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Convergence of level sets in TV denoising through variational curvatures in unbounded domains

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Abstract

We present some results of geometric convergence of level sets for solutions of total variation denoising as the regularization parameter tends to zero. The common feature among them is that they make use of explicit constructions of variational mean curvatures for general sets of finite perimeter. Consequently, no additional regularity of the level sets of the ideal data is assumed, and in particular the subgradient of the total variation at it could be empty. In exchange, other restrictions on the data or on the noise are required.

1 Introduction and main results

We aim to provide a precise analysis of the generalized Rudin-Osher-Fatemi denoising scheme based on total variation minimization in the low noise regime, in general dimension and with no source condition assumptions. More precisely, given a real function $\psi : \mathbb{R} \to \mathbb{R}$, some ideal data to be recovered $f : \mathbb{R}^d \to \mathbb{R}$ with compact support, an additive perturbation w, as well as a regularization parameter $\alpha > 0$, we consider minimizers of

$$\inf_{u \in \mathrm{BV}(\mathbb{R}^d)} \int_{\mathbb{R}^d} \psi(u - f - w) + \alpha \,\mathrm{TV}(u). \tag{1.1}$$

We make the following assumptions on the function ψ appearing in the data term and its Fenchel conjugate ψ^*:

$$\begin{gathered}\psi \text{ is strictly convex and even with } \psi(0) = 0,\ \psi^* \text{ is uniformly convex and} \\ |\psi(s)| \leqslant C|s|^{d/(d-1)} \text{ for some } C > 0.\end{gathered} \tag{A}$$

If $1 < p \leqslant 2$ the functions $t \mapsto |t|^p/p$ satisfy these convexity properties [9, Example 5.3.10], so the case $p = d/(d-1)$ satisfies all the conditions of Assumption (A).

Remark 1.1. Since ψ is strictly convex, even and vanishes at zero, we have that $\psi(t) > 0$ for $t \neq 0$. Moreover, ψ^* being uniformly convex implies that ψ is differentiable with ψ' uniformly continuous [9, Thm. 5.3.17, Prop. 4.2.14], in particular $\psi \in \mathcal{C}^1(\mathbb{R})$. Moreover, strict convexity of ψ implies that ψ^* is also differentiable [9, Thm. 5.3.7]. We will use both of these properties in the sequel.

We study the regime in which α and w tend to zero simultaneously, for which under natural assumptions it is easy to prove (see Proposition 1.5 below) that the unique minimizers $u_{\alpha,w}$ of (1.1) converge to f in the strong L^1_{loc} topology. In particular, if along a sequence the solutions

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u_{α_n,w_n} have a common compact support, we have, using Fubini's theorem [24, Thm. 2], for a.e. $s > 0$ that

$$\left|\{u_{\alpha_n,w_n} > s\}\Delta\{f > s\}\right| \to 0.$$

Moreover, this can in some cases be improved to convergence in Hausdorff distance, which can be interpreted as geometric uniform convergence. This type of convergence has been proved in [15] for classical ROF denoising in the plane, and in [24, 23] for linear inverse problems, bounded domains, Banach space measurements and general dimensions. All of these results (with the exception of when explicit dual certificates for $u_{\alpha_n,0}$ are known [15, Sec. 8]) assume a source condition implying $\partial\,\mathrm{TV}(f) \neq \emptyset$. On the one hand this source condition guarantees, in particular, that the level sets of the minimizers $u_{\alpha,w}$ satisfy uniform density estimates independent of α and w, as long as these are related through an adequate parameter choice. On the other, this subgradient condition excludes cases of interest where geometric convergence is still expected, like the case when f is the indicatrix of a planar polygon [15, Sec. 3.3].

Our main goal is to obtain this improved mode of convergence while assuming as little regularity of $\{f > s\}$ as possible, and this is achieved in two different situations. The first is when f is the characteristic function of a bounded finite perimeter set, and admitting noisy measurements with a natural parameter choice. The second concerns a generic class of BV functions in which "flat regions are controlled" and including piecewise constant functions, but with noiseless measurements. The techniques used have as a central point the variational mean curvatures for general finite perimeter sets introduced in [8, 6] which, through comparison arguments, are used as a lower integrability replacement for the missing dual certificates for f.

1.1 Main results

Theorem 1.2. *Assume that $f = 1_D$, the indicatrix of a bounded finite perimeter set $D \subset \mathbb{R}^d$, and that the sequences $\alpha_n \to 0$ and w_n are such that*

$$\frac{\|w_n\|_{L^{d/(d-1)}(\mathbb{R}^d)}}{\alpha_n} \leqslant C_{\psi,d} < \frac{m_{\psi^*}(\Theta_d)}{\Theta_d}, \tag{1.2}$$

where Θ_d denotes the isoperimetric constant in $\mathbb{R}^d$ and m_{ψ^*} is the largest modulus of uniform convexity for ψ^*. Then we have, up to a not relabelled subsequence, the convergence in Hausdorff distance

$$d_{\mathcal{H}}(\partial\{u_{\alpha_n,w_n} > s\}, \partial D) \to 0 \text{ for a.e. } s \in (0,1).$$

Theorem 1.3. *Let $f \in \mathrm{BV}(\mathbb{R}^d)$ with bounded support. Denote $E^s_\alpha = \{u_{\alpha,0} > s\}$ and $E^s_0 = \{f > s\}$ if $s > 0$, and $E^s_\alpha = \{u_{\alpha,0} < s\}$ and $E^s_0 = \{f < s\}$ for $s < 0$. Assume that s satisfies that $|E^s_\alpha \Delta E^s_0| \to 0$, and also*

$$\lim_{\nu\to 0} d_{\mathcal{H}}\left(E^s_0, E^{s+\nu}_0\right) = 0, \text{ and } \lim_{\nu\to 0} d_{\mathcal{H}}\left(\mathbb{R}^d \setminus E^s_0, \mathbb{R}^d \setminus E^{s+\nu}_0\right) = 0. \tag{1.3}$$

Then in the absence of noise ($w = 0$) we have the convergence $d_{\mathcal{H}}(\partial E^s_\alpha, \partial E^s_0) \to 0$.

Remark 1.4. Any piecewise constant function satisfies trivially (1.3) at all the values that it does not attain, and therefore the conclusions of Theorem 1.3 are valid. It also holds for almost every s under the source condition of [15, 23], a fact we prove in Corollary 5.9. In contrast, Example 5.10 provides a function f for which the set of values where (1.3) does not hold is of full measure.

1.2 Structure of the paper

We start with some preliminary results in Subsection 1.3. Section 2 is dedicated to auxiliary results about convergence in Hausdorff distance of bounded subsets of $\mathbb{R}^d$. In Section 3 we are concerned with variational mean curvatures, with the main goal of pinning down the construction of such a curvature on the outside of any finite perimeter set, and also stating basic comparison results. Section 4 aims at the proof of Theorem 1.2 using density estimates that degenerate as the set D is approached and dual stability estimates with respect to the noise, which are themselves proved in Appendices A and B. Then, Section 5 is devoted to the proof of Theorem 1.3 by approximation of finite perimeter sets with the level sets of their optimal variational mean curvatures. Finally, in Section 6 we explore whether it is possible to recover uniform density estimates without the source condition; it turns out that this is possible for indicatrices of some planar polygons.

1.3 Preliminaries

The total variation TV appearing in (1.1) is the norm of the distributional derivative as a Radon measure, that is

$$\mathrm{TV}(u) := |Du|(\mathbb{R}^d) = \sup\left\{ \int_{\mathbb{R}^d} u \operatorname{div} z \, \mathrm{d}x \,\middle|\, z \in \mathcal{C}_c^\infty(\mathbb{R}^d\,;\,\mathbb{R}^d), \|z\|_{L^\infty(\mathbb{R}^d)} \leqslant 1 \right\}.$$

Correspondingly we say that $u : \mathbb{R}^d \to \mathbb{R}$ is of bounded variation when it belongs to

$$\mathrm{BV}(\mathbb{R}^d) := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{R}^d) \,\middle|\, \mathrm{TV}(u) < +\infty \right\},$$

where we remark that we only require such functions to be locally summable. Likewise, the space $\mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ consists of those functions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for which $|Du|(K) < +\infty$ for each compact set K. A set E is called of finite perimeter whenever its indicatrix 1_E is of bounded variation, and the perimeter is defined as

$$\mathrm{Per}(E) := \mathrm{TV}(1_E).$$

Since this notion is invariant with respect to zero Lebesgue measure modification of E, we need a notion of boundary which satisfies this invariance as well. For this purpose, we can take a representative of E for which the topological boundary equals the support of the derivative of 1_E, which can be described [30, Prop. 12.19] as

$$\partial E = \operatorname{Supp} D1_E = \left\{ x \in \mathbb{R}^d \,\middle|\, 0 < \frac{|E \cap B(x,r)|}{|B(x,r)|} < 1 \text{ for all } r > 0 \right\},$$

and this choice will be assumed in all what follows. Notice that we might have $|\partial E| > 0$ (see [30, Example 12.25] for an example), so particular care is needed when combining topological and measure-theoretic arguments for this boundary. The topological interior of a set E will be denoted by $\mathring{E}$, and by $E^{(1)}$ its subset of points of full density in E. Moreover, convex hulls are denoted by $\operatorname{Conv} E$ and complements by $E^c := \mathbb{R}^d \setminus E$.

From general properties of the space $\mathrm{BV}(\mathbb{R}^d)$ one can deduce the following basic result on existence and convergence of minimizers for (1.1):

Proposition 1.5. *Assuming* (A) *and* $\int \psi(w) < +\infty$, *the minimization problem* (1.1) *admits a unique solution* $u_{\alpha,w}$. *Furthermore, if* $\alpha_n \to 0$ *and* w_n *are such that*

$$\frac{1}{\alpha_n} \int \psi(w_n) \leqslant C, \tag{1.4}$$

then $u_{\alpha_n,w_n} \to f$ *weakly in* $L^{d/(d-1)}$ *and strongly in* L^q_{loc} *for* $1 \leqslant q < d/(d-1)$.

Proof. Let u_k be minimizing sequence for (1.1). Discarding some elements of the sequence if necessary and using the symmetry of ψ we have the estimate

$$\frac{1}{\alpha} \int \psi(u_k - f - w) + \mathrm{TV}(u_k) \leqslant \frac{1}{\alpha} \int \psi(w) + \mathrm{TV}(f). \tag{1.5}$$

On the other hand we also have the Sobolev inequality [4, Thm. 3.47]

$$\|u_k - c_k\|_{L^{d/(d-1)}} \leqslant C\,\mathrm{TV}(u_k)$$

for some constants $c_k \in \mathbb{R}$. Since f is compactly supported and $\psi(t) > 0$ for $t \neq 0$, we have that (1.5) and $\int \psi(w)$ being finite imply $c_k = 0$ for all n. Therefore, using weak-* compactness in BV [4, Thm. 3.23] and weak compactness in $L^{d/(d-1)}$ we can extract a limit $u_{\alpha,w}$ in those topologies. Moreover, we have lower semicontinuity of TV with respect to L^1_{loc} convergence [4, Rem. 3.5], while positivity and convexity of ψ implies that the first term of (1.1) is also lower semicontinuous with respect to weak $L^{d/(d-1)}$ convergence [18, Thm. 3.20], so $u_{\alpha,w}$ must be a minimizer of (1.1).

In view of (1.5) and (1.4), one can apply the same compactness arguments to u_{α_n,w_n} to obtain a subsequence converging in weakly in $L^{d/(d-1)}$ and strongly in L^q_{loc}. Moreover since ψ is strictly convex, $\psi(0) = 0$ and $\psi(t) > 0$ if $t > 0$ it must be increasing on $[0, +\infty)$, so (1.4) implies that $w_n \to 0$ in measure, which in turn implies that [21, Thm. 2.30] up to possibly taking a further subsequence also $w_n(x) \to 0$ for a.e. x. Finally (1.5) also gives that $\int \psi(u_{\alpha_n,w_n} - f - w_n) \to 0$, so the limit must be f. Since for any subsequence we are able to find a further subsequence converging to the fixed limit f, the whole sequence u_{α_n,w_n} must converge to it. □

We recall that for any E, F with finite perimeter we have [4, Prop. 3.38(d)]

$$\mathrm{Per}(E \cap F) + \mathrm{Per}(E \cup F) \leqslant \mathrm{Per}(E) + \mathrm{Per}(F), \tag{1.6}$$

and the isoperimetric inequality [4, Thm. 3.46]

$$\mathrm{Per}(F) \geqslant \Theta_d \min\left(|F|^{(d-1)/d}, |\mathbb{R}^d \setminus F|^{(d-1)/d}\right), \text{ with } \Theta_d = \frac{\mathrm{Per}(B(0,1))}{|B(0,1)|^{(d-1)/d}}. \tag{1.7}$$

Below we study in detail problem (1.1) with $f = 1_D$ and $w = 0$, for which the minimizer $u_\alpha := u_{\alpha,0}$ has level sets $E^s_\alpha := \{u_\alpha > s\}$ that minimize for a.e. $s \in (0,1)$

$$E \mapsto \mathrm{Per}(E) + \frac{1}{\alpha}\int_E \psi'(s - f(x))\,\mathrm{d}x = \mathrm{Per}(E) - \frac{\psi'(1-s)}{\alpha}|E \cap D| + \frac{\psi'(s)}{\alpha}|E \setminus D|,$$

as can be seen from (A.2), the coarea formula for BV functions [4, Thm. 3.40] and the general layer cake formula [29, Thm. 1.13]. More generally we have:

Proposition 1.6. *Let u minimize (1.1). Then for $s \in \mathbb{R}$ its upper level sets $E^s := \{u > s\}$ minimize, among sets of finite mass, the functional*

$$E \mapsto \mathrm{Per}(E) + \frac{1}{\alpha}\int_E \psi'(s - f - w),$$

and moreover we have

$$\mathrm{Per}(E^s) = \frac{1}{\alpha}\int_{E^s} \psi'(f + w - s).$$

For the lower level sets $\{u < s\}$ analogous statements hold by changing the sign on the integral terms.

Proof. The proof of the first statement can be found in [25, Prop. 2.3.14]. The second is proved in [15, Prop. 3]. □

Remark 1.7. Note that if $s < 0$ it is often convenient to work with the lower level sets $\{u < s\}$ since then $|\{u < s\}| < +\infty$, in which case the integral terms change sign. This will be useful in some results below.

2 Density estimates and Hausdorff convergence

We begin with some auxiliary results on convergence in the Hausdorff distance, defined for $E, F \subset \mathbb{R}^d$ as

$$d_{\mathcal{H}}(E,F) := \max\left\{\sup_{x\in E}\operatorname{dist}(x,F),\ \sup_{y\in F}\operatorname{dist}(y,E)\right\}, \tag{2.1}$$

and its relation with L^1 convergence when density estimates are available, which will be used in the proof of the main results.

Definition 2.1. Let $\{E_\gamma\}_\gamma$ be a family of finite perimeter sets of uniformly bounded measure, that is, there is $M > 0$ such that $|E_\gamma| < M$ for all γ. If there are constants $r_0 > 0$ and $C \in (0,1)$ such that for all γ and all $x \in \partial E_\gamma$ we have for all $r < r_0$ that

$$\frac{|E_\gamma \cap B(x,r)|}{|B(x,r)|} \geqslant C, \tag{2.2}$$

we say that this family satisfies *uniform inner density estimates* with constant C at scale r_0. Similarly, if instead we have for $r \leqslant r_0$

$$\frac{|B(x,r) \setminus E_\gamma|}{|B(x,r)|} \geqslant C \tag{2.3}$$

we say that this family satisfies *uniform outer density estimates*, again with constant C at scale r_0. When speaking of *uniform density estimates*, we understand that both estimates hold with the same constants.

First, in [24, 23] the following result is claimed, although with some flaws in its presentation:

Proposition 2.2. *Assume we have $\{E_n\}_n, E_0$ are subsets of $\mathbb{R}^d$ satisfying uniform inner density estimates with some scale r_0 and constant C, and such that $|E_n \Delta E_0| \to 0$. Then $d_{\mathcal{H}}(E_n, E_0) \to 0$.*

Proof. First, we notice that if we have the estimate

$$|E_n \cap B(x,r)| \geqslant C|B(x,r)| \text{ for } x \in \partial E_n \text{ and } r \leqslant r_0, \tag{2.4}$$

then we also have

$$|E_n \cap B(y,\tilde r)| \geqslant \frac{C}{2^d}|B(y,\tilde r)| \text{ for } y \in \overline{E_n}, \text{ and } \tilde r \leqslant 2r_0. \tag{2.5}$$

To see this, first set $r = \tilde r/2$. Then, if $\operatorname{dist}(y, \partial E_n) \geqslant r$, the whole ball $B(y,r) \subset E_n$, so that $|E_n \cap B(y,\tilde r)| \geqslant |B(y,r)| = |B(y,\tilde r)|/2^d$ and (2.5) holds. If $0 \leqslant \operatorname{dist}(y,\partial E) < r$, then there is at least one boundary point $x_y \in \partial E_n$ for which $B(x_y, r) \subset B(y,\tilde r)$, and applying (2.4) to x_y and r we get (2.5).

