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Oblique projection output-based feedback exponential stabilization of nonautonomous parabolic equations

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OBLIQUE PROJECTION OUTPUT-BASED FEEDBACK EXPONENTIAL STABILIZATION OF NONAUTONOMOUS PARABOLIC EQUATIONS

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ABSTRACT. A new output injection operator and a new feedback control operator are proposed to estimate and stabilize the state of linear nonautonomous parabolic equations. The proposed operators are time-independent and involve appropriately constructed oblique projections. Firstly, it is shown the estimating property of the dynamical Luenberger observer and the stabilizing property of the feedback controller. Then, it is shown the stability of the observer-controller coupled system, where the control input is dynamically constructed from the measured output. The numbers of actuators and sensors are finite. Finally, results of simulations are presented showing the stability of the closed-loop observer-controller coupled system.

1. INTRODUCTION

We consider evolutionary linear parabolic-like equations, for time $t \geq 0$, as

$$\dot{y} + Ay + A_{\rm rc}y = 0, \qquad y(0) = y_0, \tag{1.1}$$

evolving in a Hilbert space H. Here A and $A_{\rm rc} = A_{\rm rc}(t)$ are, respectively, a time-independent linear diffusion-like operator and a time-dependent linear reaction-convection-like operator. System (1.1) can be unstable, that is, the norm $|y(t)|_H$ of its solution may diverge to $+\infty$ as $t \to +\infty$.

Our primary goal concerns the feedback stabilization of (1.1). We want to find a set of M_σ actuators

$$U_M \coloneqq \{\Phi_j \mid \Phi_j \in H, \quad 1 \leq j \leq M_\sigma\},$$

where M_σ is a positive integer, and a feedback operator

$$\mathcal{K}_M = \mathcal{K}_M(t) \colon H \to \mathcal{U}_M, \qquad \mathcal{U}_M \coloneqq \operatorname{span} U_M,$$

such that the system

$$\dot{y} + Ay + A_{\rm rc}y = \mathcal{K}_M y, \qquad y(0) = y_0, \tag{1.2}$$

is exponentially stable. In other words, for suitable real numbers $C \geq 1$ and $\mu > 0$, the solution of (1.2) satisfies

$$|y(t)|_H \leq C\mathrm{e}^{-\mu t} |y_0|_H, \quad \text{for all } y_0 \in H, \ \ t \geq 0. \tag{1.3}$$

As we can see, the stabilizing input $u(t) \coloneqq \mathcal{K}_M(t)y(t)$ depends on $y(t)$, which means that even if we are able to find $\mathcal{K}_M$, we still need $y(t)$ to be able to compute $u(t)$.

For parabolic equations, the state $y(t)$, at a given instant of time $t \geq 0$, lives in an infinite-dimensional space, often in the function space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^d$ is a

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given open connected subset. Thus we cannot expect that we will be able to know $y(t)$ exactly. This leads us to the need of a state estimate $\widehat{y}(t)$ for $y(t)$.

Though we cannot measure the entire state $y(t)$, we will be able to obtain partial information through measurements obtained by a finite number of sensors. Then, to construct an estimate for the state of a system as (1.2), we will design a Luenberger observer. Suppose we are given a system as

$$\dot{y} + Ay + A_{\mathrm{rc}}y = f, \qquad w = \mathcal{Z}y, \tag{1.4}$$

where the initial condition is unknown, and f is an external force. At our disposal we have the output $w(t) = \mathcal{Z}y(t) \in \mathbb{R}^{S_\varsigma}$, which consists of a set of measurements made by S_ς sensors in $W_S := \{\mathfrak{w}_i \mid 1 \le i \le S_\varsigma\}$, where S_ς is a positive integer. More precisely, we will assume that the measurements $w_i(t)$ are of the form

$$w_i(t) := (\mathfrak{w}_i, y(t))_H, \text{ with } \mathfrak{w}_i \in H, \ \ 1 \le i \le S_\varsigma. \tag{1.5}$$

We will also assume that:

$$W_S = \{\mathfrak{w}_i \mid 1 \le i \le S_\varsigma\} \text{ is linearly independent.} \tag{1.6}$$

Thus, for $\mathcal{W}_S := \operatorname{span} W_S$ it holds $\dim \mathcal{W}_S = S_\varsigma$.

We want to find such set of sensors and an output injection operator

$$\mathfrak{I}_S = \mathfrak{I}_S(t) \colon \mathbb{R}^{S_\varsigma} \to H,$$

such that the solution of the Luenberger-type observer

$$\dot{\widehat{y}} + A\widehat{y} + A_{\mathrm{rc}}\widehat{y} = f + \mathfrak{I}_S(\mathcal{Z}\widehat{y} - \mathcal{Z}y), \qquad \widehat{y}(0) = \widehat{y}_0, \tag{1.7}$$

converges to $y(t)$, for all $(\widehat{y}_0, y(0)) \in H \times H$, as

$$|\widehat{y}(t) - y(t)|_H \le C_1 \mathrm{e}^{-\mu_1 t} |\widehat{y}_0 - y(0)|_H, \qquad t \ge 0. \tag{1.8}$$

For the error $z := \widehat{y} - y$, with $z_0 = \widehat{y}_0 - y(0)$, we obtain

$$\dot{z} + Az + A_{\mathrm{rc}}z = \mathfrak{I}_S\mathcal{Z}z, \qquad z(0) = z_0, \tag{1.9}$$

hence we want to construct the injection operator $\mathfrak{I}_S$ such that system (1.9) is stable,

$$|z(t)|_H \le C_1 \mathrm{e}^{-\mu_1 t} |z(0)|_H, \text{ for all } z_0 \in H, \ t \ge 0. \tag{1.10}$$

Finally, once we have a feedback operator $\mathcal{K}_M$ stabilizing (1.2), and an output injection operator $\mathfrak{I}_S$ so that the observer (1.7) is estimating the state of systems as (1.4), the next question is the following. Can we still stabilize (1.2) when we use the estimate provided by the observer to construct the control input? Note that since $\widehat{y}(t)$ is an estimate for the true state $y(t)$, then $\mathcal{K}_M(t)\widehat{y}(t)$ can be seen as an estimate for the true input control $\mathcal{K}_M(t)y(t)$ stabilizing (1.2). Our primary goal is that the component y of the coupled closed-loop system

$$\begin{aligned} &\dot{\widehat{y}} + A\widehat{y} + A_{\mathrm{rc}}\widehat{y} = \mathcal{K}_M\widehat{y} + \mathfrak{I}_S(\mathcal{Z}\widehat{y} - \mathcal{Z}y), && \widehat{y}(0) = \widehat{y}_0,\\ &\dot{y} + Ay + A_{\mathrm{rc}}y = \mathcal{K}_M\widehat{y}, && y(0) = y_0, \end{aligned}$$

satisfies (1.3). In the coordinates $(z, y) = (\widehat{y} - y, y)$, we arrive to the system

$$\dot{z} + Az + A_{\mathrm{rc}}z = \mathfrak{I}_S\mathcal{Z}z, \qquad z(0) = z_0, \tag{1.12a}$$

$$\dot{y} + Ay + A_{\mathrm{rc}}y = \mathcal{K}_My + \mathcal{K}_Mz, \qquad y(0) = y_0. \tag{1.12b}$$

We have that (1.3) follows from the stability of (1.12), which will follow as a consequence of the stability of systems (1.2) and (1.9).

To derive the stabilizability and estimatability results we will need large enough numbers of actuators and sensors. To deal with concrete examples it may be convenient to increase the number of actuators and sensors in an appropriate way, this is the reason why we denote the number of sensors and actuators by M_σ and S_ς, and not simply by M and S. Hence, we will have sequences

$$(M_\sigma)_{M\in\mathbb{N}_0} \text{ and } (S_\varsigma)_{S\in\mathbb{N}_0},$$

with $M_\sigma \coloneqq \sigma(M)$ and $S_\varsigma \coloneqq \varsigma(S)$, where

$$\sigma\colon \mathbb{N}_0 \to \mathbb{N}_0 \quad\text{and}\quad \varsigma\colon \mathbb{N}_0 \to \mathbb{N}_0$$

are strictly increasing functions. Here $\mathbb{N}_0$ is the set of positive integers.

Remark 1.1. *In examples we shall present later on, the functions σ and ς will be appropriately chosen. Throughout most of the paper, for simplicity, the reader may think of $\sigma = \varsigma = \mathbf{1}$, that is, of $M_\sigma = M$ and $S_\sigma = S$.*

1.1. **The main results.** Hereafter, we present stabilizing feedback operators and output injection operators which are based in suitable oblique projections.

Let us be given a pair $\left(\mathfrak{n}, (F_N, \widetilde{F}_N)_{N\in\mathbb{N}_0}\right)$ satisfying:

$$\mathfrak{n}\colon \mathbb{N}_0 \to \mathbb{N}_0 \text{ is strictly increasing} \tag{1.13a}$$

and, with $N_\mathfrak{n} \coloneqq \mathfrak{n}(N)$,

$$F_N \coloneqq \{f_j \mid 1 \le j \le N_\mathfrak{n}\} \subset H, \tag{1.13b}$$

$$\widetilde{F}_N \coloneqq \{\widetilde{f}_j \mid 1 \le j \le N_\mathfrak{n}\} \subset V \subset H. \tag{1.13c}$$

Further, with $\mathcal{F}_N \coloneqq \operatorname{span} F_N$ and $\widetilde{\mathcal{F}}_N \coloneqq \operatorname{span} \widetilde{F}_N$,

$$\dim \mathcal{F}_N = N_\mathfrak{n} = \dim \widetilde{\mathcal{F}}_N \text{ and } H = \mathcal{F}_N \oplus \widetilde{\mathcal{F}}_N^\perp. \tag{1.13d}$$

We will further assume general properties for the operators A and $A_{\rm rc}$ and a particular property for the sequence $(\mathcal{F}_N, \widetilde{\mathcal{F}}_N)_{N\in\mathbb{N}_0}$. Without entering into the details at this point, we will show that under such assumptions the following result holds true.

Theorem 1.2. *For N and λ large enough, the system*

$$\dot y + Ay + A_{\rm rc} y = -\lambda P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp} A^\rho P_{\widetilde{\mathcal{F}}_N}^{\mathcal{F}_N^\perp} y, \qquad y(0) = y_0, \tag{1.14}$$

is exponentially stable. Where $\rho \in \{0, 1\}$.

In (1.14), the operator $A^0 \coloneqq \mathbf{1}$ stands for the identity operator and $A^1 \coloneqq A$. Further, by P_X^Y we denote the oblique projection in H onto X along Y. Here we assume that, $X \subset H$ and $Y \subset H$ are closed subspaces satisfying $H = X \oplus Y$.

We will apply Theorem 1.2, to derive:

- stability of (1.2) for $(\mathcal{F}_M, \widetilde{\mathcal{F}}_M) = (\mathcal{U}_M, \widetilde{\mathcal{U}}_M)$,
- stability of (1.9) for $(\mathcal{F}_S, \widetilde{\mathcal{F}}_S) = (\mathcal{W}_S, \widetilde{\mathcal{W}_S})$,

for suitable auxiliary subspaces $\widetilde{\mathcal{U}}_M \subset V$ and $\widetilde{\mathcal{W}_S} \subset V$. In this way we will obtain, in (1.2) and in (1.9), the control input and the output injection as follows

$$\mathcal{K}_M y = \mathcal{K}_M^{\lambda,\rho} y = -\lambda P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^\rho P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp} y \in \mathcal{U}_M \subset H, \tag{1.15}$$

$$\mathfrak{I}_S \mathcal{Z} z = \mathfrak{I}_S^{\lambda,\rho} \mathcal{Z} z - \lambda P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}_S}^\perp} A^\rho P_{\widetilde{\mathcal{W}_S}}^{\mathcal{W}_S^\perp} z \in \mathcal{W}_S \subset H. \tag{1.16}$$

The injection operator $\mathfrak{I}_S$, is indeed well defined by (1.16). In fact, we can write

$$P_{\widetilde{\mathcal{W}_S}}^{\mathcal{W}_S^\perp} z = P_{\widetilde{\mathcal{W}_S}}^{\mathcal{W}_S^\perp} P_{\mathcal{W}_S} z, \tag{1.17a}$$

where $P_{\mathcal{W}_S} = P_{\mathcal{W}_S}^{\mathcal{W}_S^\perp}$ stands for the orthogonal projection onto $\mathcal{W}_S$, and from [8, Lem. 2.8] (cf. [12, Sect. 2]), for outputs in the form (1.5) we can write

$$P_{\mathcal{W}_S} z = \mathbf{Z}^{W_S} \mathcal{Z} z \tag{1.17b}$$

for a suitable operator $\mathbf{Z}^{W_S}\colon \mathbb{R}^{S_\varsigma} \to \mathcal{W}_S$. Therefore, from (1.16) and (1.17) we find that

$$\mathfrak{I}_S \mathcal{Z} z = -\lambda P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp} A^\rho P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp} \mathbf{Z}^{W_S}(\mathcal{Z}\widehat{y} - \mathcal{Z}y).$$

Hence, we arrive at the output injection operator

$$\mathfrak{I}_S = -\lambda P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp} A^\rho P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp} \mathbf{Z}^{W_S}. \tag{1.18}$$

We shall use Theorem 1.2 to derive the following.

