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Abstract

In this paper we introduce a $C^1$ spline space over mixed meshes composed of triangles and quadrilaterals. The space combines the Argyris triangle, cf. [1], with the $C^1$ quadrilateral element introduced in [8, 31] for polynomial degrees $p \geq 5$. The space is assumed to be $C^2$ at all vertices and $C^1$ across edges, and the splines are uniquely determined by $C^2$-data at the vertices, values and normal derivatives at chosen points on the edges, and values at some additional points in the interior of the elements.

The motivation for combining the Argyris triangle element with a recent $C^1$ quadrilateral construction, inspired by isogeometric analysis, is two-fold: On one hand, the ability to connect triangle and quadrilateral finite elements in a $C^1$ fashion is non-trivial and of theoretical interest. We provide not only approximation error bounds but also numerical tests verifying the results. On the other hand, the construction facilitates the meshing process by allowing more flexibility while remaining $C^1$ everywhere. This is for instance relevant when trimming of tensor-product B-splines is performed.

In the presented construction we assume to have (bi)linear element mappings and piecewise polynomial function spaces of arbitrary degree $p \geq 5$. The basis is simple to implement and the obtained results are optimal with respect to the mesh size for $L^\infty$, $L^2$ as well as Sobolev norms $H^1$ and $H^2$.

Keywords: $C^1$ discretization, Argyris triangle, $C^1$ quadrilateral element, mixed triangle and quadrilateral mesh

1. Introduction

Isogeometric analysis (IGA) was introduced in [21] to apply numerical analysis directly to the B-spline or NURBS representation of CAD models. Since IGA is based on B-spline representations, it is capable to generate smooth discretizations, which are used to discretize higher order PDEs, e.g. [51]. Applications which need higher order smoothness (at least $C^1$) are, e.g. Kirchhoff–Love shell formulations [33, 34], the Navier–Stokes–Korteweg equation [16], or the Cahn–Hilliard equation [15].

CAD models are composed of several B-spline or NURBS patches, which are smooth in the interior. To obtain higher order smoothness for complicated geometries one has to additionally impose smoothness across patch interfaces. One possibility is a manifold-like setting, which merges two patches across an interface in a $C^k$ fashion, creating overlapping charts, and remains $C^0$ near extraordinary vertices, where several patches meet, see [9, 46]. Such approaches need to be modified to increase the smoothness at extraordinary points. One may introduce singularities and define a suitable, locally modified space [39, 40, 53].

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These approaches are strongly related to constructions based on subdivision surfaces [43], such as [45, 55] based on Catmull–Clark subdivision over quadrilateral meshes or [12, 24] based on Loop subdivision over triangle meshes.

Smooth function spaces over general quadrilateral meshes for surface design predate IGA, such as [17, 41, 44]. These constructions rely on the concept of geometric continuity, that is, in the case of $G^1$, surfaces that are tangent continuous without having a $C^1$ parametrization. Also before IGA (or around the same time), several approaches were developed for numerical analysis of higher order problems over quad meshes, such as the Bogner–Fox–Schmit element [6], the elements developed by Brenner and Sung [8] for $p \geq 6$, or the constructions in [3, 37]. Recently, a family of $C^1$ quadrilateral finite elements was described in [31].

Due to the increased interest in IGA, the connection between $C^1$ isoparametric functions and $G^1$ surfaces was (re)discovered in the IGA context by [13, 18, 32]. As a consequence, $C^1$ isogeometric spaces over quadrilateral meshes or multi-patch B-spline configurations were studied extensively, see also [4, 5, 10, 25, 27–30, 38].

Already before the introduction of IGA, $C^1$ splines over triangles were introduced and also used for numerical analysis. The first approach to obtain $C^1$ discretizations for analysis was the Argyris finite element [1]. Other constructions for splines over triangulations followed [14, 19, 54], see also the book [36]. Recently, also due to IGA, there is more interest in splines over triangulations [23, 26, 49, 50].

There are some straightforward connections between splines over triangulations and splines over quadrangulations. Triangular and tensor-product Bézier patches are two alternative generalizations of Bézier curves and are related in the following way. Triangular patches can be interpreted as singular tensor-product patches, where one edge is collapsed to a single point [20, 52].

In this paper we combine the $C^1$ constructions for triangles and quadrilaterals for degrees $p \geq 5$. We extend the idea of Hermite interpolation with Argyris triangle elements from [22] as well as the framework from [31] for quad meshes to mixed quad-triangle meshes. The approach is similar to the most general setting in [38]. However, we focus on having given a physical mesh instead of a topological one and we provide a construction which is local to the elements.

Such a mixed triangle and quadrilateral mesh can be relevant for many applications. Fluid-structure interaction problems are often solved by combining two different PDE formulations, discretized differently, within a single setup [2]. Mixed meshes may also arise from trimming [35, 47]. Most CAD software relies on trimming procedures to perform Boolean operations. In that case a B-spline patch is modified by modifying its parameter domain, which is usually a box. The parameter domain is then given as a part of the full box, where some parts are cut out by so-called trimming curves. These trimming curves divide the Bézier (polynomial) elements of the spline patch into inner, outer and cut elements, where the outer elements and outer parts of cut elements are discarded. In that case the resulting mesh is composed mostly of quadrilaterals, with (in general) triangles, quadrilaterals and pentagons as cut elements near the trimming boundary. For practical purposes (e.g. to simplify quadrature) the cut elements are often split into triangles. Thus, this procedure results in a mixed triangle and quadrilateral mesh (see Figure 1). We also want to point out [48], where the authors combine tensor-product volumes as an outer layer with tetrahedral Bézier elements inside the domain, with an extra layer of pyramidal elements in between.

Figure 1: An example of a trimmed patch, where the trimming curve is prescribed in the parameter domain. Note that some of the cut elements are triangles, some are quadrilaterals and some are pentagons.
The paper is organized as follows. In Section 2 we first introduce the notation and mesh configuration which will be used throughout the paper. After that we define the $C^1$ space over mixed triangle and quadrilateral meshes. Section 3 is devoted to the investigation of continuity conditions across interfaces that need to be considered in the construction of splines from such space. This allows us to analyze properties of the space in Section 4 by introducing and studying a projection operator onto the space. The operator is defined via an interpolation problem and serves us to show that the space is of optimal approximation order. The latter is also verified numerically in Section 5. We present the conclusions and possible extensions in Section 6.

2. The Argyris-like space $\mathcal{A}_p$

The aim of this section is to introduce necessary notation to describe a domain partition consisting of triangles and quadrilaterals. Over such a mixed mesh we then define a spline space, which can be regarded as an extension of the well-known Argyris space.

2.1. Mixed triangle and quadrilateral meshes

We consider an open domain $\Omega \subset \mathbb{R}^2$, whose closure $\overline{\Omega}$ is the disjoint union of triangular or quadrilateral elements $\Omega^{(i)}$, $i \in \mathcal{I}_\Omega$, edges $\mathcal{E}^{(i)}$, $i \in \mathcal{I}_\mathcal{E}$, and vertices $V^{(i)}$, $i \in \mathcal{I}_\mathcal{V}$, that is

$$\overline{\Omega} = \left( \bigcup_{i \in \mathcal{I}_\Omega} \Omega^{(i)} \right) \cup \left( \bigcup_{i \in \mathcal{I}_\mathcal{E}} \mathcal{E}^{(i)} \right) \cup \left( \bigcup_{i \in \mathcal{I}_\mathcal{V}} V^{(i)} \right). \tag{1}$$

Each vertex $V^{(i)}$ is a point in the plane,

$$V^{(i)} \in \mathbb{R}^2, \quad \text{for all } i \in \mathcal{I}_\mathcal{V},$$

and each edge is given by two vertices, i.e.