With these facts, let us assume that there is $\delta > 0$ such that $d_{\mathcal{H}}(E_n, E_0) > \delta$ for infinitely many n, and derive a contradiction. Reducing δ if necessary, we can assume that $\delta \leqslant 2r_0$. In view of the definition (2.1) we must then have a subsequence n_k for which either $\sup_{x\in E_{n_k}} \operatorname{dist}(x, E_0) > \delta$ or $\sup_{x\in E_0} \operatorname{dist}(x, E_{n_k}) > \delta$. For the first case, we have a sequence of points $x_{n_k} \in E_{n_k}$ for which $\operatorname{dist}(x, E_0) > \delta$. Then (2.5) applied to E_{n_k} and with $\tilde r = \delta$ gives

$$|E_{n_k} \Delta E_0| \geqslant |E_{n_k} \setminus E_0| \geqslant |E_{n_k} \cap B(x_{n_k}, \delta)| \geqslant \frac{C}{2^d}\delta^d |B(0,1)|,$$

a contradiction with $|E_n \Delta E_0| \to 0$. For the second case, we obtain $x_{n_k} \in E_0$ for which $\operatorname{dist}(x_{n_k}, E_{n_k}) > \delta$. In this case, we use (2.5) for E_0 to end up as before with

$$|E_{n_k} \Delta E_0| \geqslant |E_0 \setminus E_{n_k}| \geqslant |E_0 \cap B(x_{n_k}, \delta)| \geqslant \frac{C}{2^d}\delta^d |B(0,1)|,$$

again a contradiction. ∎

In Proposition 2.2 we only used the inner density estimates. However, for level sets of total variation minimizers and imaging applications one is mostly interested in convergence of their boundaries. The latter is not implied by the convergence of the sets themselves, even under other modes of convergence assumed, as demonstrated in the following example. We will see later that to obtain convergence of the boundaries, the outer density estimates also need to be used.

Example 2.3. Consider the unit square $E_0 := (0,1)^2$ and a sequence obtained by removing from it thin triangles:

$$E_n := (0,1)^2 \setminus \operatorname{Conv}\left(\left\{\left(\frac{1}{2}-\frac{1}{n+2},0\right),\ \left(\frac{1}{2}+\frac{1}{n+2},0\right),\ \left(\frac{1}{2},\frac{1}{2}\right)\right\}\right),$$

which admits uniform inner density estimates, but with the outer densities not being uniform at $(1/2,1/2)$. We have $|E_n \Delta E_0| \to 0$ and $D1_{E_n} \overset{*}{\rightharpoonup} D1_{E_0}$. To see the latter, just notice that $D1_{E_n} = \nu_{E_n}\mathcal{H}^1 \llcorner \partial^* E_n$, and that each of the non-vanishing sides of the triangle converge to the same vertical segment, but with opposite orientations. Moreover, since $E_n \subset E_0$ also

$$d_{\mathcal{H}}(E_n, E_0) = \sup_{x\in E_0} \operatorname{dist}(x, E_n) \leqslant \frac{1}{n+2} \to 0, \text{ but } d_{\mathcal{H}}(\partial E_n, \partial E_0) = \frac{1}{2}.$$

Remark 2.4. In general, the Hausdorff distances $d_{\mathcal{H}}(E,F)$ and $d_{\mathcal{H}}(\partial E, \partial F)$ are not related. In [34, Thm. 14] it is proven that these are equal for bounded closed convex sets, and in [34, Examples 6 and 13] examples are given for pairs of planar sets where both possible strict inequalities hold.

Under L^1 convergence, the Hausdorff convergence of boundaries is in fact stronger:

Proposition 2.5. *Assume that $\{E_n\}_n, E_0$ are subsets of $\mathbb{R}^d$ such that we have the convergences*

$$|E_n \Delta E_0| \to 0 \text{ and } d_{\mathcal{H}}(\partial E_n, \partial E_0) \to 0.$$

Then also $d_{\mathcal{H}}(E_n, E_0) \to 0$.

Proof. Assume that the hypotheses are satisfied but $d_{\mathcal{H}}(E_n, E_0) \not\to 0$. Then there is $\delta > 0$ with $d_{\mathcal{H}}(E_{n_k}, E_0) > \delta$ for some subsequence n_k. Removing leading terms if needed, we can assume that

$$d_{\mathcal{H}}(\partial E_{n_k}, \partial E_0) < \frac{\delta}{2}. \tag{2.6}$$

Now, we have

$$d_{\mathcal{H}}(E_{n_k}, E_0) = \max\left(\sup_{x\in E_{n_k}} \operatorname{dist}(x, E_0),\ \sup_{x\in E_0} \operatorname{dist}(x, E_{n_k})\right) > \delta,$$

so at least one of the arguments in the supremum must be larger than δ for infinitely many k. Assume that it is the first one, and relabel the subsequence n_k so that

$$\sup_{x\in E_{n_k}} \operatorname{dist}(x, E_0) > \delta,$$

implying that there is a sequence $x_{n_k} \in E_{n_k}$ for which $\operatorname{dist}(x_{n_k}, E_0) > \delta$. In consequence for all $y \in \overline{E_0}$, and in particular for all $y \in \partial E_0$, we have $|x_{n_k} - y| \geqslant \delta$. Therefore, for each k we must have $\operatorname{dist}(x_{n_k}, \partial E_{n_k}) \geqslant \delta/2$, since otherwise (2.6) and the triangle inequality would lead to a contradiction. We have then

$$B\left(x_{n_k}, \frac{\delta}{3}\right) \subset E_{n_k} \text{ and } d\left(x_{n_k}, E_0\right) > \delta, \text{ so } B\left(x_{n_k}, \frac{\delta}{3}\right) \subset E_{n_k} \setminus E_0,$$

a contradiction with $|E_n \Delta E_0| \to 0$. The other case is dealt with similarly. □

Proposition 2.6. *Let $E, F \subset \mathbb{R}^d$. Then*

$$d_{\mathcal{H}}(\partial E, \partial F) \leqslant \max\left(d_{\mathcal{H}}(E,F), d_{\mathcal{H}}(E^c,F^c)\right). \tag{2.7}$$

Proof. We use the characterization (often used as definition of $d_{\mathcal{H}}$, see for example [31, Sec. 4.C])

$$d_{\mathcal{H}}(A,B) = \inf\left\{r \geqslant 0 \,|\, A \subset U_r B \text{ and } B \subset U_r A\right\}, \tag{2.8}$$

for the dilations $U_r A = \{\mathrm{dist}(\cdot, A) \leqslant r\}$. Now, if the inequality to be proved failed, denoting

$$r = \max\left(d_{\mathcal{H}}(E,F), d_{\mathcal{H}}(E^c,F^c)\right)$$

we would have that either $U_r\partial F \setminus \partial E \neq \emptyset$ or $U_r\partial E \setminus \partial F \neq \emptyset$. Without loss of generality assume that the first case holds, so that there is $x \in \partial E$ for which $\mathrm{dist}(x, \partial F) > r$. If $x \in F^c$, by the properties of the boundary we can produce $\hat{x} \in E \setminus F$ with $\mathrm{dist}(\hat{x}, \partial F) > r$, which since $\hat{x} \in F^c$ also implies

$$\mathrm{dist}(\hat{x}, F) = \mathrm{dist}(\hat{x}, \partial F) > r,$$

contradicting $r \geqslant d_{\mathcal{H}}(E,F)$. Similarly, if $x \in \{\mathrm{dist}(\cdot, \partial F) > r\} \cap F$, we can find $\check{x} \in E^c$ with $\mathrm{dist}(\check{x}, \partial F) > r$ as well, and as before since $\check{x} \in F \setminus E$ we have

$$\mathrm{dist}(\check{x}, F^c) = \mathrm{dist}(\check{x}, \partial F) > r,$$

a contradiction with $r \geqslant d_{\mathcal{H}}(E^c, F^c)$. □

Combining Propositions 2.5 and 2.6 we obtain

Theorem 2.7. *Assume that $\{E_n\}_n, E_0$ are subsets of $\mathbb{R}^d$ such that $|E_n \Delta E_0| \to 0$. Then $d_{\mathcal{H}}(\partial E_n, \partial E_0) \to 0$ if and only if $d_{\mathcal{H}}(E_n, E_0) \to 0$ and $d_{\mathcal{H}}(E_n^c, E_0^c) \to 0$ simultaneously.*

We can conclude Hausdorff convergence of the boundaries without the need of derivatives, by using both density estimates:

Theorem 2.8. *Assume $\{E_n\}_n, E_0$ are subsets of $\mathbb{R}^d$ satisfying uniform density estimates with some scale r_0 and constant C, and such that $|E_n \Delta E_0| \to 0$. Then*

$$d_{\mathcal{H}}(\partial E_n, \partial E_0) \to 0.$$

Proof. We notice that

$$\begin{aligned} E_n \Delta E_0 &= (E_n \setminus E_0) \cup (E_0 \setminus E_n) = (E_n \cap E_0^c) \cup (E_0 \cap E_n^c) \\ &= (E_0^c \setminus E_n^c) \cup (E_n^c \setminus E_0^c) = E_o^c \Delta E_n^c, \end{aligned}$$

and that by taking complements the roles of (2.2) and (2.3) are reversed. Therefore, using both we can apply Proposition 2.2 for E_n and for E_n^c so that $d_{\mathcal{H}}(E_n, E_0) \to 0$ and $d_{\mathcal{H}}(E_n^c, E_0^c) \to 0$. Proposition 2.6 gives then the conclusion. □

As a direct consequence we get the following result, proved but not explicitly stated in [15], which also applies to the cases treated in [24, 23]:

Proposition 2.9. *Assume we have $\{E_n\}_n, E_0$ finite perimeter sets satisfying uniform density estimates with some scale r_0 and constant C, and such that the characteristic functions $1_{E_n} \overset{*}{\rightharpoonup} 1_E$ in* BV*. Then $d_{\mathcal{H}}(\partial E_n, \partial E_0) \to 0$.*

3 A few results on variational mean curvatures

We now turn our attention to the weak notion of mean curvature for boundaries which will be our main tool to describe the behaviour of level sets of minimizers of (1.1):

Definition 3.1. We say that a set $A \subset \mathbb{R}^d$ has a variational mean curvature $\kappa : \mathbb{R}^d \to \mathbb{R}$ if it minimizes, among $E \subset \mathbb{R}^d$, the functional

$$E \mapsto \operatorname{Per}(E) - \int_E \kappa. \tag{3.1}$$

If the set A has a smooth boundary and κ is continuous, this minimization property implies that the restriction of κ to the boundary ∂A is, up to a multiplicative factor, the usual mean curvature of ∂A. To see this, just notice [30, Rem. 17.6] that if ∂A is $\mathcal{C}^2$, the first variation of the perimeter along the flow generated by a vector field $V \in \mathcal{C}_c^\infty(\mathbb{R}^d; \mathbb{R}^d)$ is

$$\int_{\partial A} \operatorname{div}_{\partial A} V \, \mathrm{d}\mathcal{H}^{d-1} = \int_{\partial A} (d-1) H_A \, V \cdot \nu_A \, \mathrm{d}\mathcal{H}^{d-1}, \tag{3.2}$$

for $\operatorname{div}_{\partial A}$ the surface divergence, ν_A the outward normal vector and H_A the usual mean curvature of ∂A, while that of the integral term in (3.1) for continuous κ amounts to

$$-\int_{\partial A} \kappa \, V \cdot \nu_A \, \mathrm{d}\mathcal{H}^{d-1},$$

from which we conclude by noticing that A is a minimizer of (3.1) and T is arbitrary. Analogously, if we had $u \in \mathcal{C}^2$ a minimizer of (1.1) with $w = 0$ and f continuous, using the implicit function theorem and Proposition 1.6, we would find for $E^s := \{u > s\}$ that $H_{E^s} = -\psi'(s-f)/\alpha$. If additionally $\nabla u(x) \neq 0$ for all, x taking the first variation of $\operatorname{TV}(u)$, which under this assumption is differentiable and equals $\int |\nabla u|$, leads to

$$-\frac{1}{\alpha}\psi'\big(u(x) - f(x)\big) = (d-1)H_{E^s}(x) = \operatorname{div}\left(\frac{\nabla u(x)}{|\nabla u(x)|}\right) \text{ for } x \in \partial E^s, \text{ so } u(x) = s.$$

We recall that there is a natural weak notion of mean curvature based on (3.2), the *distributional* mean curvature, which can be defined not just for boundaries of finite perimeter sets but also for most notions of non-regular surfaces (e.g. varifolds). The distributional and variational mean curvatures coincide in the very regular case just described, but it is not quite clear whether they do on less regular cases where both are available; some positive results are given in [7].

From the definition one sees that variational mean curvatures for a given set, as functions defined in $\mathbb{R}^d$, contain "too much information" and one can not expect them to be unique. In fact, if κ is a variational curvature for A, any other function κ' with $\kappa' \geqslant \kappa$ on A and $\kappa' \leqslant \kappa$ on $\mathbb{R}^d \setminus A$ is another variational mean curvature for A as well.

Remark 3.2. Using the coarea and layer-cake formulas as for Proposition 1.6, it is straightforward to check that if we have $v \in \partial \operatorname{TV}(f)$ for some $f \in L^{d/(d-1)}$ and $v \in L^d$, almost all of the upper level sets of f at positive values are minimizers of (3.1) with $\kappa = v$, making v a variational curvature for all of them. For negative values, one switches to lower level sets and the curvature sign to $-v$, cf. Remark 1.7.

Being a minimizer of (3.1) is stable by intersection and union, which in particular enables speaking about maximal and minimal minimizers:

Proposition 3.3. *Let E_1 and E_2 be two minimizers of* (3.1)*. Then $E_1 \cap E_2$ and $E_1 \cup E_2$ are also minimizers of* (3.1)*.*

Proof. One can write, using the minimality of E_1 and E_2 and noting that $E_1 \cap E_2$ as well as $E_1 \cup E_2$ are admissible for that problem,

$$\operatorname{Per}(E_1 \cap E_2) - \int_{E_1 \cap E_2} \kappa \geqslant \operatorname{Per}(E_1) - \int_{E_1} \kappa,$$

and

$$\operatorname{Per}(E_1 \cup E_2) - \int_{E_1 \cup E_2} \kappa \geqslant \operatorname{Per}(E_2) - \int_{E_2} \kappa.$$

Summing these inequalities and noticing that the volume terms exactly compensate, we obtain

$$\operatorname{Per}(E_1 \cap E_2) + \operatorname{Per}(E_1 \cup E_2) \geqslant \operatorname{Per}(E_1) + \operatorname{Per}(E_2).$$

Since the reverse inequality is also true by (1.6), we must have an equality, which also implies that the two first inequalities are equalities and the expected conclusion. □

We make extensive use of the following basic but fundamental comparison lemma for variational mean curvatures:

Lemma 3.4. *Assume that the finite perimeter sets E_1 and E_2 admit variational mean curvatures κ_1 and κ_2 respectively, and such that $\kappa_1 < \kappa_2$ in $E_1 \setminus E_2$. Then $|E_1 \setminus E_2| = 0$, that is $E_1 \subseteq E_2$ up to Lebesgue measure zero.*

Proof. We can write

$$\operatorname{Per}(E_1) - \int_{E_1} \kappa_1 \leqslant \operatorname{Per}(E_1 \cap E_2) - \int_{E_1 \cap E_2} \kappa_1,$$

$$\operatorname{Per}(E_2) - \int_{E_2} \kappa_2 \leqslant \operatorname{Per}(E_1 \cup E_2) - \int_{E_1 \cup E_2} \kappa_2.$$

Summing and using (1.6), we arrive at

$$\int_{E_1 \setminus E_2} \kappa_2 \leqslant \int_{E_1 \setminus E_2} \kappa_1,$$

which implies the result. □

We will repeatedly use the previous lemma to compare with balls:

Example 3.5. For $x_0 \in \mathbb{R}^d$ and $r > 0$, any function $v_{B(x_0,r)} \in L^1(\mathbb{R}^d)$ with

$$\begin{aligned} v_{B(x_0,r)} = \frac{d}{r} \text{ in } B(x_0,r),\ \ v_{B(x_0,r)} < 0 \text{ in } \mathbb{R}^d \setminus B(x_0,r), \text{ and} \\ \int_{\mathbb{R}^d \setminus B(x_0,r)} v_{B(x_0,r)} = -\operatorname{Per}(B(x_0,r)) \end{aligned} \tag{3.3}$$

is a variational mean curvature for $B(x_0,r)$. To check this, first we notice that

$$\operatorname{Per}(B(x_0,r)) - \int_{B(x_0,r)} v_{B(x_0,r)}(x)\,\mathrm{d}x = 0.$$

Moreover, for any other finite perimeter set with $|E| < \infty$, we have by the isoperimetric inequality (1.7) that for arbitrary $y \in \mathbb{R}^d$

$$\operatorname{Per}\left(B\left(y, r_E\right)\right) \leqslant \operatorname{Per}(E), \text{ with } r_E := \left(\frac{|E|}{|B(0,1)|}\right)^{1/d},$$

and clearly $|B(y,r_E)\cap B(x_0,r)|$ is maximized by picking $y = x_0$. If $r_E > r$ then $\operatorname{Per}(B(x_0,r_E)) > \operatorname{Per}(B(x_0,r_0))$ but

$$\int_{B(x_0,r_E)} v_{B(x_0,r)}(x)\,\mathrm{d}x < \int_{B(x_0,r)} v_{B(x_0,r)}(x)\,\mathrm{d}x,$$

so E could not be a minimizer. If $r_E \leqslant r$, then

$$\begin{aligned}\operatorname{Per}(B(x_0,r_E)) &= \left(\frac{r_E}{r}\right)^{d-1}\operatorname{Per}(B(x_0,r)) = \left(\frac{r_E}{r}\right)^{d-1}\int_{B(x_0,r)} v_{B(x_0,r)}(x)\,\mathrm{d}x\\ &= \frac{r}{r_E}\int_{B(x_0,r_E)} v_{B(x_0,r)}(x)\,\mathrm{d}x \geqslant \int_{B(x_0,r_E)} v_{B(x_0,r)}(x)\,\mathrm{d}x,\end{aligned}$$

with equality if and only if $r_E = r$. The case in which $|\mathbb{R}^d\setminus E| < +\infty$, implying that E must be of the form $\mathbb{R}^d\setminus B(y,\tilde r_E)$ for some $\tilde r_E > 0$, is handled with similar computations once we notice that condition (3.3) prevents the full space $\mathbb{R}^d$ from having negative energy.

