\tau}} \right) \\ &= \prod_{j=1}^s \left(1 + 2\zeta(\alpha\tau)\gamma_j^\tau\right) \qquad (9) \\ &\le \exp\left(2\zeta(\alpha\tau) \sum_{j=1}^\infty \gamma_j^\tau \right) < \infty, \end{aligned}$$

where we also used that $\tau > 1/\alpha$ and hence $\zeta(\alpha\tau) < \infty$. This implies that we have strong polynomial tractability and that

$$\tau^*(\Lambda^{\mathrm{all}}) \le 2\max(s_{\boldsymbol{\gamma}}, \tfrac{1}{\alpha}). \tag{10}$$

On the other hand, assume we have strong polynomial tractability. Then there exists a finite τ such that (8) holds true. From (9) we see that we obviously require that $\tau > \frac{1}{\alpha}$. Then, again using (9), we obtain that

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^s} (r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^\tau = \prod_{j=1}^s (1 + 2\zeta(\alpha\tau)\gamma_j^\tau) \ge 2\zeta(\alpha\tau) \sum_{j=1}^s \gamma_j^\tau .$$

Again, since (8) holds true, we require that $\sum_{j=1}^\infty \gamma_j^\tau < \infty$ and hence $s_{\boldsymbol{\gamma}} < \tau < \infty$. Combining both results yields that $\tau > \max(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha})$ and hence also

$$\tau^*(\Lambda^{\mathrm{all}}) \ge 2\max(s_{\boldsymbol{\gamma}}, \tfrac{1}{\alpha}). \tag{11}$$

Equations (10) and (11) then imply that $\tau^*(\Lambda^{\mathrm{all}}) = 2\max(s_{\boldsymbol{\gamma}}, \frac{1}{\alpha})$.

2. We use ideas from [12]. In order to prove the equivalence of strong polynomial tractability and polynomial tractability it suffices to prove that polynomial tractability implies strong polynomial tractability. So let us assume that APP is polynomially tractable, i.e., there exist numbers $C, p > 0$ and $q \geq 0$ such that

$$n(\varepsilon, \mathrm{APP}_s, \Lambda^{\mathrm{all}}) \leq C\, s^q\, \varepsilon^{-p} \quad \text{for all } \varepsilon \in (0,1] \text{ and } s \in \mathbb{N}.$$

Without loss of generality we may assume that q is an integer. Take $s \in \mathbb{N}$ such that $s \geq q+1$ and choose vectors $\boldsymbol{k} \in \mathbb{Z}^s$ with $s-q-1$ components equal to 0 and $q+1$ components equal to 1. The total number of such vectors is $\binom{s}{q+1}$. Now choose $\varepsilon_* = \frac{1}{2}\gamma_s^{(q+1)/2}$. Assume that $\boldsymbol{k} \in \mathbb{Z}^s$ is of the form as mentioned above and denote by $\mathfrak{u} \subseteq \{1, \ldots, s\}$ the set of indices of $\boldsymbol{k}$ which are equal to 1. Then we have

$$r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j \in \mathfrak{u}} \gamma_j \geq \gamma_s^{q+1} > \varepsilon_*^2.$$

Hence all the $\binom{s}{q+1}$ vectors $\boldsymbol{k}$ of the form mentioned above belong to $\mathcal{A}(\varepsilon_*, s)$ and this implies that

$$|\mathcal{A}(\varepsilon_*, s)| \geq \binom{s}{q+1} \geq \frac{(s-q)^{q+1}}{(q+1)!} \geq \frac{s^{q+1}}{(q+1)!(q+1)^{q+1}} =: s^{q+1} c_q.$$

This now yields

$$s^{q+1} c_q \leq |\mathcal{A}(\varepsilon_*, s)| = n(\varepsilon_*, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \leq C\, s^q\, \varepsilon_*^{-p} = 2^p\, C\, s^q\, \gamma_s^{-(q+1)p/2},$$

which in turn implies that

$$\gamma_s^{(q+1)p/2} \ll_{p,q} \frac{1}{s}$$

and hence

$$\gamma_s \ll_{p,q} \frac{1}{s^{2/((q+1)p)}}.$$

This estimate holds for all $s \geq q+1$. Hence the sum exponent $s_{\boldsymbol{\gamma}}$ of the sequence $\boldsymbol{\gamma} = (\gamma_j)_{j \geq 1}$ is finite, $s_{\boldsymbol{\gamma}} < \infty$, and this implies by the first statement that we have strong polynomial tractability.