Theorem 1.3. *For S, M, λ_1, and λ_2 large enough, the coupled system*

$$\dot{\widehat{y}} + A\widehat{y} + A_{\mathrm{rc}}\widehat{y} = -\lambda_1 P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp} A^{\rho_1} P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp} \mathbf{Z}^{W_S}(\mathcal{Z}\widehat{y} - \mathcal{Z}y), \tag{1.19a}$$

$$\dot{y} + Ay + A_{\mathrm{rc}}y = -\lambda_2 P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^{\rho_2} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp} \widehat{y}, \tag{1.19b}$$

$$(y(0), \widehat{y}(0)) = (y_0, \widehat{y}_0), \tag{1.19c}$$

is exponentially stable. With $\rho_i \in \{0, 1\}$, $i \in \{1, 2\}$.

Remark 1.4. *Above, together with the span of the actuators $\mathcal{U}_M \subset H$ and the span of the sensors $\mathcal{W}_M \subset H$, we consider also auxiliary subspaces of $\widetilde{\mathcal{U}}_M \subset V$ and $\widetilde{\mathcal{W}}_M \subset V$ of a suitable subspace $V \subset H$. This is due to a technical reason related to our proofs, which cannot be repeated with $\widetilde{\mathcal{U}}_M \subset H$ and $\widetilde{\mathcal{W}}_M \subset H$ (cf. Rem. 4.3). Further, if $\mathcal{U}_M \subset V$ we can take $\widetilde{\mathcal{U}}_M = \mathcal{U}_M$, and if $\mathcal{W}_S \subset V$ we can take $\widetilde{\mathcal{W}}_S = \mathcal{W}_S$. However, for the case of parabolic equations we will be particularly interested in the case the actuators and sensors are indicator functions of small subdomains $\omega \subset \Omega$, which are elements of $H \setminus V$.*

1.2. **Illustrating example. Scalar parabolic equations.** The results will follow under general assumptions on the plant dynamics operators A and A_{rc}. Such assumptions will be presented later on and will be satisfied, in particular, for a general class of standard linear parabolic equations, under either Dirichlet or Neumann boundary conditions. Namely, as an application of Theorem 1.3, it will follow that the system

$$\tfrac{\partial}{\partial t}\widehat{y} - \nu\Delta\widehat{y} + a\widehat{y} + b\cdot\nabla\widehat{y} = -\lambda_1 P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp} P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp} \mathbf{Z}^{W_S}(\mathcal{Z}\widehat{y} - \mathcal{Z}y), \tag{1.20a}$$

$$\tfrac{\partial}{\partial t}y - \nu\Delta y + ay + b\cdot\nabla y = -\lambda_2 P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp} \widehat{y}, \tag{1.20b}$$

$$\mathfrak{B}y\big|_{\partial\Omega} = 0 = \mathfrak{B}\widehat{y}\big|_{\partial\Omega}, \quad (y(0), \widehat{y}(0)) = (y_0, \widehat{y}_0). \tag{1.20c}$$

is exponentially stable, for large enough λ_1, λ_2, S, and M. Here the state y is assumed to be defined in a bounded connected open spatial subset $\Omega \in \mathbb{R}^d$, where d is a positive integer (in applications, often $d \in \{1, 2, 3\}$). The domain Ω is assumed to be either smooth or a convex polygon. The state is a function $y = y(x, t)$, defined for $(x, t) \in \Omega \times (0, +\infty)$. The operator $\mathfrak{B}$ imposes the boundary conditions at the boundary $\partial\Omega$ of Ω,

$$\mathfrak{B} = \mathbf{1}, \qquad \text{for Dirichlet boundary conditions,} \tag{1.21}$$

$$\mathfrak{B} = \mathbf{n}\cdot\nabla = \tfrac{\partial}{\partial\mathbf{n}}, \qquad \text{for Neumann boundary conditions,} \tag{1.22}$$

where $\mathbf{n} = \mathbf{n}(\bar{x})$ stands for the outward unit normal vector to $\partial\Omega$, at $\bar{x} \in \partial\Omega$.

The functions $a = a(x, t)$ and $b = b(x, t)$ are defined in $\Omega \times (0, +\infty)$. Thus (a, b) is allowed to depend on both space and time variables. We assume that

$$a \in L^\infty(\Omega \times (0, +\infty)), \qquad b \in L^\infty(\Omega \times (0, +\infty))^d. \tag{1.23}$$

By defining, for both Dirichlet and Neumann boundary conditions, the spaces

$$H^2_{\mathfrak{B}}(\Omega) \coloneqq \{h \in H^2(\Omega) \mid \mathfrak{B}h\big|_{\partial\Omega} = 0\}, \text{ for } \mathfrak{B} \in \{\mathbf{1}, \tfrac{\partial}{\partial\mathbf{n}}\},$$

and

$$H^1_{\mathbf{1}}(\Omega) := \{h \in H^1(\Omega) \mid h|_{\partial\Omega} = 0\}, \qquad H^1_{\frac{\partial}{\partial \mathbf{n}}}(\Omega) := H^1(\Omega),$$

in order to apply Theorem 1.3 we simply write (1.20) in the abstract form (1.19). For this purpose, we set

$$H := L^2(\Omega), \quad V := H^1_{\mathfrak{B}}(\Omega), \quad \text{and} \quad \mathrm{D}(A) := H^2_{\mathfrak{B}}(\Omega),$$

and the linear operators

$$A := -\nu\Delta + \mathbf{1} \quad \text{and} \quad A_{\mathrm{rc}} := (a-1)\mathbf{1} + b\cdot\nabla.$$

Therefore, what remains is to choose the sets of actuators U_M and sensors W_S, and also to choose auxiliary sets $\widetilde{U}_M$ and $\widetilde{W}_S$ satisfying a particular condition that we shall present later on. We shall also give examples of such spaces, where actuators and sensors are suitable indicator functions of small spatial subdomains.

1.3. **On the literature.** Oblique projection based feedbacks have been introduced in [8]. There, the feedback control uses the dynamics of the system and is given by

$$\overline{\mathcal{K}}_M = P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp}\Big(Ay + A_{\mathrm{rc}}y - \lambda y\Big), \tag{1.24}$$

where the auxiliary space $\widetilde{\mathcal{U}}_M = \mathcal{E}_M$ is spanned by a suitable set of M_σ eigenfunctions of the diffusion operator A. Actually, in [8] it is taken for simplicity $M_\sigma = M$, but it is also mentioned that in some settings it is more convenient to consider a more general strictly increasing sequence M_σ, namely in the setting of parabolic equations evolving in higher dimensional rectangles $\Omega = \mathcal{R} \subset \mathbb{R}^d$, $d \geq 2$. See [8, Rem. 3.9 and Sect. 4.8].

Appropriate variations of such linear feedback operator were used in [9] to stabilize coupled parabolic-ODE systems, and in [2] to stabilize damped wave equations.

A particular property of the feedback in (1.24) is that it imposes the dynamics of the projection on the auxiliary finite-dimensional space $\mathcal{E}_M$, namely the orthogonal projection $q := P_{\mathcal{E}_M} y$ of the solution onto $\mathcal{E}_M$ satisfies

$$\dot{q} = -\lambda q, \qquad q(0) = q_0 \in \mathcal{E}_M. \tag{1.25}$$

In particular, the dynamics of q does not depend on the infinite-dimensional component $Q = y - q = P_{\mathcal{E}_M^\perp} y$, and the component q converges exponentially to zero. This two facts (together with a suitable condition on the pair $(\mathcal{U}_M, \mathcal{E}_M)$) can then be used to show that the component Q also goes exponentially to zero.

Here, we follow a different approach. Note that:

- *in general, with the feedback as* (1.15)*, the dynamics for* $q := P_{\widetilde{\mathcal{U}}_M} y$ *is not so simple as* (1.25)*, such dynamics is also not anymore independent of* $Q := P_{\widetilde{\mathcal{U}}_M^\perp} y$*.*

This can be seen as a disadvantage of the feedback (1.15) when compared to the feedback (1.24). However, on the other side we note that:

- *the feedback* (1.15) *does not use* A_{rc}*,*

which can be seen as an advantage of the new feedback, because it needs less knowledge of the system dynamics. Indeed, stabilization of nonautonomous systems appear, for example when we want to (locally) stabilize a nonlinear parabolic system to a desired trajectory, see for example [11, Introduction]. In that case a time-dependent reaction-convection term $A_{\mathrm{rc}} = A_{\mathrm{rc}}(t)$ will appear, which involves the linearization of the nonlinearity around the reference trajectory. This means that to compute the control input in (1.24) we will need to know the reference trajectory, which can be difficult for applications. On the other side to compute the input as in (1.15) we do not need to know the reference trajectory, because we do not use $A_{\mathrm{rc}}(t)$.

Concerning the output injection operators. In [12] a dynamic Luenberger observer is presented in the form

$$\overline{\mathcal{I}}_S\mathcal{Z}z = AP_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp}z + A_{\rm rc}P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp}z - \lambda P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp}P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp}z,$$

where, $\widetilde{\mathcal{W}}_S = \mathcal{E}_S$ is again spanned by suitable eigenfunctions. We see that the output injection operator uses the operator $A_{\rm rc}$, while the output injection operator (1.18), proposed in this manuscript does not. However, we still need to know $A_{\rm rc}$ to implement the Luenberger-type observer (1.7).

Recall that the more classical Riccati based feedbacks and observers also use the sum $A + A_{\rm rc}(t)$, indeed to find such feedback we need to solve a suitable time-backwards operator differential Riccati equation where $A + A_{\rm rc}(t)$ appears as a coefficient. See [4, 11]. In reality, for nonautonomous systems the computation of such feedback is infeasible in the infinite time horizon, but we can still compute it in a (large) finite time interval $[0, T]$ if we "assume" that $A_{\rm rc}$ is time-independent for time $t \ge T$, which corresponds to assuming that our reference trajectory is time-independent for time $t \ge T$. This "assumption" is proposed in [7]. Recall that if $A_{\rm rc}$ is time-independent, then the Riccati feedback is given by the solution of an *algebraic* Riccati equation, thus independent of time. In this way we find the operator solution at time $t = T$, and we can then solve the equation backwards for time $t \in [0, T]$. However such computations can be a difficult numerical task for accurate discretizations of the parabolic equations.

The computation of an oblique projection operator is essentially independent of the accuracy of the discretization, because the computation of an oblique projection $P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N}$ involves essentially the inversion of a matrix in $\mathbb{R}^{N_{\mathfrak{n}}\times N_{\mathfrak{n}}}$. Thus depends essentially on the number of actuators and/or sensors.