$$\mathcal{E}^{(i)} = \mathcal{E}(V^{(i_1)}, V^{(i_2)}) = \left\{ (1-v)V^{(i_1)} + vV^{(i_2)} : v \in [0,1] \right\},$$

for all $i \in \mathcal{I}_\mathcal{E}$, where $i_1, i_2 \in \mathcal{I}_\mathcal{V}$. Each element $\Omega^{(i)}$, with $i \in \mathcal{I}_\Omega = \mathcal{I}_\triangle \cup \mathcal{I}_\square$, is either a triangle or a quadrilateral, where $\mathcal{I}_\triangle$ and $\mathcal{I}_\square$ are the sets of indices of the triangle and quadrilateral elements $\Omega^{(i)}$, respectively. We assume that elements are always open sets and use the notation $\Omega^{(i)} = \mathcal{T}(V^{(i_1)}, V^{(i_2)}, V^{(i_3)}, V^{(i_4)})$ for triangle elements and $\Omega^{(i)} = \mathcal{Q}(V^{(i_1)}, V^{(i_2)}, V^{(i_3)}, V^{(i_4)})$ for quadrilateral elements. For all $i \in \mathcal{I}_\triangle$, we have

$$\overline{\Omega} = \{(1-u-v)V^{(i_1)} + uV^{(i_2)} + vV^{(i_3)} : (u,v) \in \triangle_0 \},$$

where $i_1, i_2, i_3 \in \mathcal{I}_\mathcal{V}$ and

$$\triangle_0 := \{(u,v) \in \mathbb{R}^2 : u \in [0,1], \ v \leq 1 - u \},$$

whereas for all $i \in \mathcal{I}_\square$ we have

$$\overline{\Omega} = \{(1-u)(1-v)V^{(i_1)} + u(1-v)V^{(i_2)} + uvV^{(i_3)} + (1-u)vV^{(i_4)} : (u,v) \in \square_0 \},$$

where $i_1, i_2, i_3, i_4 \in \mathcal{I}_\mathcal{V}$ and $\square_0 = [0,1]^2$. We call such a collection of vertices, edges and triangles as well as quadrilateral elements a mixed triangle and quadrilateral mesh, or in short a mixed mesh.

We denote by $F^{(i)} : \triangle_0 \to \overline{\Omega}^{(i)}$, with $i \in \mathcal{I}_\triangle$, and $F^{(i)} : \square_0 \to \overline{\Omega}^{(i)}$, with $i \in \mathcal{I}_\square$, the parametrizations of the elements, which are linear mappings in case of triangles and bilinear in case of quadrilaterals. The parametrizations are always assumed to be regular. An example of a mixed mesh, together with the mappings $F^{(i)}$, is shown in Fig. 2.

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2.2. The Argyris-like space $\mathcal{A}_p$ over mixed meshes

Given an integer $p \geq 1$, let $\mathbb{P}_p$ be the space of univariate polynomials of degree $p$ on the unit interval $[0, 1]$, let $\mathbb{P}_p^{2}$ denote the space of bivariate polynomials of total degree $p$ on the parameter triangle $\triangle_0$ and let $\mathbb{P}_{p,p}^{2}$ be the space of bivariate polynomials of bi-degree $(p, p)$ on the unit square $\square_0$. We denote by $B^p_i$, $B^\triangle_{i,j,k}$ and $B^{p,p}_{i,j}$ the corresponding univariate, triangle and tensor-product Bernstein bases given by

$$B^p_i(u) = \binom{p}{i} u^i (1-u)^{p-i}, \quad i = 0, 1, \ldots, p,$$

$$B^\triangle_{i,j,k}(u,v) = \frac{p!}{i!j!k!} u^i v^j (1-u-v)^k, \quad i,j,k = 0, 1, \ldots, p, \quad i+j+k = p,$$

and

$$B^{p,p}_{i,j}(u,v) = B^p_i(u) B^p_j(v), \quad i,j = 0, 1, \ldots, p,$$

respectively.

We are interested in the construction and study of a particular super-smooth $C^1$ spline space on the mixed multi–patch domain $\Omega$ defined as

$$\mathcal{A}_p := \left\{ \varphi \in C^1(\overline{\Omega}) : \varphi \circ F^{(i)} \in \begin{cases} \mathbb{P}_p^2, & i \in \mathcal{I}_\Delta; \\ \mathbb{P}_{p,p}^2, & i \in \mathcal{I}_\square \end{cases}, \varphi \in C^2\left(\mathcal{V}^{(i)}\right), \varphi \mid_{E^{(i)}} \in \mathbb{P}_{p-1} \right\}. \quad (2)$$

Here $D_{n_i}\varphi$ denotes the derivative in the direction of the unit vector $n_i$ orthogonal to the edge $E^{(i)}$. In what follows, we denote the orthogonal vectors by $(x, y)^T := (y, -x)$. In case of a triangle mesh, that is $\mathcal{I}_\Delta = \mathcal{I}_2$, the space $\mathcal{A}_p$ corresponds to the classical Argyris triangle finite element space of degree $p$, cf. [1]. Therefore, we refer to the space $\mathcal{A}_p$ as the (mixed triangle and quadrilateral) Argyris-like space of degree $p$. Note that...
the space $\mathcal{A}_p$ was also discussed in [30] for the case of a quadrilateral mesh (i.e. $\mathcal{I}_\Omega = \mathcal{I}_6$) for $p = 5, 6$. Moreover, in [8] the corresponding quadrilateral element was introduced for $p \geq 6$. These constructions were summarized, and extended on the one hand to $p = 5$ and on the other hand to specific macro-elements for $p = 3, 4$ in [31].

3. Continuity conditions

We study the $C^1$ continuity conditions relating two neighboring elements from a mixed mesh. After presenting some general results, we devote attention to three specific cases of interest, i.e., a quadrilateral–triangle, triangle–triangle, and quadrilateral–quadrilateral join.

3.1. General conditions across the interface

Without loss of generality let two neighboring elements from the mixed mesh be denoted by $\Omega^{(1)}$ and $\Omega^{(2)}$ and parameterized by (bi)linear geometry mappings

$$\mathbf{F}^{(1)} : \mathcal{D}^{(1)} \to \Omega^{(1)} \quad \text{and} \quad \mathbf{F}^{(2)} : \mathcal{D}^{(2)} \to \Omega^{(2)},$$

where $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ denote \(\ominus_0\) or \(\ominus_4\). Suppose that $\Omega^{(1)}$ and $\Omega^{(2)}$ have a common interface $\mathcal{E}$ attained at $\mathbf{F}^{(1)}(0, v) = \mathbf{F}^{(2)}(0, v), \: v \in [0, 1]$. The graph $\Phi \subset (\overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}}) \times \mathbb{R}$ of any function

$$\varphi : \overline{\Omega^{(1)}} \cup \overline{\Omega^{(2)}} \to \mathbb{R}, \quad \varphi(x, y) = \begin{cases} \varphi^{(1)}(x, y), & (x, y) \in \overline{\Omega^{(1)}} \\ \varphi^{(2)}(x, y), & (x, y) \in \overline{\Omega^{(2)}} \end{cases},$$

is composed of two patches given by parameterizations

$$\Phi^{(1)} := \left[ F^{(1)} \right] : \mathcal{D}^{(1)} \to \mathbb{R}^3, \quad \Phi^{(2)} := \left[ F^{(2)} \right] : \mathcal{D}^{(2)} \to \mathbb{R}^3,$$

where $f^{(\ell)} = \varphi^{(\ell)} \circ \mathbf{F}^{(\ell)}, \: \ell = 1, 2$. Along the common interface the function $\varphi$ is $C^1$ continuous if and only if its graph $\Phi$ is $C^1$ continuous, i.e.

$$\Phi^{(1)}(0, v) = \Phi^{(2)}(0, v), \quad \det \left[ D_u \Phi^{(2)}(0, v), D_v \Phi^{(1)}(0, v), D_v \Phi^{(1)}(0, v) \right] = 0,$$

cf. [13, 18, 32]. Equivalently, it must hold that

$$f^{(1)}(0, v) = f^{(2)}(0, v), \quad (4)$$

$$\alpha_1(v) D_u f^{(2)}(0, v) - \alpha_2(v) D_u f^{(1)}(0, v) + \alpha_3(v) D_v f^{(1)}(0, v) = 0, (5)$$

where

$$\alpha_1(v) := \det JF^{(1)}(0, v), \quad \alpha_2(v) := \det JF^{(2)}(0, v), \quad \alpha_3(v) := \det \left[ D_u F^{(2)}(0, v), D_u F^{(1)}(0, v) \right]$$

are the so called "gluing functions" for the interface $\mathcal{E}$. Note that $\alpha_i(v) \neq 0$ for $v \in [0, 1], \: i = 1, 2$. The next two lemmas will further be needed. Their proofs follow by straightforward computations.