3. We use the following criterion for QPT taken from [9, Sec. 23.1.1] (see also [3]), which states that QPT holds if and only if there exists a $\tau > 0$ such that

$$C := \sup_{s \in \mathbb{N}} \frac{1}{s^2} \left(\sum_{j=1}^{\infty} \lambda_{s,j}^{\tau(1+\ln s)} \right)^{1/\tau} < \infty, \tag{12}$$

where $\lambda_{s,j}$ is the j-th eigenvalue of the operator W_s in non-increasing order.

Assume that we have $\gamma_* \in (0,1)$ and $j_0 \in \mathbb{N}$ such that

$$\gamma_1 = \cdots = \gamma_{j_0} = 1 \quad \text{and} \quad \gamma_j \leq \gamma_* < 1 \text{ for all } j > j_0. \tag{13}$$

For the weighted Korobov space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ we have by Lemma 4 that

$$\begin{aligned}
\sum_{j=1}^{\infty} \lambda_{s,j}^{\tau(1+\ln s)} &= \sum_{\boldsymbol{k}\in\mathbb{Z}^s} (r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}))^{\tau(1+\ln s)} \\
&= \prod_{j=1}^{s}\left(1+2\sum_{k=1}^{\infty}(r_{\alpha,\gamma_j}(k))^{\tau(1+\ln s)}\right) \\
&= \prod_{j=1}^{s}\left(1+2\zeta(\alpha\tau(1+\ln s))\gamma_j^{\tau(1+\ln s)}\right).
\end{aligned}$$

In order that $\zeta_s := \zeta(\alpha\tau(1+\ln s)) < \infty$ for all $s \in \mathbb{N}$, we need to require from now on that $\tau > 1/\alpha$. Furthermore, we have that

$$\begin{aligned}
\frac{1}{s^2}\left(\sum_{j=1}^{\infty}\lambda_{s,j}^{\tau(1+\ln s)}\right)^{1/\tau} &= \frac{1}{s^2}\left(\prod_{j=1}^{s}\left(1+2\zeta_s\gamma_j^{\tau(1+\ln s)}\right)\right)^{1/\tau} \\
&= \exp\left(\frac{1}{\tau}\sum_{j=1}^{s}\ln\left(1+2\zeta_s\gamma_j^{\tau(1+\ln s)}\right)-2\ln s\right) \\
&\le \exp\left(\frac{1}{\tau}2\zeta_s\sum_{j=1}^{s}\gamma_j^{\tau(1+\ln s)}-2\ln s\right),
\end{aligned}$$

where we used that $\ln(1+x) \le x$ for all $x \ge 0$. Now we use the well-known fact that $\zeta(x) \le 1+\frac{1}{x-1}$ for all $x > 1$ and thus

$$\zeta_s \le 1+\frac{1}{(\alpha\tau-1)+\alpha\tau\ln s}.$$

Then we obtain

$$\frac{1}{s^2}\left(\sum_{j=1}^{\infty}\lambda_{s,j}^{\tau(1+\ln s)}\right)^{1/\tau} \le \exp\left(\frac{2}{\tau}\left(1+\frac{1}{(\alpha\tau-1)+\alpha\tau\ln s}\right)\sum_{j=1}^{s}\gamma_j^{\tau(1+\ln s)}-2\ln s\right).$$