As we have said, in previous works the auxiliary spaces $\widetilde{\mathcal{U}}_M$ and $\widetilde{\mathcal{W}}_S$, have been chosen as the span of suitable eigenfunctions. Here, we show that, for the feedback we present, we have alternative choices for the auxiliary spaces, in particular, we do not need to compute the eigenfunctions to implement the proposed feedback and observer in applications.

Our oblique projection approach applies to the case of internal sensors and actuators. The extension to the case of boundary sensors are actuators remain an open question. For such controls, we can still use, for example, the classical Riccati [11] approach or the backstepping approach which makes use of particular transformations. We refer the reader to [1, 5, 10, 14] and references therein.

We mention also the use of matrix inequalities in the problems of stabilization and state estimation. We refer the reader to [6, 15] and references therein.

For stabilization of autonomous systems see also the feedback proposed in [3], using the spectral properties of the time-independent operator $A + A_{\rm rc}$.

1.4. **Contents and notation.** In Section 2 we present the assumptions we require for the dynamics' operators A and $A_{\rm rc}$ and for all the "parameters" involved in the output injection operator. Auxiliary results are gathered in Section 3, which are used later in Section 4 to prove the stabilizing property of the oblique projection operator. The stability of the coupled closed-loop system in proven in Section 5. The applicability to parabolic equations is showed in Section 6 and corresponding numerical results are presented in Section 7. Finally, Section 8 gathers the computations of particular Poincaré-like constants, and Section 9 gathers short final remarks.

Concerning the notation, we write $\mathbb{R}$ and $\mathbb{N}$ for the sets of real numbers and nonnegative integers, respectively, and we set $\mathbb{R}_r := (r, +\infty)$, $r \in \mathbb{R}$, and $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$.

Given two Banach spaces X and Y, if the inclusion $X \subseteq Y$ is continuous, we write $X \hookrightarrow Y$. We write $X \overset{\mathrm{d}}{\hookrightarrow} Y$, respectively $X \overset{\mathrm{c}}{\hookrightarrow} Y$, if the inclusion is also dense, respectively compact.

Let $X \subseteq Z$ and $Y \subseteq Z$ be continuous inclusions, where Z is a Hausdorff topological space. Then we can define the Banach spaces $X\times Y$, $X\cap Y$, and $X+Y$, endowed with the norms $|(h,g)|_{X\times Y} := \left(|h|_X^2 + |g|_Y^2\right)^{\frac{1}{2}}$, $|\hat{h}|_{X\cap Y} := |(\hat{h},\hat{h})|_{X\times Y}$, and $|\tilde{h}|_{X+Y} := \inf\limits_{(h,g)\in X\times Y} \left\{|(h,g)|_{X\times Y} \mid \tilde{h} = h+g\right\}$, respectively. In case we know that $X\cap Y = \{0\}$, we say that $X+Y$ is a direct sum and we write $X \oplus Y$ instead.

The space of continuous linear mappings from X into Y is denoted by $\mathcal{L}(X,Y)$. In case $X = Y$ we write $\mathcal{L}(X) := \mathcal{L}(X,X)$. The continuous dual of X is denoted $X' := \mathcal{L}(X,\mathbb{R})$. The adjoint of an operator $L \in \mathcal{L}(X,Y)$ will be denoted $L^* \in \mathcal{L}(Y',X')$.

The space of continuous functions from X into Y is denoted by $\mathcal{C}(X,Y)$.

The orthogonal complement to a given subset $B \subset H$ of a Hilbert space H, with scalar product $(\cdot,\cdot)_H$, is denoted $B^\perp := \{h \in H \mid (h,s)_H = 0 \text{ for all } s \in B\}$.

Given two closed subspaces $F \subseteq H$ and $G \subseteq H$ of the Hilbert space $H = F \oplus G$, we denote by $P_F^G \in \mathcal{L}(H,F)$ the oblique projection in H onto F along G. That is, writing $h \in H$ as $h = h_F + h_G$ with $(h_F,h_G) \in F \times G$, we have $P_F^G h := h_F$. The orthogonal projection in H onto F is denoted by $P_F \in \mathcal{L}(H,F)$. Notice that $P_F = P_F^{F^\perp}$.

Given a sequence $(a_j)_{j\in\{1,2,\ldots,n\}}$ of real nonnegative constants, $n \in \mathbb{N}_0$, $a_i \geq 0$, we denote $\|a\| := \max\limits_{1\leq j\leq n} a_j$.

By $\overline{C}_{[a_1,\ldots,a_n]}$ we denote a nonnegative function that increases in each of its nonnegative arguments a_i, $1 \leq i \leq n$.

Finally, C, C_i, $i = 0, 1, \ldots$, stand for unessential positive constants.

2. Assumptions

The results will follow under general assumptions on the plant dynamics operators A and A_{rc}. We will also need a particular assumption on the pair $(\mathcal{F}_N, \widetilde{\mathcal{F}}_N)$ defining the oblique projection $P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp}$ in H onto $\mathcal{F}_N$ along $\widetilde{\mathcal{F}}_N^\perp$. Below we will need several Hilbert spaces, so we must precise that here, and hereafter, the symbol $X^\perp$ stands for the orthogonal of the set $X \subset H$ in the Hilbert space H, where system (1.1) is evolving in, and which will be set as a pivot space, that is, we identify, $H' = H$. Let V be another Hilbert space with $V \subset H$.

Assumption 2.1. *$A \in \mathcal{L}(V,V')$ is symmetric and $(y,z) \mapsto \langle Ay,z\rangle_{V',V}$ is a complete scalar product in V.*

Hereafter, we suppose that V is endowed with the scalar product $(y,z)_V := \langle Ay,z\rangle_{V',V}$, which again makes V a Hilbert space. Necessarily, $A\colon V \to V'$ is an isometry.

Assumption 2.2. *The inclusion $V \subseteq H$ is dense, continuous, and compact.*

Necessarily, we have that

$$\langle y,z\rangle_{V',V} = (y,z)_H, \quad \text{for all } (y,z) \in H \times V,$$

and also that the operator A is densely defined in H, with domain $\mathrm{D}(A)$ satisfying

$$\mathrm{D}(A) \xhookrightarrow{\mathrm{d,\,c}} V \xhookrightarrow{\mathrm{d,\,c}} H \xhookrightarrow{\mathrm{d,\,c}} V' \xhookrightarrow{\mathrm{d,\,c}} \mathrm{D}(A)'.$$

Further, A has a compact inverse $A^{-1}\colon H \to \mathrm{D}(A)$, and we can find a nondecreasing system of (repeated accordingly to their multiplicity) eigenvalues $(\alpha_n)_{n\in\mathbb{N}_0}$ and a corresponding complete basis of eigenfunctions $(e_n)_{n\in\mathbb{N}_0}$:

$$0<\alpha_1\le\alpha_2\le\dots\le\alpha_n\to+\infty,\quad Ae_n=\alpha_n e_n. \tag{2.1}$$

Assumption 2.3. *For almost every $t>0$, $A_{\rm rc}(t)\in\mathcal{L}(H,V')+\mathcal{L}(V,H)$ and we have a uniform bound as $|A_{\rm rc}|_{L^\infty(\mathbb{R}_0,\mathcal{L}(H,V')+\mathcal{L}(V,H))}=:C_{\rm rc}<+\infty$.*

Assumption 2.4. *There exists a strictly increasing function $\mathfrak{n}\colon\mathbb{N}_0\to\mathbb{N}_0$, $N_{\mathfrak{n}}:=\mathfrak{n}(N)$, and a sequence of pairs $(\mathcal{F}_N,\widetilde{\mathcal{F}}_N)_{N\in\mathbb{N}_0}$ satisfying* (1.13).

Next, for each $N\in\mathbb{N}_0$, we define the Poincaré-like constants

$$\xi_{N_+}=\xi_{N_+}^{\mathcal{F}}:=\inf_{\Theta\in(V\bigcap\mathcal{F}_N^\perp)\setminus\{0\}}\tfrac{|\Theta|_V^2}{|\Theta|_H^2}, \tag{2.2a}$$

$$\xi_N=\xi_N^{\widetilde{\mathcal{F}}}:=\sup_{\theta\in\widetilde{\mathcal{F}}_N\setminus\{0\}}\tfrac{|\theta|_V^2}{|\theta|_H^2}, \tag{2.2b}$$

Assumption 2.5. *The pair $(\mathfrak{n},(\mathcal{F}_N,\widetilde{\mathcal{F}}_N)_{N\in\mathbb{N}_0})$, as in Assumption 2.4, satisfy $\lim\limits_{N\to+\infty}\xi_{N_+}=+\infty$.*

The last assumption concerns the type of outputs.

Assumption 2.6. *The output $w=\mathcal{Z}y\in\mathbb{R}^{S_\varsigma}$ of the measurements made by our set of sensors $W_S=\{\mathfrak{w}_i\mid 1\le i\le S_\varsigma\}\subset H$ is of the form*

$$w(t)=(w_1(t),w_2(t),\dots,w_{S_\varsigma}(t)),\quad w_i(t):=(\mathfrak{w}_i,y(t))_H.$$

Remark 2.7. *We shall show that Assumptions 2.1–2.6 are satisfiable for parabolic systems as in Section 1.2, evolving in rectangular domains $\Omega\subset\mathbb{R}^d$. An analogous argument can be used to extend the satisfiability of the assumptions for parabolic systems evolving in convex polygonal domains.*

3. Auxiliary results

Here we gather auxiliary lemmas we will need later.

Lemma 3.1. *Let F and G be closed subspaces of H, such that $H=F\oplus G^\perp$. Then the adjoint $(P_F^{G^\perp})^*$, in $\mathcal{L}(H)$, of the oblique projection $P_F^{G^\perp}$ is the oblique projection $P_G^{F^\perp}$.*

For a proof of Lemma 3.1 we refer to [13, Lem. 3.8].

Lemma 3.2. *If Assumption 2.6 holds true and the set of sensors is linearly independent, then there exists an operator $\mathbf{Z}^{W_S}\colon\mathbb{R}^{S_\varsigma}\to\mathcal{W}_S$ such that $P_{\mathcal{W}_S}=\mathbf{Z}^{W_S}\mathcal{Z}$.*

For the proof of Lemma 3.2 see [12, Sect. 2].

Lemma 3.3. *Let $z\in\mathcal{C}([0,+\infty),[0,+\infty))$ satisfy $z(t_2)\le \mathrm{e}^{-\mu(t_2-t_1)}z(t_1)$ for all $t_2\ge t_1\ge0$. If $z(s)>0$, then z strictly decreases at time $t=s$.*

Proof. Let $s\ge0$ and $z(s)>0$. If $s_2>s>s_1\ge0$, it follows that

$$z(s_2)-z(s)\le(\mathrm{e}^{-\mu(s_2-s)}-1)z(s)<0,\quad s_2>s,$$

and $0<z(s)\le\mathrm{e}^{-\mu(s-s_1)}z(s_1)$, where the latter implies

$$z(s_1)-z(s)\ge(1-\mathrm{e}^{-\mu(s-s_1)})z(s_1)>0,\quad s>s_1.$$

Hence z is strictly decreasing at time $t=s$. □

Lemma 3.4. *Let $\mu_1>\mu>0$. Then we have*

$$\int_s^t\mathrm{e}^{-\mu(t-\tau)}\mathrm{e}^{-\mu_1(\tau-s)}\,\mathrm{d}\tau\le\tfrac{1-\mathrm{e}^{-(\mu_1-\mu)(t-s)}}{\mu_1-\mu}\mathrm{e}^{-\mu(t-s)}.$$

For a proof of Lemma 3.4 we refer to [2, Prop. 3.2].

Lemma 3.5. *Let $\mathcal{A} \in \mathcal{L}(H, V') + \mathcal{L}(V, H)$. Then, for every $\gamma > 0$, and every $(v_1, v_2) \in V \times V$ it holds*

$$2\left|\langle \mathcal{A}v_1, v_2\rangle_{V',V}\right|_{\mathbb{R}} \le \gamma(|v_2|_V^2 + |v_1|_V^2) + \gamma^{-1}\|\mathcal{A}\|^2 (|v_1|_H^2 + |v_2|_H^2), \tag{3.1}$$

with $\|\mathcal{A}\| := |\mathcal{A}|_{\mathcal{L}(H,V')+\mathcal{L}(V,H)}$.