Furthermore, Lemma 3.4 combines with the strict convexity of ψ to give a comparison principle for denoised solutions:

Proposition 3.6. *Let $f \leqslant g$ and $u^f_{\alpha,0}$, $u^g_{\alpha,0}$ be the corresponding minimizers of* (1.1) *with $w = 0$. Then one has $u^f_{\alpha,0} \leqslant u^g_{\alpha,0}$.*

Proof. To simplify the notation we drop the subindices that remain arbitrary, but fixed, in what follows. By Proposition 1.6, one can see that the level sets $\{u^f \geqslant s\}$ and $\{u^g \geqslant s\}$ are the maximal minimizers among E of respectively

$$\operatorname{Per}(E) + \frac{1}{\alpha}\int_E \psi'(s-f), \quad\text{and}\quad \operatorname{Per}(E) + \frac{1}{\alpha}\int_E \psi'(s-g).$$

Since ψ is strictly convex, we then have, for $s' < s$,

$$\psi'(s-f) > \psi'(s'-g),$$

which implies by Lemma 3.4 that $|\{u^g \geqslant s'\}\setminus\{u^f \geqslant s\}| = 0$. Since $s' < s$ was arbitrary and these sets are nested with respect to s', we infer

$$|\{u^g \geqslant s\}\setminus\{u^f \geqslant s\}| = \left|\bigcap_n \left\{u^g \geqslant s - \frac{1}{n}\right\}\setminus\{u^f \geqslant s\}\right| = 0.$$

Denoting the set

$$A := \bigcup_{s\in\mathbb{R}}\{u^g \geqslant s\}\setminus\{u^f \geqslant s\} = \bigcup_{s\in\mathbb{R}}\{u^g \geqslant s\}\cap\{u^f < s\},$$

we would like to see that $|A| = 0$ so that $u^g \geqslant u^f$ almost everywhere. We cannot immediately conclude since the union is over an uncountable index set. To proceed, define

$$A_{\mathbb{Q}} := \bigcup_{r\in\mathbb{Q}}\{u^g \geqslant r\}\cap\{u^f < r\}$$

with $|A_{\mathbb{Q}}| = 0$, and let $x \in A\setminus A_{\mathbb{Q}}$. Then there is some $s_0 \in \mathbb{R}$ for which both $u^g(x) \geqslant s_0$ and $u^f(x) < s_0$ hold. However, for all $r \in \mathbb{Q}$ we have either $u^g(x) < r$ or $u^f(x) \geqslant r$. Let $\{r_n\}_n \subset \mathbb{Q}$ with $r_n < s_0$ and $r_n \to s_0$. If we had that $u^g(x) < r_n$ for some n, then $u^g(x) < r_n < s_0$, a contradiction. So we must have $u^f(x) \geqslant r_n$ for all n, implying that $u^f(x) \geqslant s_0$, which is again a contradiction. Therefore $A = A_{\mathbb{Q}}$. □

3.1 Construction of variational mean curvatures for bounded sets

A natural question is whether a variational mean curvature can be found for a given set. The following crucial result proven in [8, 6] provides a positive answer:

Theorem 3.7. *Let D be a bounded set with finite perimeter. Then, D has variational mean curvatures in $L^1(\mathbb{R}^d)$. In addition, there exists a variational mean curvature κ_D for D which minimizes the $L^p(D)$ norm for all $p > 1$ among such curvatures. There might be $p > 1$ for which this minimal norm is not finite.*

The construction of κ_D in [8, 6] involves an arbitrary positive function $g \in L^1(\mathbb{R}^d)$ and minimizers of the problems

$$\min_{E \subset D} \operatorname{Per}(E) - \lambda \int_E g, \text{ and} \tag{3.4}$$

$$\min_{F \subset \mathbb{R}^d \setminus D} \operatorname{Per}(F) - \lambda \int_F g. \tag{3.5}$$

Namely, for $x \in D$ one defines

$$\kappa_D(x) := \inf \left\{ \lambda g(x) \;\middle|\; \lambda > 0 \text{ and } x \in E^\lambda, \text{ for } E^\lambda \text{ any minimizer of (3.4)} \right\}, \tag{3.6}$$

and for $x \in \mathbb{R}^d \setminus D$

$$\kappa_D(x) := -\inf \left\{ \lambda g(x) \;\middle|\; \lambda > 0 \text{ and } x \in F^\lambda, \text{ for } F^\lambda \text{ any minimizer of (3.5)} \right\}. \tag{3.7}$$

By definition $\kappa_D > 0$ in D and $\kappa_D < 0$ in $\mathbb{R}^d \setminus D$, consistent with the lack of uniqueness for variational mean curvatures described above. Moreover, the proof of Proposition 3.3 is also valid with inclusion constraints, so one can speak of maximal and minimal E^λ and F^λ. For completeness, we check that κ_D is well defined:

Proposition 3.8. *The problems* (3.4) *and* (3.5) *admit at least one minimizer. Moreover if for every compact set $K \subset \mathbb{R}^d$ one can find c_K such that*

$$g(x) \geqslant c_K > 0 \text{ for a.e. } x \in K, \tag{3.8}$$

then for almost every $x \in D$, we have that $x \in E^{\lambda_x}$ for some $\lambda_x > 0$ and E^{λ_x} a minimizer of (3.4)*, and similarly for a.e. $x \in \mathbb{R}^d \setminus \overline{D}$ and a corresponding minimizer of* (3.5)*.*

Proof. Let us focus first on minimizers of (3.5), for which we consider the equivalent problem for the complement

$$\min_{D \subset E} \operatorname{Per}(E) + \lambda \int_E g.$$

Let $\{E_n\}_n$ be a minimizing sequence for this problem. The objective is nonnegative, so comparing with any fixed nonempty set we have an upper bound for $\operatorname{Per}(E_n)$. Since $1_{E_n}(x) \in \{0, 1\}$ and hence bounded in L^1_{loc}, we can then apply compactness in BV_{loc} [4, Thm. 3.23] to obtain $v \in \mathrm{BV}_{\text{loc}}(\mathbb{R}^d)$ such that $1_{E_n} \to v$ in L^1_{loc} and in consequence also almost everywhere. Since $\operatorname{Per}(E_n) = |D1_{E_n}|(\mathbb{R}^d) \leqslant C$ we have in fact that $v \in \mathrm{BV}(\mathbb{R}^d)$, and since the convergence is in L^1_{loc} strong, there must be a finite perimeter set E_0 for which $v = 1_{E_0}$. Furthermore, by the lower semicontinuity of the total variation [4, Rem. 3.5] with respect to L^1_{loc} convergence and since $g \in L^1(\mathbb{R}^d)$, using the dominated convergence theorem we have

$$\operatorname{Per}(E_0) \leqslant \liminf_n \operatorname{Per}(E_n), \text{ and } \int_{E_0} g = \liminf_n \int_{E_n} g,$$

so $\mathbb{R}^d \setminus E_0$ is a minimizer of (3.5). For (3.4) one proceeds similarly, with the difference that under the constraint $E \subset D$ the fact that D is bounded allows to obtain full L^1 convergence of a minimizing sequence $\{E_n\}_n$.

To see the second part, we treat the inside and outside problems separately. First, notice that D is admissible in (3.4), so we have that

$$\operatorname{Per}(E^\lambda) - \lambda \int_{E^\lambda} g \leqslant \operatorname{Per}(D) - \lambda \int_D g,$$

or equivalently

$$\lambda \left(\int_D g - \int_{E^\lambda} g \right) \leqslant - \operatorname{Per}(F^\lambda) + \operatorname{Per}(D) \leqslant \operatorname{Per}(D),$$

where since $g > 0$ and $E^\lambda \subset D$ the left hand side is positive, and using (3.8) for $\overline{D}$ we get

$$|D \setminus E^\lambda| \leqslant \frac{1}{c_{\overline{D}}} \left(\int_D g - \int_{E^\lambda} g \right) \xrightarrow[\lambda \to \infty]{} 0,$$

so for a.e. $x \in D$ we must have $x \in E^{\lambda_x}$ for some λ_x. Similarly $\mathbb{R}^d \setminus D$ is admissible in (3.5), so using $\operatorname{Per}(\mathbb{R}^d \setminus D) = \operatorname{Per}(D)$ we have for F^λ any minimizer of (3.5) the bound

$$\lambda \left(\int_{\mathbb{R}^d \setminus D} g - \int_{F^\lambda} g \right) \leqslant \operatorname{Per}(D).$$

This time, to be able to use (3.8) we would need to see that $(\mathbb{R}^d \setminus D) \setminus F^\lambda = \mathbb{R}^d \setminus F^\lambda$ is bounded, which is not a priori obvious. For large enough λ we prove in Lemma 3.9 below that $\mathbb{R}^d \setminus F^\lambda$ is indeed bounded, allowing us to conclude. □

Lemma 3.9. *Assume that for every compact set $K \subset \mathbb{R}^d$ one can find c_K such that* (3.8) *holds, and that $D \subset B(0,1)$. Then there is some λ_1 such that if $\lambda > \lambda_1$ all minimizers F^λ of* (3.5) *satisfy $\mathbb{R}^d \setminus F^\lambda \subset \overline{B(0,1)}$, and in particular $|\mathbb{R}^d \setminus F^\lambda| < +\infty$. Moreover in that case $F^\lambda \cap B(0,2)$ is also a minimizer of*

$$\min_{F \subset B(0,2) \setminus D} \operatorname{Per}\big(F;\, B(0,2)\big) - \lambda \int_F g. \tag{3.9}$$

Proof. Let us define the compact set

$$K_1 := \overline{B(0,2)} \setminus B(0,1) = \bigcup_{x \in \partial B(0,3/2)} \overline{B\,(x, 1/2)}.$$

Using Lemma 3.4, Example 3.5, this expression and the condition on g we see that if

$$\lambda > \frac{2d}{c_{K_1}} =: \lambda_1, \text{ then } \mathring{K}_1 = B(0,2) \setminus \overline{B(0,1)} \subset F^\lambda.$$

But since $g > 0$ and $\partial B(0,3/2) \subset \mathring{K}_1$ this means that

$$\operatorname{Per}\Big(F^\lambda \cup \big(\mathbb{R}^d \setminus B(0,3/2)\big)\Big) \leqslant \operatorname{Per}(F^\lambda) \text{ and } \int_{F^\lambda \cup\, (\mathbb{R}^d \setminus B(0,3/2))} g \geqslant \int_{F^\lambda} g,$$

so necessarily $\mathbb{R}^d \setminus B(0,3/2) \subset F^\lambda$ for all $\lambda > \lambda_1$ as well, hence $\mathbb{R}^d \setminus \overline{B(0,1)} \subset F^\lambda$. These considerations also directly prove that $F^\lambda \cap B(0,2)$ minimizes (3.9). □

The curvatures arising from this construction are in fact not independent of the choice of the density g, as is shown in Proposition 3.10 below. As has been noted in previous works [6, 23], since we work with bounded D, this ambiguity can be mitigated by choosing $g(x) = 1$ for all $x \in D$. However $g \in L^1(\mathbb{R}^d \setminus D)$ is required to make sense of the unbounded problem (3.5), and there is no canonical choice for it outside of D. Moreover, as opposed to most other works using this variational mean curvature, we plan to make explicit use of κ_D on $\mathbb{R}^d \setminus D$ and the corresponding minimizers of (3.5).

Proposition 3.10. *For any bounded D and any positive $g \in L^1(\mathbb{R}^d)$, there exists some $\lambda_g > 0$ such that if $\lambda \leqslant \lambda_g$, the only minimizer of (3.5) is the empty set, and in consequence $\kappa_D(x) \geqslant -\lambda g(x)$ for a.e. x. Moreover, if additionally $|\mathbb{R}^d \setminus F^\lambda| < +\infty$ for all λ then*

$$\kappa_D(x) = -\lambda_g\, g(x) \text{ for a.e. } x \in \mathbb{R}^d \setminus \operatorname{Conv} D.$$

Proof. Let us define

$$G_D := \operatorname*{arg\,min}_{D \subset E} \operatorname{Per}(E),$$

which exists by the same compactness arguments as in Proposition 3.8 (if $d = 2$ then in fact $G_D = \operatorname{Conv} D$ [20]). Then any minimizer $F \neq \emptyset$ of (3.5) must have $\operatorname{Per}(F) \geqslant \operatorname{Per}(G_D)$, so that

$$\operatorname{Per}(F) - \lambda \int_F g \geqslant \operatorname{Per}(G_D) - \lambda \int_{\mathbb{R}^d \setminus D} g. \tag{3.10}$$

But whenever

$$\lambda < \lambda_c := \frac{\operatorname{Per}(G_D)}{\int_{\mathbb{R}^d \setminus D} g}$$

we have that the right hand side of (3.10) is positive, making F is a worse competitor than the empty set. We can then define

$$\lambda_g := \sup \left\{ \lambda > 0 \;\middle|\; \inf_{F \subset \mathbb{R}^d \setminus D} \operatorname{Per}(F) - \lambda \int_F g = 0 \right\} \geqslant \lambda_c > 0,$$

and notice that $\lambda_g < +\infty$ by Proposition 3.8.

To prove the second part, notice that having $\kappa_D(x) < -\lambda_g\, g(x)$ means that $x \notin F^{\lambda_g + \varepsilon}$ for some $\varepsilon > 0$, or equivalently, that x belongs to the minimal minimizer $E^{\lambda_g+\varepsilon}$ of

$$\min_{D \subset E} \operatorname{Per}(E) + (\lambda_g + \varepsilon) \int_E g.$$

However since by assumption we have $|E^{\lambda_g+\varepsilon}| = |\mathbb{R}^d \setminus F^{\lambda_g+\varepsilon}| < \infty$, taking its intersection with a convex set cannot increase the perimeter (see [12, Lem. 3.5] for a proof in the general setting) and we get

$$\operatorname{Per}(E^{\lambda_g+\varepsilon}) + (\lambda_g + \varepsilon) \int_{E^{\lambda_g+\varepsilon}} g \geqslant \operatorname{Per}(E^{\lambda_g+\varepsilon} \cap \operatorname{Conv} D) + (\lambda_g + \varepsilon) \int_{E^{\lambda_g+\varepsilon} \cap \operatorname{Conv} D} g,$$

and the inequality would be strict if $|E^{\lambda_g+\varepsilon} \setminus \operatorname{Conv} D| > 0$, so necessarily

$$\left| \left\{ x \in \mathbb{R}^d \setminus \operatorname{Conv} D \;\middle|\; \kappa_D(x) < -\lambda_g\, g(x) \right\} \right| = \left| \bigcap_{\varepsilon > 0} E^{\lambda_g+\varepsilon} \setminus \operatorname{Conv} D \right| = 0.$$

□

We see that the concrete choice of density g affects the values of κ_D. We now introduce one such choice which at least allows for a purely geometric description of minimizers for λ large enough.

Definition 3.11. Assume $D \subseteq B(0,1)$. For any $R > 1$ we define g_R by

$$g_R(x) := \begin{cases} 1 & \text{if } 0 \leqslant |x| \leqslant R, \\ g_f & \text{if } |x| > R, \end{cases} \tag{3.11}$$

for some $g_f \in L^1(\mathbb{R}^d \setminus B(0,R))$ with $0 < g_f \leqslant 1$ and satisfying (3.8).

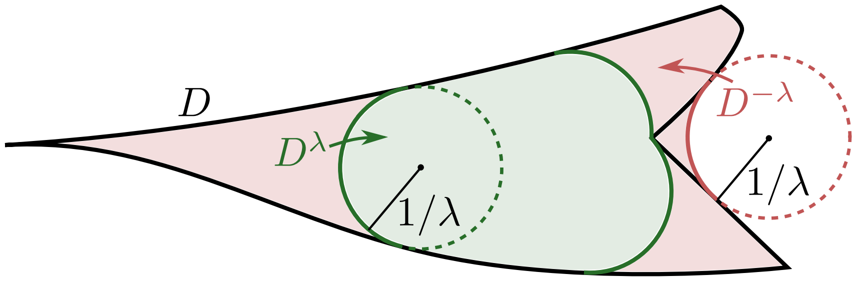

Figure 1: Approximation of the set D with the level sets of κ_D. For fixed $\lambda > 0$, we always have $D^\lambda \subset E \subset D^{-\lambda}$, and $d_{\mathcal{H}}(\partial D^\lambda, \partial D^{-\lambda}) \to 0$ as $\lambda \to +\infty$. With respect to their outer normals, the free boundaries of D^λ and $D^{-\lambda}$ have curvature λ and $-\lambda$ respectively.