Using assumption (13) we obtain for every $s \in \mathbb{N}$ that

$$\begin{aligned}
\sum_{j=1}^{s}\gamma_j^{\tau(1+\ln s)} &\le j_0+\gamma_*^{\tau(1+\ln s)}\max(s-j_0,0) \\
&= j_0+\frac{\gamma_*^{\tau}\max(s-j_0,0)}{s^{\tau\ln\gamma_*^{-1}}} \le j_0+1,
\end{aligned}$$

as long as $\tau \ge (\ln\gamma_*^{-1})^{-1}$. Thus, if $\tau > 1/\alpha$ and $\tau \ge (\ln\gamma_*^{-1})^{-1}$ we have

$$\begin{aligned}
\frac{1}{s^2}\left(\sum_{j=1}^{\infty}\lambda_{s,j}^{\tau(1+\ln s)}\right)^{1/\tau} &\le \exp\left(\frac{2}{\tau}\left(1+\frac{1}{(\alpha\tau-1)+\alpha\tau\ln s}\right)(j_0+1)-2\ln s\right) \\
&= \exp(\mathcal{O}(1)) < \infty,
\end{aligned}$$

for all $s \in \mathbb{N}$. By the characterization in (12), this implies quasi-polynomial tractability. Of course, quasi-polynomial tractability implies uniform weak tractability, which in turn implies weak tractability.

It remains to show that weak tractability implies condition (13). Assume on the contrary that (13) does not hold, i.e., $\gamma_j = 1$ for all $j \in \mathbb{N}$. Then we have for all $\boldsymbol{k} \in \{-1,0,1\}^s$ that $r_{s,\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = 1$. This means that for all $\varepsilon \in (0,1)$ we have $\{-1,0,1\}^s \subseteq \mathcal{A}(\varepsilon, s)$ and hence $n(\varepsilon, \mathrm{APP}_s; \Lambda^{\mathrm{all}}) \geq 3^s$. This means that the approximation problem suffers from the curse of dimensionality and, in particular, we cannot have weak tractability. This concludes the proof of item 3.

4. Again from [9, Theorem 23.2] we know that the exponent of quasi-polynomial tractability is
$$t^*(\Lambda^{\mathrm{all}}) = 2 \inf\{\tau \ : \ \tau \text{ for which (12) holds}\}.$$
From the above part of the proof we already know that τ satisfies (12) as long as $\tau > 1/\alpha$ and $\tau \geq (\ln \gamma_*^{-1})^{-1}$, where we put $(\ln \gamma_*^{-1})^{-1} := 0$ whenever $\gamma_* = 0$. Therefore,
$$t^*(\Lambda^{\mathrm{all}}) \leq 2 \max\left(\frac{1}{\alpha}, \frac{1}{\ln \gamma_*^{-1}}\right).$$
Assume now that we have quasi-polynomial tractability. Then (12) holds true for some $\tau > 0$. Considering the special instance $s = 1$ this means
$$C \geq \left(\sum_{j=1}^{\infty} \lambda_{1,j}^{\tau}\right)^{1/\tau} = (1 + 2\zeta(\alpha\tau)\gamma_1^{\tau})^{1/\tau}$$
and hence we must have $\tau > 1/\alpha$.

Now, again according to (12), there exists a $\tau > 1/\alpha$ such that for all $s \in \mathbb{N}$ we have
$$\begin{aligned} C &\geq \frac{1}{s^2}\left(\prod_{j=1}^{s}\left(1 + 2\zeta(\alpha\tau(1+\ln s))\gamma_j^{\tau(1+\ln s)}\right)\right)^{1/\tau} \\ &\geq \exp\left(\frac{1}{\tau}\sum_{j=1}^{s} \ln\left(1 + \gamma_j^{\tau(1+\ln s)}\right) - 2\ln s\right). \end{aligned}$$
Taking the logarithm leads to
$$\ln C \geq \frac{1}{\tau}\sum_{j=1}^{s} \ln\left(1 + \gamma_j^{\tau(1+\ln s)}\right) - 2\ln s$$
for all $s \in \mathbb{N}$. According to item 3, quasi-polynomial tractability forces the weights γ_j to be of the form (13). Therefore we also have QPT for the weight sequence
$$(\underbrace{1, 1, \ldots, 1}_{j_0 \text{ times}}, \gamma_*, \gamma_*, \ldots).$$