For a proof of Lemma 3.5 we refer to [12, Sect. 2].

4. Proof of the main stability result

We give the proof of Theorem 1.2, which we can now state more precisely as follows.

Theorem 4.1. *Let Assumptions 2.1–2.5 hold true, and let $\mu > 0$. Then there are $N \in \mathbb{N}_0$ and $\lambda > 0$ large enough, such that the solution of system* (1.14) *satisfies*

$$|y(t)|_H \le \mathrm{e}^{-\mu(t-s)} |y(s)|_H\,, \text{ for } t \ge s \ge 0 \text{ and } y_0 \in H.$$

In particular, $t \mapsto |y(t)|_H$ is strictly decreasing at time $t = s$, if $|y(s)|_H \ne 0$. Furthermore, N and λ can be chosen as:

- $N = \overline{C}_{[C_{\mathrm{rc}},\mu]}$ *and* $\lambda = \overline{C}_{[C_{\mathrm{rc}},\mu,\xi_N]}$, *in case* $\rho = 0$,
- $N = \overline{C}_{[C_{\mathrm{rc}},\mu]}$ *and* $\lambda = \overline{C}_{[C_{\mathrm{rc}},\mu]}$, *in case* $\rho = 1$.

Proof. We decompose the solution y into the oblique components as follows,

$$y = \theta + \Theta, \quad \text{with} \quad \theta = P_{\widetilde{\mathcal{F}}_N}^{\mathcal{F}_N^\perp} y \quad \text{and} \quad \Theta = P_{\mathcal{F}_N^\perp}^{\widetilde{\mathcal{F}}_N} y.$$

We consider the cases $\rho = 0$ and $\rho = 1$ separately.

$\triangleright$ *The case $\rho = 0$.* Multiplying the dynamics in (1.14) by y, with $\rho = 0$, we find

$$\begin{aligned}\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}|y|_H^2 &= -\langle Ay + A_{\mathrm{rc}}y, y\rangle_{V',V} - \lambda\langle P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp}\theta, y\rangle_{V',V}\\ &= -|y|_V^2 - \langle A_{\mathrm{rc}}y, y\rangle_{V',V} - \lambda|\theta|_H^2\,,\end{aligned} \tag{4.1}$$

where we have used $\langle P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp}\theta, y\rangle_{V',V} = (P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp}\theta, y)_H = (\theta, (P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp})^* y)_H$, and Lemma 3.1.

Using Assumption 2.3 and Lemma 3.5 we obtain, for every $\gamma_1 > 0$,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}|y|_H^2 &\le -2|y|_V^2 + 2\gamma_1|y|_V^2 + 2\gamma_1^{-1}C_{\mathrm{rc}}^2|y|_H^2 - 2\lambda|\theta|_H^2\\ &= -(2 - 2\gamma_1)|\Theta + \theta|_V^2 + 2\gamma_1^{-1}C_{\mathrm{rc}}^2|\Theta + \theta|_H^2 - 2\lambda|\theta|_H^2\,.\end{aligned} \tag{4.2}$$

For any Hilbert space Z and $(z_1, z_2) \in Z \times Z$, we have

$$\begin{aligned}|z_1 + z_2|_Z^2 &= |z_1|_Z^2 + |z_2|_Z^2 + 2(z_1, z_2)_Z,\\ 2\left|(z_1, z_2)_Z\right|_{\mathbb{R}} &\le \gamma_0|z_1|_Z^2 + \gamma_0^{-1}|z_2|_Z^2\,, \quad \text{for all} \quad \gamma_0 > 0,\end{aligned}$$

hence, from (4.2) it follows that, if $1 - \gamma_1 \ge 0$, we have that for every $\gamma_2 > 0$ and $\gamma_3 > 0$,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}|y|_H^2 &\le -2(1-\gamma_1)\left(|\Theta|_V^2 + |\theta|_V^2 + 2(\Theta, \theta)_V\right)\\ &\quad + \gamma_1^{-1}C_{\mathrm{rc}}^2\left((1+\gamma_2)|\Theta|_H^2 + (1+\gamma_2^{-1})|\theta|_H^2\right) - 2\lambda|\theta|_H^2\\ &\le -2(1-\gamma_1)\left(|\Theta|_V^2 + |\theta|_V^2\right) - 2\lambda|\theta|_H^2 + 2(1-\gamma_1)\left(\gamma_3|\Theta|_V^2 + \gamma_3^{-1}|\theta|_V^2\right)\\ &\quad + \gamma_1^{-1}C_{\mathrm{rc}}^2\left((1+\gamma_2)|\Theta|_H^2 + (1+\gamma_2^{-1})|\theta|_H^2\right)\\ &= -2(1-\gamma_1)(1-\gamma_3)|\Theta|_V^2 + \gamma_1^{-1}C_{\mathrm{rc}}^2(1+\gamma_2)|\Theta|_H^2\\ &\quad - 2(1-\gamma_1)(1-\gamma_3^{-1})|\theta|_V^2 + \gamma_1^{-1}C_{\mathrm{rc}}^2(1+\gamma_2^{-1})|\theta|_H^2 - 2\lambda|\theta|_H^2\,.\end{aligned}$$

We can choose $\gamma_1 \in (0,1)$, $\gamma_2 > 0$ and $\gamma_3 \in (0,1)$, and use (2.2) to obtain

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} |y|_H^2 &\le -2\lambda |\theta|_H^2 + 2(1-\gamma_1)(\gamma_3^{-1}-1)\xi_N |\theta|_H^2 + C_{\mathrm{rc}}^2(1+\gamma_2^{-1})\gamma_1^{-1} |\theta|_H^2\\ &\quad - 2(1-\gamma_1)(1-\gamma_3)\xi_{N_+} |\varTheta|_H^2 + C_{\mathrm{rc}}^2(1+\gamma_2)\gamma_1^{-1} |\varTheta|_H^2\\ &= -\Xi_1(N) |\varTheta|_H^2 - \Xi_2(N,\lambda) |\theta|_H^2, \end{aligned}\tag{4.3a}$$

with

$$(\gamma_1,\gamma_2,\gamma_3) \in (0,1)\times\mathbb{R}_0\times(0,1), \tag{4.3b}$$

$$\Xi_1(N) \coloneqq 2(1-\gamma_1)(1-\gamma_3)\xi_{N_+} - C_{\mathrm{rc}}^2(1+\gamma_2)\gamma_1^{-1}, \tag{4.3c}$$

$$\Xi_2(N,\lambda) \coloneqq 2\lambda - 2(1-\gamma_1)(\gamma_3^{-1}-1)\xi_N - C_{\mathrm{rc}}^2(1+\gamma_2^{-1})\gamma_1^{-1}. \tag{4.3d}$$

Observe that for a fixed triple $(\gamma_1,\gamma_2,\gamma_3) \in (0,1)\times\mathbb{R}_0\times(0,1)$ and for an arbitrary given $\mu > 0$, by Assumption 2.5, we can choose N large enough so that

$$\Xi_1(N) \ge 4\mu \tag{4.4a}$$

and, subsequently, we can choose λ large enough so that

$$\Xi_2(N,\lambda) \ge 4\mu. \tag{4.4b}$$

Now, for such choices we obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t} |y|_H^2 \le -4\mu \left(|\varTheta|_H^2 + |\theta|_H^2\right) \le -2\mu |y|_H^2$$

and, by the Gronwall Lemma it follows that

$$|y(t)|_H^2 \le \mathrm{e}^{-2\mu(t-s)} |y(s)|_H^2, \quad \text{for all} \quad t \ge s \ge 0.$$

Note that we have chosen N and λ as $N = \overline{C}_{[C_{\mathrm{rc}},\mu]}$ and $\lambda = \overline{C}_{[C_{\mathrm{rc}},\mu,\xi_N]}$.

▷ *The case $\rho = 1$.* As in the case $\rho = 0$ we multiply (1.14) by y. Now, for $\rho = 1$, we find the analogous of (4.1),

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} |y|_H^2 &= -2\langle Ay + A_{\mathrm{rc}}y, y\rangle_{V',V} - 2\lambda\langle P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp} A\theta, y\rangle_{V',V}\\ &= -2 |y|_V^2 - 2\langle A_{\mathrm{rc}}y, y\rangle_{V',V} - 2\lambda |\theta|_V^2, \end{aligned}$$

where now it appears $|\theta|_V^2$ instead of $|\theta|_H^2$. Note that since $\theta = P_{\widetilde{\mathcal{F}}_N}^{\mathcal{F}_N^\perp} y \in \widetilde{\mathcal{F}}_N = \operatorname{span} \widetilde{F}_N \subset V$, then $A\theta \in V'$, thus $P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp}$ here must be seen as the extension of $P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp}$ to V', namely as $P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp} A\theta \in \mathcal{F}_N \subset H \subset V'$ defined by

$$\langle P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp} A\theta, v\rangle_{V',V} \coloneqq \langle A\theta, P_{\widetilde{\mathcal{F}}_N}^{\mathcal{F}_N^\perp} v\rangle_{V',V}, \quad \text{for all} \quad v \in V,$$

which is well defined, since $P_{\widetilde{\mathcal{F}}_N}^{\mathcal{F}_N^\perp} v \in \widetilde{\mathcal{F}}_N \subset V$. Note that $\langle P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp} A\theta, v\rangle_{V',V} = \langle P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp} A\theta, P_{\mathcal{F}_N} v\rangle_{V',V}$, by definition.

We can write $\langle P_{\mathcal{F}_N}^{\widetilde{\mathcal{F}}_N^\perp} A\theta, y\rangle_{V',V} = \langle A\theta, \theta\rangle_{V',V} = |\theta|_V^2$, and then arrive at the analogous of (4.3), as follows

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} |y|_H^2 &\le -2(1-\gamma_1)(1-\gamma_3)\xi_{N_+} |\varTheta|_H^2\\ &\quad + C_{\mathrm{rc}}^2(1+\gamma_2)\gamma_1^{-1} |\varTheta|_H^2 + C_{\mathrm{rc}}^2(1+\gamma_2^{-1})\gamma_1^{-1} |\theta|_H^2\\ &\quad - 2\lambda |\theta|_V^2 + 2(1-\gamma_1)(\gamma_3^{-1}-1) |\theta|_V^2, \end{aligned}$$

and for λ large enough so that $2\lambda - 2(1-\gamma_1)(\gamma_3^{-1}-1) > 0$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} |y|_H^2 &\le -2(1-\gamma_1)(1-\gamma_3)\xi_{N_+} |\Theta|_H^2 \\
&\quad + C_{\mathrm{rc}}^2(1+\gamma_2)\gamma_1^{-1} |\Theta|_H^2 + C_{\mathrm{rc}}^2(1+\gamma_2^{-1})\gamma_1^{-1} |\theta|_H^2 \\
&\quad - (2\lambda - 2(1-\gamma_1)(\gamma_3^{-1}-1))\alpha_1 |\theta|_H^2 \\
&= -\Xi_1(N) |\Theta|_H^2 - \widehat{\Xi}_2(\lambda) |\theta|_H^2 ,
\end{aligned} \tag{4.5a}$$

where $\alpha_1 = |\mathbf{1}|_{\mathcal{L}(V,H)}^{-2}$ is the first eigenvalue of A, and

$$(\gamma_1,\gamma_2,\gamma_3) \in (0,1)\times\mathbb{R}_0\times(0,1), \tag{4.5b}$$

$$\Xi_1(N) \coloneqq 2(1-\gamma_1)(1-\gamma_3)\xi_{N_+} - C_{\mathrm{rc}}^2(1+\gamma_2)\gamma_1^{-1}, \tag{4.5c}$$

$$\widehat{\Xi}_2(\lambda) \coloneqq (2\lambda - 2(1-\gamma_1)(\gamma_3^{-1}-1))\alpha_1 - C_{\mathrm{rc}}^2(1+\gamma_2^{-1})\gamma_1^{-1}, \tag{4.5d}$$

Now, we can choose N and λ large enough so that

$$\lambda - (1-\gamma_1)(\gamma_3^{-1}-1) > 0, \tag{4.6a}$$

$$\Xi_1(N) \ge 4\mu, \quad \text{and} \quad \widehat{\Xi}_2(\lambda) \ge 4\mu, \tag{4.6b}$$

which leads us to

$$|y(t)|_H^2 \le \mathrm{e}^{-2\mu(t-s)} |y(s)|_H^2 , \quad \text{for all} \quad t \ge s \ge 0.$$

Note that we have chosen N and λ as $N = \overline{C}_{[C_{\mathrm{rc}},\mu]}$ and $\lambda = \overline{C}_{[C_{\mathrm{rc}},\mu]}$.