Since we will make extensive use of minimizers of (3.4) and (3.5) with this particular choice of density, we introduce some notation for them.

Definition 3.12. Let $\lambda > 0$ and $D \subset B(0,1)$ of finite perimeter. We denote by D^λ the maximal (in the sense of inclusion) minimizer of (3.4) with density g_2, that is, of

$$\min_{E \subset D} \operatorname{Per}(E) - \lambda |E|. \tag{3.12}$$

We also define $D^{-\lambda}$ as

$$D^{-\lambda} := \mathbb{R}^d \setminus F^\lambda,$$

where F^λ is the maximal minimizer of (3.5) with density g_2. In view of the proof of Lemma 3.9, whenever $\lambda > 2d$ we have that F^λ can be determined from (3.9), which since $g_2 \equiv 1$ on $B(0,2)$ turns into

$$\min_{F \subset B(0,2) \setminus D} \operatorname{Per}\big(F;\, B(0,2)\big) - \lambda |F|.$$

Moreover, notice that $D^{-\lambda}$ can also be found directly as the minimal minimizer of

$$\min_{E \supset D} \operatorname{Per}(E) + \lambda \int_E g_2.$$

Remark 3.13. We have chosen $D \subset B(0,1)$ but other bounded sets can be treated by rescaling. If for any set E and $q > 0$ we consider the rescaled set qE we have

$$\operatorname{Per}(qE) - \lambda \int_{qE} g_R(x)\,\mathrm{d}x = q^{d-1} \operatorname{Per}(E) - q^d \lambda \int_E g_{R/q}(y)\,\mathrm{d}y,$$

so the minimization problem for these rescaled sets is equivalent to the original one with λ replaced by $q\lambda$ and R replaced by R/q.

The choice of signs in the notation is motivated by (3.6) and (3.7), and by the fact that the free boundaries of D^λ and $D^{-\lambda}$ have curvature λ and $-\lambda$ respectively with respect to their outer normals, see (3.2).

Remark 3.14. From now on, whenever we use the variational mean curvature κ_D for some $D \subset B(0,1)$, we will always assume that the density used is g_2, as in Definition 3.12 above.

We will see in later sections that for large values of λ, the sets D^λ and $D^{-\lambda}$ provide us with an approximation of D in Hausdorff distance from the inside and outside respectively, motivating the notation. Moreover, they also determine the curvature κ_D through (3.6) and (3.7).

3.2 Bounds and examples of variational mean curvatures

Lemma 3.15. *Assume that* x_0, r *are such that* $B(x_0, r) \subseteq D$ *up to measure zero, that is,* $|B(x_0, r) \setminus D| = 0$*. Then the optimal variational mean curvature* κ_D *of* D *satisfies*

$$\kappa_D\big|_{B(x_0,r)} \leqslant \frac{d}{r}. \tag{3.13}$$

In consequence, for any interior point $x \in \mathring{D}$*, we have*

$$\kappa_D(x) \leqslant \frac{d}{\operatorname{dist}(x, \partial D)}. \tag{3.14}$$

Similarly, for $x \in \mathbb{R}^d \setminus \overline{D}$ *we have* $-\kappa_D(x) \leqslant d/\operatorname{dist}(x, \partial D)$*. Therefore, for any* $K \subset \mathbb{R}^d$ *we have*

$$\|\kappa_D\|_{L^\infty(K)} \leqslant \frac{d}{\operatorname{dist}(K, \partial D)}, \tag{3.15}$$

where $\operatorname{dist}(K, \partial D) := \inf_{x \in K} \operatorname{dist}(x, \partial D)$*.*

Proof. By the definition of D^λ as the maximal solution of (3.12) and that of κ_D in (3.6), $x \in D^\lambda$ implies $\kappa_D(x) \leqslant \lambda$. On the other hand, by Lemma 3.4 and since by Example 3.5 we know that we can find a variational mean curvature for $B(x_0, r)$ with value d/r in $B(x_0, r)$, we have that $\lambda > d/r$ implies $B(x_0, r) \subseteq D^\lambda$, so for $x \in B(x_0, r)$ we get $\kappa_D(x) \leqslant \lambda$ for every $\lambda > d/r$, which is (3.13). To see (3.14), just notice that since $x \in \mathring{D}$, we have that $B(x, r) \subset D$ for each $r < \operatorname{dist}(x, \partial D)$.

If $x \in \mathbb{R}^d \setminus \overline{D}$, we can proceed similarly using $F^\lambda = \mathbb{R}^d \setminus D^{-\lambda}$ and its variational problem (3.5). These two cases prove (3.15), since the bound is trivial when $\operatorname{dist}(K, \partial D) = 0$. □

Lemma 3.16. *Let* $D \subset \mathbb{R}^d$ *be bounded. Denote by*

$$h(D) := \min_{E \subset D} \frac{\operatorname{Per}(E)}{|E|}$$

the Cheeger constant of D*, the minimum being attained at Cheeger sets of* D*. Then* $\kappa_D(x) \geqslant h(D)$ *for* $x \in D$*, with equality for* $x \in C_D$*, the maximal Cheeger set.*

Proof. We again consider the problem

$$\min_{E \subseteq D} \operatorname{Per}(E) - \lambda |E|, \tag{3.16}$$

with D^λ its maximal solution. Assume $\lambda > 0$ is such that $|D^\lambda| \neq 0$, then by comparing with the empty set we have

$$\operatorname{Per}(D^\lambda) - \lambda |D^\lambda| \leqslant 0,$$

which implies

$$\lambda \geqslant \frac{\operatorname{Per}(D^\lambda)}{|D^\lambda|} \geqslant \min_{E \subset D} \frac{\operatorname{Per}(E)}{|E|} = h(D),$$

proving that $\kappa_D(x) \geqslant h(D)$ for all $x \in D$. Similarly, considering (3.16) for $\lambda = h(D)$ we get

$$\operatorname{Per}(D^{h(D)}) - h(D)|D^{h(D)}| \leqslant 0,$$

so that $D^{h(D)}$ is a Cheeger set, and in fact by maximality $D^{h(D)} = C_D$, which in turn implies $\kappa_D(x) = h(D)$ for $x \in C_D$. □

Proposition 3.17. *Let $S = (0,1) \times (0,1)$ be the unit square in $\mathbb{R}^2$. Then denoting by $Q^1 := (0,1/2) \times (0,1/2)$ the lower left quadrant we have*

$$\kappa_S(x) = \begin{cases} h(S) = 1/r_S := 2 + \sqrt{\pi} & \text{if } x \in C_S, \\ (x_1 + x_2 + \sqrt{2x_1x_2})^{-1} & \text{if } x \in (S \setminus C_S) \cap Q^1, \end{cases} \tag{3.17}$$

and similarly for the other quadrants. The Cheeger set C_S (unique by convexity, see [3]) is given by

$$C_S = \left\{ x \in S \,\middle|\, \operatorname{dist}\bigl(x, (r_S, 1 - r_S) \times (r_S, 1 - r_S)\bigr) < r_S \right\}.$$

Proof. If $x \in C_S$ we use Lemma 3.16; the value of r_S can be found for example in [26, Thm. 3]. If $x \notin C_S$, without loss of generality we can assume that $x = (x_1, x_2)$ is in $(S \setminus C_S) \cap Q^1$. Now, x belongs to the circle centered at $(R(x), R(x))$ with radius $R(x)$, that is

$$(x_1 - R(x))^2 + (x_2 - R(x))^2 = R(x)^2.$$

This is a quadratic equation for $R(x)$ that we can solve to find $\kappa_S(x) = 1/R(x)$. □

Remark 3.18. It is proved in [32, Thm. 3.32(i)] that for a convex planar set E and λ small enough, E^λ can be written as a union of balls of radius $1/\lambda$. In this situation, the proof of [26, Thm. 1] building up on this characterization as a union implies in particular that

$$E^\lambda = [E]_i^{1/\lambda} + \frac{1}{\lambda} B(0,1),$$

where $[E]_i^{1/\lambda} \subset E$ is the $1/\lambda$-offset of E in the direction of its inner normal, so that E^λ is obtained by a "rolling ball" procedure. Furthermore, it has been recently proven [27] that the same characterization holds for more general planar sets, namely Jordan domains with no necks.

4 Convergence for indicatrices with noise

In this section we prove Theorem 1.2 on Hausdorff convergence of level sets for denoising the indicatrix of a bounded finite perimeter set $D \subset B(0,1)$, with the variational mean curvature $\kappa_D \in L^1(\mathbb{R}^d)$ constructed with the choices of Definition 3.12, and with noise and parameter choice controlled by (1.2). To this end, let $u_{\alpha,0}$ be the precise representative of the minimizer of (1.1) with $f = 1_D$ and $w = 0$, that is, with no noise added. We denote by

$$\kappa_\alpha := v_{\alpha,0} = \frac{1}{\alpha} \psi'(1_D - u_{\alpha,0})$$

the corresponding variational mean curvature associated to $u_{\alpha,0}$ through duality in Proposition A.1, and by E_α^s its upper level sets. The definition of κ_D also provides us with a natural precise representative for it; we will implicitly use these precise representatives in the rest of the section.

We implement the local strategy of [15, Thm. 2], which requires that $v_{\alpha,0}$ is L^d-equiintegrable only on $K_\delta = \{\operatorname{dist}(\cdot, \partial D) \geqslant \delta\}$ for each $\delta > 0$. This equiintegrability is in turn a consequence of Lemma 3.15 and Proposition 4.1 below, which combine to give the bound

$$\|v_{\alpha,0}\|_{L^\infty(K_\delta)} \leqslant \frac{d}{\delta}. \tag{4.1}$$

Proposition 4.1. *The noiseless dual variable κ_α satisfies $|\kappa_\alpha| \leqslant |\kappa_D|$ almost everywhere.*

To prove this proposition, we will use the following lemmas:

Lemma 4.2. *The denoised solution $u_{\alpha,0}$ with $f = 1_D$ and $w = 0$ satisfies*

$$u_{\alpha,0}(x) = 0 \text{ for a.e. } x \in \mathbb{R}^d \setminus \operatorname{Conv} D.$$

Proof. Denote $u := u_{\alpha,0}$, and assume for the sake of contradiction that

$$|\{u \neq 0\} \setminus \operatorname{Conv} D| > 0.$$

This implies, defining $u_c := u 1_{\operatorname{Conv} D}$, that

$$\begin{aligned}\int_{\mathbb{R}^d} \psi(u - 1_D) &= \int_{\operatorname{Conv} D} \psi(u - 1_D) + \int_{\mathbb{R}^d \setminus \operatorname{Conv} D} \psi(u) \\ &> \int_{\operatorname{Conv} D} \psi(u - 1_D) = \int_{\mathbb{R}^d} \psi(u_c - 1_D).\end{aligned}$$

Moreover we can write the coarea formula for u as

$$\begin{aligned}\operatorname{TV}(u) &= \int_0^{+\infty} \operatorname{Per}(\{u > s\})\, \mathrm{d}s + \int_{-\infty}^0 \operatorname{Per}(\{u < s\})\, \mathrm{d}s \\ &\geqslant \int_0^{+\infty} \operatorname{Per}(\{u > s\} \cap \operatorname{Conv} D)\, \mathrm{d}s + \int_{-\infty}^0 \operatorname{Per}(\{u < s\} \cap \operatorname{Conv} D)\, \mathrm{d}s \\ &= \int_0^{+\infty} \operatorname{Per}(\{u_c > s\})\, \mathrm{d}s + \int_{-\infty}^0 \operatorname{Per}(\{u_c < s\})\, \mathrm{d}s = \operatorname{TV}(u_c),\end{aligned}$$

where we have used that the level sets $\{u > s\}$ for $s > 0$ and $\{u < s\}$ for $s < 0$ must have finite mass since $u \in L^{d/(d-1)}$, and the convexity of $\operatorname{Conv} D$. These two inequalities mean that u could not be a minimizer. □

Lemma 4.3. *Let $0 < s \leqslant 1$. Then, for $0 < \lambda < \psi'(1-s)/\alpha$, one has $D^\lambda \subset E_\alpha^s$ whereas for $0 > -\lambda > -\psi'(s)/\alpha$, one has $E_\alpha^s \subset D^{-\lambda}$.*

Proof. First, notice that the problems satisfied by D^λ and $D^{-\lambda}$ are of obstacle type, so that as in [24, Lem. 9], one can lift the obstacle constraint and conclude that D^λ minimizes

$$E \mapsto \operatorname{Per}(E) - \int_E \kappa_i^\lambda \quad \text{with} \quad \kappa_i^\lambda = \lambda 1_D + \kappa_D 1_{\mathbb{R}^d \setminus D} \tag{4.2}$$

whereas $D^{-\lambda}$ minimizes

$$E \mapsto \operatorname{Per}(E) - \int_E \kappa_o^\lambda \quad \text{with} \quad \kappa_o^\lambda = -\lambda g_2 1_{\mathbb{R}^d \setminus D} - \kappa_D 1_D.$$

Therefore, we can write

$$\operatorname{Per}(E_\alpha^s \cap D^\lambda) - \int_{E_\alpha^s \cap D^\lambda} \kappa_i^\lambda \geqslant \operatorname{Per}(D^\lambda) - \int_{D^\lambda} \kappa_i^\lambda$$

and

$$\operatorname{Per}(E_\alpha^s \cup D^\lambda) - \int_{E_\alpha^s \cup D^\lambda} \kappa_\alpha \geqslant \operatorname{Per}(E_\alpha^s) - \int_{E_\alpha^s} \kappa_\alpha.$$

Summing these two inequalities, we obtain

$$\int_{D^\lambda \setminus E_\alpha^s} \kappa_i^\lambda \geqslant \int_{D^\lambda \setminus E_\alpha^s} \kappa_\alpha$$

that rewrites, since $D^\lambda \subset D$ and using the definition of κ_i^λ in (4.2),

$$\int_{D^\lambda \setminus E_\alpha^s} \left(\lambda - \frac{\psi'(1 - u_\alpha)}{\alpha} \right) \geqslant 0. \tag{4.3}$$

Now, on $(E^s_\alpha)^c$ by definition $u_\alpha \leqslant s$ which, since ψ' is strictly increasing, implies $\lambda - \psi'(1-u_\alpha)/\alpha \leqslant \lambda - \psi'(1-s)/\alpha$. Therefore, for $\lambda < \psi'(1-s)/\alpha$, (4.3) can hold only if $|D^\lambda \setminus E^s_\alpha| = 0$, that is $D^\lambda \subset E^s_\alpha$ a.e.

Similarly, we can write

$$\mathrm{Per}(E^s_\alpha \cup D^{-\lambda}) - \int_{E^s_\alpha \cup D^{-\lambda}} \kappa^\lambda_o \geqslant \mathrm{Per}(D^{-\lambda}) - \int_{D^{-\lambda}} \kappa^\lambda_o,$$

$$\mathrm{Per}(E^s_\alpha \cap D^{-\lambda}) - \int_{E^s_\alpha \cap D^{-\lambda}} \kappa_\alpha \geqslant \mathrm{Per}(E^s_\alpha) - \int_{E^s_\alpha} \kappa_\alpha,$$

to sum these inequalities and, using $(D^{-\lambda})^c \subset D^c$ and $u_\alpha > s$ on E^s_α, obtain

$$0 \geqslant \int_{E^s_\alpha \setminus D^{-\lambda}} \left(\frac{\psi'(u_\alpha)}{\alpha} - \lambda g_2 \right) > \int_{E^s_\alpha \setminus D^{-\lambda}} \left(\frac{\psi'(s)}{\alpha} - \lambda g_2 \right).$$

Hence, since we have $g_2 \leqslant 1$, as soon as $0 < \lambda < \psi'(s)/\alpha$, we obtain $E^s_\alpha \subset D^{-\lambda}$ a.e. □

Proof of Proposition 4.1. First we take $x \in D$ and define $s := u_{\alpha,0}(x)$, implying $\kappa_\alpha(x) = \psi'(1-s)/\alpha$ and $\kappa_D(x) \geqslant 0$, and assume that for some $\varepsilon > 0$

$$\kappa_\alpha(x) = \frac{\psi'(1-s)}{\alpha} \geqslant \kappa_D(x) + \varepsilon, \tag{4.4}$$

to then use Lemma 4.3 to derive a contradiction. By definition of the level sets we have that for all $\delta > 0$,

$$x \in E^{s-\delta}_\alpha, \text{ and } x \notin E^{s+\delta}_\alpha. \tag{4.5}$$

This, combined with Lemma 4.3 implies that $x \notin D^\lambda$, whenever $0 < \lambda < \psi'(1-s-\delta)/\alpha$. On the other hand, the construction of κ_D and (4.4) give $x \in D^\lambda$ for all $\lambda \geqslant \psi'(1-s)/\alpha - \varepsilon > 0$, where for the last inequality we have used (4.4). Choosing δ such that

$$\psi'(1-s) - \psi'(1-s-\delta) \leqslant \alpha\varepsilon,$$

which is possible since $\psi \in \mathcal{C}^1(\mathbb{R})$, these two statements are contradictory and therefore we must have $\kappa_\alpha(x) \leqslant \kappa_D(x)$ for all $x \in D$.