Hence, we have for $s \geq j_0$ that

$$\ln C \geq \frac{1}{\tau}(s - j_0) \ln\left(1 + \gamma_*^{\tau(1+\ln s)}\right) - 2\ln s.$$

Since $\gamma_* \in (0,1)$ and since $\ln(1+x) \geq x \ln 2$ for all $x \in [0,1]$, it follows that for all $s \geq j_0$ we have

$$\ln C \geq \frac{(s-j_0)\ln 2}{\tau} \gamma_*^{\tau(1+\ln s)} - 2\ln s = \frac{\gamma_*^\tau (s-j_0)\ln 2}{\tau\, s^{\tau \ln \gamma_*^{-1}}} - 2\ln s.$$

This implies that $\tau \geq (\ln \gamma_*^{-1})^{-1}$. Therefore, we also have that

$$t^*(\Lambda^{\mathrm{all}}) \geq 2 \max\left(\frac{1}{\alpha}, \frac{1}{\ln \gamma_*^{-1}}\right)$$

and the claimed result follows. □

The information class Λ^{std}

Below, we provide the remaining proof of Theorem 3.

Proof of Theorem 3. The necessary and sufficient conditions for polynomial and strong polynomial tractability (items 1 and 2) have already been proved in [5]. See also [7, p. 215ff.], where the exact exponent of strong polynomial tractability $\tau^*(\Lambda^{\mathrm{std}})$ is given. We will therefore only provide proofs for items 3 to 6.

We start with a preliminary remark about the relation between integration and approximation. It is well known that multivariate approximation is not easier than multivariate integration $\mathrm{INT}_s(f) = \int_{[0,1]^s} f(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ for $f \in \mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$, see, e.g., [5]. In particular, necessary conditions for some notion of tractability for the integration problem are also necessary for the approximation problem. We will use this basic observation later on. Now we present the proof of item 3.

3. Obviously, it suffices to prove that quasi-polynomial tractability implies polynomial tractability. Assume therefore that quasi-polynomial tractability for the class Λ^{std} holds for approximation. Then we also have quasi-polynomial tractability for the integration problem. Now we apply [8, Theorem 16.16] which states that integration is T-tractable if and only if

$$\limsup_{s+\varepsilon^{-1}\to\infty} \frac{\sum_{j=1}^s \gamma_j + \ln \varepsilon^{-1}}{1 + \ln T(\varepsilon^{-1}, s)} < \infty. \tag{14}$$

We do not require the definition of T-tractability here (see, e.g., [7, p. 291]). For our purpose it suffices to know that the special case $T(\varepsilon^{-1}, s) = \exp((1+\ln s)(1+\ln \varepsilon^{-1}))$ corresponds to quasi-polynomial tractability. But for this instance condition (14) is equivalent to the criterion

$$\limsup_{s\to\infty} \frac{1}{\ln s} \sum_{j=1}^s \gamma_j < \infty. \tag{15}$$

From item 2, we know that condition (15) implies polynomial tractability and this completes the proof of item 3.

For the remaining conditions in items 4 to 6, note that since $\alpha > 1$ the trace of W_s, denoted by $\mathrm{trace}(W_s)$, is finite for all $s \in \mathbb{N}$. Indeed, we have

$$\mathrm{trace}(W_s) = \sum_{\boldsymbol{k} \in \mathbb{Z}^s} r_{\alpha,\boldsymbol{\gamma}}(\boldsymbol{k}) = \prod_{j=1}^{s} (1 + 2\gamma_j \zeta(\alpha)) < \infty. \tag{16}$$

In this case, we can use relations between notions of tractability for Λ^{all} and Λ^{std} which were first proved in [13] (see also [9, Section 26.4.1]).