Finally, by Lemma 3.3 we have that when $|y(s)|_H^2 \ne 0$ the norm $|y(t)|_H^2$ strictly decreases at time $t = s$. □

Remark 4.2. *One difference between the case $\rho = 0$ and $\rho = 1$ is the choice of λ. In the latter λ can be chosen independently of N, and in particular of ξ_N.*

Remark 4.3. *Note that in the computations in the proof of Theorem 4.1, we have used the fact that, for $y \in V$, we have $(\theta,\Theta) \in V\times V$, which holds because $\widetilde{\mathcal{F}}_N \subset V$.*

5. The closed-loop observer-feedback system

In this section we investigate the output based feedback control problem. We will use Theorem 4.1 to prove Theorem 1.3, or more precisely to prove the following.

Theorem 5.1. *Let Assumptions 2.1–2.3 hold true. Let pairs $(\sigma,(\mathcal{U}_M,\widetilde{\mathcal{U}}_M)_{M\in\mathbb{N}_0})$ and $(\varsigma,(\mathcal{W}_S,\widetilde{\mathcal{W}}_S)_{S\in\mathbb{N}_0})$ be both satisfying Assumptions 2.4–2.5. Let the output $\mathcal{Z}y$ of the measurements made by the sensors in $\mathcal{W}_S$ be given as in Assumption 2.6 and let $\mathbf{Z}^{\mathcal{W}_S}$ be given by Lemma 3.2. Finally, let $\mu > 0$. Then, for S, M, λ_1, and λ_2 large enough, the solution of system* (1.19) *satisfies*

$$|(y(t),\widehat{y}(t))|_{H\times H} \le \varpi \mathrm{e}^{-\mu(t-s)} |(y(s),\widehat{y}(s))|_{H\times H} ,$$

for a suitable constant $\varpi > 1$ and for all $t \ge s \ge 0$ and all $(y_0,\widehat{y}_0) \in H\times H$.

Proof. We set the observer estimate error $z \coloneqq \widehat{y} - y$, and from (1.19) we find

$$\dot{z} + Az + A_{\mathrm{rc}}z = -\lambda_1 P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp} A^{\rho_1} P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp} z, \tag{5.1a}$$

$$\dot{y} + Ay + A_{\mathrm{rc}}y = -\lambda_2 P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^{\rho_2} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp} y - \lambda_2 P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^{\rho_2} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp} z, \tag{5.1b}$$

with $y(0) = y_0$ and $z(0) = z_0 \coloneqq \widehat{y}_0 - y_0$.

Let us fix $\mu_1 > \mu$. From Theorem 4.1, with μ_1 in the role of μ, we have that for λ_1 and S large enough it follows that system (5.1a) is stable, and

$$|z(t)|_H \le \mathrm{e}^{-\mu_1(t-s)} |z(s)|_H , \quad \text{for} \quad t \ge s \ge 0. \tag{5.2}$$

Analogously we have that the solution of the unperturbed system (5.1b)

$$\dot{\overline{y}} + A\overline{y} + A_{\mathrm{rc}}\overline{y} = -\lambda_2 P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^{\rho_2} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp} \overline{y} \tag{5.3}$$

satisfies, for λ_2 and M large enough,

$$|\overline{y}(t)|_H \le \mathrm{e}^{-\mu(t-s)} |\overline{y}(s)|_H, \quad \text{for} \quad t \ge s \ge 0.$$

By Duhamel formula we obtain that the solution of (5.1b) satisfies, with $\mathbf{P}_{\rho_2} := P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^{\rho_2} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp}$,

$$\begin{aligned}
|y(t)|_H &\le \mathrm{e}^{-\mu(t-s)} |y(s)|_H + \int_s^t \mathrm{e}^{-\mu(t-\tau)} |\lambda_2 \mathbf{P}_{\rho_2} z(\tau)|_H \,\mathrm{d}\tau \\
&\le \mathrm{e}^{-\mu(t-s)} |y(s)|_H + \lambda_2 |\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)} |z(s)|_H \mathcal{I}
\end{aligned}$$

with $\mathcal{I} := \int_s^t \mathrm{e}^{-\mu(t-\tau)} \mathrm{e}^{-\mu_1(\tau-s)} \,\mathrm{d}\tau$. Hence, by Lemma 3.4 it follows that

$$|y(t)|_H \le \mathrm{e}^{-\mu(t-s)} |y(s)|_H + \lambda_2 |\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)} \tfrac{1}{\mu_1-\mu} \mathrm{e}^{-\mu(t-s)} |z(s)|_H, \tag{5.4}$$

and from (5.2), (5.4), and

$$|(y(t), \widehat{y}(t))|_{H\times H}^2 = |y(t)|_H^2 + |y(t) + z(t)|_H^2 \le 3 |y(t)|_H^2 + 2 |z(t)|_H^2, \tag{5.5}$$

we obtain

$$\begin{aligned}
|(y(t), \widehat{y}(t))|_{H\times H}^2 &\le 3\mathrm{e}^{-2\mu(t-s)} |y(s)|_H^2 + \left(3\lambda_2^2 |\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)}^2 (\tfrac{1}{\mu_1-\mu})^2 + 2\right) \mathrm{e}^{-2\mu(t-s)} |z(s)|_H^2 \\
&\le \varpi^2 \mathrm{e}^{-2\mu(t-s)} |(y(s), \widehat{y}(s))|_H^2,
\end{aligned}$$

with $\varpi^2 = \overline{C}_{\left[\frac{\lambda_2 |\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)}}{\mu_1-\mu}\right]}$, which finishes the proof. □

Remark 5.2. *Note that the transient bound constant ϖ depends (or may depend) on M. Indeed, in the case $\rho_2 = 0$ it depends on $\lambda_2 = \lambda_2(\xi_M)$, and in the case $\rho_2 = 1$ it depends on $|\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)} = |\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)}(M)$.*

Remark 5.3. *If the auxiliary set satisfies in addition $\widetilde{U}_M \subset \mathrm{D}(A) \subset V$, we can find the estimates*

$$\begin{aligned}
|\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)} &\le \left| P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} \right|_{\mathcal{L}(H)}^2, \quad \textit{for } \rho_2 = 0, \\
|\mathbf{P}_{\rho_2}|_{\mathcal{L}(H)} &\le \left| P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} \right|_{\mathcal{L}(H)}^2 \left| P_{\widetilde{\mathcal{U}}_M} A P_{\widetilde{\mathcal{U}}_M} \right|_{\mathcal{L}(H)}, \quad \textit{for } \rho_2 = 1.
\end{aligned}$$

6. Example of application

We show that the parabolic coupled system (1.20), evolving in spatial rectangular domains, is stable for large enough S, M, λ_1, and λ_2, and for suitable chosen sets of actuators and sensors. The same arguments can be extended to parabolic equations evolving in triangular domains and consequently to parabolic equations evolving in general convex polygonal domains.

It is enough to show that our Assumptions 2.1–2.6 are satisfied. Assumptions 2.1–2.3 are satisfied with A and A_{rc} as in Section 1.2. Hence it remains to show that we can choose the set of actuators U_M and sensors W_S, together with auxiliary sets $\widetilde{U}_M$ and $\widetilde{W}_S$ such that Assumptions 2.4–2.6 are satisfied.

An option is simply to choose both actuators and sensors as indicator functions uniformly distributed as in [12, Sect. 4], and also to choose the auxiliary sets as sets of suitable eigenfunctions, again as in [12, Sect. 4].

Here, we present an alternative choice, where the actuators are chosen in a slightly different way, but the auxiliary sets are chosen in a considerably different way, namely in a more ad-hoc way depending on the actuators and sensors.

For simplicity we will consider here only the case of a two-dimensional rectangle. The argument is analogous for other dimensions.

Also for simplicity, we will consider the same number $N_\mathfrak{n}$ of actuators and sensors. Again both actuators and sensors are indicator functions which we place as illustrated in Figure 1 for a planar rectangle $\Omega^\times \in \mathbb{R}^2$, where the number of actuators and sensors is given by $N_\mathfrak{n} = \mathfrak{n}(N) = 2N^2$, $N \in \mathbb{N}_0$.

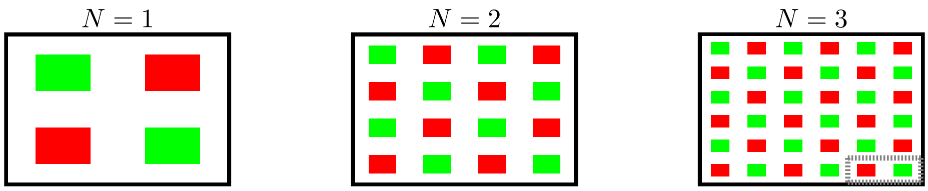

FIGURE 1. Supports of actuators (in red) and sensors (in green).

The set of actuators and the set of sensors cover each the same percentage of the domain, which is independent of the number $N_\mathfrak{n}$ of the actuators and sensors.

Note that if we consider only the actuators or only the sensors, then for a given N, we have that our planar rectangular domain is partitioned into $2N^2$ rescaled copies of the reference rectangle $\mathcal{R}$ in Figure 2 (up to a rotation). See one of these copies highlighted in Figure 1, at the right-bottom corner of the case $N = 3$.

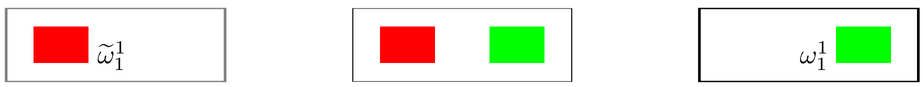

FIGURE 2. Reference rectangle $\mathcal{R}$ (center). Actuator reference rectangle $\mathcal{R}_\mathrm{a}$ (left), and sensor reference rectangle $\mathcal{R}_\mathrm{s}$ (right).

Considering the sensors first, it is clear that we have a Poincaré-like constant satisfying

$$\inf_{\Theta\in(H^1(\mathcal{R}_\mathrm{s})\bigcap\mathcal{W}_1^{r\perp})\setminus\{0\}} \frac{|\Theta|^2_{H^1(\mathcal{R}_\mathrm{s})}}{|\Theta|^2_{L^2(\mathcal{R}_\mathrm{s})}} =: \xi_{1+} > 0. \tag{6.1}$$

where $W_1^r = \{1_{\omega_1^1}|_{\mathcal{R}_\mathrm{s}}\}$ and $\mathcal{W}_1^r = \operatorname{span} W_1^r \subset L^2(\mathcal{R}_\mathrm{s})$, and $\omega_1^1 = (p, p + L_1) \times (q, q + L_2) \subset \mathcal{R}_\mathrm{s}$ is the open sub-rectangle as in Figure 2.

Then, proceeding as in [12, Sect. 4] we can arrive at

$$\lim_{N\to+\infty} \inf_{\Theta\in(V\bigcap\mathcal{W}_N^\perp)\setminus\{0\}} \frac{|\Theta|^2_V}{|\Theta|^2_H} = +\infty. \tag{6.2}$$

where $W_N = \{1_{\omega_1^N}, \ldots, 1_{\omega_{N_\mathfrak{n}}^N}\}$ and $\mathcal{W}_N = \operatorname{span} W_N \subset L^2(\Omega^\times)$, where the ω_j^Ns are the open sub-rectangles as in Figure 1 corresponding to the sensors. Thus, the divergent limit in Assumption 2.5 holds.