Lemma 1. For any two bijective and regular $C^1$ geometry mappings $\mathbf{F}^{(1)}$ and $\mathbf{F}^{(2)}$, such that

$$\mathbf{F}^{(1)}(0, v) = \mathbf{F}^{(2)}(0, v), \quad D_u \mathbf{F}^{(1)}(0, v) = D_u \mathbf{F}^{(2)}(0, v)$$

it holds that

$$\det J\mathbf{F}^{(2)}(0, v) \left( D_u \mathbf{F}^{(1)}(0, v) \right)^\perp - \det J\mathbf{F}^{(1)}(0, v) \left( D_u \mathbf{F}^{(2)}(0, v) \right)^\perp =$$

$$\det \left[ D_u \mathbf{F}^{(2)}(0, v), D_u \mathbf{F}^{(1)}(0, v) \right] \left( D_u \mathbf{F}^{(1)}(0, v) \right)^\perp. \quad (6)$$

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Lemma 2. Suppose that $f^{(l)} = \varphi^{(l)} \circ F^{(l)}$ and let $n$ be any chosen unit vector. The directional derivative $D_n \varphi^{(l)}(x,y)$ at a point $(x,y) = F^{(l)}(u,v)$ is in local coordinates equal to

\[
\omega_n^{(l)}(u,v) := \left\langle n, G^{(l)}(u,v) \right\rangle, \\
G^{(l)}(u,v) := \frac{1}{\det JF^{(l)}(u,v)} \left( D_u f^{(l)}(u,v) \left\langle D_v F^{(l)}(u,v) \right\rangle - D_v f^{(l)}(u,v) \left\langle D_u F^{(l)}(u,v) \right\rangle \right).
\]

Let us now choose a unit vector $n$ orthogonal to the common edge $E$. Multiplying (6) by $\langle n, \cdot \rangle$ we get that

\[
\alpha_3(v) = \alpha_2(v)\beta_1(v) - \alpha_1(v)\beta_2(v),
\]

where

\[
\beta_\ell(v) = \frac{1}{\beta(v)} \left\langle n, D_u F^{(\ell)}(0,v) \right\rangle, \quad \beta(v) := \left\langle n, \left\langle D_v F^{(1)}(0,v) \right\rangle \right\rangle = \left\langle n, \left\langle D_v F^{(2)}(0,v) \right\rangle \right\rangle.
\]

The assumption that parameterizations $F^{(1)}$ and $F^{(2)}$ are in $\mathbb{P}_2$ or in $\mathbb{P}_2^{1,1}$ implies that $\beta \in \mathbb{P}_0$ is a nonzero constant, the degree of $\alpha_1, \alpha_2, \beta_1, \beta_2$ is less or equal to 1, and the degree of $\alpha_3$ is less or equal to 2. By (8) the condition (5) rewrites to

\[
\alpha_1(v) \left( D_u f^{(2)}(0,v) - \beta_2(v)D_v f^{(1)}(0,v) \right) = \alpha_2(v) \left( D_u f^{(1)}(0,v) - \beta_1(v)D_v f^{(1)}(0,v) \right). 
\]

Further, using Lemma 2 one can see that the directional derivative $D_n \varphi^{(l)}$ at points on the boundary is in local coordinates equal to

\[
\omega_n^{(l)}(0,v) = \frac{\beta}{\alpha_\ell(v)} \left( D_u f^{(l)}(0,v) - \beta_\ell(v)D_v f^{(l)}(0,v) \right), \quad \ell = 1, 2.
\]

So, from (9) and (10) it follows that the $G^1$ continuity condition (9) is simply equal to

\[
\omega_n^{(1)}(0,v) = \omega_n^{(2)}(0,v), \quad v \in [0,1],
\]

cf. [42]. The additional assumption $D_n \varphi^{(l)}|_E \in \mathbb{P}_{p-1}$ implies that

\[
\omega_n^{(1)}(0,v) = \omega_n^{(2)}(0,v) = \sum_{j=0}^{p-1} d_j B_j^{p-1}(v), \quad v \in [0,1],
\]

for some coefficients $d_j$, $j = 0, 1, \ldots, p-1$.

In the following subsections we analyze the continuity conditions across the common edge in more detail for three different types of element pairs: quadrilateral–triangle, triangle–triangle and quadrilateral–quadrilateral. According to the simplified notation introduced in this section, we denote the common edge of $\Omega^{(1)}$ and $\Omega^{(2)}$ by $E = E(V^{(1)}, V^{(2)})$, and we choose $n$ as the vector orthogonal to $E$, i.e.,

\[
n = \left( V^{(2)} - V^{(1)} \right) / \| V^{(2)} - V^{(1)} \|.
\]

3.2. Quadrilateral–triangle

Suppose that $\Omega^{(1)}$ is a quadrilateral and $\Omega^{(2)}$ a triangle,

\[
\Omega^{(1)} = Q \left( V^{(1)}, V^{(2)}, V^{(3)}, V^{(4)} \right), \quad \Omega^{(2)} = T \left( V^{(5)}, V^{(2)}, V^{(1)} \right),
\]

and let the geometry mappings be equal to

\[
F^{(1)}(u,v) = (1-u)(1-v)V^{(1)} + (1-u)vV^{(2)} + uvV^{(3)} + u(1-v)V^{(4)}, \\
F^{(2)}(u,v) = uV^{(5)} + vV^{(2)} + (1-u-v)V^{(1)}.
\]

This configuration is visualized in Figure 3.
Remark 1. Note that the parameters of the geometry mapping $F^{(2)}$ define the triple $(u, v, 1 - u - v)$ which represent the barycentric coordinates of the point $F^{(2)}(u, v)$ with respect to the triangle $T \left( V^{(5)}, V^{(2)}, V^{(1)} \right)$.

The gluing functions $\alpha_1$, $\alpha_3$ are in this case linear polynomials $\alpha_i(v) = (1 - v) \alpha_{i,0} + v \alpha_{i,1}$, $i = 1, 3$, with coefficients

$$
\begin{align*}
\alpha_{1,0} &= \det \left[ V^{(4)} - V^{(1)}, V^{(2)} - V^{(1)} \right], \\
\alpha_{1,1} &= \det \left[ V^{(3)} - V^{(2)}, V^{(2)} - V^{(1)} \right], \\
\alpha_{3,0} &= \det \left[ V^{(5)} - V^{(1)}, V^{(4)} - V^{(1)} \right], \\
\alpha_{3,1} &= \det \left[ V^{(5)} - V^{(1)}, V^{(3)} - V^{(2)} \right],
\end{align*}
$$

while $\alpha_2$ reduces to a nonzero constant, $\alpha_2 = \det \left[ V^{(5)} - V^{(1)}, V^{(2)} - V^{(1)} \right]$. Moreover, from (10) we see that $\beta, \beta_1 \in \mathbb{P}_0$, $\beta_1 \in \mathbb{P}_1$,

$$
\beta = \| V^{(2)} - V^{(1)} \|, \quad \beta_2 = \frac{\left\langle V^{(2)} - V^{(1)}, V^{(5)} - V^{(1)} \right\rangle}{\| V^{(2)} - V^{(1)} \|^2},
$$

$$
\beta_1(v) = (1 - v) \beta_{1,0} + v \beta_{1,1}, \quad \beta_{1,0} = \frac{\left\langle V^{(2)} - V^{(1)}, V^{(4)} - V^{(1)} \right\rangle}{\| V^{(2)} - V^{(1)} \|^2}, \quad \beta_{1,1} = \frac{\left\langle V^{(2)} - V^{(1)}, V^{(3)} - V^{(2)} \right\rangle}{\| V^{(2)} - V^{(1)} \|^2}.
$$

Let $f^{(1)}$ be a bivariate polynomial of bi-degree $(p, p)$ and let $f^{(2)}$ be a bivariate polynomial of total degree $p$, expressed in a Bernstein basis as

$$
f^{(1)}(u, v) = \sum_{i,j=0}^{p} b^{(1)}_{i,j} B^p_i(u) B^p_j(v), \quad f^{(2)}(u, v) = \sum_{i+j+k=p} b^{(2)}_{i,j,k} B^p_{i,j,k}(u, v). \quad (14)
$$