Now, if $x \in \mathbb{R}^d \setminus \overline{D}$, by Lemma 4.2 we can assume $x \in \mathrm{Conv}\, D \setminus \overline{D}$, since otherwise we would have $\kappa_\alpha(x) = 0$ and the inequality is trivially satisfied. This implies in particular that $g_2(x) = 1$. We have $\kappa_\alpha(x) = \psi'(-s)/\alpha$ and $\kappa_D(x) \leqslant 0$, and failure of the statement means that for some $\varepsilon > 0$ we have

$$\kappa_\alpha(x) = \frac{\psi'(-s)}{\alpha} \leqslant \kappa_D(x) - \varepsilon.$$

As before, for any δ (4.5) holds and Lemma 4.3 then implies that $x \in D^{-\lambda}$ as soon as $0 > -\lambda > \psi'(-s+\delta)/\alpha$. However the definition of κ_D and that $g_2(x) = 1$ imply that $x \notin D^{-\lambda}$ if $-\lambda < \kappa_D(x) \leqslant \psi'(-s)/\alpha + \varepsilon < 0$, so that if δ is such that $\psi'(-s+\delta) - \psi'(-s) \leqslant \alpha\varepsilon$ we again derive a contradiction. □

Remark 4.4. It might seem slightly surprising that even though the construction of κ_D depends on the density g, as has been seen in Proposition 3.10, we can still obtain the inequality $|\kappa_\alpha| \leqslant |\kappa_D|$. There are two reasons for this. First, we were able to bound the support of u_α in Lemma 4.2, allowing to avoid the unintuitive behaviour of κ_D for small negative values and far away from D. Second, with our particular choice (3.11) we have $g_2 = 1$ in $\mathrm{Conv}\, D$, so we can still obtain the desired comparison without distorting the values of κ_α in question.

As a consequence of (4.1) we can obtain uniform density estimates for $E^s_{\alpha,w}$ outside of K_δ with scale r_δ and constant C_δ possibly degenerating as $\delta \to 0$. These are proved in Proposition B.1 of Appendix B. A further consequence of these density estimates is the following compact support result.

Proposition 4.5. *Assume a parameter choice such that*

$$\|v_{\alpha,w} - v_{\alpha,0}\|_{L^d(\mathbb{R}^d)} \leqslant C_0 < \Theta_d, \tag{4.6}$$

where $v_{\alpha,w}$ *is the duality certificate for* (1.1) *of Proposition A.1, and that* $f = 1_D$ *for* D *bounded. Then there is* $R > 0$ *such that*

$$\operatorname{Supp} u_{\alpha,w} \subset B(0,R),$$

with R *depending on* C_0 *but not on the specific* α *and* w.

Proof. Denote $E := E^s_{\alpha,w}$ where α, w are fixed. Since $\operatorname{Per}(E) = \int_E v_{\alpha,w}$ by Proposition 1.6, using the Hölder inequality, (4.6), the isoperimetric inequality (1.7) and Proposition 4.1 we get

$$\begin{aligned}\operatorname{Per}(E) &\leqslant \left|\int_E (v_{\alpha,w} - v_{\alpha,0})\right| + \left|\int_E v_{\alpha,0}\right| \\ &\leqslant C_0 |E|^{(d-1)/d} + \int_E |v_{\alpha,0}| \leqslant \Theta_d^{-1} C_0 \operatorname{Per}(E) + \|\kappa_D\|_{L^1},\end{aligned}$$

which provides a uniform bound for $\operatorname{Per}(E)$, and by the isoperimetric inequality also for $|E|$.

Moreover, we have that by (4.1) the hypotheses of Proposition B.1 are satisfied with $K = \{\operatorname{dist}(\cdot, \partial D) \geqslant 1\}$ and

$$r_{K,\varepsilon} = \frac{\varepsilon^{1/d}}{d|B(0,1)|^{1/d}},$$

so the E satisfy uniform density estimates at some scale r_K and with constant C_K outside the bounded set K, which combined with the mass bound implies also a uniform bound for $\operatorname{diam}(E)$. To see the last claim, consider points $x_n \in \partial E \setminus K$, $n = 1, \ldots, N$ with $|x_i - x_j| > r_0$ for $i \neq j$. The inner density estimate implies $|E \cap B(x_n, r)| > C|B(x_n, r_0)|$ combined with the uniform bound for $|E|$ gives a uniform upper bound for N, hence also for $\operatorname{diam}(E)$. □

We are now ready to prove the main result of this section.

4.1 Proof of Theorem 1.2

Proof of Theorem 1.2. First, we notice that the definition of the Hausdorff distance reads

$$d_{\mathcal{H}}(\partial E^s_{\alpha,w}, \partial D) = \max\left(\sup_{x \in \partial E^s_{\alpha,w}} \operatorname{dist}(x, \partial D), \sup_{x \in \partial D} \operatorname{dist}(x, \partial E^s_{\alpha,w})\right).$$

Therefore, we need to prove the two statements

$$\sup_{x \in \partial D} \operatorname{dist}(x, \partial E^s_{\alpha_n,w_n}) \to 0, \text{ and} \tag{4.7}$$

$$\sup_{x \in \partial E^s_{\alpha_n,w_n}} \operatorname{dist}(x, \partial D) \to 0. \tag{4.8}$$

Let us start with (4.7), for which the argument follows closely that of [15, Prop. 9], which we reproduce for completeness. Since the parameter choice (1.2) implies in particular condition (1.4), arguing as in the proof of Proposition 1.5 we have, up to a subsequence, that $Du_{\alpha_n,w_n} \overset{*}{\rightharpoonup} D1_D$. Now the coarea formula, as formulated for example in [5, Thm. 10.3.3], tells us that we can slice these measures (and not just their total variations) so that for a.e. $s \in (0,1)$ we in fact have

$$D1_{E^s_{\alpha_n,w_n}} \overset{*}{\rightharpoonup} D1_D. \tag{4.9}$$

Now let $x \in \operatorname{Supp} D1_D$, then for any $r > 0$ using (4.9) we get

$$0 < |D1_D|(B(x,r)) \leqslant \liminf_n |D1_{E^s_{\alpha_n,w_n}}|(B(x,r)),$$

which implies that $\limsup_n \operatorname{dist}(x, \operatorname{Supp} D1_{E_{\alpha_n,w_n}}) \leqslant r$. Since $r > 0$ was arbitrary we conclude $\operatorname{dist}(x, \operatorname{Supp} D1_{E_{\alpha_n,w_n}}) \to 0$, and in particular

$$\sup_{x \in \operatorname{Supp} D1_D} \operatorname{dist}(x, \operatorname{Supp} D1_{E_{\alpha_n,w_n}}) \to 0.$$

Finally, as mentioned in Section 1.3, we always use representatives for E_{α_n,w_n} and D for which

$$\partial E_{\alpha_n,w_n} = \operatorname{Supp} D1_{E_{\alpha_n,w_n}} \text{ and } \partial D = \operatorname{Supp} D1_D,$$

completing the proof of (4.7).

To prove (4.8) we assume it does not hold to reach a contradiction. One the one hand, we notice that using the parameter choice (1.2) and Proposition A.2, we can apply Proposition 4.5 to see that the u_{α_n,w_n} have a common compact support. This, combined with the convergence in L^1_{loc} of u_{α_n,w_n} proved in Proposition 1.5, implies that $|E^s_{\alpha_n,w_n} \Delta D| \to 0$. On the other, if (4.8) fails, from Proposition 2.6 and its proof we see that we must have either

$$\sup_{x \in E^s_{\alpha_n,w_n}} \operatorname{dist}(x, D) \not\to 0, \text{ or } \sup_{x \in \mathbb{R}^d \setminus E^s_{\alpha_n,w_n}} \operatorname{dist}\Big(x, \mathbb{R}^d \setminus D\Big) \not\to 0. \tag{4.10}$$

Assume the first is true, so that there is $\delta > 0$ and a subsequence $x_n \in E^s_{\alpha_n,w_n}$ for which $\operatorname{dist}(x_n, D) > \delta$. In that case, by the inner density estimates proved above, we have that

$$|E^s_{\alpha_n,w_n} \Delta D| \geqslant |E^s_{\alpha_n,w_n} \setminus D| \geqslant |B(x_n, \delta) \cap E^s_{\alpha_n,w_n}| \geqslant C_\delta |B(0, \delta)|,$$

a contradiction. If the second case of (4.10) was true, we instead use the outer density estimate to again contradict the L^1 convergence. □

5 Convergence for generic BV functions without noise

This section is aimed at the proof of Theorem 1.3 and simplified versions of it for piecewise constant functions. We are concerned with the noiseless situation, that is, we assume $w = 0$ throughout. Moreover, we always assume $\operatorname{Supp} f \subset B(0,1)$ which, arguing as in Lemma 4.2, implies

$$\operatorname{Supp} u_{\alpha,0} \subset \operatorname{Conv}(\operatorname{Supp} f) \subset \overline{B(0,1)}. \tag{5.1}$$

5.1 Approximation with the level sets of the optimal curvature of D

The key to the results of this section will be to know that we can approximate any $D \subset B(0,1)$ with the sets D^λ and $D^{-\lambda}$ as $\lambda \to \infty$ arising from the choices of Definition 3.12. First we note that this approximation happens in mass:

Lemma 5.1. *For $D \subset \mathbb{R}^d$ bounded of finite perimeter, we have as $\lambda \to +\infty$ that*

$$\Big|D \setminus D^\lambda\Big| \to 0, \text{ and } \Big|D^{-\lambda} \setminus D\Big| \to 0.$$

Proof. It is contained in the proof of Proposition 3.8. For the inside approximants D^λ, the result is proven also in [33, Thm. 2.3(ii)]. □

Moreover, this two-sided approximation also holds in Hausdorff distance of the corresponding boundaries:

Lemma 5.2. *For $D \subset \mathbb{R}^d$ bounded of finite perimeter and every $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that $D^{\lambda_\varepsilon} \subset D \subset D^{-\lambda_\varepsilon}$, $d_{\mathcal{H}}(\partial D^{\lambda_\varepsilon}, \partial D) \leqslant \varepsilon$ and $d_{\mathcal{H}}(\partial D^{-\lambda_\varepsilon}, \partial D) \leqslant \varepsilon$.*

Proof. The interior approximation is proved in [33, Thm. 2.3(iv)]; we reproduce their argument here, and see that it can also be applied for the exterior approximation with $D^{-\lambda}$.

To start, let $x \in D$ with $\operatorname{dist}(x, \partial D) > \varepsilon$. Then we have that $B(x,\varepsilon) \subset D$, and as in the proof of Lemma 3.15 we must have $B(x,\varepsilon) \subset D^\lambda$ for all $\lambda > \varepsilon/d$, in particular $x \in D^\lambda \setminus \partial D^\lambda$, implying

$$\sup_{x \in \partial D^\lambda} \operatorname{dist}(x, \partial D) \leqslant \varepsilon \text{ for all } \lambda > \varepsilon/d.$$

For the other term of the Hausdorff distance, the strategy is to cover ∂D with finitely many balls $B(x_j, \varepsilon)$ with $j = 1, \ldots, N_\varepsilon$ and $x_j \in \partial D$, which is possible since ∂D is bounded. Then, since Lemma 5.1 implies that $|D \setminus D^\lambda| \to 0$ as $\lambda \to \infty$, we can choose λ_ε such that $|D^{\lambda_\varepsilon} \cap B(x_j, \varepsilon)| > 0$ for all j. Since these balls cover ∂D, we have that

$$\sup_{x \in \partial D} \operatorname{dist}(x, \partial D^\lambda) \leqslant \varepsilon \text{ for all } \lambda > \lambda_\varepsilon.$$

Since it was only used that $\partial D = \partial(\mathbb{R}^d \setminus D)$ is bounded, we can proceed in the same way for the approximation with $\partial D^{-\lambda}$. For the first part, it suffices to notice that by definition $F^\lambda = \mathbb{R}^d \setminus D^{-\lambda}$ are minimizers of (3.5), so if $x \in \mathbb{R}^d \setminus D$ with $\operatorname{dist}(x, \partial D) > \varepsilon$ we must also have $B(x,\varepsilon) \subset F^\lambda = \mathbb{R}^d \setminus D^{-\lambda}$ for all $\lambda > \varepsilon/d$. Moreover, we have $|D^{-\lambda} \setminus D| \to 0$ by Lemma 5.1, which allows to repeat the covering argument. □

Corollary 5.3. *For every bounded finite perimeter set $D \subset \mathbb{R}^d$ and every $\varepsilon > 0$ there exists $\lambda_\varepsilon > 0$ such that*

$$d_{\mathcal{H}}(D^{\lambda_\varepsilon}, D) \leqslant \varepsilon \textit{ and } d_{\mathcal{H}}(D^{-\lambda_\varepsilon}, D) \leqslant \varepsilon,$$

and also

$$d_{\mathcal{H}}(\mathbb{R}^d \setminus D^{\lambda_\varepsilon}, \mathbb{R}^d \setminus D) \leqslant \varepsilon \textit{ and } d_{\mathcal{H}}(\mathbb{R}^d \setminus D^{-\lambda_\varepsilon}, \mathbb{R}^d \setminus D) \leqslant \varepsilon.$$

Proof. It follows by Lemmas 5.1 and 5.2 combined with Theorem 2.7. Note that the latter theorem is not quantitative and we could get different values of λ_ε from it for the different convergences, but we can then just use the largest of the two. □

Example 5.4. A result like Lemma 5.2 can only hold for bounded sets. As a counterexample, consider D defined by

$$D := \bigcup_{j=0}^{\infty} B\left((j,0), \frac{1}{2^{j+1}}\right).$$

Clearly we have $|D| < \infty$ and $\operatorname{Per}(D) < \infty$, but D^λ must be a union of finitely many balls, so $d_{\mathcal{H}}(\partial D, \partial D^\lambda) = \infty$ for all $\lambda > 0$.

To obtain Hausdorff convergence in the noiseless case, we do not need to use density estimates; for indicatrices it is enough to combine Lemmas 4.3 and 5.2:

Proposition 5.5. *Let $f = 1_D$ with $D \subset B(0,1)$, $w = 0$, and denote by $u_{\alpha,0}$ the coresponding minimizers of* (1.1). *Then for almost every $s \in (0,1)$, the boundary ∂E^s_α of the level set $E^s_\alpha = \{u_{\alpha,0} > s\}$ converges in Hausdorff distance to $\partial E^s_0 = \partial\{f > s\}$ as $\alpha \to 0$.*

5.2 Convergence for piecewise constant data

The approach of Proposition 5.5, that is comparison using Lemma 4.3, can be extended to piecewise constant data. We assume $f \in \mathrm{BV}(\mathbb{R}^d)$ attains exactly n nonzero values $0 < f_1 < \ldots < f_n$, so that

$$f = \sum_{j=1}^{n} f_j \chi_{\Omega_j} \text{ for } \Omega_j \subset B(0,1) \text{ with } \Omega_j \cap \Omega_k = \emptyset \text{ if } j \neq k.$$

Assuming $s \in (f_k, f_{k+1})$ for $k \in \{1, \dots, n-1\}$, we want to prove that

$$\partial E_\alpha^s \xrightarrow{d_\mathcal{H}} \partial \mathcal{U}_k = \partial\{f > s\}, \text{ for } \mathcal{U}_k := \bigcup_{\ell=k+1}^{n} \Omega_\ell.$$

Considering the set $\mathcal{U}_k^{\lambda_\varepsilon}$ defined as in Lemma 5.2 to approximate $\mathcal{U}_k$ from the inside with $d_\mathcal{H}(\partial\mathcal{U}_k, \partial\mathcal{U}_k^{\lambda_\varepsilon}) < \varepsilon$, we have by minimality of E_α^s that

$$\begin{aligned}
&\mathrm{Per}(E_\alpha^s) + \frac{1}{\alpha}\sum_{\ell=1}^{n} |\Omega_\ell \cap E_\alpha^s| \psi'(s - f_\ell) \\
&\qquad \leqslant \mathrm{Per}(E_\alpha^s \cup \mathcal{U}_k^{\lambda_\varepsilon}) + \frac{1}{\alpha}\sum_{\ell=1}^{n} \left|\Omega_\ell \cap \left(E_\alpha^s \cup \mathcal{U}_k^{\lambda_\varepsilon}\right)\right| \psi'(s - f_\ell),
\end{aligned}$$

which since $\mathcal{U}_k^{\lambda_\varepsilon} \subset \mathcal{U}_k = \cup_{\ell=k+1}^n \Omega_\ell$ and since ψ' is increasing brings us to

$$\begin{aligned}
\mathrm{Per}(E_\alpha^s) &\leqslant \mathrm{Per}(E_\alpha^s \cup \mathcal{U}_k^{\lambda_\varepsilon}) + \frac{1}{\alpha}\sum_{\ell=1}^{n} \left(\left|\Omega_\ell \cap \left(E_\alpha^s \cup \mathcal{U}_k^{\lambda_\varepsilon}\right)\right| - |\Omega_\ell \cap E_\alpha^s|\right) \psi'(s - f_\ell) \\
&\leqslant \mathrm{Per}(E_\alpha^s \cup \mathcal{U}_k^{\lambda_\varepsilon}) + \frac{1}{\alpha}\sum_{\ell=k+1}^{n} \left|\Omega_\ell \cap \left(\mathcal{U}_k^{\lambda_\varepsilon} \setminus E_\alpha^s\right)\right| \psi'(s - f_\ell) \\
&\leqslant \mathrm{Per}(E_\alpha^s \cup \mathcal{U}_k^{\lambda_\varepsilon}) + \frac{1}{\alpha}\left|\mathcal{U}_k^{\lambda_\varepsilon} \setminus E_\alpha^s\right| \psi'(s - f_{k+1}).
\end{aligned}$$

Since $\mathcal{U}_k^{\lambda_\varepsilon}$ has a variational mean curvature $\kappa_{i,k}^{\lambda_\varepsilon}$ (defined as in (4.2) of Lemma 4.3) we have

$$\mathrm{Per}(\mathcal{U}_k^{\lambda_\varepsilon}) - \int_{\mathcal{U}_k^{\lambda_\varepsilon}} \kappa_{i,k}^{\lambda_\varepsilon} \leqslant \mathrm{Per}(E_\alpha^s \cap \mathcal{U}_k^{\lambda_\varepsilon}) - \int_{E_\alpha^s \cap \mathcal{U}_k^{\lambda_\varepsilon}} \kappa_{i,k}^{\lambda_\varepsilon},$$

and summing we end up with

$$-\int_{\mathcal{U}_k^{\lambda_\varepsilon} \setminus E_\alpha^s} \kappa_{i,k}^{\lambda_\varepsilon} \leqslant \frac{\psi'(s - f_{k+1})}{\alpha}\left|\mathcal{U}_k^{\lambda_\varepsilon} \setminus E_\alpha^s\right|,$$

which since $\kappa_{i,k}^{\lambda_\varepsilon} = \lambda_\varepsilon > 0$ in $B(0,1) \supset \mathcal{U}_k \supset \mathcal{U}_k^{\lambda_\varepsilon}$ and ψ has even symmetry implies $|\mathcal{U}_k^{\lambda_\varepsilon} \setminus E_\alpha^s| = 0$ when $\alpha < \psi'(f_{k+1} - s)/\lambda_\varepsilon$, which can always be attained by choosing α small enough since $s < f_{k+1}$.