Observe that, for a suitable left-bottom bound pair (p_j^N, q_j^N) we have

$$\omega_j^N = (p_j^N, p_j^N + \tfrac{L_1}{N}) \times (q_j^N, q_j^N + \tfrac{L_2}{N}), \quad j = 1, 2, \ldots, N_\mathfrak{n}.$$

6.1. **Auxiliary set in V.** We propose to take the auxiliary set $\widetilde{W}_N \subset V$ as $\widetilde{W}_N = \{\phi_{\omega_1^N}, \dots, \phi_{\omega_{N_\mathfrak{n}}^N}\}$ with

$$\phi_{\omega_j^N} = \sin\left(N\tfrac{\pi(x_1-p_j^N)}{L_1}\right)\sin\left(N\tfrac{\pi(x_2-q_j^N)}{L_2}\right). \tag{6.3}$$

The equality $L^2(\Omega) = \mathcal{W}_N \oplus \widetilde{\mathcal{W}}_N^\perp$ can be concluded from [8, Lem. 2.7] and from

$$\begin{cases} (\overline{1}_{\omega_j^N}, \overline{\phi}_{\omega_i^N})_{L^2(\Omega)} = 0, & \text{if} \quad i \neq j,\\ (\overline{1}_{\omega_j^N}, \overline{\phi}_{\omega_i^N})_{L^2(\Omega)} = \frac{8}{\pi^2}, & \text{for} \quad 1 \le j \le N_\mathfrak{n}, \end{cases}$$

for normalized sensors $\overline{1}_{\omega_j^N} = (\frac{1}{L_1L_2})^{\frac12}1_{\omega_j^N}$ and functions $\overline{\phi}_{\omega_j^N} = (\frac{4}{L_1L_2})^{\frac12}\phi_{\omega_j^N}$. Now, by applying [8, Cor. 2.9] we also obtain that

$$\left|P_{\mathcal{W}_N}^{\widetilde{\mathcal{W}}_N^\perp}\right|_{\mathcal{L}(L^2(\Omega))} = \left|P_{\widetilde{\mathcal{W}}_N}^{\mathcal{W}_N^\perp}\right|_{\mathcal{L}(L^2(\Omega))} = \tfrac{\pi^2}{8}.$$

We choose the auxiliary space $\widetilde{U}_N$ in a similar way, using now the supports of the actuators, and arrive at

$$L^2(\Omega) = \mathcal{U}_N \oplus \widetilde{\mathcal{U}}_N^\perp, \qquad \left|P_{\mathcal{U}_N}^{\widetilde{\mathcal{U}}_N^\perp}\right|_{\mathcal{L}(L^2(\Omega))} = \tfrac{\pi^2}{8}.$$

Therefore, Assumptions 2.4–2.5 are satisfied.

Finally, we simply assume that our output is given as $\mathcal{Z}y(t) = \mathcal{Z}^N y(t) = (w_1^N(t), w_2^N(t), \dots, w_{N_\mathfrak{n}}^N(t))$, with $w_j^N(t) \coloneqq (1_{\omega_j^N}, y(t))_{L^2(\Omega)}$, thus Assumption (2.6) is trivially satisfied.

Theorem 6.1. *For pairwise disjoint sensor/actuator rectangular regions ω_i^N, with the choice* (6.3) *above, the Poincaré-like constant ξ_N in* (2.2)*, satisfies*

$$\xi_N = \sup_{\theta \in \widetilde{\mathcal{U}}_N \setminus \{0\}} \tfrac{\nu|\nabla\theta|_H^2 + |\theta|_H^2}{|\theta|_H^2} = N^2\tfrac{\nu\pi^2(L_1^2+L_2^2)}{4L_1L_2} + 1.$$

Thus, inequalities in (4.4) suggest that as N increases we may need to take larger values of λ_1 and λ_2. The proof of Theorem 6.1 follows by direct computations. For the sake of completeness, we give the details in Section 8.

6.2. **Auxiliary set in $\mathrm{D}(A)$.** The auxiliary functions $\phi_{\omega_j^N}$ in (6.3) are in $V \setminus \mathrm{D}(A)$. Here we show that we can take smoother functions for the auxiliary set (cf. Remark 5.3).

We can choose $\underline{\widetilde{W}}_N = \{\phi^2_{\omega_1^N}, \dots, \phi^2_{\omega_{N_\mathfrak{n}}^N}\} \subset \mathrm{D}(A)$. Note that, by normalizing

$$\phi^2_{\omega_j^N} = \sin^2\left(N\tfrac{\pi(x_1-p_j^N)}{L_1}\right)\sin^2\left(N\tfrac{\pi(x_2-q_j^N)}{L_2}\right) \tag{6.4}$$

we obtain $\overline{\phi}^2_{\omega_j^N} \coloneqq (\frac{64}{9L_1L_2})^{\frac12}\phi^2_{\omega_j^N}$,

$$\begin{cases} (\overline{1}_{\omega_j^N}, \overline{\phi}^2_{\omega_i^N})_{L^2(\Omega)} = 0, & \text{if} \quad i \neq j,\\ (\overline{1}_{\omega_j^N}, \overline{\phi}^2_{\omega_j^N})_{L^2(\Omega)} = \frac{2}{3}, & \text{for} \quad 1 \le j \le N_\mathfrak{n}, \end{cases}$$

and $\left|P_{\mathcal{W}_N}^{\underline{\widetilde{\mathcal{W}}}_N^\perp}\right|_{\mathcal{L}(L^2(\Omega))} = \left|P_{\underline{\widetilde{\mathcal{W}}}_N}^{\mathcal{W}_N^\perp}\right|_{\mathcal{L}(L^2(\Omega))} = \frac{3}{2}$. Observe also that by taking (6.4) we have an oblique projection with a larger norm, $\frac32 > \frac{\pi^2}{8} \approx 1.2337$. Recall that the transient bound constant ϖ in Theorems 5.1 increases with such norm (cf. Remark 5.2). However, we underline that $\frac32$ is considerably smaller than the values we obtain when we take eigenfunctions as auxiliary sets. See [8] and [13, Theorems 4.4

and 5.2]. In particular, in those previous results the norm of the oblique projection increases as the total volume covered by the actuators decreases. Here, we propose auxiliary functions which are constructed depending on the actuators and sensors, and due to such ad-hoc construction the norm of the oblique projections is independent of the number and volume of the actuators and sensors.

For this choice, the Poincaré constant ξ_N is slightly smaller than the one in Theorem 6.1.

Theorem 6.2. *For pairwise disjoint sensor/actuator rectangular regions ω_i^N, with the choice* (6.4) *above, the Poincaré-like constant ξ_N in* (2.2)*, satisfies*

$$\xi_N = \sup_{\theta\in\widetilde{\mathcal{U}}_N\setminus\{0\}} \frac{\nu|\nabla\theta|_H^2+|\theta|_H^2}{|\theta|_H^2} = N^2\frac{3\nu\pi^2(L_1^2+L_2^2)}{16L_1L_2}+1.$$

The details of the proof are given in Section 8.

7. Numerical simulations

We show the results of simulations concerning parabolic systems as (5.1). We also consider the case our measured output is subject to error measurements, in this case the "real" (error free) output is still denoted by $\mathcal{Z}y(t)$, but the measured output at our disposal is

$$\mathrm{e}^{\eta}\zeta(t)+\mathcal{Z}y(t), \qquad \zeta(t)\in\mathbb{R}^{S_\varsigma}, \qquad t\ge 0.$$

where $\mathrm{e}^{\eta}\zeta(t)$ is the measurement error, we shall refer to $\mathrm{e}^{\eta}\zeta(t)$ as the noise. The nonnegative factor e^{η} will be used to investigate the behavior as the magnitude of noise decreases, that is, as $\eta\to-\infty$.

We have run the simulations, for several time intervals $t\in[0,T]$, $T>0$, for systems as

$$\dot z + Az + A_{\rm rc}z = -\chi_1\lambda_1 P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp} A^{\rho_1} P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp}\mathbf{Z}^{W_S}(\mathcal{Z}z+\mathrm{e}^{\eta}\zeta), \tag{7.1a}$$

$$\dot y + Ay + A_{\rm rc}y = -\chi_2\lambda_2 P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^{\rho_2} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp}(y+\kappa z), \tag{7.1b}$$

where $(\chi_1,\chi_2,\rho_1,\rho_2,\kappa)\in\{0,1\}^5$ and $\eta\in\mathbb{R}\cup\{-\infty\}$.

Therefore, $\kappa=1$ correspond to our coupled system, and $\kappa=0$ correspond to the uncoupled system. We consider the latter just to test our feedback in the case of a perfect estimate $z=\widehat y - y = 0$. Analogously $\rho_i=1$, $i\in\{1,2\}$, corresponds to the use of diffusion in the injection/feedback operator, and $\chi_i=1$, $i\in\{1,2\}$, corresponds to the case the injection/feedback is active. Noisy measurements correspond to $\eta\in\mathbb{R}$ and perfect measurements correspond to (or, are denoted by) $\eta=-\infty$.

For the operators above we have taken

$$Aw = (-0.1\Delta+\mathbf{1})w, \qquad A_{\rm rc}w = aw + b\cdot\nabla w, \tag{7.2}$$

with $a=-2+x_1-|\sin(t+x_1)|_{\mathbb{R}}$ and $b=\begin{bmatrix} x_1+x_2\\ \cos(t)x_1x_2\end{bmatrix}$.

The spatial domain Ω has been taken as the unit square $\mathbf{R}_1=(0,1)\times(0,1)$ in $\mathbb{R}^2$, $(x_1,x_2)\in\Omega=\mathbf{R}_1$, and we have considered Neumann boundary conditions on the boundary $\partial\Omega$, $\mathbf{n}\cdot\nabla z|_{\partial\Omega}=0=\mathbf{n}\cdot\nabla y|_{\partial\Omega}$.

As initial conditions we have taken

$$z(0)=z_0=-1-2x_2^2 \quad\text{and}\quad y(0)=y_0=0.1-0.2x_1x_2,$$

and we have taken 18 sensors and 18 actuators as in Figure 1 (case $N=3$). The sensors and actuators regions, are shown in Figure 3, where moreover we show the triangulation we have used in our simulations. As auxiliary functions we take functions as (6.4).

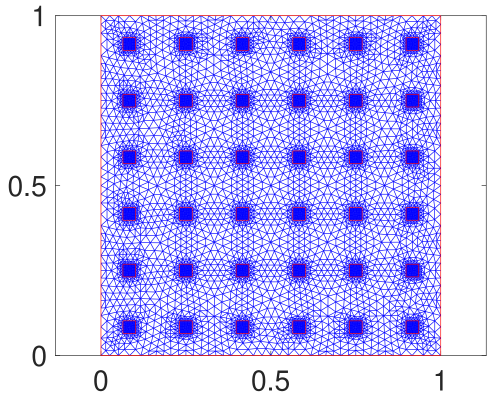

FIGURE 3. Mesh, actuators, and sensors.

In space variable we consider a piecewise linear finite element approximation based on such (unstructured) triangulation in Figure 3, which is refined in the actuators and sensors rectangular regions (in order to have better approximations for the actuators, the sensors, and the functions (6.4)). Then, we discretize the time interval $[0, +\infty)$, with uniform time step

$$k = 0.001, \qquad t_j = (j-1)k, \quad j = 1, 2, 3, \ldots$$

and used the Crank–Nicolson scheme. In the figures below we denote $H := L^2(\mathbf{R}_1)$ and $-\mathrm{Inf} := -\infty$.

7.1. **Inactive/active injection and feedback.** Figure 4 shows that when the injection operator is inactive the estimate error z dynamics is unstable, and when the feedback control is inactive the state y dynamics is unstable as well.