4.-6. We prove the three statements in one combined argument. If any of the three conditions (5), (6) or (7) holds, then this implies that the weights $(\gamma_j)_{j \geq 1}$ have to become eventually less than 1 since otherwise, for every $\sigma \in (0, 1]$,

$$\lim_{s \to \infty} \frac{1}{s^\sigma} \sum_{j=1}^{s} \gamma_j = \lim_{s \to \infty} \frac{s}{s^\sigma} = \lim_{s \to \infty} s^{1-\sigma} \geq 1.$$

Therefore, we have by Theorem 1 that uniform weak tractability (and even quasi-polynomial tractability) holds for the class Λ^{all}. Furthermore, from (16) we obtain

$$\begin{aligned} \frac{\ln(\mathrm{trace}(W_s))}{s^\sigma} &= \frac{1}{s^\sigma} \ln\left(\prod_{j=1}^{s} (1 + 2\gamma_j \zeta(\alpha)) \right) \\ &= \frac{1}{s^\sigma} \sum_{j=1}^{s} \ln(1 + 2\gamma_j \zeta(\alpha)) \leq \frac{2\zeta(\alpha)}{s^\sigma} \sum_{j=1}^{s} \gamma_j, \end{aligned}$$

and thus if $\frac{1}{s^\sigma} \sum_{j=1}^{s} \gamma_j$ converges to 0 as s goes to infinity, with $\sigma \in (0, 1]$, then

$$\lim_{s \to \infty} \frac{\ln(\mathrm{trace}(W_s))}{s^\sigma} \leq \lim_{s \to \infty} \frac{2\zeta(\alpha)}{s^\sigma} \sum_{j=1}^{s} \gamma_j = 0.$$

By the same argument as in the proof of [9, Theorem 26.11], we obtain that (5) implies weak tractability for the class Λ^{std}. The proof for the other two notions of weak tractability can be obtained analogously by appropriately modifying the argument used in the proof of [9, Theorem 26.11].

It remains to prove the necessary conditions for the three notions of weak tractability. From our preliminary remark we know that necessary conditions on tractability for integration are also necessary conditions for approximation. Hence it suffices to study integration INT_s.

Due to, e.g., [14], we know that weak tractability of integration for $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$ holds if and only if

$$\lim_{s \to \infty} \frac{1}{s} \sum_{j=1}^{s} \gamma_j = 0$$

and thus this is also a necessary condition for weak tractability of approximation.

We are left to prove the necessity of the respective conditions for uniform weak tractability and (σ,τ)-weak tractability for integration. These follow from a similar approach as used in [14] for weak tractability. We just sketch the argument which is more or less an application and combination of results from [2] and [6].

In [2, Theorem 4.2] Hickernell and Woźniakowski showed that integration in a suitably constructed weighted Sobolev space $\mathcal{H}^{\mathrm{Sob}}_{s,r,\widehat{\gamma}}$ of smoothness $r=\lceil \alpha/2\rceil$ and with product weights $\widehat{\boldsymbol{\gamma}}$ is no harder than in the weighted Korobov space $\mathcal{H}_{s,\alpha,\boldsymbol{\gamma}}$. The product weights of the Korobov and Sobolev spaces are related by $\gamma_j=\widehat{\gamma}_j G_r$ with a multiplicative non-negative factor G_r. Hence, it suffices to study necessary conditions for tractability of integration in $\mathcal{H}^{\mathrm{Sob}}_{s,r,\widehat{\gamma}}$. To this end we proceed as in [2, Section 5].

The univariate reproducing kernel $K_{1,\widehat{\gamma}}$ of $\mathcal{H}^{\mathrm{Sob}}_{1,r,\widehat{\gamma}}$ (case $s=1$) can be decomposed as

$$K_{1,\widehat{\gamma}}=R_1+\widehat{\gamma}(R_2+R_3),$$

where each R_j is a reproducing kernel of a Hilbert space $\mathcal{H}(R_j)$ of univariate functions. In our specific case, we have $R_1=1$ and $\mathcal{H}(R_1)=\mathrm{span}(1)$ (cf. [2, p. 679]). It is then shown in [2, Section 5] that all requirements of [6, Theorem 4] are satisfied. For the involved parameter α_1, we have $\alpha_1=\|h_{1,1}\|^2_{\mathcal{H}(R_1)}=1$ (this is easily shown, since $R_1=1$). Furthermore, we have that the parameter α in [6, Theorem 4] (not to be confused with the smoothness parameter α of the Korobov space) satisfies $\alpha\in[1/2,1)$, since $h_{1,2,(0)}\neq 0$ and $h_{1,2,(1)}\neq 0$, as shown in [2, p. 681] (where $h_{1,2,(j)}$ is called $\eta_{1,2,(j)}$ for $j\in\{0,1\}$). In order to avoid any misunderstanding, we denote the α in [6, Theorem 4] by $\widetilde{\alpha}$ from now on. Then, we apply [6, Theorem 4] and obtain for the squared n-th minimal integration error in the considered Sobolev space that