It is straightforward to compute

$$
f^{(1)}(0, v) = \sum_{j=0}^{p} b^{(1)}_{0,j} B^p_j(v), \quad f^{(2)}(0, v) = \sum_{j=0}^{p} b^{(2)}_{0,j,p-j} B^p_j(v),
$$

$$
D_u f^{(1)}(0, v) = p \sum_{j=0}^{p} \left( b^{(1)}_{1,j} - b^{(1)}_{0,j} \right) B^p_j(v), \quad D_v f^{(1)}(0, v) = p \sum_{j=0}^{p-1} \left( b^{(1)}_{0,j+1} - b^{(1)}_{0,j} \right) B^{p-1}_j(v),
$$

$$
D_u f^{(2)}(0, v) = p \sum_{j=0}^{p-1} \left( b^{(2)}_{1,j,p-1-j} - b^{(2)}_{0,j,p-j} \right) B^{p-1}_j(v), \quad D_v f^{(2)}(0, v) = p \sum_{j=0}^{p-1} \left( b^{(2)}_{0,j+1,p-1-j} - b^{(2)}_{0,j,p-j} \right) B^{p-1}_j(v).
$$

Thus the condition (4) is fulfilled if

$$
b^{(1)}_{0,j} = b^{(2)}_{0,j,p-j} = c_j, \quad j = 0, 1, \ldots, p, \quad (15)
$$

Figure 3: A quadrilateral–triangle pair with parameter directions for $u$ (green) and $v$ (blue).
for any chosen values $c_j$. Since in this case $\alpha_2$ and $\beta_2$ are constants, we can see that $w^{(2)}_n(0, \cdot) \in \mathbb{P}_{p-1}$, so the condition $D_n \varphi^{(1)}(\cdot) \in \mathbb{P}_{p-1}$ is automatically fulfilled. Relations (10) and (11) imply

$$D_u f^{(1)}(0, v) - \beta_1(v) D_v f^{(1)}(0, v) = \alpha_1(v) \sum_{j=0}^{p-1} d_j B_{j}^{p-1}(v)$$

(16a)

and

$$D_u f^{(2)}(0, v) - \beta_2 D_v f^{(1)}(0, v) = \alpha_2 \sum_{j=0}^{p-1} d_j B_{j}^{p-1}(v),$$

(16b)

which together with (15), (16) and

$$\alpha_1(v) \sum_{j=0}^{p-1} d_j B_{j}^{p-1}(v) = \sum_{j=0}^{p-1} \frac{1}{p} ((p - j) \alpha_{1,0} d_j + j \alpha_{1,1} d_{j-1}) B_j^p(v),$$

$$\beta_1(v) D_v f^{(1)}(0, v) = \sum_{j=0}^{p} \left((p - j) \beta_{1,0} \left(b_{1,j+1}^{(1)} - b_{0,j}^{(1)}\right) + j \beta_{1,1} \left(b_{0,j}^{(1)} - b_{0,j-1}^{(1)}\right)\right) B_j^p(v),$$

determines the Bézier ordinates

$$b_{1,j}^{(1)} = c_j + \frac{1}{p} \left((p - j) \alpha_{1,0} d_j + j \alpha_{1,1} d_{j-1}\right) + \frac{1}{p} \left((p - j) \beta_{1,0} \left(b_{1,j+1}^{(1)} - b_{0,j}^{(1)}\right) + j \beta_{1,1} \left(b_{0,j}^{(1)} - b_{0,j-1}^{(1)}\right)\right)$$

(17)

for $j = 0, 1, \ldots, p$, and

$$b_{1,j,p-j}^{(2)} = c_j + \beta_2 \left(b_{j+1}^{(1)} - c_j\right) + \frac{1}{p} \alpha_2 d_j, \quad j = 0, 1, \ldots, p - 1,$$

(18)

where $c_{-1} := 0, c_{p+1} := 0$. We can summarize the obtained results in a following lemma.

**Proposition 1.** Assume that the two neighboring patches, corresponding geometry mappings and functions $f^{(1)}$, $f^{(2)}$ are given by (12), (13) and (14). Then the isoparametric function (3) is $C^1$ continuous across the common interface if and only if the control ordinates satisfy (15), (17) and (18) for any chosen $2p + 1$ coefficients $(c_i)_i=0^{p}$ and $(d_j)_j=0^{p-1}$.

### 3.3. Triangle–triangle

Suppose that $\Omega^{(1)}$ and $\Omega^{(2)}$ are both triangles,

$$\Omega^{(1)} = \mathcal{T} \left(V^{(3)}, V^{(2)}, V^{(1)}\right), \quad \Omega^{(2)} = \mathcal{T} \left(V^{(5)}, V^{(2)}, V^{(1)}\right),$$

(19)

and the geometry mappings equal

$$F^{(1)}(u, v) = u V^{(3)} + v V^{(2)} + (1 - u - v) V^{(1)}, \quad F^{(2)}(u, v) = u V^{(5)} + v V^{(2)} + (1 - u - v) V^{(1)}.$$  

(20)

This configuration is visualized in Figure 4.

The gluing functions $\alpha_i$, $i = 1, 2, 3$, as well as $\beta_1, \beta_2$ are in this case constants,

$$\alpha_1 = \det \left[ V^{(3)} - V^{(1)}, V^{(2)} - V^{(1)} \right], \quad \alpha_2 = \det \left[ V^{(5)} - V^{(1)}, V^{(2)} - V^{(1)} \right],$$

$$\alpha_3 = \det \left[ V^{(5)} - V^{(1)}, V^{(3)} - V^{(1)} \right],$$

$$\beta_1 = \frac{1}{\|V^{(2)} - V^{(1)}\|} \left\langle V^{(2)} - V^{(1)}, V^{(3)} - V^{(1)} \right\rangle, \quad \beta_2 = \frac{1}{\|V^{(2)} - V^{(1)}\|} \left\langle V^{(2)} - V^{(1)}, V^{(5)} - V^{(1)} \right\rangle.$$
Further, let
\[ f^{(\ell)}(u, v) = \sum_{i+j+k=p} b^{(\ell)}_{i,j,k} B_{i,j,k}(u, v), \quad \ell = 1, 2. \] (21)

Also in this case the condition \( D_{n}\varphi^{(\ell)}|_{E} \in P_{p-1} \) is fulfilled automatically and independently of the geometry of a mesh, and it is straightforward to derive the following result.

**Proposition 2.** Assume that the two neighboring patches, corresponding geometry mappings and functions \( f^{(1)}, f^{(2)} \) are given by (19), (20) and (21). Then the isoparametric function (3) is \( C^1 \) continuous across the common interface iff the control ordinates satisfy
\[
\begin{align*}
&b^{(1)}_{0,j,p-j} = b^{(2)}_{0,j,p-j} = c_j, \quad j = 0, 1, \ldots, p, \\
&b^{(\ell)}_{1,j,p-1-j} = c_j + \beta\ell (c_{j+1} - c_j) + \frac{1}{p}\alpha\ell d_j, \quad j = 0, 1, \ldots, p-1, \quad \ell = 1, 2,
\end{align*}
\]
for any chosen \( 2p + 1 \) coefficients \( c_i^p \) and \( d_i^{p-1} \).