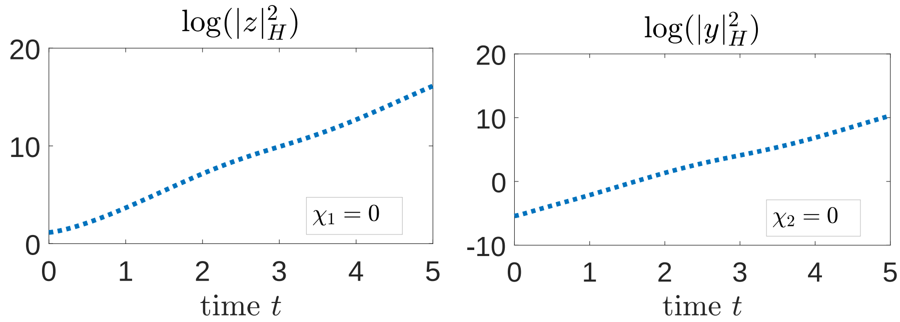

FIGURE 4. Unstable free dynamics.

Figures 5 and 6 show the stabilizing performance of the injection and feedback operators. Furthermore the estimate error norm $|z|_H$ and the state norm $|y|_H$ are strictly decreasing as stated in Theorem 4.1.

We also observe that by using the diffusion operator we obtain a faster exponential convergence of the estimate error $z = \widehat{y} - y$ and of the state y to zero.

Snapshots of the estimate error and state at selected instants of time, namely $t \in \{0.05, 5\}$, are shown in Figures 7, 8, 9, and 10.

7.2. **Values of λ_1 and λ_2.** Observe that when using the diffusion operator in the injection (resp. feedback) operator we have taken smaller values for λ_1 (resp. λ_2). However, as shown in Figure 11, observe that for the choices above, the largest

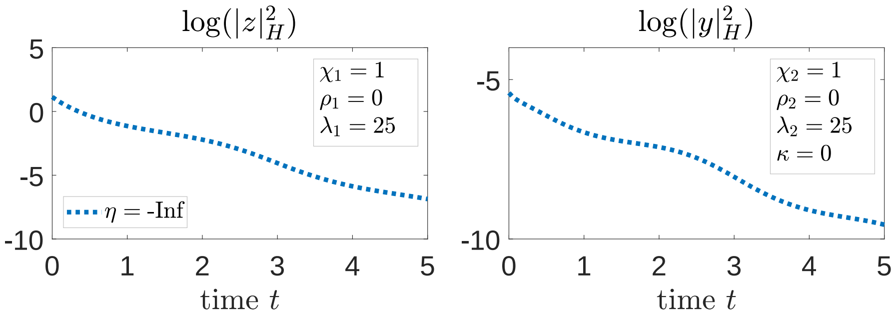

FIGURE 5. Stable error and state dynamics. No use of diffusion.

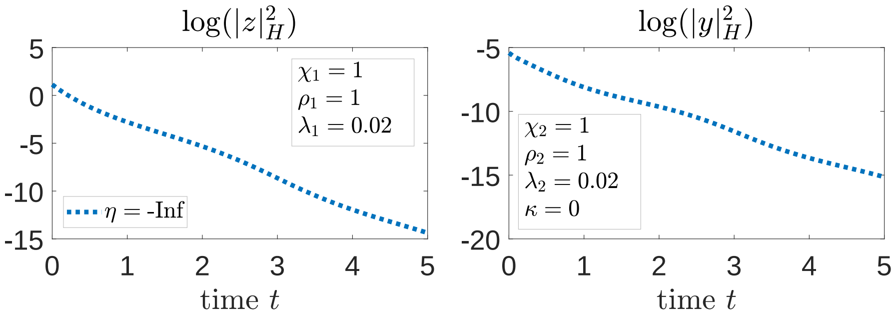

FIGURE 6. Stable error and state dynamics. Using diffusion.

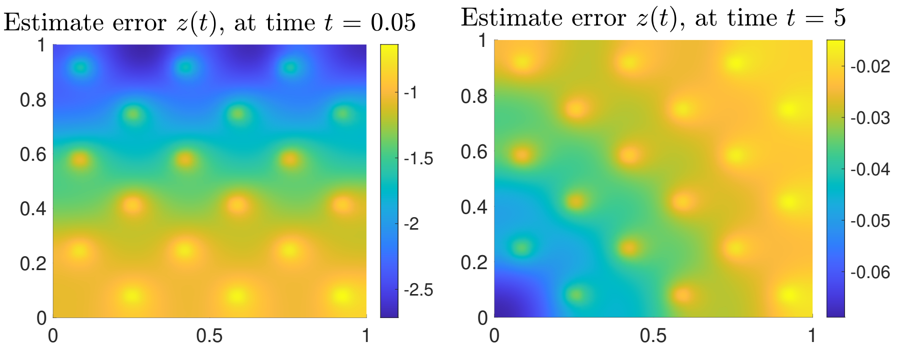

FIGURE 7. Snapshots of $z(t)$. $(\chi_1, \rho_1, \lambda_1) = (1, 0, 25)$.

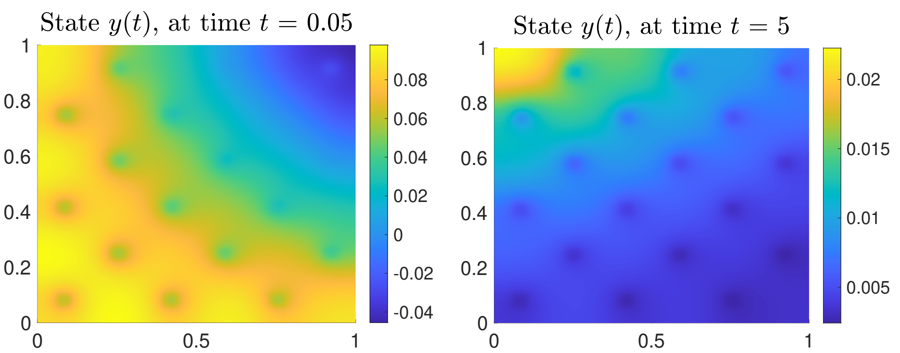

FIGURE 8. Snapshots of $y(t)$. $(\chi_2, \rho_2, \lambda_2, \kappa) = (1, 0, 25, 0)$.

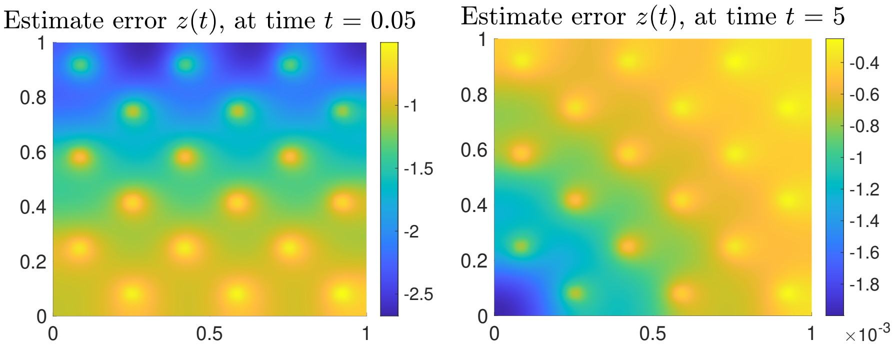

FIGURE 9. Snapshots of $z(t)$. $(\chi_1, \rho_1, \lambda_1) = (1, 1, 0.02)$.

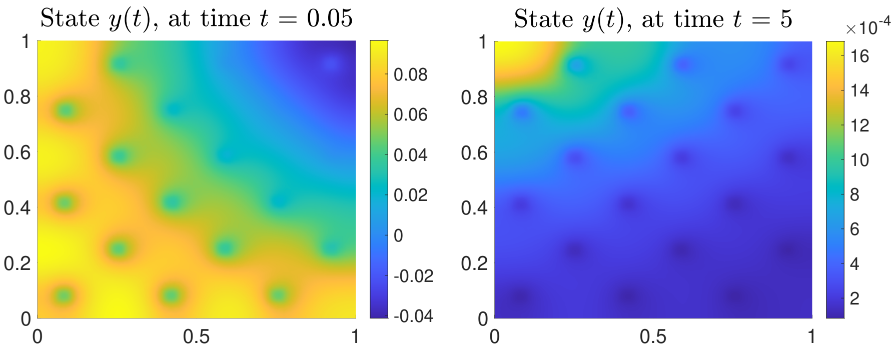

FIGURE 10. Snapshots of $y(t)$. $(\chi_2, \rho_2, \lambda_2, \kappa) = (1, 1, 0.02, 0)$.

magnitudes of the diffusion based injection (resp. input control) are still larger (for small time). Such larger magnitudes could (partially) explain why we get faster convergence to zero, when using the diffusion. In Figure 11, by simplicity we denoted

$$\mathbf{I}z = -\chi_1\lambda_1 P_{\mathcal{W}_S}^{\widetilde{\mathcal{W}}_S^\perp} A^{\rho_1} P_{\widetilde{\mathcal{W}}_S}^{\mathcal{W}_S^\perp} \mathbf{Z}^{\mathcal{W}_S}(\mathcal{Z}z + e^\eta \zeta),$$
$$\mathbf{K}y = -\chi_2\lambda_2 P_{\mathcal{U}_M}^{\widetilde{\mathcal{U}}_M^\perp} A^{\rho_2} P_{\widetilde{\mathcal{U}}_M}^{\mathcal{U}_M^\perp}(y + \kappa z),$$

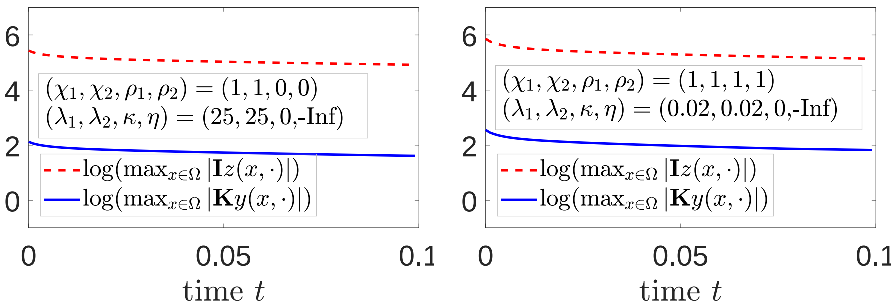

FIGURE 11. Magnitudes of output injection and feedback control.

7.3. **Closed-loop coupled observer-feedback system.** Here we present the simulations for the coupled system (7.1), that is, for the case $\kappa = 1$. We also check the behavior under the presence of measurements errors $e^\eta\zeta$.

In the simulations the noise $\zeta(t_j)$ at the discrete time instants $t_j \coloneqq (j-1)k$, $j = 1, 2, \dots$, have been chosen as

$$\zeta(t_j) = 2\mathtt{rand}^{S_\varsigma\times 1}(t_j) - 0.5^{S_\varsigma\times 1} \in \mathbb{R}^{S_\varsigma},$$

which is a "random" vector $\zeta(t_j) \in (-\frac{1}{2}, \frac{3}{2})^{S_\varsigma}$. Above $0.5^{S_\varsigma\times 1}$ stands for the vector whose coordinates are all equal to 0.5. The random vector $\mathtt{rand}^{S_\varsigma\times 1}(t_j) \in \mathbb{R}^{S_\varsigma\times 1}$ has been generated by the Matlab function `rand`.

In Figure 12 we consider the case where neither the injection operator nor the feedback operator make use of the diffusion operator.

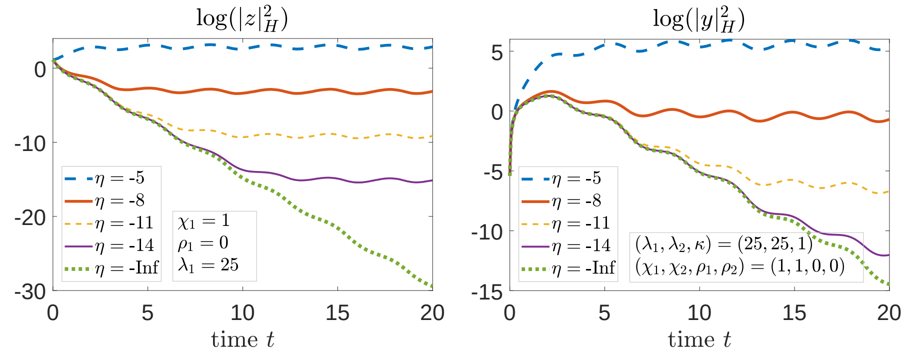

FIGURE 12. Coupled system. Case $(\rho_1, \rho_2) = (0, 0)$.