$$e^2(n,\mathrm{INT}_s)\geq \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}}(1-n\widetilde{\alpha}^{|\mathfrak{u}|})_+\,\alpha_2^{|\mathfrak{u}|}\prod_{j\in\mathfrak{u}}\widehat{\gamma}_j\prod_{j\notin\mathfrak{u}}(1+\widehat{\gamma}_j\alpha_3),$$

where α_2,α_3 are positive numbers (cf. [6, p. 425]) and $(x)_+:=\max(x,0)$. This implies

$$\begin{aligned}e^2(n,\mathrm{INT}_s)&\geq \sum_{\mathfrak{u}\subseteq\{1,\ldots,s\}}(1-n\widetilde{\alpha}^{|\mathfrak{u}|})\,\alpha_2^{|\mathfrak{u}|}\prod_{j\in\mathfrak{u}}\widehat{\gamma}_j\\&=\prod_{j=1}^{s}(1+\alpha_2\widehat{\gamma}_j)-n\prod_{j=1}^{s}(1+\alpha_2\,\widetilde{\alpha}\,\widehat{\gamma}_j),\end{aligned}$$

which in turn yields that

$$n(\varepsilon,\mathrm{INT}_s)\geq\frac{\prod_{j=1}^{s}(1+\alpha_2\widehat{\gamma}_j)-\varepsilon^2}{\prod_{j=1}^{s}(1+\alpha_2\,\widetilde{\alpha}\,\widehat{\gamma}_j)}.$$

Taking the logarithm, we obtain

$$\begin{aligned}
\ln n(\varepsilon, \mathrm{INT}_s) &\geq \ln\left(\prod_{j=1}^{s}(1+\alpha_2\widehat{\gamma}_j)\right) + \ln\left(1 - \frac{\varepsilon^2}{\prod_{j=1}^{s}(1+\alpha_2\widehat{\gamma}_j)}\right) \\
&\quad - \ln\left(\prod_{j=1}^{s}(1+\alpha_2\,\widetilde{\alpha}\,\widehat{\gamma}_j)\right) \\
&\geq \sum_{j=1}^{s}\ln(1+\alpha_2\widehat{\gamma}_j) - \alpha_2\widetilde{\alpha}\sum_{j=1}^{s}\widehat{\gamma}_j + \ln(1-\varepsilon^2),
\end{aligned}$$

where we used that $\ln(1+x) \leq x$ for any $x \in \mathbb{R}$.

Recall that $\widetilde{\alpha} < 1$ and set $c := (1+\widetilde{\alpha})/2$. Then $c \in (\widetilde{\alpha}, 1)$ and since

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1,$$

it follows that $\ln(1+x) \geq cx$ for sufficiently small $x > 0$.

Next, assume that we have (σ,τ)-weak tractability for integration in the considered Sobolev space. Then the weights $\widehat{\gamma}_j$ necessarily tend to zero for $j \to \infty$ (see [6, Theorem 4, Item 4]). In particular, there exists an index $j_0 > 0$, such that for all $j \geq j_0$ we have $\ln(1+\alpha_2\widehat{\gamma}_j) \geq c\,\alpha_2\,\widehat{\gamma}_j$. Hence for $s \geq j_0$, we have

$$\ln n(\varepsilon, \mathrm{INT}_s) \geq \alpha_2(c-\widetilde{\alpha})\sum_{j=j_0}^{s}\widehat{\gamma}_j + \ln(1-\varepsilon^2) + \mathcal{O}(1).$$