### 3.4. Quadrilateral–quadrilateral

Suppose that \( \Omega^{(1)} \) and \( \Omega^{(2)} \) are both quadrilaterals,
\[
\Omega^{(1)} = Q(V^{(1)}, V^{(2)}, V^{(3)}, V^{(4)}), \quad \Omega^{(2)} = Q(V^{(1)}, V^{(2)}, V^{(5)}, V^{(6)}),
\] (22)
and the geometry mappings equal
\[
\begin{align*}
F^{(1)}(u, v) &= (1 - u)(1 - v)V^{(1)} + (1 - u)vV^{(2)} + uvV^{(3)} + u(1 - v)V^{(4)}, \\
F^{(2)}(u, v) &= (1 - u)(1 - v)V^{(1)} + (1 - u)vV^{(2)} + uvV^{(5)} + u(1 - v)V^{(6)}.
\end{align*}
\] (23)
This configuration is visualized in Figure 5.
Now, the functions $\alpha_i, \beta_i, i = 1, 2$, are linear polynomials

$$
\alpha_i(v) = (1 - v) \alpha_{i,0} + v \alpha_{i,1},
$$

$$
\alpha_{i,0} = \det \left[ V^{(2+2i)} - V^{(1)}, V^{(2)} - V^{(1)} \right], \quad \alpha_{i,1} = \det \left[ V^{(1+2i)} - V^{(2)}, V^{(2)} - V^{(1)} \right],
$$

$$
\beta_i(v) = (1 - v) \beta_{i,0} + v \beta_{i,1},
$$

$$
\beta_{i,0} = \frac{\left\langle V^{(2)} - V^{(1)}, V^{(2+2i)} - V^{(1)} \right\rangle}{\|V^{(2)} - V^{(1)}\|^2}, \quad \beta_{i,1} = \frac{\left\langle V^{(2)} - V^{(1)}, V^{(1+2i)} - V^{(2)} \right\rangle}{\|V^{(2)} - V^{(1)}\|^2},
$$

while $\alpha_3 \in \mathbb{P}_2$. Let $f^{(1)}$ and $f^{(2)}$ be two bivariate polynomials of bi-degree $(p, p)$,

$$
f^{(\ell)}(u, v) = \sum_{i,j=0}^p b_{i,j}^{(\ell)} B_i^p(u) B_j^p(v), \quad \ell = 1, 2. \tag{24}
$$

In this case it could happen that $D_n^{\varphi^{(\ell)}}|_{\mathcal{E}}$ would be of degree $p$, not $p - 1$. In particular, this can happen if $\alpha_\ell$ reduces to a constant, which, for $\ell = 1$, happens if $\mathcal{E}(V^{(1)}, V^{(2)})$ is parallel to $\mathcal{E}(V^{(3)}, V^{(4)})$, and for $\ell = 2$ if $\mathcal{E}(V^{(1)}, V^{(2)})$ is parallel to $\mathcal{E}(V^{(5)}, V^{(6)})$. Moreover, in certain configurations, $\alpha_1$ and $\alpha_2$ are linearly dependent, with $\alpha_2(v) = \lambda \alpha_1(v)$ and can thus be replaced by $\alpha_1' \equiv 1$ and $\alpha_2' \equiv \lambda$. E.g. if both elements are rectangles, we have constant $\alpha_\ell$ and $\beta_\ell \equiv 0$. See [3, 27] for a more detailed study of the possible cases. So, the additional condition $D_n^{\varphi^{(\ell)}}|_{\mathcal{E}} \in \mathbb{P}_{p-1}$ is included to make the proceeding construction independent of the geometry of the mesh.

**Proposition 3.** Assume that the two neighboring elements, corresponding geometry mappings and functions $f^{(1)}$, $f^{(2)}$ are given by (22), (23) and (24). Then the isoparametric function (3) is $C^1$ continuous across the common interface and satisfies the additional condition that $D_n^{\varphi^{(\ell)}}|_{\mathcal{E}} \in \mathbb{P}_{p-1}$, iff the control ordinates satisfy

$$
b_{0,j}^{(1)} = b_{0,j}^{(2)} = c_j,
$$

$$
b_{1,j}^{(\ell)} = c_j + \frac{1}{p} \left( \frac{1}{p} (p-j) \alpha_{\ell,0} d_j + j \alpha_{\ell,1} d_{j-1} + (p-j) \beta_{\ell,0} (c_{j+1} - c_j) + j \beta_{\ell,1} (c_j - c_{j-1}) \right), \quad \ell = 1, 2,
$$

$$
j = 0, 1, \ldots, p, \text{ for any chosen } 2p + 1 \text{ coefficients } (c_j)_{j=0}^{p} \text{ and } (d_i)_{i=0}^{p-1}.
$$

In Figure 6 we plot pairs of elements, triangle–quadrilateral (left), triangle–triangle (center) as well as quadrilateral–quadrilateral (right). For some coefficients $c_i$ and $d_i$ we plot the relevant, non-vanishing Bézier ordinates in blue and green, respectively. The figure is to be interpreted in the following way: If all coefficients $c_i$ and $d_i$ are set to zero, except for $c_1$, then only the Bézier ordinates depicted in blue are non-vanishing. On the other hand, if all coefficients except for $d_4$ are set to zero, then only the green ordinates are non-vanishing.

![Figure 6: Pairs of elements with non-vanishing Bézier ordinates for given coefficients $c_1$ (in blue) and $d_4$ (in green). Note that the structure of non-vanishing ordinates is always the same, only shifted by the given index. In the given configurations we have $p = 6$.](image-url)
As one can see in Figure 6, the $C^1$ functions across an interface couple degrees of freedom in a non-trivial way. The dimension of the $C^1$ space around a given vertex and the construction of a basis depends on the geometry, i.e. on the exact configuration of elements around the vertex. In order to simplify the construction, we demand $C^2$ continuity at vertices, thus fixing the dimension of the space and avoiding special cases. This strategy of imposing super-smoothness is a common tool for triangle meshes, see [36] or [30] for spline patches. This leads us to the interpolation problem as described in the following section.

4. Analysis of the Argyris-like space

In order to analyze the properties of the space $A_p$ (defined in (2)), we formulate an interpolation problem that uniquely characterizes the elements of the spline space. The interpolation problem provides the dimension formula for $A_p$ and gives rise to a projection operator that is used to prove the approximation properties of the space.

4.1. Interpolation problem

The following theorem states how the elements of $A_p$ can be described in terms of interpolation data provided at the vertices, along the edges and in the interior of the mixed mesh.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^2$ be an open domain on which a mixed mesh, satisfying (1), is defined. Then there exists a unique isoparametric spline function $\varphi \in A_p$ that satisfies the following interpolation conditions:

(A) For every vertex $V^{(i)}$, $i \in I_V$, let

$$D_x^a D_y^b \varphi (V^{(i)}) = \sigma_{i,a,b}, \quad 0 \leq a + b \leq 2,$$

for some given values $\sigma_{i,a,b} \in \mathbb{R}$.

(B) For every edge $E^{(i)}$, $i \in I_E$, choose $p - 5$ pairwise different points $R^{(i)}_{\ell} \in E^{(i)}$, $\ell = 1, 2, \ldots, p - 5$, as well as $p - 4$ pairwise different points $S^{(i)}_{\ell} \in E^{(i)}$, $\ell = 1, 2, \ldots, p - 4$, and let

$$\varphi (R^{(i)}_{\ell}) = \sigma_{i,\ell}, \quad \ell = 1, 2, \ldots, p - 5, \quad D_n \varphi (S^{(i)}_{\ell}) = w_{i,\ell}, \quad \ell = 1, 2, \ldots, p - 4,$$

for some given values $\sigma_{i,\ell}, w_{i,\ell} \in \mathbb{R}$.

(C) For every quadrilateral element $\Omega^{(i)}$, $i \in I_\Omega$, choose $(p - 3)^2$ pairwise different points

$$Q^{(i)}_{\ell,k} = F^{(i)} \left( \hat{Q}_{\ell,k}^{(i)} \right), \quad \ell, k = 1, 2, \ldots, p - 3,$$

where $\hat{Q}_{\ell,k}^{(i)}$ are unisolvent in $\text{span}_{2 \leq j_1, j_2 \leq p - 2} (B_{j_1,j_2}^{p,p})$, and let

$$\varphi (Q^{(i)}_{\ell,k}) = \sigma^{(\triangle)}_{\ell,k}, \quad \ell, k = 1, 2, \ldots, p - 3,$$

for some given values $\sigma^{(\triangle)}_{\ell,k} \in \mathbb{R}$.