We also have an outside approximation $\mathcal{U}_k^{-\lambda_\varepsilon} \supset \mathcal{U}_k$ for which $d_\mathcal{H}(\partial\mathcal{U}_k, \partial\mathcal{U}_k^{-\lambda_\varepsilon}) < \varepsilon$, possibly increasing λ_ε in comparison to the value used in the previous paragraph. By minimality of E_α^s we can write

$$\begin{aligned}
&\mathrm{Per}(E_\alpha^s) + \frac{1}{\alpha}\sum_{\ell=1}^{n} |\Omega_\ell \cap E_\alpha^s| \psi'(s - f_\ell) \\
&\qquad \leqslant \mathrm{Per}(E_\alpha^s \cap \mathcal{U}_j^{-\lambda_\varepsilon}) + \frac{1}{\alpha}\sum_{\ell=1}^{n} \left|\Omega_\ell \cap \left(E_\alpha^s \cap \mathcal{U}_k^{-\lambda_\varepsilon}\right)\right| \psi'(s - f_\ell),
\end{aligned}$$

which combined with $\cup_{\ell=k+1}^n \Omega_\ell = \mathcal{U}_k \subset \mathcal{U}_k^{-\lambda_\varepsilon}$ and monotonicity of ψ' leads to

$$\begin{aligned}
&\mathrm{Per}(E_\alpha^s) + \frac{1}{\alpha}\left|E_\alpha^s \setminus \mathcal{U}_k^{-\lambda_\varepsilon}\right| \psi'(s - f_k) \\
&\quad = \mathrm{Per}(E_\alpha^s) + \frac{1}{\alpha}\sum_{\ell=1}^{k} \left|\Omega_\ell \cap \left(E_\alpha^s \setminus \mathcal{U}_k^{-\lambda_\varepsilon}\right)\right| \psi'(s - f_k) \\
&\quad \leqslant \mathrm{Per}(E_\alpha^s) + \frac{1}{\alpha}\sum_{\ell=1}^{k} \left|\Omega_\ell \cap \left(E_\alpha^s \setminus \mathcal{U}_k^{-\lambda_\varepsilon}\right)\right| \psi'(s - f_\ell) \\
&\quad \leqslant \mathrm{Per}(E_\alpha^s \cap \mathcal{U}_k^{-\lambda_\varepsilon}).
\end{aligned}$$

Denoting by $\kappa_{o,k}^{\lambda_\varepsilon}$ the corresponding variational mean curvature for $\mathcal{U}_k^{-\lambda_\varepsilon}$, we also have

$$\mathrm{Per}(\mathcal{U}_k^{-\lambda_\varepsilon}) - \int_{\mathcal{U}_k^{-\lambda_\varepsilon}} \kappa_{o,k}^{\lambda_\varepsilon} \leqslant \mathrm{Per}(E_\alpha^s \cup \mathcal{U}_k^{-\lambda_\varepsilon}) - \int_{E_\alpha^s \cup \mathcal{U}_k^{-\lambda_\varepsilon}} \kappa_{o,k}^{\lambda_\varepsilon},$$

and summing we get

$$\frac{\psi'(s-f_k)}{\alpha}\left|E_\alpha^s \setminus \mathcal{U}_k^{-\lambda_\varepsilon}\right| \leqslant -\int_{E_\alpha^s \setminus \mathcal{U}_k^{-\lambda_\varepsilon}} \kappa_{o,k}^{\lambda_\varepsilon},$$

which since $\kappa_{o,k}^{\lambda_\varepsilon} = -\lambda_\varepsilon$ outside $\mathcal{U}_k$ implies $|E_\alpha^s \setminus \mathcal{U}_k^{-\lambda_\varepsilon}| = 0$ when $\alpha < \psi'(s-f_k)/\lambda_\varepsilon$, which is possible by reducing α because $s > f_k$.

Therefore, we end up with $\mathcal{U}_k^{\lambda_\varepsilon} \subset E_\alpha^s \subset \mathcal{U}_k^{-\lambda_\varepsilon}$ a.e. and by the triangle inequality for the Hausdorff distance (which can be easily proved from characterization (2.8), see [13, Prop. 7.3.3]) also $d_{\mathcal{H}}(\partial\mathcal{U}_k^{\lambda_\varepsilon}, \partial\mathcal{U}_k^{-\lambda_\varepsilon}) < 2\varepsilon$, providing the result as ε (and hence λ_ε and α) converge to zero.

5.3 Denoising of a generic BV function. Proof of Theorem 1.3

Let now f be a generic BV function supported in $B(0,1)$ and let u_α the minimizer of (1.1) with $w = 0$. One wants to reproduce the construction of Section 5.1 for every level set of u_α. We denote $E_0^s := \{f > s\}$ for $s > 0$ and $E_0^s := \{f < s\}$ for $s < 0$ the level sets of f and similarly $E_\alpha^s = \{u_\alpha > s\}$ for $s > 0$, $E_\alpha^s = \{u_\alpha < s\}$ for $s < 0$ the ones of u_α. The sets E_α^s minimize

$$E \mapsto \mathrm{Per}(E) + \frac{\mathrm{sign}(s)}{\alpha}\int_E \psi'(s-f). \tag{5.2}$$

In comparison with Section 5.2, since the number of attained values is not finite anymore, we cannot take a uniform approximation parameter for all level sets at once. Moreover, in contrast to the situation in Sections 4 and 5.2, the functions involved may take negative values, but by using lower level sets for $s < 0$ we ensure that these are also contained in $B(0,1)$. With this in view, we have for each s an approximation $(E_0^s)^{\lambda_{\varepsilon,s}}$ (we will write $E_s^{\lambda_{\varepsilon,s}}$ to make the notation slightly lighter) of E_0^s from inside with $d_{\mathcal{H}}(\partial E_s^{\lambda_{\varepsilon,s}}, \partial E_0^s) < \varepsilon$ and curvature $\kappa_{i,s}^{\lambda_{\varepsilon,s}}$ bounded above by $\lambda_{\varepsilon,s}$. Similarly we denote by $E_s^{-\lambda_{\varepsilon,s}}$ the approximation $(E_0^s)^{-\lambda_{\varepsilon,s}}$ of E_0^s from outside with $d_{\mathcal{H}}(\partial E_s^{-\lambda_{\varepsilon,s}}, \partial E_0^s) < \varepsilon$ and curvature $\kappa_{o,s}^{-\lambda_{\varepsilon,s}}$ bounded below by $-\lambda_{\varepsilon,s}$ on $B(0,1)$.

Lemma 5.6. *Let $\delta > 0$. Then for α small enough (depending on s, δ and ε),*

$$|E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \setminus E_\alpha^s| = 0 \text{ and } |E_\alpha^s \setminus E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}| = 0 \text{ for } s > 0,$$

and analogously

$$|E_{s-\delta}^{\lambda_{\varepsilon,s-\delta}} \setminus E_\alpha^s| = 0 \text{ and } |E_\alpha^s \setminus E_{s+\delta}^{-\lambda_{\varepsilon,s+\delta}}| = 0 \text{ for } s < 0.$$

Proof. We assume that $s > 0$, since the case $s < 0$ follows in a completely analogous way after noticing that using lower level sets induces a change of sign in (5.2) as well as a change in the direction of inclusions with respect to s.

Therefore, let $s > 0$ and $\delta > 0$ be fixed. Using the minimality of E_α^s in (5.2), one can write

$$\mathrm{Per}(E_\alpha^s) + \int_{E_\alpha^s} \frac{\psi'(s-f)}{\alpha} \leqslant \mathrm{Per}(E_\alpha^s \cup E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}) + \int_{E_\alpha^s \cup E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}} \frac{\psi'(s-f)}{\alpha}.$$

On the other hand, by definition of $\kappa_{i,s+\delta}^{\lambda_{\varepsilon,s+\delta}}$, one has

$$\mathrm{Per}(E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}) - \int_{E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}} \kappa_{i,s+\delta}^{\lambda_{\varepsilon,s+\delta}} \leqslant \mathrm{Per}(E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \cap E_\alpha^s) - \int_{E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \cap E_\alpha^s} \kappa_{i,s+\delta}^{\lambda_{\varepsilon,s+\delta}}.$$

Summing these two inequalities and using (1.6), we get

$$-\int_{E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}\setminus E_\alpha^s} \kappa_{i,s+\delta}^{\lambda_{\varepsilon,s+\delta}} \leqslant \int_{E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}\setminus E_\alpha^s} \frac{\psi'(s-f)}{\alpha}. \tag{5.3}$$

Now, recall that $E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \subset E_0^{s+\delta}$, meaning that $f \geqslant s+\delta$ on this set. Hence $s-f \leqslant -\delta$ and, since ψ' is increasing and ψ is even, (5.3) implies

$$\int_{E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}\setminus E_\alpha^s} \left(-\kappa_{i,s+\delta}^{\lambda_{\varepsilon,s+\delta}} + \frac{\psi'(\delta)}{\alpha}\right) \leqslant 0.$$

Since $\kappa_{i,s+\delta}^{\lambda_{\varepsilon,s+\delta}} \leqslant \lambda_{\varepsilon,s+\delta}$, as soon as $\alpha \leqslant \psi'(\delta)/\lambda_{\varepsilon,s+\delta}$, which is always possible since $\psi'(\delta) > 0$ by strict monotonicity, one must have $|E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \setminus E_\alpha^s| = 0$.

Similarly, the equality $|E_\alpha^s \setminus E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}| = 0$ is obtained writing

$$\operatorname{Per}(E_\alpha^s) + \int_{E_\alpha^s} \frac{\psi'(s-f)}{\alpha} \leqslant \operatorname{Per}(E_\alpha^s \cap E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}) + \int_{E_\alpha^s\cap E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}} \frac{\psi'(s-f)}{\alpha}$$

and

$$\operatorname{Per}(E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}) - \int_{E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}} \kappa_{o,s-\delta}^{-\lambda_{\varepsilon,s-\delta}} \leqslant \operatorname{Per}(E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}} \cup E_\alpha^s) - \int_{E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}\cup E_\alpha^s} \kappa_{o,s-\delta}^{-\lambda_{\varepsilon,s-\delta}}.$$

Summing these inequalities we obtain

$$\int_{E_\alpha^s\setminus E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}} \frac{\psi'(s-f)}{\alpha} \leqslant -\int_{E_\alpha^s\setminus E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}} \kappa_{o,s-\delta}^{-\lambda_{\varepsilon,s-\delta}}.$$

Now, the complement of $E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}$ contains the complement of $E_0^{s-\delta}$, therefore on this set, one has $f \leqslant s-\delta$, which implies

$$\int_{E_\alpha^s\setminus E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}} \left(\frac{\psi'(\delta)}{\alpha} + \kappa_{o,s-\delta}^{-\lambda_{\varepsilon,s-\delta}}\right) \leqslant 0,$$

which, together with $\kappa_{o,s-\delta}^{-\lambda_{\varepsilon,s-\delta}} \geqslant -\lambda_{\varepsilon,s-\delta}$ on $B(0,1)$ and (5.1) forces the expected equality as soon as $\alpha \leqslant \psi'(\delta)/\lambda_{\varepsilon,s-\delta}$. □

We can now prove the main result of this section:

Proof of Theorem 1.3. The proof strongly relies on Lemma 5.6, and once again we assume without loss of generality that $s > 0$ is fixed. Let $\eta > 0$ and $\varepsilon = \eta/2$. First, we show that one can find α_0 such that $d_{\mathcal{H}}(E_\alpha^s, E_0^s) \leqslant \eta$ for every $\alpha \leqslant \alpha_0$. Using assumption (1.3), there exists $\delta > 0$ such that

$$d_{\mathcal{H}}(E_0^{s\pm\delta}, E_0^s) \leqslant \varepsilon. \tag{5.4}$$

Then, Lemma 5.6 ensures the existence of α_0 such that for $\alpha \leqslant \alpha_0$, we have up to measure zero $E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \subset E_\alpha^s \subset E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}$. Now, we just have to note that

$$d_{\mathcal{H}}(E_\alpha^s, E_0^s) = \max\left\{\sup_{x\in E_\alpha^s} \operatorname{dist}(x, E_0^s),\ \sup_{x\in E_0^s} \operatorname{dist}(x, E_\alpha^s)\right\},$$

for which

$$E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \subset E_\alpha^s \Rightarrow \sup_{x\in E_0^s} \operatorname{dist}(x, E_\alpha^s) \leqslant \sup_{x\in E_0^s} \operatorname{dist}(x, E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}})$$

and

$$E_\alpha^s \subset E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}} \Rightarrow \sup_{x\in E_\alpha^s} \operatorname{dist}(x, E_0^s) \leqslant \sup_{x\in E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}} \operatorname{dist}(x, E_0^s).$$

Now, the triangle inequality for the Hausdorff distance, (5.4), the Hausdorff convergence of Lemma 5.2 and Theorem 2.7 imply

$$\begin{aligned}\sup_{x\in E_0^s} \operatorname{dist}(x, E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}) &\leqslant d_{\mathcal{H}}(E_0^s, E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}})\\ &\leqslant d_{\mathcal{H}}(E_0^s, E_0^{s+\delta}) + d_{\mathcal{H}}(E_0^{s+\delta}, E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}}) \leqslant 2\varepsilon = \eta,\end{aligned}$$

and

$$\begin{aligned}\sup_{x\in E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}} \operatorname{dist}(x, E_0^s) &\leqslant d_{\mathcal{H}}(E_0^s, E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}})\\ &\leqslant d_{\mathcal{H}}(E_0^s, E_0^{s-\delta}) + d_{\mathcal{H}}(E_0^{s-\delta}, E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}}) \leqslant 2\varepsilon = \eta.\end{aligned}$$

We therefore conclude

$$d_{\mathcal{H}}(E_0^s, E_\alpha^s) \leqslant 2\varepsilon = \eta \text{ for } \alpha \leqslant \alpha_0,$$

as claimed.

Similarly, using the second part of (1.3) we notice that Lemma 5.6 also provides us with the reverse inclusions for the complements

$$\mathbb{R}^d \setminus E_{s+\delta}^{\lambda_{\varepsilon,s+\delta}} \supset \mathbb{R}^d \setminus E_\alpha^s \supset \mathbb{R}^d \setminus E_{s-\delta}^{-\lambda_{\varepsilon,s-\delta}},$$

so we find, possibly reducing α_0, that also

$$d_{\mathcal{H}}(\mathbb{R}^d \setminus E_\alpha^s, \mathbb{R}^d \setminus E_0^s) \leqslant \eta \text{ for } \alpha \leqslant \alpha_0.$$

Using inequality (2.7) of Proposition 2.6, convergence in Hausdorff distance of the sets E_α^s and their complements as $\alpha \to 0$ implies convergence of the boundaries. □

Remark 5.7. We recall that $|E_\alpha^s \Delta E_0^s| \to 0$ holds for a.e. s because of the strong L^1 convergence $u_\alpha \to u$, which is implied by the support bound (5.1) and the L^1_{loc} convergence proved in Proposition 1.5.