In Figure 13 we consider the case where both the injection operator and the feedback operator make use of the diffusion operator.

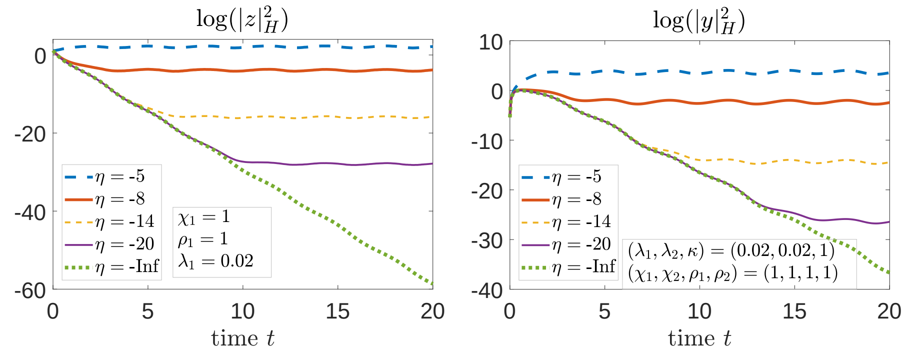

FIGURE 13. Coupled system. Case $(\rho_1, \rho_2) = (1, 1)$.

We see that for both $(\rho_1, \rho_2) = (0, 0)$ and $(\rho_1, \rho_2) = (1, 1)$, the norms of the error estimate and of the state go exponentially fast to zero, under the absence of noise.

Next, in Figure 14 we consider the cases where we use the diffusion operator either only for the feedback operator or only for the injection operator. There we show only the behavior of the norm of the state y. The behavior for the corresponding norm of the observer estimate error can be seen in Figures 12 and 13, observe that the observer estimate z depends on the noise $\mathrm{e}^{\eta}\zeta$, but is independent of the real state y, as we can see in (7.1).

In all Figures 12, 13, and 14, both the estimate error norm $|z|_H$ and the state norm $|y|_H$ converge to a neighborhood of 0, and remain, for large time, in a ball centered at 0 whose radius converges to 0 as the magnitude of the noise $\mathrm{e}^{\eta} \max |\zeta|_{\mathbb{R}}$,

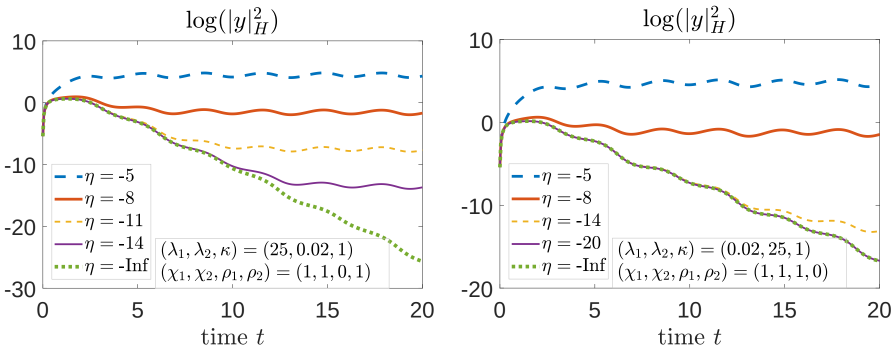

FIGURE 14. Coupled system. Norm of the state. Cases $(\rho_1, \rho_2) = (0, 1)$ and $(\rho_1, \rho_2) = (1, 0)$.

which is bounded by $1.5\mathrm{e}^{\eta}$, converges to 0. In this sense, we can say that the injection and feedback operators are robust against small errors in sensor measurements.

8. On the Poincaré-like constants

We show the computations leading to the Poincaré constants ξ_N in Theorems 6.1 and 6.2. We start by recalling the following integrals, for positive integer k,

$$\int_0^{L_1} \cos(\tfrac{k\pi x_1}{L_1})\mathrm{d}x_1 = 0, \qquad \int_0^{L_1} \sin(\tfrac{k\pi x_1}{L_1})\mathrm{d}x_1 = \tfrac{2L_1}{k\pi},$$

Recalling that $\sin^2(z) = \frac{1-\cos(2z)}{2}$, $\cos^2(z) = \frac{1+\cos(2z)}{2}$, and $\int_0^{L_1} 1\mathrm{d}x_1 = L_1$, we also find, for positive integer k,

$$\begin{aligned}
&\int_0^{L_1} \sin^2(\tfrac{k\pi x_1}{L_1})\mathrm{d}x_1 = \tfrac{L_1}{2} = \int_0^{L_1} \cos^2(\tfrac{k\pi x_1}{L_1})\mathrm{d}x_1,\\
&\int_0^{L_1} \sin^4(\tfrac{\pi x_1}{L_1})\mathrm{d}x_1 = \int_0^{L_1} \tfrac{(1-\cos(\frac{2\pi x_1}{L_1}))^2}{4}\mathrm{d}x_1\\
&\qquad = \tfrac{1}{4}\int_0^{L_1} 1 - 2\cos(\tfrac{2\pi x_1}{L_1}) + \cos(\tfrac{2\pi x_1}{L_1})^2\mathrm{d}x_1,\\
&\qquad = \tfrac{L_1}{4} + \tfrac{1}{4}\int_0^{L_1} \cos(\tfrac{2\pi x_1}{L_1})^2\mathrm{d}x_1 = \tfrac{L_1}{4} + \tfrac{L_1}{8} = \tfrac{3L_1}{8}.
\end{aligned}$$

By using the above integrals, together with suitable translations in space variable, we can compute the Poincaré-like constants in Theorems 6.1 and 6.2,

$$\xi_N = \sup_{\theta\in\widetilde{\mathcal{U}}_N\setminus\{0\}} \tfrac{|\theta|_V^2}{|\theta|_H^2} = \sup_{\theta\in\widetilde{\mathcal{U}}_N\setminus\{0\}} \tfrac{\nu|\nabla\theta|_H^2+|\theta|_H^2}{|\theta|_H^2}.$$

8.1. **Proof of Theorem 6.1.** Recall that $H = L^2(\Omega)$. With the choice as in (6.3) we find, for $\theta = \sum\limits_{j=1}^{N_{\mathrm{n}}} \theta_j\phi_{\widetilde{\omega}_j^N} \in \widetilde{\mathcal{U}}_N$,

$$\begin{aligned}
|\nabla\theta|_H^2 &= \left|\nabla \sum_{j=1}^{N_{\mathrm{n}}} \theta_j\phi_{\widetilde{\omega}_j^N}\right|_H^2 = \sum_{j=1}^{N_{\mathrm{n}}} \theta_j^2 \left|\nabla\phi_{\widetilde{\omega}_j^N}\right|_H^2 = \sum_{j=1}^{N_{\mathrm{n}}} \theta_j^2 \sum_{i=1}^{2} \left|\tfrac{\partial}{\partial x_i}\phi_{\widetilde{\omega}_j^N}\right|_H^2\\
&= N^2\pi^2 \sum_{j=1}^{N_{\mathrm{n}}} \theta_j^2 \sum_{i=1}^{2} \tfrac{1}{L_i^2}\Psi_i,
\end{aligned}$$

with

$$\begin{aligned}
\Psi_1 &= \left|\cos\left(\tfrac{N\pi(x_1-\widetilde{p}_j^N)}{L_1}\right)\sin\left(\tfrac{N\pi(x_2-\widetilde{q}_j^N)}{L_2}\right)\right|_H^2 = \tfrac{L_1L_2}{4},\\
\Psi_2 &= \left|\sin\left(\tfrac{N\pi(x_1-\widetilde{p}_j^N)}{L_1}\right)\cos\left(\tfrac{N\pi(x_2-\widetilde{q}_j^N)}{L_2}\right)\right|_H^2 = \tfrac{L_1L_2}{4}.
\end{aligned}$$

Therefore, we arrive at

$$|\nabla\theta|_H^2 = N^2\pi^2 \sum_{j=1}^{N_\mathfrak{n}} \theta_j^2 \sum_{i=1}^{2} \tfrac{1}{L_i^2}\tfrac{L_1L_2}{4} = N^2\tfrac{\pi^2(L_1^2+L_2^2)}{4L_1L_2}\,|\theta|_H^2$$

and $\xi_N = N^2\frac{\nu\pi^2(L_1^2+L_2^2)}{4L_1L_2} + 1$. □

8.2. **Proof of Theorem 6.2.** With (6.4) we find, for $\theta = \sum_{j=1}^{N_\mathfrak{n}} \theta_j \phi^2_{\widetilde{\omega}_j^N} \in \widetilde{\mathcal{U}}_N$,

$$\begin{aligned}|\nabla\theta|_H^2 &= \left|\nabla \sum_{j=1}^{N_\mathfrak{n}} \theta_j\phi^2_{\widetilde{\omega}_j^N}\right|_H^2 = \sum_{j=1}^{N_\mathfrak{n}} \theta_j^2 \left|\nabla\phi^2_{\widetilde{\omega}_j^N}\right|_H^2 = \sum_{j=1}^{N_\mathfrak{n}} \theta_j^2 \sum_{i=1}^{2}\left|2\phi_{\widetilde{\omega}_j^N}\tfrac{\partial}{\partial x_i}\phi_{\widetilde{\omega}_j^N}\right|_H^2\\ &= N^2\pi^2 \sum_{j=1}^{N_\mathfrak{n}} \theta_j^2 \sum_{i=1}^{2} \tfrac{1}{L_i^2}\varPhi_i,\end{aligned}$$

with

$$\begin{aligned}\varPhi_1 &= \left|\sin\left(\tfrac{2N\pi(x_1-\widetilde{p}_j^N)}{L_1}\right)\sin^2\left(\tfrac{N\pi(x_2-\widetilde{q}_j^N)}{L_2}\right)\right|_H^2 = \tfrac{3L_1L_2}{16},\\ \varPhi_2 &= \left|\sin^2\left(\tfrac{N\pi(x_1-\widetilde{p}_j^N)}{L_1}\right)\sin\left(\tfrac{2N\pi(x_2-\widetilde{q}_j^N)}{L_2}\right)\right|_H^2 = \tfrac{3L_1L_2}{16}.\end{aligned}$$

Therefore, we arrive at

$$|\nabla\theta|_H^2 = N^2\pi^2 \sum_{j=1}^{N_\mathfrak{n}} \theta_j^2 \sum_{i=1}^{2} \tfrac{1}{L_i^2}\tfrac{3L_1L_2}{16} = N^2\tfrac{3\pi^2(L_1^2+L_2^2)}{16L_1L_2}\,|\theta|_H^2$$

and $\xi_N = N^2\frac{3\nu\pi^2(L_1^2+L_2^2)}{16L_1L_2} + 1$. □

9. Conclusions

It has been shown that the feedback control and output injection operators in the form $-\lambda P_N P_N^*$ or $-\lambda P_N A P_N^*$, where P_N is a suitable oblique projection onto finite-dimensional spaces $\mathcal{F}_N$, are able to stabilize nonautonomous linear parabolic equations with an input dynamically constructed from the output. The dimension $N_\mathfrak{n}$ of the space $\mathcal{F}_N$ and the parameter $\lambda > 0$ are to be taken large enough, depending only on suitable norms of the operator dynamics A and $A_{\rm rc}$.

We have observed that the output based feedback is robust against (small) errors in sensors measurements. The qualitative and quantitative investigation of such robustness could be the subject of a future work, possibly also including the investigation on the robustness against small disturbances (model uncertainty).

The investigation/construction of such operators for (semiglobal) stabilization of semilinear parabolic equations is also of interest for applications.

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