(D) For every triangular element $\Omega^{(i)}$, with $i \in I_\Omega$, choose $\binom{p - 3}{2}$ pairwise different points

$$Q^{(i)}_{\ell,k} = F^{(i)} \left( \hat{Q}_{\ell,k}^{(i)} \right), \quad \ell = 1, 2, \ldots, p - 5, \quad k = 1, 2, \ldots, p - 4 - \ell,$$

where $\hat{Q}_{\ell,k}^{(i)}$ are unisolvent in $\text{span}_{2 \leq j_1, j_2, j_3} (B_{j_1,j_2,j_3}^{p,p})$, and let

$$\varphi (Q^{(i)}_{\ell,k}) = \sigma^{(\triangle)}_{\ell,k}, \quad \ell = 1, 2, \ldots, p - 5, \quad k = 1, 2, \ldots, p - 4 - \ell,$$

for some given values $\sigma^{(\triangle)}_{\ell,k} \in \mathbb{R}$.
Proof. We need to show that the interpolation conditions \((A)\)–\((D)\) uniquely determine the bivariate polynomial on every patch \(\Omega^{(i)} = F^{(i)}(D^{(i)})\), \(i \in \mathcal{I}_0 \cup \mathcal{I}_1\), and that the continuity conditions are satisfied. Let \(V^{(j)}\) be a vertex of \(\Omega^{(i)}\), obtained as \(V^{(j)} = F^{(i)}(u_j^{(i)}, v_j^{(i)})\) for some

\[
u_j^{(i)} := (u_j^{(i)}, v_j^{(i)}) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}.
\]

From conditions in \((A)\) we get

\[
\text{grad} \, \varphi \left( V^{(j)} \right) = (\sigma_{j,1,0}, \sigma_{j,0,1}), \quad H \varphi \left( V^{(j)} \right) = \begin{bmatrix} \sigma_{j,2,0} & \sigma_{j,1,1} \\ \sigma_{j,1,1} & \sigma_{j,0,2} \end{bmatrix},
\]

and from

\[
\text{grad} f^{(i)} \left( u_j^{(i)} \right) = \text{grad} \, \varphi \left( V^{(j)} \right) \cdot J F^{(i)} \left( u_j^{(i)} \right) =: \left( s_{j,1,0}^{(i)}, s_{j,0,1}^{(i)} \right) =: s_j^{(i)},
\]

\[
H f^{(i)} \left( u_j^{(i)} \right) = J F^{(i)} \left( u_j^{(i)} \right) \cdot H \varphi \left( V^{(j)} \right) \cdot J F^{(i)} \left( u_j^{(i)} \right) + D_s \varphi \left( V^{(j)} \right) H F_1^{(i)} \left( u_j^{(i)} \right) + D_y \varphi \left( V^{(j)} \right) H F_2^{(i)} \left( u_j^{(i)} \right) =: \begin{bmatrix} s_{j,2,0}^{(i)} \\ s_{j,1,1}^{(i)} \\ s_{j,0,2}^{(i)} \end{bmatrix} =: s_j^{(i)},
\]

we obtain the \(C^2\) interpolation conditions for \(f^{(i)}\) at \(u_j^{(i)}\), i.e.,

\[
D_x^a D_y^b f^{(i)} \left( u_j^{(i)} \right) = s_{j,a,b}^{(i)}, \quad 0 \leq a + b \leq 2.
\]

Further, let \(E^{(k)}, k \in \mathcal{I}_E\), be any edge of \(\Omega^{(i)}\), with boundary vertices \(V^{(k_0)}, V^{(k_1)}, k_0, k_1 \in \mathcal{I}_V\), parameterized as

\[
E^{(k)} = \left\{ F^{(i)} \left( \epsilon^{(k)}(t) \right) : t \in (0, 1) \right\}, \quad \epsilon^{(k)}(t) := (1 - t) u^{(i)}_{k_0} + t u^{(i)}_{k_1},
\]

and let

\[
\theta_k(t) := \sum_{\ell=0}^{p} c_{\ell}^{(k)} B^{\ell}_p(t) = f^{(i)} \left( \epsilon^{(k)}(t) \right), \quad \omega_k(t) := \sum_{\ell=0}^{p-1} d_{\ell}^{(k)} B^{\ell+1}_p(t) = D_n \varphi \left( F^{(i)} \left( \epsilon^{(k)}(t) \right) \right), \quad t \in [0, 1],
\]

be the restriction of \(\varphi\) and \(D_n \varphi\) on the edge \(E^{(k)}\) expressed in local coordinates. Further, let \(t^{(i)}_{k,\ell}\) be the parameters, such that \(R_{k,\ell} = F^{(i)} \left( \epsilon^{(k)} \left( t^{(i)}_{k,\ell} \right) \right)\). From \((A)\) and \((B)\) we get \(p + 1\) conditions

\[
\theta_k(\ell) = \sigma_{k,0,0}, \quad \theta'_k(\ell) = \left( s_{k,0}^{(i)}, u^{(i)}_{k_0} - u^{(i)}_{k_1} \right), \quad \theta''_k(\ell) = \left( u^{(i)}_{k_1} - u^{(i)}_{k_0}, \sigma_{k,t}^{(i)} \left( u^{(i)}_{k_1} - u^{(i)}_{k_0} \right)^T \right), \quad \ell = 0, 1,
\]

\[
\theta_k \left( t^{(i)}_{k,\ell} \right) = \sigma_{k,\ell}, \quad \ell = 1, 2, \ldots, p - 5,
\]

which uniquely determine \(\theta_k\). Note that \(u^{(i)}_{k_1} - u^{(i)}_{k_0} \in \{(-1, 0), (0, 1), (0, -1)\}\). From \((7)\) it is straightforward to see that \((A)\) and \((B)\) give also the values of \(G^{(k)}, D_n G^{(k)}\) and \(D_n G^{(k)}\) at \(u^{(i)}_{k_0}, u^{(i)}_{k_1}\). The conditions

\[
\omega_k(\ell) = \left( n_k, G^{(k)} \left( u^{(i)}_{k_0} \right) \right), \quad \ell = 0, 1,
\]

\[
\omega'_k(\ell) = \left( u^{(i)}_{k_1} - u^{(i)}_{k_0} \right) \left( n_k, D_n G^{(k)} \left( u^{(i)}_{k_0} \right) \right) + \left( v^{(i)}_{k_1} - v^{(i)}_{k_0} \right) \left( n_k, D_n G^{(k)} \left( u^{(i)}_{k_0} \right) \right), \quad \ell = 0, 1,
\]

\[
\omega_k \left( t^{(i)}_{k,\ell} \right) = w_{k,\ell}, \quad \ell = 1, 2, \ldots, p - 4,
\]

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then uniquely determine \( w_k \).

Suppose now that \( i \in \mathcal{I}_2 \), and \( f^{(i)}(u,v) = \sum_{j=0}^{p} b_{j,\ell}^{(i)} B_j^p(u) B_\ell^p(v) \). Following Proposition 1, 2 and 3 we see that polynomials \( \theta_k \) and \( w_k \) uniquely determine the ordinates \( b_{j,\ell}^{(i)} \) and \( \tilde{R}_{\ell}^{(i)} \) for \( \ell = 0, 1, p - 1, p, j = 0, 1, \ldots, p \).

The remaining \((p - 3)\) ordinates \( b_{j,\ell}^{(i)}, j, \ell = 2, 3, \ldots, p - 2 \), are computed uniquely from conditions (C).

Similarly for \( i \in \mathcal{I}_3 \) and \( f^{(i)}(u,v) = \sum_{j=0}^{p} b_{j,\ell,r}^{(i)} B_j^{s+1} B_\ell^{s+1} B_r^{p}(u,v) \). Polynomials \( \theta_k \) and \( \omega_k \) uniquely determine the ordinates \( b_{j,\ell,r}^{(i)} \) with \( j, \ell, r \in \{p, p - 1\} \), while the remaining ones follow from conditions (D). Since the continuity conditions are satisfied by the construction, the proof is completed.

One possible choice for the interpolation points, which we further use in the examples, is the following.