Note that $c - \widetilde{\alpha} > 0$. Since we assume (σ,τ)-weak tractability, we have that

$$0 = \lim_{s+\varepsilon^{-1}\to\infty}\frac{\ln n(\varepsilon, \mathrm{INT}_s)}{s^{\sigma}+\varepsilon^{-\tau}} \geq \lim_{s+\varepsilon^{-1}\to\infty}\frac{\alpha_2(c-\widetilde{\alpha})\sum_{j=j_0}^{s}\widehat{\gamma}_j + \ln(1-\varepsilon^2)}{s^{\sigma}+\varepsilon^{-\tau}}.$$

This, however, implies that

$$\lim_{s\to\infty}\frac{1}{s^{\sigma}}\sum_{j=1}^{s}\widehat{\gamma}_j = 0,$$

and thus, since γ_j and $\widehat{\gamma}_j$ only differ by a multiplicative factor, that

$$\lim_{s\to\infty}\frac{1}{s^{\sigma}}\sum_{j=1}^{s}\gamma_j = 0.$$

Now the claimed results follow.

$\square$

References

[1] M. Gnewuch, H. Woźniakowski. *Quasi-polynomial tractability.* J. Complexity 27, no. 3–4, 312–330, 2011.

[2] F. Hickernell, H. Woźniakowski. *Tractability of multivariate integration for periodic functions.* J. Complexity 17, no. 4, 660–682, 2001.

[3] P. Kritzer, H. Woźniakowski. *Simple characterizations of exponential tractability for linear multivariate problems.* J. Complexity 51, 110–128, 2019.

[4] F.Y. Kuo, I.H. Sloan, H. Woźniakowski. *Lattice rules for multivariate approximation in the worst case setting.* In: H. Niederreiter, D. Talay (eds), Monte Carlo and Quasi-Monte Carlo Methods 2004, Springer, Berlin, Heidelberg, 2006.

[5] E. Novak, I.H. Sloan, H. Woźniakowski. *Tractability of approximation for weighted Korobov spaces on classical and quantum computers.* Found. Comput. Math. 4, no. 2, 121–156, 2004.

[6] E. Novak, H. Woźniakowski. *Intractability results for integration and discrepancy.* J. Complexity 17, no. 2, 388–441, 2001.

[7] E. Novak, H. Woźniakowski. *Tractability of Multivariate Problems.* Volume I: Linear Information, EMS, Zurich, 2008.

[8] E. Novak, H. Woźniakowski. *Tractability of Multivariate Problems.* Volume II: Standard Information for Functionals, EMS, Zurich, 2010.

[9] E. Novak, H. Woźniakowski. *Tractability of Multivariate Problems.* Volume III: Standard Information for Operators, EMS, Zurich, 2012.

[10] P. Siedlecki. *Uniform weak tractability.* J. Complexity 29, no. 6, 438–453, 2013.

[11] P. Siedlecki, M. Weimar. *Notes on (s,t)-weak tractability: a refined classification of problems with (sub)exponential information complexity.* J. Approx. Theory 200, 227–258, 2015.

[12] G.W. Wasilkowski, H. Woźniakowski. *Weighted tensor product algorithms for linear multivariate problems.* J. Complexity 15, no. 3, 402–447, 1999.

[13] G.W. Wasilkowski, H. Woźniakowski. *On the power of standard information for weighted approximation.* Found. Comput. Math. 1, no. 4, 417–434, 2001.

[14] H. Woźniakowski. *Tractability of multivariate integration for weighted Korobov spaces: My 15 year partnership with Ian Sloan.* In: P. L'Ecuyer, A. Owen (eds), Monte Carlo and Quasi-Monte Carlo Methods 2008, Springer, Berlin, Heidelberg, 2009.

Authors' addresses:

Adrian Ebert
Johann Radon Institute for Computational and Applied Mathematics (RICAM)
Austrian Academy of Sciences
Altenbergerstr. 69, 4040 Linz, Austria
E-mail: adrian.ebert@oeaw.ac.at

Friedrich Pillichshammer
Institut für Finanzmathematik und Angewandte Zahlentheorie
Johannes Kepler Universität Linz
Altenbergerstr. 69, 4040 Linz, Austria
E-mail: friedrich.pillichshammer@jku.at