The assumption we used for the level sets E_0^s holds for many well-behaved functions, in particular:

Proposition 5.8. *Let f be such that the level sets E_0^s satisfy uniform density estimates at some scale r_0 and constant C, independent of the level s. Then* (1.3) *holds for a.e. s.*

Proof. By the assumption and arguing as for (2.5) in Proposition 2.2, we have that at any point $x \in E_0^s$ we have the inner density estimate

$$\frac{|E_0^s \cap B(x,r)|}{|B(x,r)|} \geqslant \overline{C}, \tag{5.5}$$

for $r \leqslant \overline{r}_0 = 2r_0$ and $r_0, \overline{C} = C/2^d$ independent of x and s. Moreover, since the E_0^s are decreasing in s we may assume $\delta > 0$ in the limit, and to conclude that $d_{\mathcal{H}}(E_0^s, E^{s+\delta}) \to 0$ we just need to check

$$\sup_{x\in E_0^s} \operatorname{dist}(x, E_0^{s+\delta}) \xrightarrow{\delta\to 0} 0, \tag{5.6}$$

since the other term in the Hausdorff distance vanishes. However, if (5.6) were false, we can find $\{\delta_i\}_i$, $\rho > 0$ and $x_{\delta_i} \in E_0^s$ such that $\operatorname{dist}(x_{\delta_i}, E_0^{s+\delta_i}) > \rho$. But using (5.5) for E_0^s and x_{δ_i}, and possibly reducing ρ so that $\rho \leqslant \overline{r}_0$ that

$$|E_0^s \setminus E_0^{s+\delta_i}| \geqslant |E_0^s \cap B(x_{\delta_i}, \rho)| \geqslant \overline{C}|B(x_{\delta_i}, \rho)| = \overline{C}|B(0,\rho)|,$$

which is a contradiction with $|E_0^s \Delta E_0^{s+\delta}| \to 0$.

Moreover, we also have the outer density estimate

$$\frac{|B(x,r) \setminus E_0^s|}{|B(x,r)|} \geqslant \overline{C}, \tag{5.7}$$

again for $r \leqslant \overline{r}_0 = 2r_0$ and $\overline{C} = C/2^d$. Since the sets $\mathbb{R}^d \setminus E_0^s$ are increasing in δ, to conclude that $d_{\mathcal{H}}(\mathbb{R}^d \setminus E^s, \mathbb{R}^d \setminus E^{s+\delta}) \to 0$ we must check

$$\sup_{x \in \mathbb{R}^d \setminus E_0^{s+\delta}} \operatorname{dist}(x, \mathbb{R}^d \setminus E_0^s) \xrightarrow{\delta \to 0} 0.$$

If this does not hold, we can find $\{\delta_i\}_i$, $\rho > 0$ and $x_{\delta_i} \in \mathbb{R}^d \setminus E_0^{s+\delta_i}$ such that $\operatorname{dist}(x_{\delta_i}, \mathbb{R}^d \setminus E_0^s) > \rho$. But using (5.7) for $E_0^{s+\delta_i}$ and x_{δ_i} and with $\rho \leqslant \overline{r}_0$ we have

$$\begin{aligned}\left|\left(\mathbb{R}^d \setminus E_0^{s+\delta_i}\right) \setminus \left(\mathbb{R}^d \setminus E_0^s\right)\right| = |E_0^s \setminus E_0^{s+\delta_i}| &\geqslant |B(x_{\delta_i}, \rho) \setminus E_0^{s+\delta_i}| \\ &\geqslant \overline{C}|B(x_{\delta_i}, \rho)| = \overline{C}|B(0,\rho)|,\end{aligned}$$

leading again to a contradiction. □

The results of [15] or [23] then directly imply that this assumption is also valid when the source condition holds:

Corollary 5.9. *Let f be such that*

$$\partial\, \mathrm{TV}(f) \neq \emptyset.$$

Then the level sets E_0^s of f satisfy (1.3) *for a.e. s.*

This conclusion is however nontrivial, since it could be that (1.3) fails for a set of values of full measure:

Example 5.10. Let $\{B_i\}_{i \geqslant 0}$ be a collection of balls such that

$$B_i \subset B(0,1), \quad B_i \cap B_j = \emptyset \text{ if } i \neq j, \quad \sum_{i=0}^{\infty} \mathrm{Per}(B_i) < +\infty$$

We construct a function

$$f := \sum_{i=0}^{\infty} a_i 1_{C_i}, \text{ with } C_i = B_i + \sigma(i)\left(\frac{3}{2}, 0\right)$$

and values a_i and offset signs $\sigma(i) \in \{-1, 1\}$ that we now describe. For that, let $\mathcal{B}$ be the subset of functions in $2^{\mathbb{N}}$ with finitely many nonzero values, and let us enumerate its elements as $\{b_i\}_{i \geqslant 0}$ in an order in which the position of their last nonzero value is increasing, say

$$0,\ 1,\ 01,\ 11,\ 001,\ 011,\ 101,\ 111,\ 0001\ldots$$

The $\sigma(i)$ are defined iteratively by

$$\sigma(0) = 1,\ \sigma(1) = -1,\ \text{and } \sigma(i) = (-1)\big(\sigma \circ \iota \circ p\big)(b_i) \text{ for } i \geqslant 2,$$

where $\iota : \mathcal{B} \to \mathbb{N} \cup \{0\}$ gives the index in the enumeration described, and $p : \mathcal{B} \setminus \{0\} \to \mathcal{B}$ is the map that deletes the last nonzero element. Since $\iota \circ p\,(b_i) < i$, this process is well defined. Finally, let

$$a_i = \sum_{k=1}^{\infty} \frac{1}{2^k} b_i(k).$$

Now, for any value $s \in (0,1)$ with an infinite binary expansion and all irrationals in particular, when one denotes as s_ℓ the expansion of s up to ℓ digits (so $s_\ell = a_{i_\ell}$ for some i_ℓ) the corresponding offset signs $\sigma(i_\ell)$ alternate with ℓ, so

$$d_{\mathcal{H}}\big(\{f \geqslant s_\ell\}, \{f \geqslant s\}\big) \geqslant 1 \text{ while } s_\ell \to s \text{ as } \ell \to \infty.$$

6 Can we have uniform density estimates at fixed scale?

In Section 4 we have proved Hausdorff convergence of level sets for denoising of $1_D + w$ by using density estimates at scales that converge to 0 as $\alpha \to 0$. However, as the next example shows, often more can be expected out of the denoised solutions:

Example 6.1. Consider for $\ell_n > r_n \to 0$, with $S = (0,1)^2 \subset \mathbb{R}^2$ and $\psi(t) = t^2/2$ the situation

$$f = 1_S, w_n = 1_{B_n} \text{ with } B_n := B\big((-\ell_n, -\ell_n), r_n\big), \text{ so that } \operatorname{dist}(S, B_n) = \ell_n - r_n.$$

The nontrivial level sets of $f + w_n$ are all $S \cup B_n$, and we clearly have that $d_{\mathcal{H}}\big(\partial(S \cup B_n), \partial S\big) \to 0$, but they contain a spurious connected component not seen in the limit. Moreover we notice that if $\ell_n / r_n \to +\infty$ the sets $S \cup B_n$ fail to satisfy uniform density estimates, since in that case for any $x_n \in \partial B_n$ we have

$$\frac{\big|(S \cup B_n) \cap B(x_n, \ell_n - r_n)\big|}{|B(x_n, \ell_n - r_n)|} = \frac{|B_n|}{|B(x_n, \ell_n - r_n)|} \to 0.$$

Now, again using $\ell_n / r_n \to +\infty$ we have that $\operatorname{Per}(\operatorname{Conv}(S \cup B_n)) > \operatorname{Per}\big(S \cup B_n\big)$, so we have for the level sets of minimizers of (1.1) that $E^s_{\alpha_n, w_n} \subset S \cup B_n$. Moreover, if s and α_n are such that $(1-s)/\alpha_n < 2/r_n$ we have that $E^s_{\alpha_n, w_n} \cap B_n = \emptyset$, as can be seen from the computations done in Example 3.5. This implies, when using a linear parameter choice $\alpha_n = C\|w_n\|_{L^2} = C\sqrt{\pi}\, r_n$, that whenever $C > 1/(2\sqrt{\pi}) = 1/\Theta_2$ the effect of w_n is not seen in the solution. In that case it is easy to see that level sets admit uniform density estimates at fixed scale, since then $E^s_{\alpha_n, w_n} = S^{s/\alpha_n}$ where the notation S^{s/α_n} is understood in the sense of Definition 3.12, which are explicitly computed for the case of the square S in Proposition 3.17 and Remark 3.18.

Uniform density estimates along the sequence of level sets of minimizers provide not only Hausdorff convergence of the boundaries of level sets, but also prevent the appearance of spurious structures smaller than a certain scale. For general sets of finite perimeter as in Section 4, since the limit is not regular, we cannot in general expect uniform density for the level sets approaching it.

On the opposite side, if we knew that $\partial\,\mathrm{TV}(1_D) \neq \emptyset$, uniform density estimates for the level sets are implied by the results of [15] for $d = 2$ and [23] for $d > 2$. However, this source condition excludes large classes of sets D where we would expect the level sets of minimizers to also satisfy uniform density estimates, in particular sets D with general Lipschitz boundary and satisfying density estimates themselves, like the square in the example above. The question then arises of how to derive these estimates for the solutions in such cases. We are not able to give a complere answer but we collect some observations, specialized to the two-dimensional case and $\psi(t) = t^2/2$.

Examining the proof of the density estimates in Proposition B.1, to have a uniform scale at which the estimates hold it would be sufficient to have an inequality bounding the the integral of

κ_D on small sets by a quantity strictly less than their perimeter. In particular, since connected (indecomposable) components of D inherit the curvature of the whole set, such an inequality implies that no arbitrarily small components can be present, a property which (by the dual stability of Proposition A.2) is also true for the denoised level sets with an adequate parameter choice. We formulate this property as the following assumption:

Assumption 6.2. There is a constant $0 < \xi_D < 1$ and a scale $r_0 > 0$ such that for any $A \subset \mathbb{R}^2$ admitting a variational mean curvature in $L^2(\mathbb{R}^2)$, $x \in \mathbb{R}^2$ and $0 < r \leqslant r_0$ the following inequalities hold:

$$\begin{aligned} \int_{A \cap B(x,r)} \kappa_D^+ &\leqslant \xi_D \operatorname{Per}(B(x,r) \cap A), \text{ and} \\ \int_{A \cap B(x,r)} \kappa_D^- &\leqslant \xi_D \operatorname{Per}(B(x,r) \cap A), \end{aligned} \tag{6.1}$$

where $\kappa_D^+ = \max(\kappa_D, 0)$ and $\kappa_D^- = -\min(\kappa_D, 0)$.

Let us check that Assumption 6.2 holds for the square.

Example 6.3. Denote the unit square by $S \subset \mathbb{R}^2$ and the test set directly by $E := A \cap B(x,r)$, since we will not use its form or regularity explicitly. By definition $\kappa_S \geqslant 0$ in S, and since S is convex, Proposition 3.10 implies that $\kappa_S(x) = -\lambda_g g(x)$ for $x \in \mathbb{R}^2 \setminus S$. Let us start with the first inequality of (6.1). It is enough to prove that there is $\xi_S < 1$

$$\int_E \kappa_S \leqslant \xi_S \operatorname{Per}(E),$$

for all $E \subset S$ with $|E|$ small and $\operatorname{diam}(E) < 1/2$. Assuming $|E| < |S \setminus C_S|/4$ with C_S the Cheeger set of S, and in view of the optimal curvature of the square (3.17) we have that

$$\int_E \kappa_S \leqslant \int_{(S \setminus S^\Lambda) \cap Q^1} \kappa_S,$$

for S^Λ again in the sense of Definition 3.12, with some Λ such that $|(S \setminus S^\Lambda) \cap Q^1| = |E|$ and Q^1 the lower left quadrant. Now, for each $\lambda > 0$

$$|(S \setminus S^\lambda) \cap Q^1| = \frac{1}{\lambda^2} - \frac{\pi}{4\lambda^2} = \frac{4-\pi}{4\lambda^2}$$

Therefore

$$\begin{aligned} \int_{(S \setminus S^\Lambda) \cap Q^1} \kappa_S &= \Lambda |\{\kappa_S > \Lambda\}| + \int_\Lambda^\infty |\{\kappa_S > \lambda\}| \,\mathrm{d}\lambda \\ &= \Lambda |(S \setminus S^\Lambda) \cap Q^1| + \int_\Lambda^\infty |(S \setminus S^\lambda) \cap Q^1| \,\mathrm{d}\lambda \\ &= \frac{4-\pi}{2\Lambda} \end{aligned} \tag{6.2}$$

and on the other hand, by the isoperimetric inequality

$$\operatorname{Per}(E) \geqslant 2\sqrt{\pi |E|} = 2\sqrt{\pi \frac{4-\pi}{4\Lambda^2}} = \frac{\sqrt{\pi(4-\pi)}}{\Lambda}, \tag{6.3}$$

and we get the first inequality of (6.1) with any ξ_S such that

$$1 > \xi_S > \frac{4-\pi}{2\sqrt{\pi(4-\pi)}} \approx 0.26.$$

To prove the second part of (6.1) we notice that whenever

$$r \leqslant \frac{2\sqrt{\pi}}{\xi_S\,\lambda_g\,|B(0,1)|^{1/2}} \tag{6.4}$$

for any F with $\operatorname{diam}(F) \leqslant r$ we can write

$$\begin{aligned}\frac{1}{\operatorname{Per}(F)}\int_F \kappa_S^- &= \frac{\lambda_g}{\operatorname{Per}(F)}\int_{F\setminus S} g \leqslant \frac{\lambda_g}{\operatorname{Per}(F)}\int_F g\\ &\leqslant \frac{\lambda_g|F|}{\operatorname{Per}(F)} \leqslant \frac{\lambda_g|F|}{2\sqrt{\pi}|F|^{1/2}} = \frac{\lambda_g|F|^{1/2}}{2\sqrt{\pi}} \leqslant \frac{\lambda_g\, r\,|B(0,1)|^{1/2}}{2\sqrt{\pi}} \leqslant \xi_S,\end{aligned}$$

where we have used $g \leqslant 1$, the isoperimetric inequality and (6.4).

Remark 6.4. For a convex polygon P, one could try to repeat the proof above around a vertex with angle 2θ, and $\lambda > 0$ large enough so that the contact points of a circle of radius $1/\lambda$ lie on the two edges the vertex belongs to. The analogous formulas to (6.2) and (6.3) are then

$$\int_E \kappa_P \leqslant \frac{2}{\Lambda}\left(\frac{1}{\tan\theta}+\theta-\frac{\pi}{2}\right) \text{ and } \operatorname{Per}(E) \geqslant 2\sqrt{\pi|E|} = \frac{2\sqrt{\pi}}{\Lambda}\sqrt{\left(\frac{1}{\tan\theta}+\theta-\frac{\pi}{2}\right)}.$$

However, the quotient of these two quantities is below 1 only for θ larger than ≈ 0.219. It is very likely this is a problem of the proof method, since the isoperimetric estimate used is far from sharp, and that in fact Assumption 6.2 holds for any polygon P. Convexity is likely also not required, since a polygon cannot have arbitrarily thin necks, so by the results cited in Remark 3.18 inclusions of balls characterize P^λ for λ large enough, and we can also determine $P^{-\lambda}$ analogously.

Now we check that Assumption 6.2 indeed implies uniform density estimates at a fixed scale. Our scheme will be to work first with the solutions corresponding to noiseless data, and then comparing them with the noisy ones using Proposition 4.1.

Theorem 6.5. *Let $w \in L^2(\mathbb{R}^2)$ and α satisfying the parameter choice*

$$\frac{\|w\|_{L^2}}{\alpha} \leqslant \eta < 2\sqrt{\pi}\,(1-\xi_D)\,, \tag{6.5}$$

and let $u_{\alpha,w}$ denote the corresponding minimizers of (1.1) *with $f = 1_D$, where D satisfies Assumption 6.2 with constant ξ_D and scale r_0. Then there is $C_0 \in (0,1)$ such that for a.e. s, the level sets*

$$E^s_{\alpha,w} := \{u_{\alpha,w} > s\} \text{ if } s>0 \text{ and } E^s_{\alpha,w} := \{u_{\alpha,w} < s\} \text{ if } s<0$$

satisfy uniform density estimates at scale r_0 and with constant C_0, that is

$$\frac{|E^s_{\alpha,w}\cap B(x,r)|}{|B(x,r)|} \geqslant C_0, \text{ and } \frac{|B(x,r)\setminus E^s_{\alpha,w}|}{|B(x,r)|} \geqslant C_0$$

for all $x \in \partial E^s_{\alpha,w}$ and $0 < r \leqslant r_0$.