For a given edge \( \mathcal{E}^{(i)} = E(V^{(1)}, V^{(2)}) \) we first compute an equidistant set of points

\[
\tilde{R}_{\ell} = \frac{2}{2} - \ell \left( \frac{p - 2}{p} V^{(1)} + \frac{2}{p} V^{(2)} \right) + \frac{\ell}{2} \left( \frac{2}{p} V^{(1)} + \frac{2}{p} V^{(2)} \right), \quad \ell = 1, 2, \ldots, \left\lfloor \frac{p}{2} \right\rfloor - 3.
\]

Then for odd degree \( p \) we choose

\[
R_{\ell}^{(i)} := \tilde{R}_{\ell}, \quad \ell = 1, \ldots, \frac{p - 5}{2}, \quad R_{\ell}^{(i)} := \tilde{R}_{\ell + 1}, \quad \ell = \frac{p - 3}{2}, \ldots, p - 5,
\]

and for even \( p \)

\[
R_{\ell}^{(i)} := \tilde{R}_{\ell}, \quad \ell = 1, \ldots, \frac{p - 2}{2}, \quad R_{\ell}^{(i)} := \tilde{R}_{\ell + 2}, \quad \ell = \frac{p - 2}{2}, \ldots, p - 5,
\]

\[
S_{\ell}^{(i)} := \tilde{R}_{\ell}, \quad \ell = 1, \ldots, \frac{p - 4}{2}, \quad S_{\ell}^{(i)} := \tilde{R}_{\ell + 1}, \quad \ell = \frac{p - 2}{2}, \ldots, p - 4.
\]

Additional interpolation points in the interior are chosen as

\[
Q_{\ell,k}^{(i)} := F^{(i)} \left( \frac{\ell}{p}, \frac{k}{p} \right) \in \Omega^{(i)}, \quad \ell, k = 2, \ldots, p - 2,
\]

for quadrilaterals and

\[
Q_{\ell,k}^{(i)} := F^{(i)} \left( \frac{\ell}{p}, \frac{k}{p} \right) \in \Omega^{(i)}, \quad \ell = 2, \ldots, p - 2, \quad k = 2, \ldots, p - 2 - \ell,
\]

for triangles. For the graphical interpretation of these interpolation points in the case \( p = 8 \) and \( p = 9 \) see Fig. 11.

4.2. Properties of the space \( \mathcal{A}_p \)

Due to the interpolation conditions (A)–(D), we have the following dimension formula.

**Corollary 1.** The dimension of the space \( \mathcal{A}_p \) equals

\[
\dim \mathcal{A}_p = 6 |\mathcal{I}_V| + (2p - 9) |\mathcal{I}_E| + (p - 3)^2 |\mathcal{I}_S| + \left( \frac{p - 4}{2} \right)^2 |\mathcal{I}_D|.
\]

Moreover, the space \( \mathcal{A}_p \) contains bivariate polynomials of total degree \( p \).

**Lemma 3.** We have \( \mathbb{P}^2_p \subset \mathcal{A}_p \).

This lemma follows directly from the definition of the space, hence the local space for triangles is equal to \( \mathbb{P}^2_p \), whereas it contains \( \mathbb{P}^2_p \) for quadrilaterals (see [31]). Another useful consequence of Theorem 1 is the following: Based on this theorem we define the global projection operator

\[
\mathcal{Q}_p : C^2(\overline{\mathcal{M}}) \rightarrow \mathcal{A}_p
\]
that assigns to every function \( f \in C^2(\Omega) \) the \( C^1 \)-spline \( Q_p f \in A_p \) that satisfies the interpolation conditions (A)–(D) for data sampled from the function \( f \). We can moreover define the local projection operators

\[
Q_p^{(i)} : C^2(\Omega^{(i)}) \rightarrow A_p|_{\Gamma^{(i)}}.
\]

By definition we have

\[
(Q_p \varphi)|_{\Gamma^{(i)}} = Q_p^{(i)} \left( \varphi|_{\Gamma^{(i)}} \right).
\]

The operator \( Q_p \) is bounded, if the elements \( \Omega^{(i)} \) of the mesh are shape regular. Given a shape regularity constant \( C_{SR} \), we say that an element is shape regular, if

\[
1 \leq \frac{\max |\det \nabla \bf{F}^{(i)}|}{\min |\det \nabla \bf{F}^{(i)}|} \leq C_{SR}
\]

as well as

\[
1 \leq \frac{\max_{1 \leq i, j \leq \nu} |\bf{V}^{(i)} - \bf{V}^{(j)}|}{\min_{1 \leq i < j \leq \nu} |\bf{V}^{(i)} - \bf{V}^{(j)}|} \leq C_{SR},
\]

where \( \nu = 3 \) and \( \Omega^{(i)} = T(\bf{V}^{(1)}, \bf{V}^{(2)}, \bf{V}^{(3)}) \) for triangles and \( \nu = 4 \) and \( \Omega^{(i)} = Q(\bf{V}^{(1)}, \bf{V}^{(2)}, \bf{V}^{(3)}, \bf{V}^{(4)}) \) for quadrilaterals. As a consequence, for shape regular elements all angles are bounded between \( \epsilon \) and \( \pi - \epsilon \), where \( \epsilon > 0 \) depends only on \( C_{SR} \). In the following we denote by \( |\cdot|_{H^\ell(D)} \) the \( H^\ell \)-seminorm, where \( |\cdot|^2_{H^\ell(D)} \) is the sum of squares of \( L^2 \)-norms of all derivatives of order \( \ell \) over the domain \( D \), and by

\[
|||\varphi|||_{H^m(D)} = \left( \sum_{\ell=0}^m |\varphi|^2_{H^\ell(D)} \right)^{\frac{1}{2}}
\]

the \( H^m \)-norm. We have the following lemma for shape regular elements.

**Lemma 4.** Let \( \Omega^{(i)} \) be an element with \( \text{diam}(\Omega^{(i)}) = 1 \) and let \( R^{(i)} \), \( S^{(i)} \) and \( Q^{(i)} \) be defined by (25)–(28). Then the local projector satisfies

\[
|||Q_p^{(i)} \varphi|||_{H^2(\Omega^{(i)})} \leq \sigma |||\varphi|||_{C^2(\Gamma^{(i)})},
\]

as well as

\[
|||Q_p^{(i)} \varphi|||_{L^\infty(\Omega^{(i)})} \leq \sigma |||\varphi|||_{C^2(\Gamma^{(i)})},
\]

where \( \sigma \) depends only on the degree \( p \) and on the shape regularity of \( \Omega^{(i)} \) and where \( |||\varphi|||_{C^2(\Gamma^{(i)})} \) takes the supremum of all derivatives up to second order on the element \( \Omega^{(i)} \).

**Proof.** A proof of this lemma can be found in [7] for triangles and in [31] for quadrilaterals and \( p = 5 \). The main idea is that the degrees of freedom determine functions that are bounded in \( H^2 \) and \( L^\infty \). For this, the interpolation points need to be chosen such that the interpolation procedure is stable, e.g., by (25)–(28). Note that the extension to higher degrees follows in the same way. \( \square \)

**Theorem 2.** Let \( \Omega^{(i)} \) be a shape-regular element of the mesh on \( \Omega \), let \( 0 \leq \ell \leq 2 \) and \( 4 \leq m \leq p + 1 \). There exists a constant \( C > 0 \) such that we have for all \( \varphi \in H^m(\Omega^{(i)}) \)

\[
|\varphi - Q_p^{(i)} \varphi|_{H^\ell(\Omega^{(i)})} \leq C h_i^{m-\ell} |\varphi|_{H^m(\Omega^{(i)})},
\]

where \( h_i = \text{diam}(\Omega^{(i)}) \). The constant \( C \) depends on the shape regularity of \( \Omega^{(i)} \) and on \( p \). We moreover have

\[
|||\varphi - Q_p^{(i)} \varphi|||_{L^\infty(\Omega^{(i)})} \leq C h_i^m |\varphi|_{W^{m,\infty}(\Omega^{(i)})},
\]

for all \( \varphi \in W^{m,\infty}(\Omega^{(i)}) \), where \( |||\cdot|||_{W^{m,\infty}(D)} \) takes the essential supremum of all derivatives of order \( m \) over \( D \).
This is a simple consequence of the boundedness of the local projector, see [7, 11]. From this local error estimate we derive the following global estimate.

**Corollary 2.** We assume to have a shape-regular, mixed mesh on \( \Omega \). Let \( 0 \leq \ell \leq 2 \) and \( 4 \leq m \leq p + 1 \). There exists a constant \( C > 0 \), depending on the degree \( p \) and on the shape regularity constant \( C_{SR} \), such that we have for all \( \varphi \in H^m(\Omega) \)

\[
|\varphi - Q_p \varphi|_{H^\ell(\Omega)} \leq C h^{m-\ell} |\varphi|_{H^m(\Omega)},
\]

as well as for all \( \varphi \in W^{m,\infty}(\Omega) \)

\[
||\varphi - Q_p \varphi||_{L^{\infty}(\Omega)} \leq C h^m |\varphi|_{W^{m,\infty}(\Omega)}.
\]

Here \( h \) denotes the length of the longest edge of the mesh.

Examples of interpolants \( Q_p f \) together with numerical observation of the approximation order are provided in the next section.