Proof. Since we aim to prove both inner and outer density estimates, we can assume without loss of generality that $s > 0$, so that by Proposition 1.6 $E^s_{\alpha,w}$ admits the variational curvature $v_{\alpha,w} = 1_D + w - u_{\alpha,w}$. For such x and r we have by the Cauchy-Schwartz inequality, Proposition 4.1 and since $\operatorname{sign}(v_{\alpha,0}) = \operatorname{sign}(\kappa_D)$ that

$$\begin{aligned}\int_{E^s_{\alpha,w}\cap B(x,r)} v_{\alpha,w} &\leqslant |E^s_{\alpha,w}\cap B(x,r)|^{1/2}\|v_{\alpha,w}-v_{\alpha,0}\|_{L^2(\mathbb{R}^2)} + \int_{E^s_{\alpha,w}\cap B(x,r)} v_{\alpha,0}\\ &\leqslant |E^s_{\alpha,w}\cap B(x,r)|^{1/2}\|v_{\alpha,w}-v_{\alpha,0}\|_{L^2(\mathbb{R}^2)} + \int_{E^s_{\alpha,w}\cap B(x,r)} \kappa_D^+\\ &\leqslant |E^s_{\alpha,w}\cap B(x,r)|^{1/2}\|v_{\alpha,w}-v_{\alpha,0}\|_{L^2(\mathbb{R}^2)} + \xi_D\operatorname{Per}(E^s_{\alpha,w}\cap B(x,r))\\ &\leqslant |E^s_{\alpha,w}\cap B(x,r)|^{1/2}\eta + \xi_D\operatorname{Per}(E^s_{\alpha,w}\cap B(x,r))\end{aligned} \tag{6.6}$$

where we have used the first inequality in (6.1) for the penultimate step and for the last step the parameter choice (6.5) combined with the dual stability of Proposition A.2, since in this case $\sigma_{\psi^*} = \mathrm{Id}$. Plugging this in formula (B.3) of Appendix B, we obtain

$$\mathrm{Per}(E^s_{\alpha,w} \cap B(x,r))\,(1-\xi_D) - |E^s_{\alpha,w} \cap B(x,r)|^{1/2}\eta \leqslant 2\,\mathrm{Per}\left(B(x,r); (E^s_{\alpha,w})^{(1)}\right).$$

Using now the isoperimetric inequality, we get

$$|E^s_{\alpha,w} \cap B(x,r)|^{1/2}\left(2\sqrt{\pi}\,(1-\xi_D) - \eta\right) \leqslant 2\,\mathrm{Per}\left(B(x,r); (E^s_{\alpha,w})^{(1)}\right).$$

We can derive the inner density estimate $|E^s_{\alpha,w} \cap B(x,r)| \geqslant C_0|B(x,r)|$ with C_0 depending on η by integrating this differential inequality up to r_0.

For the outer density, one proceeds in an analogous fashion with the complements $\mathbb{R}^2 \setminus E^s_{\alpha,w}$, which switches the sign of the curvature to $-v_{\alpha,w}$ and makes κ_D^- play a role in (6.6) through the second inequality in (6.1). □

We notice that in the situation of Theorem 6.5, the convergence $d_{\mathcal{H}}(\partial E^s_{\alpha,w}, \partial D) \to 0$ for $0 < s < 1$ follows then by Proposition 4.5 and Theorem 2.8. Moreover:

Corollary 6.6. *With α_n, w_n and D satisfying the assumptions of Theorem 6.5 and for either $s > 1$ or $s < 0$ we additionally have*

$$\limsup_{n\to\infty} \partial E^s_{\alpha_n,w_n} = \emptyset,$$

where $\limsup \partial E^s_{\alpha_n,w_n}$ is defined to be [31, Def. 4.1] the set of all limits of subsequences of points in $\partial E^s_{\alpha_n,w_n}$.

Proof. If $s < 0$, or $s > 1$, by the convergence $u_{\alpha_n,w_n} \to 1_D$ in L^q we have $|E^s_{\alpha_n,w_n}| \to 0$. Assume for a contradiction that we had $x \in \limsup \partial E^s_{\alpha_n,w_n}$. Then we have a not relabelled subsequence and $x_{\alpha_n} \in \partial E^s_{\alpha_n,w_n}$ such that $x_{\alpha_n} \to x$. Now, as in the proof of Theorem 1.2, by using the inner density estimate, which now holds with constant C_0 and for scales $r \leqslant r_0$ uniformly both in α and the chosen points we get

$$|E^s_{\alpha_n,w_n}| \geqslant |B(x_{\alpha_n}, r_0) \cap E^s_{\alpha_n,w_n}| \geqslant C|B(0,r_0)|,$$

which contradicts $|E^s_{\alpha_n,w_n}| \to 0$. □

Observe that in the setting of Theorem 1.2 where the density estimates depend on the distance to ∂D, the proof we have given for this corollary fails. Indeed, with such density estimates we could only get that $\mathrm{dist}(x, \partial D) \leqslant r$ for all $r > 0$ small enough and $x \in \limsup_\alpha \partial E^s_{\alpha,w}$, or

$$\limsup_{\alpha} \partial E^s_{\alpha,w} \subset \partial D,$$

which for $s < 0$ or $s > 1$ is not a satisfactory conclusion. We conclude with some further observations about when inequality (6.1) could be expected to hold.

Remark 6.7. Although it is naturally of L^2 scaling for κ_D, Assumption 6.2 can be formulated in more general spaces with this scaling, giving some hope that it could hold for sets with Lipschitz boundary. For example, we would have (6.1) with $\xi_D < 1$ if we had $\|\kappa_D\|_{L^{2,w}} < 2\sqrt{\pi}$ for the weak L^2 norm. In fact, in the notation of [28, Def. 3.3], it is also enough to have $\|\kappa_D\|_{S(\mathbb{R}^2)} < 1$, and in [28, Thm. 3.7] it is shown that $S(\mathbb{R}^2)$ in fact coincides with the Morrey space $L^{1,1}$ (with different norms, a priori). The quantitative bounds are necessary, since the example in [28, Thm. 8.5] provides a set D without density estimates, whose curvature κ_D belongs to $L^{1,1}$.

Remark 6.8. We have by definition that

$$\|\kappa_D\|_{L^{2,w}} = \sup_\lambda \lambda \left|\{|\kappa_D| \geqslant \lambda\}\right|^{1/2}.$$

Now, if D is convex the construction of κ_D implies that $\kappa_D \geqslant h(D)$ in D, for

$$h(D) = \inf_{A\subset D} \frac{\operatorname{Per}(A)}{|A|}$$

the Cheeger constant of D, attained by the unique Cheeger set C_D. So with the isoperimetric inequality and that $C_D \subset D$ we have

$$h(D)\left|\{\kappa_D \geqslant h(D)\}\right|^{1/2} = h(D)|D|^{1/2} = \frac{\operatorname{Per}(C_D)}{|C_D|}|D|^{1/2} \geqslant 2\sqrt{\pi}\,\frac{|D|^{1/2}}{|C_D|^{1/2}} \geqslant 2\sqrt{\pi},$$

with equality if and only if D is a circle. This means that to use the language of weak norms, it would be necessary to restrict/truncate to small scales or large curvatures.

Remark 6.9. If D is convex we have that $u_{\alpha,0} = (1-\alpha\kappa_D)^+ 1_D$ (see [3, Prop. 2.2] or [14, Thm. 6]). This implies that one can construct a vector field $z \in L^\infty(\mathbb{R}^d)$ with $|z| \leqslant 1$ and divergence κ_D, and which coincides with the normal to D on ∂D. The Green formula would provide us with (6.1) if z was for example continuous in $\mathring{D}$, since then cancellations of the flux would appear.

Remark 6.10. An inequality resembling (6.1) in Assumption 6.2 also appears in some works dealing with prescribed mean curvature surfaces in periodic media, like [17] and [22]. In that case, the setting is that of a bounded cell Q and a potential $\tilde{g} \in L^d(Q)$ satisfying $\int_E \tilde{g} \leqslant (1-\delta)\operatorname{Per}(E;Q)$ for fixed $\delta \in (0,1)$ and all $E \subset Q$ is used. In fact, it is proved in [17, Prop. 4.1] using the results of [11] that in this case there is a continuous vector field $z \in C(Q;\mathbb{R}^d)$ with $|z| \leqslant 1$ for which $\operatorname{div} z = \tilde{g}$, which is also incompatible with $\tilde{g}$ being the variational mean curvature of a nonsmooth set D, since in that case we would expect that $z\big|_{\partial D} = \nu_D$ [16, Thm. 3.7]. This, after Remark 6.8, is yet more evidence that (6.1) can only be expected for small r.

A Dual problem and its stability

Proposition A.1. *Assume that $f, w \in L^{d/(d-1)}(\mathbb{R}^d)$. The Fenchel dual of* (1.1) *reads*

$$\sup_{v\in\partial TV(0)} \int v\,(f+w) - \frac{1}{\alpha}\int \psi^*(-\alpha v), \tag{A.1}$$

which has a unique maximizer $v_{\alpha,w}$ that satisfies the optimality condition

$$v_{\alpha,w} = -\frac{1}{\alpha}\psi'(u_{\alpha,w} - f - w) \in \partial\,\mathrm{TV}(u_{\alpha,w}), \tag{A.2}$$

where $u_{\alpha,w}$ is the unique minimizer of (1.1)*.*

Proof. Existence follows strong duality in Banach spaces [10, Thm. 4.4.3, p. 136] applied to the space $L^{d/(d-1)}(\mathbb{R}^d)$ with functions $\mathrm{TV}(\cdot)$, $G(\cdot) = \frac{1}{\alpha}\int_{\mathbb{R}^d}\psi(\cdot - f - w)$ and the identity operator, while uniqueness is a consequence of strict convexity of ψ^*.

To apply strong duality we need a qualification condition. Since G is up to a shift and a constant factor the functional $\int\psi$ and $f, w \in L^{d/(d-1)}$, it is enough to check that $u \mapsto \int\psi(u)$ is continuous on $L^{d/(d-1)}$, so that G is in particular continuous at 0. By [19, Prop. IV.1.1] continuity holds as soon as we can guarantee that $\psi \circ u \in L^1$ for every $u \in L^{d/(d-1)}$, which is directly implied by the inequality $|\psi(t)| \leqslant C|t|^{d/(d-1)}$ included in Assumption (A).

The Fenchel conjugate of G reads

$$G^*(v) = -\int v\,(f+w) + \frac{1}{\alpha}\int \psi^*(\alpha v).$$

As already computed in [24, Thm. 1], the conjugate of the total variation is $\mathrm{TV}^* = \chi_{\partial \mathrm{TV}(0)}$, the indicator function of the convex set $\partial\,\mathrm{TV}(0)$. In this duality setting, we have [19, Eqs. I.(4.24), I.(4.25)] the optimality conditions $v_{\alpha,w} \in \partial\,\mathrm{TV}(u_{\alpha,w})$ and $-v_{\alpha,w} \in \partial G(u_{\alpha,w})$ as well, which are exactly (A.2). □

Now we use assumption (A) to arrive at a stability result for the maximizers $v_{\alpha,w}$ of (A.1).

Proposition A.2. *We have the stability estimate*

$$\|v_{\alpha,w} - v_{\alpha,0}\|_{L^d(\mathbb{R}^d)} \leqslant \sigma_\psi\left(\frac{\|w\|_{L^{d/(d-1)}}}{\alpha}\right), \tag{A.3}$$

where σ_ψ is the inverse of the function $t \mapsto m_{\psi^*}(t)/t$, with m_{ψ^*} the largest modulus of uniform convexity for ψ^*.

Proof. The computations are analogous to the ones in [23, Prop. 3.5, Prop. 3.6], in turn originating from the methods in [1, 2], adapted to the slightly different framework here. The main idea is, for the weak-* closed and convex set $K := \partial\,\mathrm{TV}(0) \subset L^d$, to define a generalized projection $\pi : L^{d/(d-1)} \to K$ by

$$\pi(u) := \operatorname*{arg\,min}_{v\in K} \int \psi(u) - vu + \psi^*(v), \tag{A.4}$$

which is single valued by strict convexity of ψ^* and then noticing that the dual variable is obtained as

$$v_{\alpha,w} = \frac{1}{\alpha}\pi\left(f+w\right), \tag{A.5}$$

where we have used that ψ being even implies that ψ^* is also even.

Now, given any $u \in L^{d/(d-1)}$ and $v \in L^d$, differentiating the argument of the right hand side of (A.4) in direction $\pi(u) - v$ and using minimality at $\pi(u)$ we end up with

$$\int (v - \pi(u))\big(u - (\psi^*)' \circ \pi(u)\big) \geqslant 0. \tag{A.6}$$

Moreover, we have the uniform monotonicity inequality (for a proof, see [23, Lem. 1.2])

$$\begin{aligned}\int (\pi(u_1) - \pi(u_2))\big((\psi^*)' \circ \pi(u_1) - (\psi^*)' \circ \pi(u_2)\big)\\ \geqslant 2m_{\psi^*}\big(\|\pi(u_1) - \pi(u_2)\|_{L^{d/(d-1)}}\big),\end{aligned} \tag{A.7}$$

for whose left hand side we have, using (A.6) twice and Hölder inequality, that

$$\begin{aligned}&\int (\pi(u_1) - \pi(u_2))\big((\psi^*)' \circ \pi(u_1) - (\psi^*)' \circ \pi(u_2)\big)\\ &\leqslant \int (\pi(u_1) - \pi(u_2))(u_1 - u_2)\\ &\quad + \int (\pi(u_1) - \pi(u_2))\big((\psi^*)' \circ \pi(u_1) - u_1\big)\\ &\quad - \int (\pi(u_1) - \pi(u_2))\big((\psi^*)' \circ \pi(u_2) - u_2\big)\\ &\leqslant \int (\pi(u_1) - \pi(u_2))(u_1 - u_2)\\ &\leqslant \|\pi(u_1) - \pi(u_2)\|_{L^d}\,\|u_1 - u_2\|_{L^{d/(d-1)}}.\end{aligned} \tag{A.8}$$

The combination of (A.8), (A.7) and (A.5) allows us then to conclude (A.3). As already noted in [23], the property (see [9, Fact 5.3.16]) $m_{\psi^*}(ct) > c^2 m_{\psi^*}(t)$ for all $c > 1$ implies that the function $t \mapsto m_{\psi^*}(t)/t$ is strictly increasing, so its inverse is well defined. □

B Density estimates for denoised level sets

Proposition B.1. *Let $K \subset \mathbb{R}^d$ be a bounded set, assume that*

$$\|v_{\alpha,w} - v_{\alpha,0}\|_{L^d(\mathbb{R}^d)} \leqslant C_0 < \Theta_d, \tag{B.1}$$

which is possible by Proposition A.2. Furthermore, assume that for each $\varepsilon > 0$ there is $r_{K,\varepsilon} > 0$ such that for all $x \in \mathbb{R}^d \setminus K$ and all α we have the equi-integrability estimate

$$\int_{B(x,r_{K,\varepsilon})} |v_{\alpha,0}|^d \leqslant \varepsilon. \tag{B.2}$$

Then the level sets $E^s_{\alpha,w}$ satisfy uniform density estimates at some scale r_K and with constant C_K outside K, that is

$$\frac{|E^s_{\alpha,w} \cap B(x,r)|}{|B(x,r)|} \geqslant C_K, \text{ and } \frac{|B(x,r) \setminus E^s_{\alpha,w}|}{|B(x,r)|} \geqslant C_K$$

for all $x \in \partial E^s_{\alpha,w} \setminus K$ and $0 < r \leqslant r_K$.

Proof. Let $x \in \partial E^s_{\alpha,w} \setminus K$. We start from the formula

$$\operatorname{Per}(E^s_{\alpha,w} \cap B(x,r)) - \int_{E^s_{\alpha,w} \cap B(x,r)} v_{\alpha,w} \leqslant 2 \operatorname{Per}\left(B(x,r); (E^s_{\alpha,w})^{(1)}\right), \tag{B.3}$$

which can be seen to hold (for a proof, see for example [24, Lem. 8]) for almost every $r > 0$ by repeated application of the precise formulas for the perimeter of an intersection [30, Thm. 16.3], and noticing that substantial tangential contact can only happen on a set of radii of measure zero.

On the other hand we have, thanks to the Hölder inequality, the condition (B.1) and local equiintegrability (B.2) that for $0 < r \leqslant r_{K,\varepsilon}$

$$\begin{aligned}\int_{E^s_{\alpha,w} \cap B(x,r)} v_{\alpha,w} &\leqslant |E^s_{\alpha,w} \cap B(x,r)|^{(d-1)/d} \|v_{\alpha,w} - v_{\alpha,0}\|_{L^d(\mathbb{R}^d)} + \int_{E^s_{\alpha,w} \cap B(x,r)} |v_{\alpha,0}| \\ &\leqslant |E^s_{\alpha,w} \cap B(x,r)|^{(d-1)/d}\, C_0 + \int_{E^s_{\alpha,w} \cap B(x,r)} |v_{\alpha,0}| \\ &\leqslant |E^s_{\alpha,w} \cap B(x,r)|^{(d-1)/d}\, (C_0 + \varepsilon)\end{aligned}$$

Plugging this in (B.3), we obtain

$$\operatorname{Per}(E^s_{\alpha,w} \cap B(x,r)) - |E^s_{\alpha,w} \cap B(x,r)|^{(d-1)/d}(C_0 + \varepsilon) \leqslant 2 \operatorname{Per}\left(B(x,r); (E^s_{\alpha,w})^{(1)}\right).$$

Using now the isoperimetric inequality, we get

$$|E^s_{\alpha,w} \cap B(x,r)|^{(d-1)/d}\, (\Theta_d - C_0 - \varepsilon) \leqslant 2 \operatorname{Per}\left(B(x,r); (E^s_{\alpha,w})^{(1)}\right). \tag{B.4}$$

Taking some fixed $\varepsilon_0 < \Theta_d - C_0$, and since for a.e. $r > 0$

$$\operatorname{Per}\left(B(x,r); (E^s_{\alpha,w})^{(1)}\right) = \mathcal{H}^{d-1}\left(B(x,r) \cap (E^s_{\alpha,w})^{(1)}\right) = \left.\frac{d}{dt}\right|_{t=r} \left|E^s_{\alpha,w} \cap B(x,t)\right|,$$

we can derive the inner density estimate $|E^s_{\alpha,w} \cap B(x,r)| \geqslant C_K |B(x,r)|$ with C_K depending on ε_0 by integrating the differential inequality (B.4) up to $r_K := r_{K,\varepsilon_0}$.

The outer density estimate follows analogously by considering the complement $\mathbb{R}^d \setminus E^s_{\alpha,w}$, which admits the variational mean curvature $-v_{\alpha,w}$. □

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