5. Numerical examples

This section provides some numerical examples that confirm the derived theoretical results. In particular, the super-smooth \( C^1 \) Argyris-like space \( A_p \) is employed for three different applications on several mixed triangle and quadrilateral meshes. The first two applications are the interpolation of a given function, based on the projection operator \( Q_p \), and the \( L^2 \)-approximation. Both applications numerically verify the optimal approximation properties of the Argyris-like space \( A_p \). Thirdly, we solve a particular fourth order PDE given by the biharmonic equation, which requires the use of globally \( C^1 \) functions for solving the PDE via its weak form and Galerkin discretization.

5.1. Mixed meshes, refinement strategy & super-smooth \( C^1 \) spline spaces

We consider the three mixed triangle and quadrilateral meshes shown in Fig. 7–9 (first column) denoted by Mesh 1–3. The three mixed meshes are refined by splitting each triangle and each quadrilateral of the corresponding mesh into four triangles and into four quadrilaterals, respectively, as visualized in Fig. 10. As an example, the resulting refined meshes for the third level of refinement (i.e. for Level 3) are presented in Fig. 7–9 (second column). We further construct for all three meshes Argyris-like spaces \( A_p \) as described in the previous section for the levels of refinement \( L = 0, 1, \ldots , 5 \), and denote the resulting super-smooth \( C^1 \) spline spaces by \( A_{p,h} \), where \( h = O(2^{-L}) \) is the length of the longest edge in the mesh.

![Figure 7: Mesh 1 – mixed triangle and quadrilateral mesh for the initial level of refinement (i.e. Level 0) with the coordinates of the vertices and for the third level of refinement (i.e. Level 3).](image-url)
5.2. Interpolation

To test the interpolation error we choose a smooth function

\[ f(x) = f(x, y) = 4 \cos \left( \frac{2x}{3} \right) \sin \left( \frac{2y}{3} \right) \] (29)

and compute the interpolants \( Q_p f \) for degrees \( p = 5, 6, \ldots, 10 \) on mixed meshes Mesh 1–3. Fig. 11 schematically presents the interpolation data we have used for degrees \( p = 8 \) and \( p = 9 \) when constructing interpolants over Mesh 1. Interpolating splines \( Q_8 f \) over all three meshes are shown in Fig. 12. To compare between interpolants of different degrees we use the \( L^\infty \)-error \( \| f - Q_p f \|_\infty \), which we compute numerically by evaluating in \( 51^2 = 2601 \) and \( \binom{52}{2} = 1326 \) uniformly spaced points on every quadrilateral and triangle respectively. The errors for degrees \( p = 5, 6, \ldots, 10 \) for all three meshes are given in Table 1.

Further, to numerically observe the approximation order, we compute interpolants \( Q_p f \) over meshes with different refinement levels. Let us denote the \( L^\infty \)-error of the interpolant of degree \( p \) on level \( L \) by \( \text{err}_{p,L} \). For Mesh 1, these values are shown in Table 2 together with the estimated decay exponents \( \gamma_{p,L} \). The values in Table 2 numerically confirm that the approximation order for splines of degree \( p \) is optimal, i.e. \( p + 1 \). Similar results hold true for the other two example meshes as one can see in Fig. 13, where the errors of interpolants of different degrees over different refinement levels are plotted in log–log-scale in dependence on the number of degrees of freedom.
5.3. $L^2$-approximation

We use the constructed super-smooth $C^1$ spline spaces $A_{5,h}$ to approximate in a least-squares sense the function (29) on meshes Mesh 1–3. That is, we compute in each case that function $f_h \in A_{5,h}$, which minimizes the objective function

$$\int_{\Omega} (f_h(x) - f(x))^2 \, dx.$$ 

Fig. 14 (second row) reports the resulting $L^\infty$-errors as well as the resulting $L^2$-errors for different levels of refinement with respect to the number of degrees of freedom (NDOF). The obtained results indicate that both errors decrease with rates of optimal order of $O(h^6)$, which numerically verify the optimal approximation power of the super-smooth $C^1$ Argyris-like space $A_5$.

<table>
<thead>
<tr>
<th>degree $p$</th>
<th>Mesh 1</th>
<th>Mesh 2</th>
<th>Mesh 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$4.22291 \cdot 10^{-1}$</td>
<td>$1.01635 \cdot 10^{-1}$</td>
<td>$3.17295 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>6</td>
<td>$9.72889 \cdot 10^{-2}$</td>
<td>$3.98448 \cdot 10^{-3}$</td>
<td>$1.16293 \cdot 10^{-2}$</td>
</tr>
<tr>
<td>7</td>
<td>$4.65926 \cdot 10^{-3}$</td>
<td>$6.25145 \cdot 10^{-4}$</td>
<td>$1.91039 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>8</td>
<td>$1.99952 \cdot 10^{-3}$</td>
<td>$3.25673 \cdot 10^{-5}$</td>
<td>$1.17307 \cdot 10^{-4}$</td>
</tr>
<tr>
<td>9</td>
<td>$8.47799 \cdot 10^{-5}$</td>
<td>$6.50782 \cdot 10^{-6}$</td>
<td>$2.27334 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>10</td>
<td>$4.27483 \cdot 10^{-5}$</td>
<td>$2.77481 \cdot 10^{-7}$</td>
<td>$1.27041 \cdot 10^{-6}$</td>
</tr>
</tbody>
</table>

Table 1: Table of errors $\|f - Q_p f\|_\infty$ of interpolants of different degrees over meshes Mesh 1–3.
5.4. Solving the biharmonic equation

We solve a particular fourth order PDE, namely the biharmonic equation

\[
\begin{align*}
\Delta^2 u(x) &= g(x) \quad x \in \Omega \\
u(x) &= g_1(x) \quad x \in \partial\Omega \\
\frac{\partial u}{\partial n}(x) &= g_2(x) \quad x \in \partial\Omega
\end{align*}
\]  

(30)

on the Meshes 1–3 by employing the super-smooth \( C^1 \) spline spaces \( A_{5,h} \) as discretization spaces. For all three meshes, the functions \( g, g_1 \) and \( g_2 \) are derived from the same exact solution (29) as before for the case of \( L^2 \)-approximation. The biharmonic equation (30) is solved via its weak form and Galerkin projection by at first strongly imposing the Dirichlet boundary conditions to the numerical solution \( u_h \in A_{5,h} \). The resulting relative \( L^2 \)-, \( H^1 \)- and \( H^2 \)-errors for the different levels of refinement, again with respect to the number of degrees of freedom, are shown in Fig. 14 (third row). The estimated convergence rates are for all examples of optimal order of \( O(h^6), O(h^5) \) and \( O(h^4) \) with respect to the \( L^2 \)-, \( H^1 \)- and \( H^2 \)-norm, respectively.
Figure 13: $L^\infty$-errors of interpolants of different degrees $p$ over meshes Mesh 1–3 with different refinement levels. Errors are shown in log–log-scale in dependence on the number of degrees of freedom.

Figure 14: Performing $L^2$-approximation and solving the biharmonic equations for the exact solutions (29) on the Meshes 1–3 from Fig. 7–9 and the resulting errors.
6. Conclusions

We studied a construction of $C^1$ splines over mixed triangle and quadrilateral meshes for polynomial degrees $p \geq 5$. The degrees of freedom are given by $C^2$-data at the vertices, point data and normal derivative data at suitable points along the edges as well as additional point data in the interior of the elements. The resulting space is $C^1$ globally and $C^2$ at all vertices. The degrees of freedom define a stable projection operator which, together with the local polynomial reproduction, yields optimal approximation error bounds with respect to the mesh size for $L^\infty$, $L^2$ as well as Sobolev norms $H^1$ and $H^2$.

In this paper we only considered planar (bi)linear elements. Extensions to domains with curved boundaries or to surface domains were already discussed separately in case of quadrilateral meshes as well as triangle meshes. For quadrilateral meshes, [3, 37] provided extensions to elements with curved boundaries, i.e. elements where one boundary edge is curved and the other three are straight. Extensions to surface domains were briefly discussed for spline patches in [13, 28]. Constructions of $C^1$ surfaces of arbitrary topology using triangle meshes where developed in [19]. However, to work out the details of such extensions in the mixed case requires further studies which we intend to do in the future.

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References

Spline functions on triangulations


